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# ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF THE LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRIX

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**1. Introduction.** In our previous paper [14], we have proved the unbiasedness of the modified likelihood ratio (= modified LR) tests (i) for the equality of covariance matrix to a given matrix and (ii) for the equality of two covariance matrices. We have also shown the unbiasedness of the LR tests (iii) for sphericity and (iv) for the equality of mean vector and covariance matrix to a given vector and matrix.

In this paper asymptotic expansions of the distributions of the test criteria for (i) and (iv) both under hypothesis and alternatives are derived, by inverting the characteristic function directly. The asymptotic expansion of the non-null distribution of the LR criterion for sphericity (iii), is obtained by using the differential operator due to Welch [15], and also the limiting non-null distribution of the LR test for the equality of $k$ covariance matrices is derived in a similar way. This method has been shown to be useful in other problems in multivariate analysis by Ito [5], Siotani [11], Okamoto [8], and others.

All the limiting non-null distributions of these test criteria are shown to be normal distributions, whereas the limiting distributions under hypothesis are $\chi^2$–distributions as in Box [2] or Anderson ([1], Chapter 10). It may be interesting to note that the limiting non-null distribution of the likelihood ratio criterion for the multivariate linear hypothesis is noncentral $\chi^2$, the asymptotic expansion of which was obtained by Sugiura and Fujikoshi [13].

**2. Expansion of the distribution of the criterion for $\Sigma = \Sigma_0$.** Let $p \times 1$ vectors $X_1, \cdots, X_N$ be a random sample from a $p$-variate normal distribution with unknown mean vector $\mu$ and covariance matrix $\Sigma$ (positive definite). The LR criterion for testing the hypothesis $H_1 : \Sigma = \Sigma_0$ against the alternatives $K_1 : \Sigma \neq \Sigma_0$, for some given positive definite matrix $\Sigma_0$, is given by

$$\lambda = (e/N)^{Np/2} |S\Sigma_0^{-1}|^{N/2} \operatorname{etr} \{-(\tfrac{1}{2})\Sigma_0^{-1} S\}, \tag{2.1}$$

where etr means exp tr and $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$, $\bar{X} = (1/N) \sum_{\alpha=1}^{N} X_\alpha$. This LR test is not unbiased. However, if we modify this criterion by reducing the sample size $N$ to the degrees of freedom $n = N - 1$, it has some desirable property, that is, the unbiasedness is shown by Sugiura and Nagao [14] and the monotonicity of the power function with respect to $p$ characteristic roots of $\Sigma\Sigma_0^{-1}$ is established by Nagao [7] and Das Gupta [4].

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So we shall consider the asymptotic expansion of the modified LR statistic $\lambda^*$ instead of $\lambda$.

$$\lambda^* = (e/n)^{np/2}|S\Sigma_0^{-1}|^{n/2} \operatorname{etr}\{-(\tfrac{1}{2})\Sigma_0^{-1}S\}. \tag{2.2}$$

The limiting distribution of the statistic $-2\log\lambda^*$ under the hypothesis is the $\chi^2$-distribution with $p\,(p+1)/2$ degrees of freedom, which can be seen in Anderson ([1], page 267). The $h$th moment of the statistic $\lambda^*$ under alternative $K_1$ is given by

$$E[\lambda^{*h} \mid K_1] = (2e/n)^{nhp/2}[\Gamma_p(n(1+h)/2)/\Gamma_p(\tfrac{1}{2}n)] \cdot |\Sigma\Sigma_0^{-1}|^{nh/2}/|I + h\Sigma\Sigma_0^{-1}|^{n\,(1+h)/2}, \tag{2.3}$$

where $\Gamma_p(x) = \pi^{p\,(p-1)/4}\prod_{\alpha=1}^{p}\Gamma(x-(\alpha-1)/2)$. Hence the characteristic function of $-2\log\lambda^*$ under the hypothesis $H_1$ is expressed as

$$C_{H_1}(t) = (\tfrac{1}{2}n/e)^{itpn}[\Gamma_p(n(1-2it)/2)/\Gamma_p(\tfrac{1}{2}n)](1-2it)^{-np\,(1-2it)/2}. \tag{2.4}$$

We shall use the following asymptotic formula for the gamma function as in Box [2] or Anderson ([1] page 204).

$$\log\Gamma(x+h) = \log(2\pi)^{\frac{1}{2}} + (x+h-\tfrac{1}{2})\log x - x - \textstyle\sum_{r=1}^{k}(-1)^r B_{r+1}(h)/[r(r+1)x^r] + O(x^{-k-1}). \tag{2.5}$$

This holds for large values of $|x|$ with fixed $h$. The Bernoulli polynomial $B_r(h)$ of degree $r$ is given by $\tau e^{h\tau}/(e^\tau - 1) = \sum_{r=0}^{\infty}(\tau^r/r\,!)B_r(h)$. Some of these are listed below;

$$\begin{aligned} &B_1(h) = h - \tfrac{1}{2} && B_3(h) = h^3 - \tfrac{3}{2}h^2 + \tfrac{1}{2}h \\ &B_2(h) = h^2 - h + \tfrac{1}{6} && B_4(h) = h^4 - 2h^3 + h^2 - \tfrac{1}{30}. \end{aligned} \tag{2.6}$$

Applying formula (2.5) to each gamma function in (2.4), we get

$$\log C_{H_1}(t) = -\tfrac{1}{4}p(p+1)\log(1-2it) - \textstyle\sum_{r=1}^{k}(-2)^r B_{r+1}/[r(r+1)n^r]\{(1-2it)^{-r} - 1\} + O(n^{-k-1}), \tag{2.7}$$

where $B_{r+1} = \sum_{\alpha=1}^{p} B_{r+1}((1-\alpha)/2)$. This formula implies the asymptotic expansion of the characteristic function of $-2\log\lambda^*$.

$$\begin{aligned} C_{H_1}(t) = (1-2it)^{-p\,(p+1)/4}[1 &+ B_2 n^{-1}\{(1-2it)^{-1} - 1\} \\ &+ (\tfrac{1}{6})n^{-2}\{(3B_2^2 - 4B_3)(1-2it)^{-2} - 6B_2^2(1-2it)^{-1} \\ &+ (3B_2^2 + 4B_3)\} \\ &+ (\tfrac{1}{6})n^{-3}\{(4B_4 - 4B_2B_3 + B_2^3)(1-2it)^{-3} \\ &+ B_2(4B_3 - 3B_2^2)(1-2it)^{-2} + B_2(4B_3 + 3B_2^2)(1-2it)^{-1} \\ &- (4B_4 + 4B_2B_3 + B_2^3)\}] + O(n^{-4}), \end{aligned} \tag{2.8}$$

where

$$(2.9)\qquad \begin{aligned} B_2 &= p(2p^2 + 3p - 1)/24 \\ B_3 &= -p(p - 1)(p + 1)(p + 2)/32 \\ B_4 &= p(6p^4 + 15p^3 - 10p^2 - 30p + 3)/480. \end{aligned}$$

Inverting this characteristic function, using the well-known fact that $(1 - 2it)^{-f/2}$ is the characteristic function of the $\chi^2$-distribution with $f$ degrees of freedom, we have the following theorem.

THEOREM 2.1. *Under the hypothesis* $H_1: \Sigma = \Sigma_0$, *the distribution of the modified* LR *criterion* $-2 \log \lambda^*$ *defined by* (2.2) *can be expanded asymptotically as*

$$(2.10)\qquad \begin{aligned} &P(-2 \log \lambda^* \leqq z) \\ &= P(\chi_f^2 \leqq z) + B_2 n^{-1}\{P(\chi_{f+2}^2 \leqq z) - P(\chi_f^2 \leqq z)\} \\ &\quad + (\tfrac{1}{6})n^{-2}\{(3B_2^2 - 4B_3)P(\chi_{f+4}^2 \leqq z) - 6B_2^2 P(\chi_{f+2}^2 \leqq z) \\ &\quad + (3B_2^2 + 4B_3)P(\chi_f^2 \leqq z)\} \\ &\quad + (\tfrac{1}{6})n^{-3}\{(4B_4 - 4B_2B_3 + B_2^3)P(\chi_{f+6}^2 \leqq z) + B_2(4B_3 - 3B_2^2) \\ &\qquad \cdot P(\chi_{f+4}^2 \leqq z) + B_2(4B_3 + 3B_2^2)P(\chi_{f+2}^2 \leqq z) \\ &\quad - (4B_4 + 4B_2B_3 + B_2^3)P(\chi_f^2 \leqq z)\} + O(n^{-4}), \end{aligned}$$

*where* $\chi_f^2$ *means the* $\chi^2$*-variate with* $f$ *degrees of freedom and* $f = p(p + 1)/2$; *the constant* $B_r$ *is given by* (2.9).

By a slightly different asymptotic formula, Korin [6] computed the percentage points of $-2 \log \lambda^*$ recently. Now we shall consider the asymptotic expansion of the non-null distribution of this criterion $-2 \log \lambda^*$. The characteristic function of $-2n^{-\frac{1}{2}} \log \lambda^*$ under alternative $K_1$ can be written from formula (2.3) for the moment of $\lambda^*$ as

$$(2.11)\qquad \begin{aligned} C_{K_1}(t) = (\tfrac{1}{2}n/e)^{itpn^{1/2}}&[\Gamma_p(\tfrac{1}{2}n - n^{\frac{1}{2}}it)/\Gamma_p(\tfrac{1}{2}n)] \\ &\cdot |\Sigma\Sigma_0^{-1}|^{-n^{1/2}it}/|I - 2itn^{-\frac{1}{2}}\Sigma\Sigma_0^{-1}|^{\frac{1}{2}n - n^{1/2}it}. \end{aligned}$$

Applying the asymptotic formula (2.5) for the gamma function to each term of $\Gamma_p(\frac{1}{2}n - n^{\frac{1}{2}}it)/\Gamma_p(\frac{1}{2}n)$, we have

$$(2.12)\qquad \begin{aligned} &\log [\Gamma_p(\tfrac{1}{2}n - n^{\frac{1}{2}}it)/\Gamma_p(\tfrac{1}{2}n)] \\ &= n^{\frac{1}{2}}itp \log (2/n) - pt^2 \\ &\quad + p \textstyle\sum_{r=1}^{2k} n^{-r/2}(2it)^r\{2(it)^2/[(r + 1)(r + 2)] + (p + 1)/(4r)\} \\ &\quad - \textstyle\sum_{r=1}^{k} (-2)^r B_{r+1}/[r(r + 1)n^r]\{(1 - n^{-\frac{1}{2}}2it)^{-r} - 1\} + O(n^{-k-\frac{1}{2}}). \end{aligned}$$

Since the asymptotic formula $-\log |I - Z/n| = \sum_{r=1}^{k} n^{-r} \operatorname{tr} (Z^r)/r + O(n^{-k-1})$ holds for any symmetric matrix $Z/n$, whose characteristic roots are smaller than one in absolute value, we have

$$
\begin{aligned}
&-(\tfrac{1}{2}n - n^{\frac{1}{2}}it)\log|I - 2itn^{-\frac{1}{2}}\Sigma\Sigma_0^{-1}| \\
(2.13)\qquad &= n^{\frac{1}{2}}it\ \mathrm{tr}\ \Sigma\Sigma_0^{-1} - t^2\{\mathrm{tr}\ (\Sigma\Sigma_0^{-1})^2 - 2\ \mathrm{tr}\ \Sigma\Sigma_0^{-1}\} \\
&+ \textstyle\sum_{r=1}^{2k} n^{-r/2}2^{r+1}(it)^{r+2} \\
&\cdot\{\mathrm{tr}\ (\Sigma\Sigma_0^{-1})^{r+2}/(r+2) - \mathrm{tr}\ (\Sigma\Sigma_0^{-1})^{r+1}/(r+1)\} + O(n^{-k-\frac{1}{2}}).
\end{aligned}
$$

Substituting these two expressions into (2.11), we can see that

$$
\begin{aligned}
&\log C_{K_1}(t) \\
&= -n^{\frac{1}{2}}it\{\log|\Sigma\Sigma_0^{-1}| + \mathrm{tr}\ (I - \Sigma\Sigma_0^{-1})\} - t^2\ \mathrm{tr}\ (I - \Sigma\Sigma_0^{-1})^2 \\
(2.14)\qquad &+ \textstyle\sum_{r=1}^{2k} n^{-r/2}(2it)^r[p(p+1)/(4r) + 2p(it)^2/[(r+1)(r+2)] \\
&+ 2(it)^2\{\mathrm{tr}\ (\Sigma\Sigma_0^{-1})^{r+2}/(r+2) - \mathrm{tr}\ (\Sigma\Sigma_0^{-1})^{r+1}/(r+1)\}] \\
&- \textstyle\sum_{r=1}^{k}(-2)^r B_{r+1}/[r(r+1)n^r]\{(1 - n^{-\frac{1}{2}}2it)^{-r} - 1\} + O(n^{-k-\frac{1}{2}}),
\end{aligned}
$$

which implies that the statistic $\lambda^{**} = -2n^{-\frac{1}{2}}\log\lambda^* - n^{\frac{1}{2}}\{\mathrm{tr}\ (\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}|\}$ converges in law to the normal distribution with mean zero and variance $\tau^2 = 2\ \mathrm{tr}\ (I - \Sigma\Sigma_0^{-1})^2$, and further it enables us to expand the characteristic function of $\lambda^{**}/\tau$ up to any order asymptotically. We shall write it up to order $n^{-1}$.

$$
(2.15)\qquad C_{\lambda^{**}/\tau}(t) = e^{-t^2/2}\{1 + n^{-\frac{1}{2}}A_1 + n^{-1}A_2\},
$$

where the coefficients $A_1$ and $A_2$ of each term are given by

$$
\begin{aligned}
A_1 &= it\tau^{-1}p(p+1)/2 + (\tfrac{2}{3})\tau^{-3}(it)^3(p + 2\ \mathrm{tr}_3 - 3\ \mathrm{tr}_2) \\
A_2 &= (\tfrac{1}{8})\tau^{-2}(it)^2p(p+1)(p^2+p+4) + (\tfrac{1}{3})\tau^{-4}(it)^4 \\
&\quad\cdot\{p(p^2+p+2) - 3p(p+1)\mathrm{tr}_2 + 2(p^2+p-4)\mathrm{tr}_3 + 6\ \mathrm{tr}_4\} \\
&\quad + (\tfrac{2}{9})\tau^{-6}(it)^6(p + 2\ \mathrm{tr}_3 - 3\ \mathrm{tr}_2)^2,
\end{aligned}
$$

with the abbreviated notation $\mathrm{tr}_j = \mathrm{tr}\ (\Sigma\Sigma_0^{-1})^j$. By inverting this characteristic function, we can get the following theorem.

THEOREM 2.2. *Under the alternative* $K_1:\Sigma \neq \Sigma_0$, *the distribution of the modified* LR *criterion* $-2\log\lambda^*$ *defined by* (2.2) *can be expanded asymptotically as*

$$
\begin{aligned}
&P((1/n^{\frac{1}{2}}\tau)[-2\log\lambda^* - n\{\mathrm{tr}\ (\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}|\}] \leqq z) \\
&= \Phi(z) - \tfrac{1}{6}n^{-\frac{1}{2}}[3\tau^{-1}\Phi^{(1)}(z)p(p+1) + 4\tau^{-3}\Phi^{(3)}(z)(p + 2\ \mathrm{tr}_3 - 3\ \mathrm{tr}_2)] \\
&\quad + 1/(72n)[9\tau^{-2}\Phi^{(2)}(z)p(p+1)(p^2+p+4) + 24\tau^{-4}\Phi^{(4)}(z) \\
(2.16)\qquad &\quad\cdot\{p(p^2+p+2) - 3p(p+1)\mathrm{tr}_2 + 2(p^2+p-4)\mathrm{tr}_3 + 6\ \mathrm{tr}_4\} \\
&\quad + 16\tau^{-6}\Phi^{(6)}(z)(p + 2\ \mathrm{tr}_3 - 3\ \mathrm{tr}_2)^2] + O(n^{-\frac{3}{2}}),
\end{aligned}
$$

*where* $\mathrm{tr}_j = \mathrm{tr}\ (\Sigma\Sigma_0^{-1})^j$ *and* $\tau^2 = 2\ \mathrm{tr}\ (I - \Sigma\Sigma_0^{-1})^2$; $\Phi^{(r)}(z)$ *means the* $r$th *derivative of the standard normal distribution function* $\Phi(z)$.

It may be interesting to note that the asymptotic mean and variance of the statistic $-2n^{-\frac{1}{2}} \log \lambda^*$ vanish, if we put $\Sigma = \Sigma_0$ in the above theorem. This shows singularity of the limiting distribution at the hypothesis. The statistic $-2n^{-\frac{1}{2}} \log \lambda^*$ converges to zero under the hypothesis; in fact, $-2 \log \lambda^*$ converges in law to the $\chi^2$-distribution.

**3. Expansion of the distribution of the criterion for $\Sigma = \Sigma_0$ and $\mu = \mu_0$.** Let a $p$-variate random sample of size $N$ from the normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$ be denoted by $X_1, X_2, \cdots, X_N$. The LR statistic for testing the hypothesis $H_2:\Sigma = \Sigma_0$ and $\mu = \mu_0$ against alternatives $K_2:\Sigma \neq \Sigma_0$ or $\mu \neq \mu_0$, where $\Sigma_0$ and $\mu_0$ are a given positive definite matrix and a given vector, is expressed as

$$\lambda = (e/N)^{Np/2} |S\Sigma_0^{-1}|^{N/2} \operatorname{etr}[-\tfrac{1}{2}\Sigma_0^{-1}\{S + N(\bar{X} - \mu_0)(\bar{X} - \mu_0)'\}], \tag{3.1}$$

where $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$ and $\bar{X} = (1/N) \sum_{\alpha=1}^{N} X_\alpha$. For this problem, it has been shown by Sugiura and Nagao [14] and Das Gupta [4] that the LR test without modification is unbiased. So we shall consider the asymptotic expansion of the distribution of the LR statistic $\lambda$. The $h$th moment of $\lambda$ under alternative $K_2$ is expressed as

$$\begin{aligned} E[\lambda^h \mid K_2] = {} & (2e/N)^{Nhp/2}[\Gamma_p((n + Nh)/2)/\Gamma_p(\tfrac{1}{2}n)] \\ & \cdot|\Sigma\Sigma_0^{-1}|^{Nh/2}/|I + h\Sigma\Sigma_0^{-1}|^{N(1+h)/2} \operatorname{etr}[-\tfrac{1}{2}Nh\Sigma_0^{-1}(\mu - \mu_0) \\ & \cdot(\mu - \mu_0)'\{I - h\Sigma_0^{-1}(\Sigma^{-1} + h\Sigma_0^{-1})^{-1}\}], \end{aligned} \tag{3.2}$$

where $n = N - 1$. Similarly, as in Section 2, we can see that the asymptotic formula of the characteristic function of $-2 \log \lambda$ under $H_2$ is obtained by replacing $-p(p + 1)/4$, $n$, $B_{r+1}$ by $-p(p + 1)/4 - p/2$, $N$, $B'_{r+1} = \sum_{j=1}^{p} B_{r+1}(-j/2)$, respectively in the formula (2.7), which implies the following theorem.

THEOREM 3.1. *Under the hypothesis $H_2:\Sigma = \Sigma_0$ and $\mu = \mu_0$, the distribution of the* LR *criterion $-2 \log \lambda$ given in* (3.1) *can be expanded asymptotically as*

$$\begin{aligned} P(-2 \log \lambda \leqq z) = {} & P(\chi_f^2 \leqq z) + B_2'N^{-1}\{P(\chi_{f+2}^2 \leqq z) - P(\chi_f^2 \leqq z)\} \\ & + (\tfrac{1}{6})N^{-2}\{(3B_2'^2 - 4B_3')P(\chi_{f+4}^2 \leqq z) \\ & - 6B_2'^2P(\chi_{f+2}^2 \leqq z) + (3B_2'^2 + 4B_3')P(\chi_f^2 \leqq z)\} \\ & + (\tfrac{1}{6})N^{-3}\{(4B_4' - 4B_2'B_3' + B_2'^3)P(\chi_{f+6}^2 \leqq z) \\ & + B_2'(4B_3' - 3B_2'^2)P(\chi_{f+4}^2 \leqq z) + B_2'(4B_3' + 3B_2'^2) \\ & \cdot P(\chi_{f+2}^2 \leqq z) - (4B_4' + 4B_2'B_3' + B_2'^3) \\ & \cdot P(\chi_f^2 \leqq z)\} + O(n^{-4}), \end{aligned} \tag{3.3}$$

*where $f = p + p(p + 1)/2$ and the constant $B_r'$ is given by $B_2' = p(2p^2 + 9p + 11)/24$, $B_3' = -p(p + 1)(p + 2)(p + 3)/32$ and $B_4' = p(6p^4 + 45p^3 + 110p^2 + 90p + 3)/480$.*

The limiting distribution of $-2 \log \lambda$ is stated in Anderson ([1] page 268). Now we shall consider the asymptotic expansion of the distribution of $-2 \log \lambda$ under the alternative $K_2$. The characteristic function of $-2N^{-\frac{1}{2}} \log \lambda$ under $K_2$ can be obtained from the moment of $\lambda$ in (3.2) as $C_{K_2}(t) = C_{K_2}^{(1)}(t)C_{K_2}^{(2)}(t)$, where

$$C_{K_2}^{(1)}(t) = (\tfrac{1}{2}N/e)^{N^{1/2}pit}[\Gamma_p(\tfrac{1}{2}nN^{\frac{1}{2}}it)/\Gamma_p(\tfrac{1}{2}n)] \cdot |\Sigma\Sigma_0^{-1}|^{-N^{1/2}it}/|I - 2itN^{-\frac{1}{2}}\Sigma\Sigma_0^{-1}|^{(N/2)-N^{1/2}it} \tag{3.4}$$

$$C_{K_2}^{(2)}(t) = \operatorname{etr}[N^{\frac{1}{2}}it\Sigma_0^{-1}(\mu - \mu_0)(\mu - \mu_0)' \cdot \{I + 2itN^{-\frac{1}{2}}\Sigma_0^{-1}(\Sigma^{-1} - 2itN^{-\frac{1}{2}}\Sigma_0^{-1})^{-1}\}].$$

The first factor $C_{K_2}^{(1)}(t)$ has a similar expression to the characteristic function $C_{K_1}(t)$ in (2.11). The same computation gives us the following expression

$$\begin{aligned} \log C_{K_2}^{(1)}(t) &= N^{\frac{1}{2}}it\{\operatorname{tr}(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}|\} - t^2\operatorname{tr}(\Sigma\Sigma_0^{-1} - I)^2 \\ &\quad + \textstyle\sum_{r=1}^{2k} N^{-r/2}(2it)^r\{p(p+3)/(4r) + 2p(it)^2/[(r+1)(r+2)] \\ &\quad + 2(it)^2(\operatorname{tr}(\Sigma\Sigma_0^{-1})^{r+2}/(r+2) - \operatorname{tr}(\Sigma\Sigma_0^{-1})^{r+1}/(r+1))\} \\ &\quad - \textstyle\sum_{r=1}^{k} (-2)^r B_{r+1}'/[r(r+1)N^r]\{(1 - N^{-\frac{1}{2}}2it)^{-r} - 1\} + O(N^{-k-\frac{1}{2}}). \end{aligned} \tag{3.5}$$

Applying the formula $(I - N^{-\frac{1}{2}}Z)^{-1} = \sum_{r=0}^{2k+1} N^{-r/2}Z^r + O(N^{-k-1})$ to the second factor $C_{K_2}^{(2)}(t)$, we have

$$\log C_{K_2}^{(2)}(t) = N^{\frac{1}{2}}it(\mu - \mu_0)'\Sigma_0^{-1}(\mu - \mu_0) + \textstyle\sum_{r=0}^{2k} N^{-r/2}\, 2^{r+1}(it)^{r+2} \cdot (\mu - \mu_0)'\Sigma_0^{-1}(\Sigma\Sigma_0^{-1})^{r+1}(\mu - \mu_0) + O(N^{-k-\frac{1}{2}}). \tag{3.6}$$

Hence we have an asymptotic formula for the characteristic function $\log C_{K_2}(t)$ by adding the expression (3.5) and (3.6). This shows that the statistic $\lambda^* = -2N^{-\frac{1}{2}}\log\lambda - N^{\frac{1}{2}}\{\operatorname{tr}(\Sigma\Sigma_0^{-1} - I) - \log|\Sigma\Sigma_0^{-1}| + (\mu - \mu_0)'\Sigma_0^{-1}(\mu - \mu_0)\}$ is distributed asymptotically according to the normal distribution with mean zero and variance $\tau^2 = 2\{\operatorname{tr}(\Sigma\Sigma_0^{-1} - I)^2 + 2(\mu - \mu_0)'\Sigma_0^{-1}\Sigma\Sigma_0^{-1}(\mu - \mu_0)\}$ and further the characteristic function of $\lambda^*/\tau$ can be expanded asymptotically as

$$e^{-t^2/2}\{1 + N^{-\frac{1}{2}}A_1 + N^{-1}A_2 + O(N^{-\frac{3}{2}})\}, \tag{3.7}$$

where the coefficients $A_1$ and $A_2$ are given by

$$A_1 = (\tfrac{1}{6})\{3\tau^{-1}itp(p+3) + 4\tau^{-3}(it)^3(p + 6d_2 + 2\operatorname{tr}_3 - 3\operatorname{tr}_2)\}$$

$$\begin{aligned} A_2 = (\tfrac{1}{72})[&9\tau^{-2}(it)^2p(p+3)(p^2+3p+4) + 24\tau^{-4}(it)^4\{p(p+1)(p+2) \\ &+ 6p(p+3)d_2 + 24d_3 + 6\operatorname{tr}_4 + 2(p+4)(p-1)\operatorname{tr}_3 - 3p(p+3)\operatorname{tr}_2\} \\ &+ 16\tau^{-6}(it)^6(p + 6d_2 + 2\operatorname{tr}_3 - 3\operatorname{tr}_2)^2], \end{aligned}$$

with the abbreviated notations $d_r = (\mu - \mu_0)'\Sigma_0^{-1}(\Sigma\Sigma_0^{-1})^r(\mu - \mu_0)$ and $\operatorname{tr}_j = \operatorname{tr}(\Sigma\Sigma_0^{-1})^j$. Inverting this characteristic function, we immediately obtain the following theorem.

THEOREM 3.2. *Under the alternative* $K_2:\Sigma \neq \Sigma_0$ *or* $\mu \neq \mu_0$, *the distribution of the* LR *criterion* $-2 \log \lambda$ *given in* (3.1) *can be expanded asymptotically as*

$$
\begin{aligned}
&P((1/N^{\frac{1}{2}}\tau)[-2 \log \lambda - N\{\mathrm{tr}\ (\Sigma\Sigma_0^{-1} - I) - \log |\Sigma\Sigma_0^{-1}| + d_0\}] \leqq z)\\
&\quad = \Phi(z) - \tfrac{1}{6}N^{-\frac{1}{2}}\\
&\qquad \cdot[4\tau^{-3}\Phi^{(3)}(z)(p + 6d_2 + 2\ \mathrm{tr}_3 - 3\ \mathrm{tr}_2) + 3\tau^{-1}\Phi^{(1)}(z)p(p + 3)]\\
&\qquad + 1/(72N)[16\tau^{-6}\Phi^{(6)}(z)(p + 6d_2 + 2\ \mathrm{tr}_3 - 3\ \mathrm{tr}_2)^2 + 24\tau^{-4}\Phi^{(4)}(z)\\
&\qquad \cdot\{p(p+1)(p+2) + 6p(p+3)d_2 + 24d_3 + 6\ \mathrm{tr}_4 + 2(p+4)\\
&\qquad \cdot(p-1)\ \mathrm{tr}_3 - 3p(p+3)\ \mathrm{tr}_2\} + 9\tau^{-2}\Phi^{(2)}(z)p(p+3)\\
&\qquad \cdot(p^2 + 3p + 4)] + O(N^{-\frac{3}{2}}),
\end{aligned}
\tag{3.8}
$$

*where* $d_r = (\mu - \mu_0)'\Sigma_0^{-1}(\Sigma\Sigma_0^{-1})^r(\mu - \mu_0)$, $\mathrm{tr}_j = \mathrm{tr}\ (\Sigma\Sigma_0^{-1})^j$ *and* $\tau^2 = 2\{\mathrm{tr}\ (\Sigma\Sigma_0^{-1} - I)^2 + 2d_1\}$; $\Phi^{(r)}(z)$ *means the* $r$th *derivative of the standard normal distribution function* $\Phi(z)$.

**4. Expansion of the distribution of the criterion for sphericity.** The LR statistic for testing the sphericity hypothesis $H_3:\Sigma = \sigma^2 I$ against the alternatives $K_3:\Sigma \neq \sigma^2 I$, where $\sigma^2$ is unspecified, based on a $p$-variate random sample of size $N$ drawn from a normal population with covariance matrix $\Sigma$, is

$$\lambda = |S|^{N/2}(p^{-1}\ \mathrm{tr}\ S)^{-Np/2}, \tag{4.1}$$

where $S = \sum_{\alpha=1}^{N} (X_\alpha - \bar{X})(X_\alpha - \bar{X})'$. Unbiasedness of this test criterion was established by Gleser [3] and Sugiura and Nagao [14], whereas we already know that in order to get unbiasedness in the $k$-sample situation, the statistic

$$\lambda^* = |S|^{n/2}(p^{-1}\ \mathrm{tr}\ S)^{-np/2}, \tag{4.2}$$

is preferred to $\lambda$ by Sugiura and Nagao [14]. Under the hypothesis $H_3$, the $h$th moment of $\lambda^*$ is expressed as

$$E[\lambda^{*h} \mid H_3] = p^{\frac{1}{2}nhp}\Gamma(\tfrac{1}{2}np)\Gamma_p(\tfrac{1}{2}n(1+h))/[\Gamma(\tfrac{1}{2}pn(1+h))\Gamma_p(\tfrac{1}{2}n)], \tag{4.3}$$

which implies the asymptotic expansion of the null distribution of this criterion as in Anderson ([1] page 263),

$$
\begin{aligned}
&P(-2\rho \log \lambda^* \leqq z)\\
&\quad = P(\chi_f^2 \leqq z) + (1/288p^2m^2)(p+2)(p-1)(p-2)\\
&\qquad \cdot(2p^3 + 6p^2 + 3p + 2)\{P(\chi_{f+4}^2 \leqq z) - P(\chi_f^2 \leqq z)\} + O(m^{-3}),
\end{aligned}
\tag{4.4}
$$

where $f = \{p(p+1)/2\} - 1$, $m = \rho n$ and $\rho = 1 - (2p^2 + p + 2)/6pn$. The correction factor $\rho$ is so chosen that the term of order $m^{-1}$ in the expansion vanishes. It may be remarked that in the previous two sections we cannot use the correction factor $\rho$ as we can in this case. The reason is that the $h$th moment of the test statistic has two gamma functions containing $h$ in (4.3), but only one gamma function containing $h$ in (2.3) and (3.2).

We shall now consider the asymptotic expansion of the non-null distribution of the LR criterion $-2\rho \log \lambda^*$. Since the $h$th moment of $\lambda^*$ under alternatives cannot be written explicitly, some different technique is necessary. It is provided by using differential operators. Noting that $S$ has the Wishart distribution $W_p(n, \Sigma)$ under $K_3$, we can write the characteristic function of $-2\rho m^{-\frac{1}{2}} \log \lambda^*$ as

$$(4.5)\quad C_{K_3}(t) = 1/[\Gamma_p(\tfrac{1}{2}n)2^{np/2}] \cdot \int |S|^{(n-p-1)/2-m^{\frac{1}{2}}it} \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}S)/[|\Sigma|^{n/2}(\operatorname{tr} S/p)^{-m^{\frac{1}{2}}itp}]\, dS.$$

By the transformation $S \to HSH'$ for some orthogonal matrix $H$ of order $p$, we may assume $\Sigma = \Gamma = \operatorname{diag}(\lambda_1, \cdots, \lambda_p)$ where $\lambda_j$ are characteristic roots of $\Sigma$. Put $U = (1/m)S$, then the statistic $U$ converges in probability to $\Gamma$ as $m$ tends to infinity. We can express the characteristic function as

$$(4.6)\quad C_{K_3}(t) = m^{np/2}p^{-m^{\frac{1}{2}}itp}/[\Gamma_p(\tfrac{1}{2}n)2^{np/2}|\Gamma|^{n/2}] \int [|U|^{(n-p-1)/2-m^{\frac{1}{2}}it}/(\operatorname{tr} U)^{-m^{\frac{1}{2}}itp}] \cdot \operatorname{etr}(-\tfrac{1}{2}m\Gamma^{-1}U)\, dU.$$

Transform the variable $U$ to $D$ and $R$ by $U = D^{\frac{1}{2}}RD^{\frac{1}{2}}$ such that the matrix $D$ is diagonal, and the diagonal element is given by that of $U$. Then $|\partial U/\partial(D, R)| = |D|^{(p-1)/2}$ and $\operatorname{tr}\Gamma^{-1}U = \operatorname{tr}\Gamma^{-1}D$, so we have

$$(4.7)\quad \begin{aligned} C_{K_3}(t) &= m^{np/2}p^{-m^{\frac{1}{2}}itp}/[\Gamma_p(n/2)2^{np/2}|\Gamma|^{n/2}] \int |R|^{(n-p-1)/2-m^{\frac{1}{2}}it}\, dR \\ &\quad \cdot \int (\operatorname{tr} D)^{m^{\frac{1}{2}}itp}|D|^{(n/2)-m^{\frac{1}{2}}it-1} \operatorname{etr}(-(m/2)\Gamma^{-1}D)\, dD \\ &= m^{np/2}p^{-m^{\frac{1}{2}}itp}\Gamma_p(\tfrac{1}{2}n - m^{\frac{1}{2}}it)/[\Gamma_p(n/2)2^{np/2}|\Gamma|^{n/2}\Gamma((n/2) - m^{\frac{1}{2}}it)^p] \\ &\quad \cdot \int (\operatorname{tr} D)^{m^{\frac{1}{2}}itp}|D|^{(n/2)-m^{\frac{1}{2}}it-1} \operatorname{etr}(-(m/2)\Gamma^{-1}D)\, dD. \end{aligned}$$

Put $f(\Lambda) = (\operatorname{tr}\Lambda)^{m^{\frac{1}{2}}itp}$ and $\partial = \operatorname{diag}(\partial/\partial\lambda_1, \cdots, \partial/\partial\lambda_p)$, where $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_p)$. Noting that the diagonal matrix $D$ converges in probability to $\Gamma$ as $m$ tends to infinitiy, we shall expand the function $f(D)$ in the expression (4.7) about $D = \Gamma$, that is, $f(D) = \operatorname{etr}[(D - \Gamma)\partial]f(\Lambda)\,|_{\Lambda=\Gamma}$. This gives by taking the integration regarding the operator $\partial$ as constant, as in Okamoto [8], et al.

$$(4.8)\quad \begin{aligned} &\int |D|^{(n/2)-m^{\frac{1}{2}}it-1} \operatorname{etr}(-\tfrac{1}{2}m\Gamma^{-1}D)f(D)\, dD \\ &= (2/m)^{np/2-m^{\frac{1}{2}}itp}\Gamma(\tfrac{1}{2}n - m^{\frac{1}{2}}it)^p \operatorname{etr}\{-\Gamma\partial - (\tfrac{1}{2}n - m^{\frac{1}{2}}it) \\ &\quad \cdot \log|\Gamma^{-1} - 2m^{-1}\partial|\}f(\Lambda)\,|_{\Lambda=\Gamma}. \end{aligned}$$

Hence we have the following expression of the characteristic function for $-2\rho \log \lambda^*$.

$$(4.9)\quad C_{K_3}(t) = (\tfrac{1}{2}m/p)^{m^{\frac{1}{2}}itp}\Gamma_p((\tfrac{1}{2}n) - m^{\frac{1}{2}}it)/[\Gamma_p(\tfrac{1}{2}n)\,|\Gamma|^{m^{\frac{1}{2}}it}] \cdot \operatorname{etr}\{-\Gamma\partial - (\tfrac{1}{2}n - m^{\frac{1}{2}}it)\log|I - (2/m)\Gamma\partial|\}f(\Lambda)\,|_{\Lambda=\Gamma}.$$

The first factor can be expanded by using the asymptotic formula (2.5) for the gamma function with respect to $m$ instead of $n$ as

$$\begin{aligned}&\Gamma_p((n/2)-m^{\frac{1}{2}}it)/\Gamma_p(n/2)\\ (4.10)\quad &= (2/m)^{m^{\frac{1}{2}}itp}e^{-pt^2}[1+(\tfrac{1}{6}m^{-\frac{1}{2}})\{(p^2+2p-2)it+4p(it)^3\}+(\tfrac{1}{72}m^{-1})\\ &\quad\cdot\{(p^2+2p-2)(p^2+2p+10)(it)^2+8p(p^2+2p+4)(it)^4\\ &\quad+16p^2(it)^6\}+O(m^{-\frac{3}{2}})].\end{aligned}$$

The problem is how to evaluate the exponential part of $C_{K_3}(t)$ in (4.9). Since $\partial f(\Lambda)=m^{\frac{1}{2}}itp(\mathrm{tr}\,\Lambda)f(\Lambda)$, we must regard the order of $\partial$ as $m^{\frac{1}{2}}$. Applying the formula $\log|I-(2/m)\Gamma\partial|=-\sum_{r=1}^{2k}(2/m)^r\,\mathrm{tr}\,(\Gamma\partial)^r/r+O(m^{-k-\frac{1}{2}})$ to (4.9) we can write

$$\begin{aligned}(4.11)\quad &\mathrm{etr}\,\{-\Gamma\partial-((n/2)-m^{\frac{1}{2}}it)\log|I-(2/m)\Gamma\partial|\}\,f(\Lambda)\\ &\qquad=\exp\,[A_0(\partial)+m^{-\frac{1}{2}}A_1(\partial)+m^{-1}A_2(\partial)+O(m^{-\frac{3}{2}})]f(\Lambda),\end{aligned}$$

where

$$A_0(\partial)=m^{-1}\,\mathrm{tr}\,(\Gamma\partial)^2-2itm^{-\frac{1}{2}}\,\mathrm{tr}\,(\Gamma\partial)$$

$$A_1(\partial)=4/(3m^{-\frac{3}{2}})\,\mathrm{tr}\,(\Gamma\partial)^3-2itm^{-1}\,\mathrm{tr}\,(\Gamma\partial)^2+(2p^2+p+2)/(6pm^{\frac{1}{2}})\,\mathrm{tr}(\Gamma\partial)$$

$$A_2(\partial)=2m^{-2}\,\mathrm{tr}\,(\Gamma\partial)^4-8it/(3m^{\frac{3}{2}})\,\mathrm{tr}\,(\Gamma\partial)^3+(2p^2+p+2)/(6mp)\,\mathrm{tr}\,(\Gamma\partial)^2.$$

Note that the result of applying each term $A_0(\partial)$, $A_1(\partial)$ and $A_2(\partial)$ to $f(\Lambda)$ is $O(1)\cdot f(\Lambda)$. We can expand the expression (4.11) as

$$\begin{aligned}(4.12)\quad &\{1+A_1(\partial)m^{-\frac{1}{2}}+m^{-1}(A_2(\partial)+\tfrac{1}{2}A_1(\partial)^2)+O(m^{-\frac{3}{2}})\}\\ &\qquad\cdot\{\exp A_0(\partial)\}f(\Lambda)\,|_{\Lambda=\Gamma}\,.\end{aligned}$$

We now evaluate $[\exp A_0(\partial)]f(\Lambda)$.

$$\begin{aligned}&\exp\{m^{-1}\,\mathrm{tr}\,(\Gamma\partial)^2-2itm^{-\frac{1}{2}}\,\mathrm{tr}\,\Gamma\partial\}f(\Lambda)\\ (4.13)\quad &=\textstyle\sum_{r=0}^{\infty}r!^{-1}(m^{-1}\,\mathrm{tr}\,(\Gamma\partial)^2-2itm^{-\frac{1}{2}}\,\mathrm{tr}\,\Gamma\partial)^rf(\Lambda)\\ &=\{\textstyle\sum_{r=0}^{\infty}r!^{-1}\sum_{k+l=r}\binom{r}{k}(-2it)^l/m^{k+(l/2)}\sum_{(k),(l)}k!/(k_1!\cdots k_p!)\\ &\quad\cdot l!/(l_1!\cdots l_p!)\prod_{\alpha=1}^{p}\lambda_\alpha^{2k_\alpha+l_\alpha}\partial_\alpha^{2k_\alpha+l_\alpha}\}\cdot f(\Lambda),\end{aligned}$$

where $\sum_{(k),(l)}$ means the sum of all possible combinations of non-negative integers $k_1,\cdots,k_p$ and $l_1,\cdots,l_p$ such that $k_1+\cdots+k_p=k$ and $l_1+\cdots+l_p=l$. Since the equality $\prod_{\alpha=1}^{p}\partial_\alpha^{2k_\alpha+l_\alpha}f(\Lambda)=(m^{\frac{1}{2}}itp)_{2k+l}(\mathrm{tr}\,\Lambda)^{m^{\frac{1}{2}}itp-2k-l}$ holds where $(a)_k=a(a-1)\cdots(a-k+1)$, we can simplify the above expression as

$$\begin{aligned}(4.14)\quad &\textstyle\sum_{r=0}^{\infty}r!^{-1}\sum_{k+l=r}\binom{r}{k}(-2it)^l(m^{\frac{1}{2}}itp)_{r+k}/m^{k+(l/2)}\\ &\qquad\cdot\{\mathrm{tr}\,\Gamma^2/(\mathrm{tr}\,\Lambda)^2\}^k(\mathrm{tr}\,\Gamma/\mathrm{tr}\,\Lambda)^l(\mathrm{tr}\,\Lambda)^{m^{\frac{1}{2}}itp}.\end{aligned}$$

We can easily see that $(m^{\frac{1}{2}}itp)_{r+k}$ can be expanded asymptotically with respect to $m$ as

$$
\begin{aligned}
&(m^{\frac12}itp)_{r+k} \\
(4.15)\quad &= (m^{\frac12}itp)^{r+k} - (m^{\frac12}itp)^{r+k-1}(\tfrac12)\{(k)_2 + 2rk + (r)_2\} + (m^{\frac12}itp)^{r+k-2}(\tfrac{1}{24}) \\
&\cdot\{3(k)_4 + 4(3r+2)(k)_3 + 6r(3r+1)(k)_2 + 12r^2(r-1)k \\
&+ r(r-1)(3r-1)(r-2)\} + O(m^{(r+k-3)/2}),
\end{aligned}
$$

which makes it possible to simplify the summation in (4.14), giving

$$
\begin{aligned}
\{\exp A_0(\partial)\}f(\Lambda) &= \{\exp B_0(\Lambda)\}\{1 + m^{-\frac12}B_1(\Lambda) + m^{-1}B_2(\Lambda)\}f(\Lambda), \\
B_0(\Lambda) &= -pt^2(p \operatorname{tr}\Gamma^2/(\operatorname{tr}\Lambda)^2 - 2\operatorname{tr}\Gamma/\operatorname{tr}\Lambda) \\
B_1(\Lambda) &= -2(it)^3p(p\operatorname{tr}\Gamma^2/(\operatorname{tr}\Lambda)^2 - \operatorname{tr}\Gamma/\operatorname{tr}\Lambda)^2 \\
(4.16)\qquad &\quad - itp\operatorname{tr}\Gamma^2/(\operatorname{tr}\Lambda)^2 \\
B_2(\Lambda) &= (1/6)[12p^2(it)^6(p\operatorname{tr}\Gamma^2/(\operatorname{tr}\Lambda)^2 - \operatorname{tr}\Gamma/\operatorname{tr}\Lambda)^4 \\
&\quad + 4p(it)^4(p\operatorname{tr}\Gamma^2/(\operatorname{tr}\Lambda)^2 - \operatorname{tr}\Gamma/\operatorname{tr}\Lambda)^2 \\
&\quad \cdot(13p\operatorname{tr}\Gamma^2/(\operatorname{tr}\Lambda)^2 - 4\operatorname{tr}\Gamma/\operatorname{tr}\Lambda) + 3(it)^2 \\
&\quad \cdot\{11p^2(\operatorname{tr}\Gamma^2)^2/(\operatorname{tr}\Lambda)^4 - 8p\operatorname{tr}\Gamma^2\operatorname{tr}\Gamma/(\operatorname{tr}\Lambda)^3\}].
\end{aligned}
$$

Applying the operator $[1 + m^{-\frac12}A_1(\partial) + m^{-1}(A_2(\partial) + A_1(\partial)^2/2)]$ to this expression we see that the exponential part of (4.9) can be written asymptotically as

$$
\begin{aligned}
(4.17)\quad &\{1 + m^{-\frac12}(B_1(\Lambda) + A_1(\partial)) + m^{-1}(B_2(\Lambda) + A_1(\partial)B_1(\Lambda) + A_2(\partial) \\
&\qquad + A_1(\partial)^2/2) + O(m^{-\frac32})\}\cdot\{\exp B_0(\Lambda)\}f(\Lambda)\,|_{\Lambda=\Gamma}.
\end{aligned}
$$

Noting that the formulas

$$
\begin{aligned}
&m^{-l/2}\operatorname{tr}(\Gamma\partial)^l\,\{\exp B_0(\Lambda)\}f(\Lambda) \\
&= (itp)^l(\operatorname{tr}\Lambda)^{m^{\frac12}itp-l}\,\{\exp B_0(\Lambda)\}\operatorname{tr}\Gamma^l \\
&\quad\cdot[1 + (m^{\frac12}itp)^{-1}\{-l(l-1)/2 + 2lpt^2(p\operatorname{tr}\Gamma^2/(\operatorname{tr}\Lambda)^2 - \operatorname{tr}\Gamma/\operatorname{tr}\Lambda)\} + O(m^{-1})] \\
&m^{-(k+l)}\operatorname{tr}(\Gamma\partial)^k\operatorname{tr}(\Gamma\partial)^l\,\{\exp B_0(\Lambda)\}f(\Lambda) \\
&= (itp)^{k+l}\operatorname{tr}\Gamma^k\operatorname{tr}\Gamma^l\,\{\exp B_0(\Lambda)\}(\operatorname{tr}\Lambda)^{m^{\frac12}itp-k-l}\{1 + O(m^{-\frac12})\},
\end{aligned}
$$

hold for any non-negative integers $k$ and $l$, we can compute each term in (4, 17), obtaining

$$
\begin{aligned}
&\exp\{-\Gamma\partial - (\tfrac12 n - m^{\frac12}it)\log|I - 2m^{-1}\Gamma\partial|\}f(\Lambda)\,|_{\Lambda=\Gamma} \\
&\qquad = \{\exp B_0(\Gamma)\}f(\Gamma)\{1 + m^{-\frac12}C_1(\Gamma) + m^{-1}C_2(\Gamma) + O(m^{-\frac32})\} \\
&C_1(\Gamma) = (\tfrac23)(it)^3p\{-3(t_2-1)^2 + 2t_3 - 3t_2\} + (\tfrac16)it(2p^2 + p + 2 - 6t_2) \\
(4.18)\quad &C_2(\Gamma) = (\tfrac29)p^2(it)^6\{3(t_2-1)^2 + 3t_2 - 2t_3\}^2 + (\tfrac19)p(it)^4 \\
&\quad\cdot\{3(t_2-1)^2(26t_2 - 2p^2 - p - 10) + 6(t_2-1)(15t_2 - 14t_3) \\
&\quad + 18t_4 + 2(2p^2 + p - 16)t_3 - 3(2p^2 + p - 4)t_2\} + (\tfrac{1}{72})(it)^2 \\
&\quad\cdot\{396t_2^2 - 288t_3 - 24(2p^2 + p + 8)t_2 + (2p^2 + p + 2) \\
&\quad\cdot(2p^2 + p + 26)\}
\end{aligned}
$$

with the abbreviated notation $t_j = p^{j-1}(\mathrm{tr}\ \Gamma^j)/(\mathrm{tr}\ \Gamma)^j$. Combining this result with equation (4.10), we can finally obtain the asymptotic formula for the characteristic function $C_{K_3}(t)$ of the statistic $-2\rho m^{-\frac{1}{2}} \log \lambda^*$ given by (4.2).

$$C_{K_3}(t) = \exp [m^{\frac{1}{2}}it \log \{(p^{-1}\ \mathrm{tr}\ \Gamma)^p/|\Gamma|\} - pt^2(p\ \mathrm{tr}\ \Gamma^2/(\mathrm{tr}\ \Gamma)^2 - 1)]$$
$$\cdot\{1 + m^{-\frac{1}{2}}D_1(\Gamma) + m^{-1}D_2(\Gamma) + O(m^{-\frac{3}{2}})\}$$
$$D_1(\Gamma) = (2p/3)(it)^3\{-3(t_2-1)^2 + 2t_3 - 3t_2 + 1\} + (\tfrac{1}{2})it(p^2 + p - 2t_2)$$
$$(4.19)\quad D_2(\Gamma) = (\tfrac{2}{9})p^2(it)^6[\{3(t_2-1)^2 + 3t_2 - 2t_3\}^2 + 1 - 6(t_2-1)^2 + 4t_3 - 6t_2] + (p/3)(it)^4\{(t_2-1)^2(26t_2 - 3p^2 - 3p - 8) + 2(t_2-1)(15t_2 - 14t_3) + 6t_4 + 2(p+3)(p-2)t_3 - (3p^2+3p-4)t_2 + p^2 + p + 2\} + (\tfrac{1}{24})(it)^2 \cdot\{132t_2^2 - 96t_3 - 4(5p^2+4p+14)t_2 + 3p^4 + 6p^3 + 23p^2 + 16p + 8\},$$

which implies that the limiting distribution of the statistic

$$\lambda^{**} = m^{-\frac{1}{2}}[-2\rho \log \lambda^* - m \log \{\mathrm{tr}\ (\Gamma)/p)^p/|\Gamma|\}]$$

is normal with mean 0 and variance $\tau^2 = 2p\{p(\mathrm{tr}\ \Gamma^2)/(\mathrm{tr}\ \Gamma)^2 - 1\}$ as $m$ tends to infinity. This result has already been obtained by Olkin and Siotani [9]. Gleser [3] also gives a similar result, which is corrected in [3a] ($\lambda^{\frac{1}{2}}$ should be understood as $\bar{\lambda}$ in his correction).
By inverting the characteristic function of $\lambda^*/\tau$ we have the following theorem, from (4.19).

THEOREM 4.1. *Under the alternative $K_3:\Sigma \neq \sigma^2 I$, the asymptotic expansion of the non-null distribution of the* LR *criterion* $-2\rho \log \lambda^*$ *given in* (4.2) *for sphericity is expressed as*

$$P((1/m^{\frac{1}{2}}\tau)[-2\rho \log \lambda^* - m \log \{(\mathrm{tr}\ \Sigma/p)^p/|\Sigma|\}] \leqq z)$$
$$= \Phi(z) - m^{-\frac{1}{2}}[(2p/3\tau^3)\Phi^{(3)}(z)\{-3(t_2-1)^2 + 2t_3 - 3t_2 + 1\} + (\tfrac{1}{2}\tau)\Phi^{(1)}(z)(p^2 + p - 2t_2)]$$
$$(4.20)\quad + m^{-1}[(2p^2/9\tau^6)\Phi^{(6)}(z) \cdot\{[3(t_2-1)^2 + 3t_2 - 2t_3]^2 + 1 - 6(t_2-1)^2 + 4t_3 - 6t_2\} + (p/3\tau^4)\Phi^{(4)}(z)\{(t_2-1)^2(26t_2 - 3p^2 - 3p - 8) + 2(t_2-1)(15t_2 - 14t_3) + 6t_4 + 2(p+3)(p-2)t_3 - (3p^2 + 3p - 4)t_2 + p^2 + p + 2\} + (\tfrac{1}{24}\tau^{-2})\Phi^{(2)}(z) \cdot\{132t_2^2 - 96t_3 - 4(5p^2 + 4p + 14)t_2 + 3p^4 + 6p^3 + 23p^2 + 16p + 8\}] + O(m^{-\frac{3}{2}}),$$

*where* $t_j = p^{j-1}(\mathrm{tr}\,\Sigma^j)/(\mathrm{tr}\,\Sigma)^j$, $\tau^2 = 2p(t_2 - 1)$ *and* $\Phi^{(r)}(z)$ *means the* rth *derivative of the standard normal distribution function* $\Phi(z)$.

**5. Limiting non-null distribution of the LR criterion for** $\Sigma_1 = \cdots = \Sigma_k$. Let $X_{\alpha 1}, \cdots, X_{\alpha N_\alpha}$ be a random sample from $p$-variate normal distribution with mean vector $\mu_\alpha$ and covariance matrix $\Sigma_\alpha$ for $\alpha = 1, 2, \cdots, k$. The LR statistic for testing the hypothesis $H_4:\Sigma_1 = \Sigma_2 = \cdots = \Sigma_k$ against the alternatives $K_4:\Sigma_i \neq \Sigma_j$ for some $i, j\,(i \neq j)$, is given by

$$\lambda = \{\textstyle\prod_{\alpha=1}^k |S_\alpha/N_\alpha|^{N_\alpha/2}\}/|S/N|^{N/2}, \tag{5.1}$$

where $S_\alpha = \sum_{j=1}^{N_\alpha}(X_{\alpha j} - \bar{X}_\alpha)(X_{\alpha j} - \bar{X}_\alpha)'$, $\bar{X}_\alpha = \sum_{j=1}^{N_\alpha} X_{\alpha j}/N_\alpha$ and $S = \sum_{\alpha=1}^k S_\alpha$, $N = \sum_{\alpha=1}^k N_\alpha$. If we modify this criterion by reducing the sample size $N_\alpha$ to the degrees of freedom $n_\alpha = N_\alpha - 1$,

$$\lambda^* = \{\textstyle\prod_{\alpha=1}^k |S_\alpha/n_\alpha|^{n_\alpha/2}\}/|S/n|^{n/2} \tag{5.2}$$

with $n = \sum_{\alpha=1}^k n_\alpha$, we have an unbiased test in the univariate case (Pitman [10]), and in the two-sample case for arbitrary $p$ (Sugiura and Nagao [14]). The asymptotic expansion of the distribution of this criterion under the hypothesis is stated in Box [2] or Anderson ([1] page 255). The limiting non-null distribution of the statistic $-2n^{-\frac{1}{2}}\log\lambda^*$ can be obtained from the characteristic function by the same argument as in the previous section. However, so far as the limiting distribution is concerned, it is simpler to use the following lemma, which is a direct extension of Siotani and Hayakawa [12].

LEMMA. *Let* $n_\alpha U_\alpha$ *have the Wishart distribution* $W_p(n_\alpha, \Sigma_\alpha)$ *and* $n_\alpha = \rho_\alpha n$ *for fixed* $\rho_\alpha$ *such that* $\sum_{\alpha=1}^k \rho_\alpha = 1$. *Suppose a real-valued function* $f(U_1, \cdots, U_k)$ *is continuously differentiable with respect to each variable. Then the statistic*

$$n^{\frac{1}{2}}\{f(U_1, \cdots, U_k) - f(\Sigma_1, \cdots, \Sigma_k)\} \tag{5.3}$$

*is distributed asymptotically for large n according to the normal distribution with mean zero and variance* $2\sum_{\alpha=1}^k \rho_\alpha^{-1}\,\mathrm{tr}\,\{(\partial^{(\alpha)}f)\Sigma_\alpha\}^2$, *where* $\partial^{(\alpha)}f$ *means the symmetric matrix having* $\{(1+\delta_{ab})/2\}\partial f/\partial u_{ab}^{(\alpha)}$ *as its* $(a, b)$ *element for* $U_\alpha = (U_{ab}^{(\alpha)})$ *and Kronecker delta* $\delta_{ab}$.

Putting $f(U_1, \cdots, U_k) = \log|\sum_{\alpha=1}^k \rho_\alpha U_\alpha| - \sum_{\alpha=1}^k \rho_\alpha \log|U_\alpha|$ in the above lemma and noting that the equality $(\{(1+\delta_{ij})/2\}\partial\log|A|/\partial a_{ij}) = A^{-1}$ holds for any positive definite matrix $A$, we have the following theorem.

THEOREM 5.1. *For testing the hypothesis* $H_4:\Sigma_1 = \cdots = \Sigma_k$ *against all alternatives, the non-null distribution of the modified* LR *statistic*

$$-2n^{-\frac{1}{2}}\log\lambda^* - n^{\frac{1}{2}}\log\{|\tilde{\Sigma}|/\textstyle\prod_{\alpha=1}^k|\Sigma_\alpha|^{\rho_\alpha}\} \tag{5.4}$$

*is asymptotically normal with mean zero and variance* $2\sum_{\alpha=1}^k \rho_\alpha\,\mathrm{tr}\,(\Sigma_\alpha\tilde{\Sigma}^{-1} - I)^2$, *where* $\tilde{\Sigma} = \sum_{\alpha=1}^k \rho_\alpha\Sigma_\alpha$.

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## REFERENCES

[1] ANDERSON, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*. Wiley, New York.

[2] BOX, G. E. P. (1948). A general distribution theory for a class of likelihood criteria. *Biometrika* **36** 317–346.

[3] GLESER, LEON J. (1966). A note on the sphericity test. *Ann. Math. Statist.* **37** 464–467.

[3a]. GLESER, LEON J. (1968). Correction to "A note on the sphericity test." *Ann. Math. Statist.* **39** 684.

[4] SOMESH DAS GUPTA. (1969). Properties of power functions of some tests concerning dispersion matrices of multivariate normal distributions. *Ann. Math. Statist.* **40** 697–701.

[5] ITO, K. (1956). Asymptotic formulae for the distribution of Hotelling's generalized $T_0^2$ statistic, I. *Ann. Math. Statist.* **27** 1091–1105.

[6] KORIN, BASIL P. (1968). On the distribution of a statistic used for testing a covariance matrix. *Biometrika* **55** 171–178.

[7] NAGAO, H. (1967). Monotonicity of the modified likelihood ratio test for a covariance matrix. *J. Sci. Hiroshima Univ. Ser. A-I* **31** 147–150.

[8] OKAMOTO, M. (1963). An asymptotic expansion for the distribution of the linear discriminant function. *Ann. Math. Statist.* **34** 1286–1301.

[9] OLKIN, I. and SIOTANI, M. (1964). Asymptotic distribution of functions of a correlation matrix. Technical Report No. 6, Stanford Univ.

[10] PITMAN, E. J. G. (1939). Tests of hypothesis concerning location and scale parameters. *Biometrika* **31** 200–215.

[11] SIOTANI, M. (1957). On the distribution of the Hotelling's $T^2$-statistic. *Ann. Inst. Statist. Math.* **8** 1–14.

[12] SIOTANI, M. and HAYAKAWA, T. (1964). Asymptotic distribution of functions of Wishart matrix. *Proc. Inst. Statist. Math.* **12** (in Japanese) 191–198.

[13] SUGIURA, N. and FUJIKOSHI, Y. (1969). Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence. *Ann. Math. Statist.* **40** 942–952.

[14] SUGIURA, N. and NAGAO, H. (1968). Unbiasedness of some test criteria for the equality of one or two covariance matrices. *Ann. Math. Statist.* **39** 1686–1692.

[15] WELCH, B. L. (1947). The generalization of "Student's" problem when several different population variances are involved. *Biometrika* **34** 28–35.