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A REMARK ON ALMOST INVARIANCE[1]

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1. Introduction and summary. In [1], P. J. Bickel and the author presented a theorem giving sufficient conditions for the equivalence of invariant and almost-invariant functions. In the subsequent discussion, it was intimated without proof that the results extend to certain other classes of distributions that do not satisfy the hypotheses of the main theorem. A small lacuna was glossed over in the discussion; however, the conclusions are correct for the examples given there. In this note we justify those conclusions. We adopt the notation of [1] without further comment.

2. The lacuna. As in [1], $(\mathscr{X}, \mathscr{B})$ is the measurable (sample) space of the random variable X. If P is a probability measure on $(\mathscr{X}, \mathscr{B})$, $\mathscr{N}(P) = \{B \in \mathscr{B} : PB = 0\}$ is the ideal of P-null sets. If $\mathscr{P}$ is a set of probability measures on $(\mathscr{X}, \mathscr{B})$, $\mathscr{N}(\mathscr{P}) = \bigcap_{P \in \mathscr{P}} \mathscr{N}(P)$ is the ideal of $\mathscr{P}$-null sets. G is a group of 1–1 bimeasurable transformations of $\mathscr{X}$ to itself. Theorem 1 of [1] allows us to conclude for certain $\mathscr{P}$ that are preserved by G that almost-invariance and invariance are equivalent for $(G, \mathscr{P})$. I.e., if ϕ is a $\mathscr{P}$-almost-invariant (critical) function, there exists a G-invariant function ψ so that $(\phi \neq \psi) \in \mathscr{N}(\mathscr{P})$. ($\phi$ is $\mathscr{P}$-almost-invariant if for all $g \in G$, $(\phi g \neq \phi) \in \mathscr{N}(\mathscr{P})$.)

The following proposition allows us to conclude a similar fact for certain other families of measures.

PROPOSITION 1. *Suppose almost-invariance and invariance are equivalent for* $(G, \mathscr{P})$. *Let* $\mathscr{Q}$ *be another family of measures on* $(\mathscr{X}, \mathscr{B})$ *so that* $\mathscr{N}(\mathscr{P}) = \mathscr{N}(\mathscr{Q})$. *Then almost-invariance and invariance are equivalent for* $(G, \mathscr{Q})$.

PROOF. If ϕ is $\mathscr{Q}$-almost-invariant, for $g \in G$, $(\phi g \neq \phi) \in \mathscr{N}(\mathscr{Q}) = \mathscr{N}(\mathscr{P})$, so that ϕ is $\mathscr{P}$-almost-invariant. Thus there exists an invariant ψ so that $(\phi \neq \psi) \in \mathscr{N}(\mathscr{P}) = \mathscr{N}(\mathscr{Q})$. □

REMARK. In [1], it is intimated that the condition $\mathscr{N}(\mathscr{Q}) \supset \mathscr{N}(\mathscr{P})$ implies the conclusion of Proposition 1. We show, however, in Section 4 that Proposition 1 fails if the hypothesis $\mathscr{N}(\mathscr{P}) = \mathscr{N}(\mathscr{Q})$ is weakened to either $\mathscr{N}(\mathscr{P}) \subset \mathscr{N}(\mathscr{Q})$ or its converse.

We now consider a useful general condition under which $\mathscr{N}(\mathscr{Q}) = \mathscr{N}(\mathscr{P})$. Following [1], we say that $\mathscr{Q} \ll \mathscr{P}$ if $\mathscr{N}(\mathscr{Q}) \supset \mathscr{N}(\mathscr{P})$. Lemma 4 of [1] gives a simple sufficient condition that $\mathscr{Q} \ll \mathscr{P}$.

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PROPOSITION 2. *If $\mathscr{Q} \supset \mathscr{P}$ and $\mathscr{Q} \ll \mathscr{P}$, then $\mathscr{N}(\mathscr{Q}) = \mathscr{N}(\mathscr{P})$.*

PROOF. Follows immediately upon noting that $\mathscr{Q} \supset \mathscr{P} \Rightarrow \mathscr{N}(\mathscr{Q}) \subset \mathscr{N}(\mathscr{P})$. □

3. Examples. We reconsider the examples in Section 3 of [1] in the light of the preceding. The first example there concerns the group of shift transformations. If $\mathscr{P}$ is the normal family with unit variance, it satisfies the conditions of Theorem 1 of [1], as noted there. Hence if $\mathscr{Q}$ is the set of all absolutely continuous distributions, we have $\mathscr{Q} \supset \mathscr{P}$ and $\mathscr{Q} \ll \mathscr{P}$; hence Proposition 2 and also Proposition 1 apply. The conclusion stated in [1] is thus seen to be correct. This remark holds in any number of dimensions.

The conclusions drawn for the nonparametric examples discussed in [1] are also correct, as may be seen from Proposition 1 and Proposition 2. In particular, if $\mathscr{P}$ is generated by the set of all distributions on $(-\infty, \infty)$ having strictly increasing continuous distribution functions and $\mathscr{Q}$, by all those having continuous distributions, then since $\mathscr{Q} \ll \mathscr{P}$, $\mathscr{N}(\mathscr{Q}) = \mathscr{N}(\mathscr{P})$ (again, in any number of dimensions). This $\mathscr{Q}$ may be further enlarged as discussed in [1], Section 3.

A more interesting question arises if $\mathscr{Q}$ is the location family of a power-product distribution $\mathscr{Q}$ that is absolutely continuous with respect to Lebesgue measure λ (in n-dimensions, say). If $\mathscr{P}$ is the normal location family discussed above, since $\mathscr{P} \equiv \lambda$, if also $\mathscr{Q} \equiv \lambda$, it follows that $\mathscr{N}(\mathscr{Q}) = \mathscr{N}(\mathscr{P})$, so that Proposition 1 applies. One need only consider the location family of the uniform distribution (on [0, 1], say) in two or more dimensions to realize that in general, $\mathscr{N}(\mathscr{Q}) \supset \mathscr{N}(\mathscr{P})$. However, the following consideration can be applied in this case: For $B \in \mathscr{B}$, let $\mathscr{P}_B$ denote the elements of $\mathscr{P}$ restricted to B.

PROPOSITION 3. *Suppose there is a G-invariant set $A \in \mathscr{B}$ so that for every $Q \in \mathscr{Q}$, $QA = 1$ and that $\mathscr{N}(\mathscr{Q}_A) = \mathscr{N}(\mathscr{P}_A)$. Then if $\mathscr{N}(\mathscr{Q}) \supset \mathscr{N}(\mathscr{P})$ and if almost-invariance and invariance are equivalent for $\mathscr{P}$, the same is true of $\mathscr{Q}$.*

PROOF. Let ϕ be $\mathscr{Q}$-almost-invariant. Since A is invariant and $\mathscr{N}(\mathscr{P}_A) = \mathscr{N}(\mathscr{Q}_A)$, it follows that $\phi 1_A$ is $\mathscr{P}$-almost-invariant. Hence there is an invariant ψ so that $(\phi 1_A \neq \psi) \in \mathscr{N}(\mathscr{P})$. Since $\mathscr{N}(\mathscr{Q}) \supset \mathscr{N}(\mathscr{P})$, it then follows that $(\phi 1_A \neq \psi) \in \mathscr{N}(\mathscr{Q})$. However, since A supports $\mathscr{Q}$, it also follows that $(\phi \neq \psi) \in \mathscr{N}(\mathscr{Q})$. □

Thus if $\mathscr{Q}$ is uniform on [0, 1] in n-dimensions, the shift-invariant set $A = \{(x_1, \cdots, x_n): |x_i - x_j| \leqq 1, \forall i, j\}$ is easily seen to satisfy the hypothesis of Proposition 3. The desired conclusion may be obtained in this way for the location family of the uniform distribution. For a location family on R^n generated by an arbitrary absolutely continuous Q with i.i.d. components, $(dQ/d\lambda > 0)$ may be taken to be a square (a measurable rectangle with equal sides). It is not difficult to show that $A = G(dQ/d\lambda > 0)$ is a measurable shift-invariant set satisfying the hypothesis of Proposition 3. The details are omitted. Whether this construction extends to other groups is an open question.

4. Counter examples. We show by example that Proposition 1 fails in general if the condition $\mathscr{N}(\mathscr{P}) = \mathscr{N}(Q)$ is weakened to $\mathscr{N}(\mathscr{P}) \subset \mathscr{N}(\mathscr{Q})$ or its converse. In both examples, $\mathscr{X} = [0, 1]$, $\mathscr{B}$ is its usual Borel structure and λ denotes Lebesgue measure on $[0, 1]$.

EXAMPLE 1. $\mathscr{N}(\mathscr{P}) \subset \mathscr{N}(\mathscr{Q})$. Let G be all 1–1 transformations of $[0, 1]$ onto itself that fix all but a finite number of points. All G-invariant functions are constant. Let $\mathscr{P}$ be the set of all distributions on $[0, 1]$ and let $\mathscr{Q} = \{\lambda\}$. $\mathscr{N}(\mathscr{P}) \subset \mathscr{N}(\mathscr{Q})$. A function φ on $[0, 1]$ is $(G, \mathscr{P})$ almost-invariant iff it is constant. Hence almost-invariance and invariance are equivalent (in fact, coincide) for $(G, \mathscr{P})$. However, any function on $[0, 1]$ is $(G, \mathscr{Q})$ almost-invariant, but is not $\mathscr{Q}$-equivalent to a G-invariant function unless it is constant a.s. $[\lambda]$.

EXAMPLE 2. $\mathscr{N}(\mathscr{P}) \supset \mathscr{N}(\mathscr{Q})$. Let G consist of all 1–1 transformations of $[0, 1]$ onto itself that fix 0 and 1 and all but a finite number of points of $(0, 1)$, plus the transformation fixing $(0, 1)$ pointwise and interchanging 0 and 1. All G-invariant functions are constant on the sets $(0, 1)$ and $\{0, 1\}$. Let $\mathscr{P} = \{p\}$, where p assigns probability $\frac{1}{2}$ to 0 and 1 and let $\mathscr{Q} = \{p, \lambda\}$. $\mathscr{N}(\mathscr{P}) \supset \mathscr{N}(\mathscr{Q})$. A function φ on $[0, 1]$ is $(G, \mathscr{P})$ almost-invariant iff $\varphi(0) = \varphi(1)$. Moreover, such a φ is $\mathscr{P}$-equivalent to any G-invariant function agreeing with φ on $\{0, 1\}$ (e.g., the constant function with value $\varphi(0)$). Hence almost-invariance and invariance are equivalent for $(G, \mathscr{P})$. A function φ on $[0, 1]$ is also $(G, \mathscr{Q})$ almost-invariant iff $\varphi(0) = \varphi(1)$. However, unless φ is a.s. $[\lambda]$ constant on $(0, 1)$, it is not $\mathscr{Q}$-equivalent to a G-invariant function.

We note in passing that in both examples, $\mathscr{P}$ and $\mathscr{Q}$ are preserved by G (although this condition is not required in Proposition 1). Finally, we mention that in the proof of the main theorem in [1], Lemma 3 should be cited, not Lemma 2.

REFERENCE

[1] BERK, R. H. and BICKEL, P. J. (1968). On invariance and almost invariance. *Ann. Math. Statist.* **39** 1573–1576.