The Annals of Mathematical Statistics
1970, Vol. 41, No. 2, 648–650

ON RELATIONSHIP ALGEBRAS OF INCOMPLETE BLOCK DESIGNS

By J. Robinson

University of Connecticut

In the analyses of the relationship algebras of balanced incomplete block designs given by James (1957) and of partially balanced incomplete block designs given by Ogawa and Ishii (1965) it has been pointed out that the idempotent matrices obtained were not all unique. Those obtained were shown to correspond to sums of squares in the analysis of variance with the treatment sum of squares adjusted for blocks. It seems worthwhile to obtain explicitly from these relationship algebras the other partition of interest, namely that giving the block sum of squares adjusted for treatments.

We will use the notation of Ogawa and Ishii (1965) and obtain this alternative analysis of the relationship algebra for a partially blanced incomplete block design. We may, of course, consider the balanced incomplete block design to be a special case of this with one associate class. Consider a partially balanced incomplete block design with m associate classes and let $\mathbf{N}$ be the incidence matrix of the design. Let

$$\mathbf{NN}' = \textstyle\sum_{u=0}^{m} \rho_u \mathbf{A}_u^{\#}$$

be the spectral decomposition of $\mathbf{NN}'$. The matrices $\mathbf{A}_u^{\#}$ are the idempotents of the association algebra of the design. If $\mathbf{\Phi}$ is the incidence matrix for the treatments then we can set up an isomorphism between the elements of the association algebra and a subalgebra of the relationship algebra by the transformation

$$\mathbf{T}_u^{\#} = \mathbf{\Phi}\mathbf{A}_u^{\#}\mathbf{\Phi}', \qquad u = 1, \cdots, m.$$

The set of $4m+3$ linearly independent matrices

$$\mathbf{I}, \mathbf{G}, \mathbf{B}, \mathbf{T}_u^{\#}, \mathbf{BT}_u^{\#}, \mathbf{T}_u^{\#}\mathbf{B}, \mathbf{BT}_u^{\#}\mathbf{B}, \qquad u = 1, \cdots, m.$$

may be considered as a basis of the relationship algebra. The algebra can be expressed as a direct sum of $m+3$ minimum two-sided ideals. Three of these ideals are one-dimensional and the others are four-dimensional. We will write the principal idempotents of these ideals as $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ and $\mathbf{E}_{4u}$, $u = 1, \cdots, m$.

The three principal idempotents $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ have been obtained by James (1957) for the case of the balanced incomplete block design and can easily be found in the case of the partially balanced incomplete block design. They may be written

$$\mathbf{E}_1 = w^{-1}\mathbf{G}$$

$$\mathbf{E}_2 = k^{-1}\mathbf{B} - \textstyle\sum_{u=1}^{m} k^{-1}\rho_u^{-1}\mathbf{BT}_u^{\#}\mathbf{B}$$

$$\mathbf{E}_3 = \mathbf{I} - k^{-1}\mathbf{B} - \textstyle\sum_{u=1}^{m} \mathbf{V}_u^{\#}$$

Received June 16, 1969.

where $V_u^{\#} = [k/r(rk-\rho_u)](\mathbf{T}_u^{\#}-k^{-1}\mathbf{B}\mathbf{T}_u^{\#})(\mathbf{T}_u^{\#}-k^{-1}\mathbf{T}_u^{\#}\mathbf{B})$, $u = 1, \cdots, m$. These idempotents correspond in the analysis of variance to the correction term, the remainder of the block sum of squares after removing the treatment component and the error sum of squares, respectively. Further, $\mathbf{E}_{4u}$, $u = 1, \cdots, m$, may be written in a form similar to that of $\mathbf{E}_4$ of James (1957) as

$$\mathbf{E}_{4u} = (rk-\rho_u)^{-1}(k\mathbf{T}_u^{\#}+r\rho_u^{-1}\mathbf{B}\mathbf{T}_u^{\#}\mathbf{B}-\mathbf{B}\mathbf{T}_u^{\#}-\mathbf{T}_u^{\#}\mathbf{B}), \qquad u = 1, \cdots, m.$$

We can obtain a non-unique decomposition of $\mathbf{E}_{4u}$ into

$$\mathbf{E}_{4u} = \mathbf{F}_{2u}+\mathbf{F}_{3u}, \qquad u = 1, \cdots, m.$$

James (1957) and Ogawa and Ishii (1965) obtained the particular decomposition with

$$\mathbf{F}_{2u} = k^{-1}\rho_u^{-1}\mathbf{B}\mathbf{T}_u^{\#}\mathbf{B}$$

and

$$\mathbf{F}_{3u} = \mathbf{V}_u^{\#}, \qquad u = 1, \cdots, m.$$

These idempotents correspond to the treatment component of the block sum of squares ignoring treatments and to the sum of squares for treatments adjusted for blocks in the analysis of variance. In the isomorphism of the relationship algebra with the direct sum of three 1×1 matrix algebras and $m\,2\times 2$ matrix algebras we have the maps

$$\mathbf{E}_{4u} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{F}_{2u} \leftrightarrow \begin{bmatrix} 0 & 0 \\ k^{-1} & 1 \end{bmatrix}, \quad \mathbf{F}_{3u} \leftrightarrow \begin{bmatrix} 1 & 0 \\ -k^{-1} & 0 \end{bmatrix}, \qquad u = 1, \cdots, m.$$

We have another decomposition of $\mathbf{E}_{4u}$ into

$$\mathbf{E}_{4u} = \mathbf{F}_{2u}^{*}+\mathbf{F}_{3u}^{*}, \qquad u = 1, \cdots, m,$$

with

$$\mathbf{F}_{3u}^{*} = r^{-1}\mathbf{T}_u^{\#}$$

and so

$$\begin{aligned}\mathbf{F}_{2u}^{*} &= \mathbf{E}_{4u}-\mathbf{F}_{3u}^{*} \\ &= (rk-\rho_u)^{-1}(r^{-1}\rho_u\mathbf{T}_u^{\#}+r\rho_u^{-1}\mathbf{B}\mathbf{T}_u^{\#}\mathbf{B}-\mathbf{B}\mathbf{T}_u^{\#}-\mathbf{T}_u^{\#}\mathbf{B}), \qquad u = 1, \cdots, m.\end{aligned}$$

In the isomorphism with the direct sum of matrix algebras mentioned above we have the maps

$$\mathbf{F}_{2u}^{*} \leftrightarrow \begin{bmatrix} 0 & -r^{-1}\rho_u \\ 0 & 1 \end{bmatrix}, \quad \mathbf{F}_{3u}^{*} \leftrightarrow \begin{bmatrix} 1 & r^{-1}\rho_u \\ 0 & 0 \end{bmatrix}, \qquad u = 1, \cdots, m.$$

These idempotents correspond to the treatment component of the block sum of squares adjusted for treatments and to the treatment sum of squares ignoring blocks in the analysis of variance.

It may be noted that we can write

$$\mathbf{F}_{2u}^{*} = (rk-\rho_u)^{-1}r^{-2}\rho_u(\mathbf{T}_u^{\#}-r\rho_u^{-1}\mathbf{B}\mathbf{T}_u^{\#})(\mathbf{T}_u^{\#}-r\rho_u^{-1}\mathbf{T}_u^{\#}\mathbf{B}), \qquad u = 1, \cdots, m$$

and in the case of a balanced incomplete block design this is the idempotent matrix of the sum of squares for the treatment component of the block sum of squares eliminating treatments,

$$[rv(v-k)(k-1)]^{-1}\sum_{j=1}^{v}[(v-k)V_j-(t-1)T_j+(k-1)G]^2,$$

where V_j, T_j and G are the totals for the jth treatment, the sum of block totals for blocks containing the jth treatment and the grand total, respectively.

REFERENCES

[1] JAMES, A. T. (1957). The relationship algebra of an experimental design. *Ann. Math. Statist.* **28** 993–1002.

[2] OGAWA, J. and ISHII, G. (1965). The relationship algebra and the analysis of variance of a partially balanced incomplete block design. *Ann. Math. Statist.* **36** 1815–1828.