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BETTING SYSTEMS IN FAVORABLE GAMES[1]

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1. Introduction and summary. A favorable game is one in which, loosely speaking, there are opportunities for favorable bets. We consider only games where there is a "double or nothing" return of the stake, the win probability p being drawn independently each day from a distribution $F(p)$, and told to the player before he makes his bet.

We are interested, not in maximizing the expected fortune over a number of plays, but in minimizing the probability of eventual ruin. In order that the gambler cannot stand pat, we use two devices.

In Model I, he must pay a fee of \$1 after each play. In Model II, he must bet a minimum of \$1. His bet may never exceed his current fortune, x.

Let b denote an allowable betting strategy. We define the *ruin function* by $q(x) = \inf_b P(\text{ruin} \mid b, \text{starting fortune } x)$. In both models, if ever $x < 1$, ruin is inevitable.

In Section 3, we introduce (following Ferguson, [3]) the idea of a *loss function*, $q_0(x)$, satisfying certain conditions. A "natural" choice for loss function is $q_0^*(x) = 1$ for $x < 1$, $= 0$ otherwise. We define the *minimal expected loss after n games* by $q_n(x) = \inf_b Eq_0(X_n)$ where X_n is the fortune after the nth game.

In Lemmas 3.1–3.5, we construct a suitable Borel-measurable betting function to show that $q_n(x)$ satisfies the dynamic programming relationship; if $x \to X_1(b_1)$ after bet b_1, then $q_n(x) = \inf_{b_1} Eq_{n-1}(X_1(b_1))$. The results of Section 3 do not depend on the assumption that the game is favorable, but from now on this assumption is required.

Using the natural loss function $q_0^*(x)$, we show in Lemma 4.1 that $q_n^*(x) \to q(x)$ as $n \to \infty$, uniformly in x (in passing, Theorem 5.1 shows that for *any* loss function, $q_n(x) \to q(x)$ without uniformity) and we employ this result in Section 6 to show that $q(x)$ is continuous, for Model I only. For both models, $q(x)$ is already known [1] to be lower semi-continuous. These results are sufficient to establish [3] that a particular stationary Markov betting function, $b^*(x,p)$, is optimal (although not necessarily unique).

For large fortunes x, asymptotic forms for the ruin function $q(x)$, and as a consequence for $b^*(x,p)$, are already known [1], [3] under a slight restriction on $F(p)$, namely that it have no mass in a neighborhood of 1. On removing the restriction, some weaker results are obtained, and summarized in Section 8.

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2. Preliminaries. Starting with fortune x_0, the gambler is presented with an infinite sequence of independent games $G_1, G_2 \cdots G_j \cdots$.

Before each game G_j, a probability of win (for the gambler) p_j is drawn from a distribution $F(p)$ [the same one each time] and announced. The distribution $F(p)$ is known to the gambler. He may bet any amount b_j where

$$(2.1) \qquad 0 \leqq b_j \leqq x_{j-1} \quad \text{(Model I)}$$
$$1 \leqq b_j \leqq x_{j-1} \quad \text{(Model II)}$$

and x_{j-1} is his current fortune. The odds are 1 : 1, so that he receives back double his bet, if he wins, nothing if he loses.

For Model I, a fixed charge is imposed after each game. We take this charge as \$1. Thus the fortune after game G_j is, for Model I,

$$\begin{aligned} x_j &= \max(X_{j-1}+b_j-1, 0) && \text{probability } p_j \\ &= \max(X_{j-1}-b_j-1, 0) && \text{probability } (1-p_j) \end{aligned}$$

where $X_0 = x_0$.

Note that whenever $X_{j-1}+b_j-1 < 0$, or $X_{j-1}-b_j-1 < 0$, we arbitrarily put $X_j = 0$, in which case we shall have $X_j = X_{j+1} = \cdots = 0$.

For Model II, we have

$$\begin{aligned} X_j &= X_{j-1}+b_j && \text{probability } p_j \\ &= X_{j-1}-b_j && \text{probability } (1-p_j) \end{aligned} \Bigg\} \text{ if } X_{j-1} \geqq 1$$
$$= X_{j-1} \qquad \text{otherwise;}$$

where $X_0 = x_0$.

Note that if $X_j < 1$ for some j, then $X_j = X_{j+1} = \cdots$. This convention avoids the necessity of specifying a bet when the fortune has dropped below 1.

DEFINITIONS. For both Models, the event *ruin by game n* occurs if and only if $X_n < 1$. The event *ruin* occurs if and only if *ruin by game n* occurs for some n.

A *partial history* is a sequence $(x_0, p_1, x_1, \cdots x_{n-1}, p_n)$.

A *betting strategy* for the gambler is a set b of functions $\{b_j\}$ where, at game j, b_j associates a bet with each partial history, i.e.

$$b_j = b_j(x_0, p_1, x_1, \cdots, x_{j-1}, p_j), \qquad j \geqq 1$$

and b_j satisfies (2.1). We shall consider only strategies belonging to a certain class S, where $b = \{b_j\} \in S$ if and only if for fixed x_0, and for all $j \geqq 1, b_j = b_j(x_0, p_1, x_1, \cdots, x_{j-1}, p_j)$ is Borel-measurable in $p_1, x_1, \cdots, x_{j-1}, p_j$ (the term "optimal strategy" used subsequently refers to optimality within S, $\inf_b$ will mean the infimum over $b \in S$, etc.).

Stationary strategies will be written $b(x,p)$ where x, p denote the current fortune and win-probability.

A *favorable game* is one in which there exists a betting strategy for which the probability of ruin is less than one for some value $x > 0$ of the initial fortune.

The *ruin function for strategy b* is defined by

$$q^b(x) = P(X_n < 1 \quad \text{some} \quad n \mid b, X_0 = x) \tag{2.2}$$
$$= \lim P(X_n < 1 \mid b, X_0 = x) \quad \text{as} \quad n \to \infty.$$

and the term *ruin function*, without reference to a specific strategy, is defined by

$$q(x) = \inf_b q^b(x). \tag{2.3}$$

Ferguson ([3], page 811) points out that a necessary and sufficient condition that a game of Model I or Model II be favorable is that $F(p)$ assign positive mass to the interval $(\frac{1}{2}, 1]$, the necessity following by the martingale convergence theorem. Ferguson also shows that $q(x) < 1$ for $x > 1$ (Model I) and for $x \geqq 1$ (Model II), and that $q(x) \to 0$ as $x \to \infty$ at least exponentially fast.

Given $x > 1$, and a betting strategy b, let $t = [x]+1$, where $[x]$ denotes the largest integer $\leqq x$; let $\theta = \int_0^1 (1-p)\, dF(p)$, < 1 if we suppose $F(p)$ not concentrated on 1. Then for $x' \in [1, t]$,

$$P(X_{n+t} < 1 \mid b, X_0 = x_0, \cdots X_n = x') \geqq \theta^t \tag{2.4}$$

since $\theta^t = P(\text{loss } t \text{ consecutive times})$. Ferguson ([3], page 802) uses (2.4) to show that for fixed x, y

$$P(X_n \in [1, x] \text{ i.o.} \mid b, X_0 = y) = 0 \tag{2.5}$$

(i.o. = infinitely often) which incidentally shows that minimizing the probability of ruin is equivalent to maximizing the probability that the fortune tends to infinity.

The results (2.4), (2.5), and the lemma following do not use the restriction $F(\frac{1}{2}) < 1$, and so are true whether or not the game is favorable.

LEMMA 2.1 (*Models I and II*). *Assume $F(p)$ not concentrated on* 1.

(i) *Given $x \geqq 1$, then for $y \leqq x$,*

$$P(X_n \in [1, x] \quad \text{for} \quad n = 1, \cdots N \mid b, X_0 = y) \to 0 \quad \text{as} \quad N \to \infty,$$

uniformly in y, b.

(ii) *For fixed x, y, b*

$$P(X_n \in [1, x] \mid b, X_0 = y) \to 0 \quad \text{as} \quad n \to \infty.$$

[*Note: In the case of Model I, this theorem may be trivially modified as follows. The restriction $x \geqq 1$ may be replaced by $x > 0$, and the interval $[1, x]$ may be replaced by $(0, x]$, throughout*].

PROOF. (i) Let $t = [x]+1$. Given $y \leqq t$, we have from (2.4), for $x' \in [1, t]$, and i any nonnegative integer,

$$\sup_{x'} P(X_{(i+1)t} \in [1, t] \mid b, X_{it} = x', X_0 = y) \leqq 1 - \theta^t$$

uniformly in $y \leqq t, b$.

Therefore

$$\begin{aligned} P(X_{(i+1)t}\in[1,t]\,|\,b, X_0=y) &\leqq (1-\theta^t)P(X_{it}\in[1,t]\,|\,b, X_0=y)\\ &\leqq (1-\theta^t)^{i+1}P(X_0\in[1,t]\,|\,b, X_0=y)\\ &\leqq (1-\theta^t)^{i+1}. \end{aligned}$$

Thus, for $n > t$,

$$\begin{aligned} &P(X_j\in[1,t] \quad \text{for} \quad j=1,\cdots,n\,|\,b, X_0=y)\\ &\leqq P(X_j\in[1,t] \quad \text{for} \quad j=1,\cdots,it\,|\,b, X_0=y) \end{aligned}$$

(where $i=[n/t]$) $\leqq (1-\theta^t)^i \to 0$ uniformly for $y \leqq t$, and in b, as $n\to\infty$. The result follows, *a fortiori*; since $x < t$.

(ii) Given $x \geqq 1$; let y and b be fixed. For any sequence A_n for which $P(A_n \text{ i.o.}) = 0$,

$$0 = P(A_n \text{ i.o.}) = P(\textstyle\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n) = \lim_{N\to\infty} P(\bigcup_{n=N}^{\infty} A_n)$$

since the sequence of sets $(\bigcup_{n=N}^{\infty} A_n)$ is decreasing as $N\to\infty$. Hence $P(\bigcup_{n=N}^{\infty} A_n)\downarrow 0$ as $N\to\infty$. But $\bigcup_{n=N}^{\infty} A_n \supseteq A_n$. Hence $P(A_N)\downarrow 0$ as $N\to\infty$. The result then follows from (2.5) with $A_n = \{X_n \in [1,x]\}$.

3. Iterative scheme, finite game, and loss function.

3.1. *Definitions.* The following scheme (also referred to as the "finite game" situation, in contrast to the "infinite games" just considered) has application in numerical calculations. It is also a useful tool in proving some of the theoretical results.

We introduce the idea of a *loss function*, $q_0(x)$, which is nonincreasing with x, and satisfies

$$q_0(x) = 1, x < 1; \qquad q_0(x) \to 0, \quad \text{as} \quad x\to\infty. \tag{3.1}$$

This function could be thought of as expressing the negative of the "utility" of fortune x to the gambler. The higher the fortune, the lower the value of the loss function. (If, for example, we are interested in minimizing the probability of ruin, we should choose $q_0(x) = 1$ for $x < 1, q_0(x) = 0$ otherwise. This choice is made below.)

Suppose we start with fortune x, play n games (Model I or II), and obtain fortune X_n. The *minimal expected loss after n games* with starting fortune x, is

$$q_n(x) = \inf_b E[q_0(X_n)\,|\,b, X_0 = x]. \tag{3.2}$$

3.2. *Preliminary results for the finite game.* (Statements and/or proofs will be given for one model, and it is to be understood that modifications for the other model are trivial unless indicated otherwise.)

LEMMA 3.1. *For each $n > 0$, $q_n(x) = 1$ for $x < 1$, $q_n(x) \to 0$ as $x\to\infty$, and $q_n(x)$ is nonincreasing in x.*

PROOF. (Model I). The first result is obvious, and the fixed strategy of betting the minimum suffices to establish the second result.

Finally, if $x_0 \geqq 0$, $x_0' = x_0 + h$, where $h > 0$, we show that $q_n(x_0) \geqq q_n(x_0')$. Let $\varepsilon > 0$, and let $b = \{b_j\} \in S$ be ε-optimal at x_0 in the sense that

$$(1+\varepsilon)q_n(x_0) \leqq E\{q_0(X_n) \mid b, X_0 = x_0\}.$$

Consider starting fortune $x_0' = x_0 + h$, and general fortune x_j'. Let b' be the strategy: put aside h, and at game j, bet, using b, as if the current fortune were $x'_{j-1} - h$. As soon as the condition $x'_{j-1} \geqq h$ is violated, arbitrarily adopt a "go-for-broke" policy ($b_j' = x'_{j-1}$, $b'_{j+1} = x_j'$, etc.) thereafter. By considering partial histories, we see that since the original b is in S, the derived strategy b' is also a well-defined member of S. Clearly, for any $x \geqq 0$,

$$P(X_n \geqq x \mid b, X_0 = x_0) \leqq P(X_n \geqq x \mid b', X_0 = x_0').$$

Hence, since q_0 is nonincreasing

$$q_n(x_0+h) \leqq E[q_0(X_n) \mid b', X_0 = x_0+h] \leqq E[q_0(X_n) \mid b, X_0 = x_0] \leqq (1+\varepsilon)q_n(x_0)$$

and the result follows since ε is arbitrary.

In Lemmas 3.2–3.4 which follow, we use the following notation:

$q(x)$ denotes a function nonincreasing with x. Also, for $p \in [0, 1]$, b nonnegative,

$$\begin{aligned} \phi(x, p, b) &= pq(x+b-1)+(1-p)q(x-b-1) \qquad \text{(Model I)} \\ &= pq(x+b)+(1-p)q(x-b) \qquad \text{(Model II)}. \end{aligned}$$

$$\begin{aligned} \Phi(x, p) &= \inf\nolimits_{0 \leqq b \leqq x} \phi(x, p, b) \qquad \text{(Model I)}, \\ &= \inf\nolimits_{1 \leqq b \leqq x} \phi(x, p, b) \qquad \text{(Model II)}. \end{aligned} \tag{3.3}$$

LEMMA 3.2 (Model I). *With q, ϕ and Φ as above, $q(\infty) - q(-\infty) \leqq 1$, $p \in [0, 1]$, and b real, we have that $\phi(x,p,b)$ is*

(i) *nonincreasing with x, for fixed p, b;*

(ii) *nonincreasing with p, for fixed x, b and further, if $0 \leqq p_1 \leqq p_2 \leqq 1$, then $0 \leqq \phi(x,p_1,b) - \phi(x,p_2,b) \leqq (p_2 - p_1)$.*

Also, $\Phi(x,p)$ is (iii) *nonincreasing with x, for fixed p;*

(iv) *nonincreasing with p, for fixed x;*

(v) *concave in p, for fixed x;*

(vi) *continuous in p, for fixed x.*

PROOF. Easy.

LEMMA 3.3. (Model I). *With q, ϕ and Φ as above, $q(x) = 1$ for $x < 1$, $q(x) \to 0$ as $x \to \infty$, $p \in [0, 1]$ and b real, we have: given $\varepsilon' > 0$, there exists a function $b^{\dagger\dagger}(x,p)$, measurable (in fact elementary) in x, for each p, $0 \leqq b^{\dagger\dagger}(x,p) \leqq x$, such that*

$$\phi(x, p, b^{\dagger\dagger}(x, p)) \leqq \Phi(x, p) + \varepsilon' \quad \textit{all} \quad x. \tag{3.4}$$

PROOF. Given $\varepsilon' > 0$, take $\varepsilon = \varepsilon'/2$, and fix $p, 0 \leqq p \leqq 1$. There are at most a finite number of discontinuities of size $\geqq \varepsilon$ for $\Phi(x, p)$, from Lemma 3.2 (iii). Construct a sequence of points, $0 = x_0 < x_1 < \cdots < x_{k+1} = \infty$ containing these discontinuities of size $\geqq \varepsilon$, such that

$$\Phi^+(x_{i-1}, p) - \Phi^-(x_i, p) < \varepsilon, \qquad i = 1, 2, \cdots, k+1$$

where

$$\Phi^+(y, p) = \lim_{x \to y^+} \Phi(x, p), \ \Phi^-(y, p) = \lim_{x \to y^-} \Phi(x, p).$$

Now for any point x, we may find $b^\dagger(x,p), 0 \leqq b^\dagger(x,p) \leqq x$, such that

$$\phi(x, p, b^\dagger(x, p)) \leqq \Phi(x, p) + \varepsilon.$$

Consider a point x, where

$$x \neq x_i \qquad (i = 0, \cdots, k+1).$$

Then x must be contained in some interval (x_{i-1}, x_i), which, if $i < k+1$, we write for convenience as $(x^*, x^* + \Delta)$ where $\Delta > 0$. [If $i = k+1$, choose any $\Delta_0 > 0$, and take $\Delta = \Delta_0$].

Suppose $x \leqq x_k + \Delta_0$, then for some m, $(m = 0, 1, \cdots)$

$$x^* + \Delta/2^{m+1} \leqq x \leqq x^* + \Delta/2^m,$$

and we set

$$x^L = x^* + \Delta/2^{m-1}.$$

If $x > x_k + \Delta_0$, set $x^L = x_k + \Delta_0$. Then

$$\phi(x, p, b^\dagger(x^L, p)) \leqq \phi(x^L, p, b^\dagger(x^L, p)) < \Phi(x^L, p) + \varepsilon < \Phi(x, p) + 2\varepsilon$$

(using (i) of Lemma 3.2).

Let us define, for $x < \infty$, where x^*, Δ, m are to be determined as above,

$$\begin{aligned} b^{\dagger\dagger}(x, p) &= b^\dagger(x^* + \Delta/2^{m+1}, p) \qquad \text{if} \quad x \neq x_i \ (i \neq 0, \cdots, k) \\ &= b^\dagger(x_i, p) \qquad \text{otherwise.} \end{aligned}$$

Then, if $x \neq x_i (i = 0, \cdots k)$, we have from the last result,

$$\phi(x, p, b^{\dagger\dagger}(x, p)) \leqq \Phi(x, p) + 2\varepsilon.$$

If $x = x_i$, some i, this remains trivially true. Hence we have obtained the desired function, which is clearly measurable, and elementary in x.

LEMMA 3.4. *With q, ϕ and Φ as above, and the additional conditions on q in the previous lemma, and with $p \in [0, 1]$; given $\varepsilon > 0$, there exists a function $b(x, p)$ Borel-measurable in x, p, (in fact elementary in x, for each p, and simple in p for each x) and satisfying $0 \leqq b(x, p) \leqq x$ such that*

$$\phi(x, p, b(x, p)) \leqq \Phi(x, p) + \varepsilon$$

for all x and p.

PROOF. Let $\varepsilon > 0$. Construct a sequence $0 = p_0 < p_1 < \cdots p_m = 1$ with $p_i - p_{i-1} < \varepsilon/2$ for $i = 1, \cdots, m$. Now let $b^{\dagger\dagger}(x, p_i)$ be as in the previous lemma (with $\varepsilon' = \varepsilon/2$) for $i = 0, \cdots, m$.
Define

$$b(x, p) = b^{\dagger\dagger}(x, p_i) \quad \text{if} \quad p_{i-1} < p \leqq p_i, \qquad i = 1, \cdots, m$$

(of course, $b(x, 0) = b^{\dagger\dagger}(x, 0)$). Thus, $b(x,p)$ is clearly Borel-measurable in (x,p), elementary in x for fixed p, and simple in p for fixed x.
Furthermore, for $p_{i-1} < p \leqq p_i$,

$$\begin{aligned} \phi(x, p, b(x, p)) &= \phi(x, p, b^{\dagger\dagger}(x, p_i)) \\ &\leqq \phi(x, p_i, b^{\dagger\dagger}(x, p_i)) + \varepsilon/2 && \text{(Lemma 3.1. (ii))} \\ &\leqq \Phi(x, p_i) + \varepsilon/2 + \varepsilon/2 && \text{(Lemma 3.2.)} \\ &\leqq \Phi(x, p) + \varepsilon && \text{(Lemma 3.1. (iv))} \end{aligned}$$

completing the proof.

We can now establish the following lemma, which shows the usual "dynamic programming" property of the q_n function. The method follows Ferguson ([3], Lemma 3), where there were no measurability difficulties.

LEMMA 3.5 (Model I). *If* $q_n(x) = \inf_{\{b_j\} \in S} E[q_0(X_n) \mid \{b_j\}, X_0 = x]$ *then for* $n = 1, 2, \cdots$

$$q_n(x) = \int \inf_{0 \leqq b \leqq x} [p q_{n-1}(x+b-1) + (1-p) q_{n-1}(x-b-1)] \, dF(p).$$

REMARK. I am indebted to the referee for the following remark. The lemma is almost a special case of Dubins and Savage [2], Theorem 2.15.2. Our problem is a gambling problem in their sense with fortunes (x, p), utility function $u(x, p) = -q_0(x)$, and gambles defined in the natural way. Also, our restriction to measurable strategies does not change q_n; this follows from Theorem 2.2 of Sudderth [5].

PROOF. Let $\{b_j^\varepsilon\} \in S$ be ε-optimal in the sense that

$$q_n(x)(1+\varepsilon) \geqq E[q_0(X_n) \mid \{b_j^\varepsilon\}, X_0 = x]. \tag{3.5}$$

Then, for general x^*,

$$\begin{aligned} &E[q_0(X_n) \mid \{b_j^\varepsilon\}, X_0 = x, p_1 = p, X_1 = x^*] \\ &\quad = E[q_0(X_{n-1}) \mid \{b_j^{(x,p)}\}, X_0 = x^*] \\ &\quad \geqq \inf_{b \in S} E[q_0(X_{n-1}) \mid b, X_0 = x^*] = q_{n-1}(x^*) \end{aligned}$$

where $b_j^{(x,p)}(x_0, p_1, \cdots, p_j) = b_{j+1}^\varepsilon(x, p, x_0, p_1, \cdots, p_j)$. Then $\{b_j^{(x,p)}\} \in S$ for all (x,p).

The inequality in the above follows since the infimum refers to all $b \in S$, and the preceding expression refers to one particular betting strategy, $\{b_j^{(x,p)}\}$.

Then, continuing from (3.5), and setting $x^* = x+b_1{}^\varepsilon(x,p)-1$, and $x-b_1{}^\varepsilon(x,p)-1$ in the result just obtained

$$(3.6)\qquad E[q_0(X_n)\,|\,\{b_j{}^\varepsilon\},\,X_0 = x]$$
$$= \int \{pE[q_0(X_n)\,|\,\{b_j{}^\varepsilon\},\,X_0 = x,\,p_1 = p,\,X_1 = x+b_1{}^\varepsilon(x,p)-1]$$
$$+(1-p)E[q_0(X_n)\,|\,\{b_j{}^\varepsilon\},\,X_0 = x,\,p_1 = p,\,X_1 = x-b_1{}^\varepsilon(x,p)-1]\}\,dF(p)$$
$$\geqq \int [pq_{n-1}(x+b_1{}^\varepsilon(x,p)-1)+(1-p)q_{n-1}(x-b_1{}^\varepsilon(x,p)-1]\,dF(p)$$
$$\geqq \int \inf_{0\leqq b\leqq x}[pq_{n-1}(x+b-1)+(1-p)q_{n-1}(x-b-1)]\,dF(p).$$

This holds for all $\varepsilon > 0$ in (3.5); hence

$$(3.7)\qquad q_n(x) \geqq \int \inf_{0\leqq b\leqq x}[pq_{n-1}(x+b-1)+(1-p)q_{n-1}(x-b-1)]\,dF(p).$$

We now prove the reverse inequality:

$$(3.8)\qquad q_n(x) \leqq \int \inf_{0\leqq b\leqq x}[pq_{n-1}(x+b-1)+(1-p)q_{n-1}(x-b-1)]\,dF(p).$$

We proceed by induction on index n; it is convenient to state the inductive hypothesis for index $n-1$:

Given $\varepsilon > 0$, there exists a betting strategy b^{**} such that, for all x,

(i) $[q_{n-1}(x) \leqq\,]E[q_0(X_{n-1})\,|\,b^{**},\,X_0 = x]$
$$\leqq \int \inf_{0\leqq b\leqq x}[pq_{n-2}(x+b-1)+(1-p)q_{n-2}(x-b-1)]\,dF(p)+\varepsilon$$
$$[\,\leqq q_{n-1}(x)+\varepsilon],$$

where the last inequality follows automatically from (3.7).

(ii) For $j = 1,\cdots n-1$, we have the following conditions on $b_j{}^{**}$.

$b_j{}^{**}(x,p_1,x_1,\cdots x_{j-1},p_j)$ is an elementary function of x.

If $x,p_1,\cdots x_{j-1}$ are fixed, $b_j{}^{**}(x,p_1,x_1,\cdots x_{j-1},p_j)$ is simple in p_j.

[Note that $b^{**}\in S$]. By Lemma 3.4, the hypothesis is clearly true for index 1. We choose, for $b_1{}^{**}$, the betting strategy $b(x,p)$ in the statement of that lemma.

Next, assume the hypothesis is true for index $n-1$. We establish it for index n. Let $\varepsilon < 0$,

$$\phi_{n-1}(x,\,p,\,b) = pq_{n-1}(x+b-1)+(1-p)q_{n-1}(x-b-1)$$
$$\Phi_{n-1}(x,\,p) = \inf_{0\leqq b\leqq x}\phi_{n-1}(x,\,p,\,b).$$

The expression $\Phi_{n-1}(x,p)$ is nonincreasing with p, for fixed x, by Lemma 3.2 part (iv), and Lemma 3.1.

Given $\varepsilon > 0$, and using Lemma 3.4 choose $b^\dagger(x,p), 0 \leqq b^\dagger(x,p) \leqq x$ so that $\phi_{n-1}(x,p,b^\dagger(x,p)) < \Phi_{n-1}(x,p)+\varepsilon$, and $b^\dagger(x,p)$ is elementary in x for fixed p, and simple in p for fixed x.

We define a betting strategy b^* with $b_1{}^*(x,p_1) = b^\dagger(x,p_1)$ and complete the definition as follows: for $j = 2,3,\cdots$

$$b_j{}^*(x,\,p_1,\,x_1,\,\cdots,\,x_{j-1},\,p_j) = b_{j-1}^{**}(x_1,\,p_2,\,\cdots,\,x_{j-1},\,p_j)$$

where b^{**} was specified in the inductive hypothesis (for index $n-1$). b^* clearly satisfies condition (i) of the inductive hypothesis. Also, for all x,

$$\begin{aligned}
&E(q_0(X_n) \mid \{b_j^*\}, X_0 = x) \\
&\quad = \int E(q_0(X_n) \mid \{b_j^*\}, X_0 = x, p_1 = p)\, dF(p) \\
&\quad = \int [pE(q_0(X_n) \mid \{b_j^*\}, X_0 = x, p_1 = p, X_1 = x + b_1^*(x, p_1) - 1) \\
&\qquad + (1-p)E(q_0(X_n) \mid \{b_j^*\}, X_0 = x, p_1 = p, x_1 = x - b_1^*(x, p_1) - 1)]\, dF(p) \\
&\quad = \int [pE(q_0(X_{n-1}) \mid \{b_j^{**}\}, X_0 = x + b^\dagger(x, p) - 1) \\
&\qquad + (1-p)E(q_0(X_{n-1}) \mid \{b_j^{**}\}, x_0 = x - b^\dagger(x, p) - 1)]\, dF(p) \\
&\quad \leqq \int [pq_{n-1}(x + b^\dagger(x, p) - 1) + (1-p)q_{n-1}(x - b^\dagger(x, p) - 1)]\, dF(p) + \varepsilon \\
&\qquad\qquad \text{[using the last inequality in (i) of the hypothesis]} \\
&\quad = \int \phi_{n-1}(x, p, b^\dagger(x, p))\, dF(p) + \varepsilon \\
&\quad \leqq \int \Phi_{n-1}(x, p)\, dF(p) + \varepsilon + \varepsilon \\
&\qquad\qquad \text{[by choice of } b^\dagger(x,p)\text{]}.
\end{aligned}$$

Hence

$$q_n(x) \leqq E[q_0(X_n) \mid \{b_j^*\}, X_0 = x] \leqq \int \Phi_{n-1}(x, p)\, dF(p) + 2\varepsilon.$$

The second inequality obviously establishes part (ii) of the induction hypothesis; and the first and last members of the chain establish (3.8), since ε is arbitrary. This completes the proof.

4. A special choice for loss function. For the remainder of the paper, we assume the game is favorable. A special choice for loss function is

$$q_0^*(x) = 1(x < 1), \quad = 0(x \geqq 1). \tag{4.1}$$

The minimal expected loss after n games is written $q_n^*(x)$ and may be identified with P(ruin by game n):

$$q_n^*(x) = \inf_b E\{q_0^*(X_n) \mid b, X_0 = x\} = \inf_b P(X_n < 1 \mid b, X_0 = x). \tag{4.2}$$

For each n, clearly $q_n^*(x) \leqq q(x)$, and is nonincreasing in x, by Lemma 3.1.

LEMMA 4.1. (i) $q_n^*(x) \leqq q_{n+1}^*(x)$ *all* x, *and* $n = 0, 1, 2, \cdots$

(ii) $q_n^*(x) \to q(x)$ *uniformly in* x *as* $n \to \infty$ [*the following equivalent form is used in the sequel: given* $\varepsilon > 0$, *there exists* N *such that for all* x *and all* $n \geqq N$,

$$q(x) - \inf_b P(X_n < 1 \mid b, X_0 = x) < \varepsilon].$$

PROOF. (i) Obvious.

(ii) Since $q_n^*(x)$ is nondecreasing in n, and bounded above, the limit as $n \to \infty$ exists and $\leqq q(x)$ for all x since $q_n(x) \leqq q(x)$.

Next, given $\varepsilon > 0$ (and without loss of generality, $\varepsilon < 1$) choose $x_1 > 1$ such that

$$q(x_1) < \varepsilon (< 1) \quad \text{(possible, since the game is favorable).} \tag{4.3}$$

We show there exists an N such that

$$|q(x)-\inf_b P(X_n < 1 \mid b, X_0 = x)| \leqq 2\varepsilon \quad \text{all} \quad n \geqq N, \quad \text{all} \quad x \in [1, x_1].$$

We need two preliminary results:

(A) Let $\underline{b}^M$ denote the stationary betting strategy

$$\begin{aligned} &\text{for} \quad x \geqq M: \quad b^M(x, p) = 1(p \leqq \tfrac{1}{2}), \qquad = M(p > \tfrac{1}{2}). \\ &\text{for} \quad x < M: \quad b^M(x, p) = x \end{aligned} \tag{4.4}$$

in which M has been chosen so that if $x \geqq M$, the expected gain on a single game is positive. Then, ([3], page 811), $q_{\underline{b}^M}(x) \to 0$ as $x \to \infty$. Then, with the ε, x_1 above, we may choose $x_2 \geqq x_1$ such that

$$\inf_{x' \geqq x_2} P(X_m \geqq 1 \quad \text{all} \quad m \mid b^M, X_0 = x') > \{1+\varepsilon[1-q(x_1)]^{-1}\}^{-1}. \tag{4.5}$$

This choice is possible since the latter quantity < 1; also note that the former quantity > 0.

(B) From Lemma 2.1.(i), given $x_1 > 1$; $x_2 \geqq x_1$; $\varepsilon < 0$, we may choose N such that

$$P(X_n \in [1, x_2] \quad \text{all} \quad n \leqq N \mid b, X_0 = x) < \varepsilon \tag{4.6}$$

for all b, all $x \in [1, x_1]$.

Given an arbitrary betting system b, construct a betting system b^* as follows: starting with $x_0 = x < x_2$, use b until (if ever) $X_n > x_2$, then use the $\underline{b}^M$ of (4.4). If $X_n > x_2$ for some $n \leqq N$, let T_1 = smallest such n. Set $T_1 = \infty$ otherwise. Then

$$\begin{aligned} &P(X_m \geqq 1 \quad \text{all} \quad m \mid b^*, X_0 = x) \\ &\quad \geqq \textstyle\sum_{n=1}^N P(X_m \geqq 1 \quad \text{all} \quad m, T_1 = n \mid b^*, X_0 = x) \\ &\quad = \textstyle\sum_{n=1}^N P(T_1 = n \mid b^*, X_0 = x) P(X_m \geqq 1 \quad \text{all} \quad m \mid T_1 = n, b^*, X_0 = x). \end{aligned} \tag{4.7}$$

But

$$\begin{aligned} &P(X_m \geqq 1 \quad \text{all} \quad m \mid T_1 = n, b^*, X_0 = x) \\ &\quad = \int P(X_m \geqq 1 \quad \text{all} \quad m \mid X_n = x', T_1 = n, b^*, X_0 = x) \\ &\qquad\qquad \cdot dP(X_n \leqq x' \mid T_1 = n, b^*, X_0 = x) \\ &\quad \geqq \inf_{x' \geqq x_2} P(X_m \geqq 1 \quad \text{all} \quad m \mid b^M, X_0 = x'). \end{aligned} \tag{4.8}$$

Combining (4.7), (4.8)

$$\begin{aligned} &P(X_m \geqq 1 \quad \text{all} \quad m \mid b^*, X_0 = x) \\ &\quad \geqq P(X_n > x_2 \quad \text{for some} \quad n \leqq N \mid b, X_0 = x) \\ &\qquad\qquad \cdot \inf_{x' \geqq x_2} P(X_m \geqq 1 \quad \text{all} \quad m \mid b^M, X_0 = x'). \end{aligned}$$

We now combine these two results (A) and (B). For $x \in [1, x_1]$,

$$(4.9) \qquad \begin{aligned} 1-q(x) &= \sup_b P(X_m \geqq 1 \quad \text{all} \quad m \mid b, X_0 = x) \\ &\geqq \sup_b P(X_n > x_2 \quad \text{for some} \quad n \leqq N \mid b, X_0 = x) \\ &\qquad \cdot \inf_{x' \geqq x_2} P(X_m \geqq 1 \quad \text{all} \quad m \mid b^M, X_0 = x'). \end{aligned}$$

Dividing through by the last expression, and using (4.5),

$$(4.10) \qquad \begin{aligned} &\sup_b P(X_n > x_2 \quad \text{for some} \quad n \leqq N \mid b, X_0 = x) \\ &\leqq [1-q(x)]\{1+\varepsilon[1-q(x_1)]^{-1}\} \leqq [1-q(x)]\{1+\varepsilon[1-q(x)]^{-1}\} \\ &= 1-q(x)+\varepsilon \end{aligned}$$

since for $x \leqq x_1, q(x) \geqq q(x_1)$.
Now, with N chosen in (B), for $x \in [1, x_1]$,

$$\begin{aligned} q_N^*(x) &= \inf_b P(X_N < 1 \mid b, X_0 = x) \\ &= 1 - \sup_b P(X_n \geqq 1 \quad \text{all} \quad n \leqq N \mid b, X_0 = x). \end{aligned}$$

But, using (4.10) and (4.6)

$$\begin{aligned} &\sup_b P(X_n \geqq 1 \quad \text{all} \quad n \leqq N \mid b, X_0 = x) \\ &\leqq \sup_b [P(X_n > x_2 \quad \text{for some} \quad n \leqq N \mid b, X_0 = x) \\ &\qquad + P(X_n \in [1, x_2] \quad \text{all} \quad n \leqq N \mid b, X_0 = x)] \\ &\leqq [1-q(x)+\varepsilon]+\varepsilon = 1-q(x)+2\varepsilon. \end{aligned}$$

Therefore

$$q_N^*(x) \geqq 1-[1-q(x)+2\varepsilon] = q(x)-2\varepsilon.$$

Thus $q(x)-q_N^*(x) \leqq 2\varepsilon$ for $x \leqq x_1$, and in fact for all x by choice of x_1 in (4.3). This proves (ii).

5. Other loss functions. In the last section we made a special choice of loss function, q_0^*. The limit as $n \to \infty$ of the derived minimal expected loss functions q_n^* was shown to be the ruin function q. We now show that this same property holds for any loss function q_0 satisfying (3.1). Note that for all x, $q_0^*(x) \leqq q_0(x)$; hence clearly $q_n^*(x) \leqq q_n(x)$ for all n. Recall the definition of the ruin function for strategy b, in (2.2).

THEOREM 5.1 (Both Models). *With any loss function chosen as in* (3.1),

(A) *for any betting strategy b we have*

$$E[q_0(X_n) \mid b, X_0 = x] \to q^b(x) \quad as \quad n \to \infty.$$

(B) $$q_n(x) \to q(x) \quad as \quad n \to \infty.$$

PROOF. Given $\varepsilon > 0$, find x_1 such that $q_0(x_1) < \varepsilon$.

(A) Given b, use Lemma 2.1 (ii) to find N such that for $n > N$, $P\{X_n \in [1, x_1] \mid b, X_0 = x\} < \varepsilon$. Then, for $n > N$,

$$\begin{aligned} P(X_n < 1 \mid b, X_0 = x) &\leqq E[q_0(X_n) \mid b, X_0 = x] \\ &\leqq P[X_n < 1 \mid b, X_0 = x] + P[X_n \in [1, x_1] \mid b, X_0 = x] \\ &\qquad + E[q_0(X_n) I_{(x_1, \infty)}(X_n) \mid b, X_0 = x] \\ &\leqq P[X_n < 1 \mid b, X_0 = x] + 2\varepsilon. \end{aligned}$$

Let $n \to \infty$; the result follows, recalling that $P(X_n < 1 \mid b, X_0 = x) \to q^b(x)$.

(B) Given $\varepsilon > 0$, choose $b(\varepsilon)$ such that

$$q^{b(\varepsilon)}(x) < q(x) + \varepsilon.$$

From (A), there exists N such that for all $n \geqq N$

$$E\{q_0(X_n) \mid b(\varepsilon), X_0 = x\} < q^{b(\varepsilon)}(x) + \varepsilon < q(x) + 2\varepsilon.$$

Therefore, for all $n \geqq N$

$$q_n^*(x) \leqq q_n(x) = \inf_b E\{q_0(X_n) \mid b, X_0 = x\} < q(x) + 2\varepsilon.$$

Let $n \to \infty$; the result follows.

6. The continuity of $q(x)$ in Model I. With q^* the special choice of the loss function in Section 4, for any closed interval $\Delta = [x_1, x_2]$, and positive integer n, define

$$\text{Var}_n(\Delta) = q_n^*(x_1) - q_n^*(x_2) \quad (\leqq 1, \text{clearly}). \tag{6.1}$$

LEMMA 6.1 (Model I). *If $F(p)$ is not concentrated on the set $\{0, 1\}$, then there exists an $\eta > 0$, not dependent on n, such that the existence of an interval Δ_0 for which*

$$\text{Var}_n(\Delta_0) = K, \ |\Delta_0| = h < \tfrac{1}{2} \tag{6.2}$$

implies the existence of an interval Δ_1 for which

$$\text{Var}_{n-1}(\Delta_1) > K(1+\eta), \qquad |\Delta_1| \leqq 2h. \tag{6.3}$$

PROOF. Choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \tfrac{1}{2}, \tag{6.4}$$

$$m = m(\varepsilon) = F(1-\varepsilon) - F(\varepsilon) > 0 \tag{6.5}$$

and set

$$\tau = (\tfrac{1}{2})\varepsilon^2 m K. \tag{6.6}$$

The choice

$$\begin{aligned} \eta &= \min(\varepsilon, (\tfrac{1}{4})\varepsilon^2 m/(1-m)), \qquad & m < 1 \\ &= \varepsilon & m = 1 \end{aligned}$$

will be shown to satisfy the statement of the lemma. It is convenient to introduce the notation

$$\phi_n(x, p, b) = pq_n^*(x+b-1)+(1-p)q_n^*(x-b-1).$$

Let $\Delta = [x, x+h]$ be such an interval as defined in (6.2). Now it is clear from the proof of Lemma 3.4, or from standard measurable selection theorems such as 6.3 of Mackey [4], that we can find a function $y_2(p)$, Borel-measurable in p, $0 \leqq y_2(p) \leqq x+h$, such that

$$\int dF(p)\phi_{n-1}(x+h, p, y_2(p)) < q_n^*(x+h)+\tau. \tag{6.7}$$

We divide the interval [0, 1] into two sets, $S_1 = \{p : 0 \leqq y_2(p) \leqq x-h\}$, $S_2 = \{p : x-h < y_2(p) \leqq x+h\}$ and specify a betting function y_1 by

$$\begin{aligned} y_1(p) &= y_2(p)+h \quad \text{if} \quad p \in S_2 \\ &= x \qquad\qquad \text{if} \quad p \in S_1. \end{aligned}$$

[Note: the proof breaks down here for Model II. Take $x < 1, x+h = 1$; $y_2(p) = 1$. Then the bet x is not permitted, since $x < 1$].

Then $y_1(p) \varepsilon [0, x]$. Now, for fixed p,

$$\phi_{n-1}(x, p, y_1(p)) \geqq \inf_{0 \leqq b \leqq x} \phi_{n-1}(x, p, b).$$

Hence, from Lemma 3.5

$$\int dF(p)\phi_{n-1}(x, p, y_1(p)) \geqq q_n^*(x). \tag{6.8}$$

Using (6.7), (6.8) we may now write

$$\begin{aligned} K &= q_n^*(x)-q_n^*(x+h) \\ &\leqq \int dF(p)[\phi_{n-1}(x, p, y_1(p) \ -\{\phi_{n-1}(x+h, p, y_2(p))-\tau\}] \\ &= \int dF(p)D(p)+\tau \end{aligned} \tag{6.9}$$

where

$$\begin{aligned} D(p) = &\{pq_{n-1}^*(x-1+y_1(p))+(1-p)q_{n-1}^*(x-1-y_1(p))\} \\ &-\{pq_{n-1}^*(x+h-1+y_2(p))+(1-p)q_{n-1}^*(x+h-1-y_2(p))\}. \end{aligned}$$

Two cases are possible:

(i) If $p \in S_2$, we get on substituting $y_1 = y_2+h$,

$$D(p) = (1-p)\operatorname{Var}_{n-1}(\Delta_1'') \tag{6.10}$$

where $\Delta_1'' = [x-h-1-y_2, \ x+h-1-y_2]$.

(ii) Now suppose $p \in S_1$, i.e. $x-h < y_2 \leqq x+h$ and y_1 is chosen equal to x.

Using the fact that $q_{n-1}^*(x)$ is nonincreasing, and the hypothesis $h < \frac{1}{2}$, we can show

$$D(p) \leqq p\operatorname{Var}_{n-1}(\Delta_1') \tag{6.11}$$

where $\Delta_1' = [2x-1, 2x+2h-1]$.

Write

$$\delta(p) = (1-p)\,\mathrm{Var}_{n-1}(\Delta_1'') \quad \text{if} \quad p \in S_2$$
$$= p\,\mathrm{Var}_{n-1}(\Delta_1') \quad \text{if} \quad p \in S_1. \tag{6.12}$$

With the value of ε chosen at the start, let p be any value of the win probability for which $\varepsilon \leqq p \leqq 1-\varepsilon$ and $\delta(p) \geqq K(1-\varepsilon^2)$, if one exists. There are two possibilities:

(a) Suppose there exists such a p. If $p \in S_2$, then $\delta_2(p) > K(1-\varepsilon^2)/(1-p) > K(1+\varepsilon)$. If $p \in S_1$, then $\delta_1(p) > K(1-\varepsilon^2)/p > K(1+\varepsilon)$. In either event, we have obtained an interval Δ_1, namely Δ_1' or Δ_1'', for which $|\Delta_1| \leqq 2h$, $\mathrm{Var}_{n-1}(\Delta_1) \geqq K(1+\eta)$, since $\eta \leqq \varepsilon$.

(b) Suppose on the other hand $\delta(p) < K(1-\varepsilon^2)$ for all $p \in [\varepsilon, 1-\varepsilon]$. Let

$$\sup_{[0,\varepsilon)\cup(1-\varepsilon,1]} \delta(p) = \delta'.$$

Then from (6.9), (6.10), (6.11), (6.12)

$$K \leqq \int dF(p)\,\delta(p) + \tau$$
$$= \int_{[0,\varepsilon)\cup(1-\varepsilon,1]} \delta(p)\,dF(p) + \int_{[\varepsilon,1-\varepsilon]} \delta(p)\,dF(p) + \tau$$
$$\leqq (1-m)\,\delta' + K(1-\varepsilon^2)m + (\tfrac{1}{2})\varepsilon^2 mK, \qquad \text{using (6.6).}$$

If $m = 1$, contradiction; so that (a) must hold. Assume $m < 1$.

$$\delta' \geqq K[1-(1-\varepsilon^2/2)m]/(1-m) = K[1+\varepsilon^2 m/2(1-m)].$$

Since δ' was the supremum, the value $\delta(p) = K(1+\eta)$ is certainly surpassed for some $p = p^*$ since

$$\eta \leqq \varepsilon^2 m/4(1-m).$$

Hence $\delta(p^*) \geqq K(1+\eta)$, which yields the required interval, as in (a).

THEOREM 6.2 (Model I). *If $F(p)$ is not concentrated on 1, $q(x)$ is continuous for all x.*

(i) Suppose $F(p)$ gives positive mass to (0, 1). Suppose there exists a discontinuity of $q(x)$ at x_1, say, of size $2K$, $K > 0$. Clearly $x_1 \geqq 1$.

Find $\eta > 0$ satisfying Lemma 6.1. Choose j, an integer, such that $(1+\eta)^j > K^{-1}$. Let $h = (\frac{1}{2})^{j+1}$. Since q is nonincreasing,

$$q(x_1-h) - q(x_1+h) \geqq 2K.$$

Since $q_n^*(x) \to q(x)$ all x, (Lemma 4.1) there exists N_0 such that

$$q_N^*(x_1-h) - q_N^*(x_1+h) > K \quad \text{all} \quad N \geqq N_0.$$

Using

$$\Delta_0 = [x_1-h,\, x_1+h], \qquad n = \max\{N_0, j\}$$

apply the result of Lemma 6.1 recursively j times. We end up with an interval Δ_j for which $\mathrm{Var}_{N-j}(\Delta_j) > K(1+\eta)^j > 1$ by choice of j. Contradiction.

(ii) It remains to show that if $F(p)$ is concentrated at $\{0, 1\}$, but not at $\{1\}$, $q(x)$ is continuous.

Suppose $F(p)$ has the distribution $P(p = 0) = L$, $P(p = 1) = W = 1-L$, where $L > 0$ (the case $L = 0$ is dealt with in the remark following).

The betting strategy $\{b^\dagger(x,p)\}$ where $b(x, 1) = x, b(x, 0) = 0$ (allowable for Model I) is clearly optimal. Suppose the largest discontinuity of $q(x)$ occurs at x and has size K. Then for all $h>0$,

$$q(x-h)-q(x+h) \geqq K.$$

However, starting at $x-h$, we may apply the first bet of our optimal strategy to obtain

$$q(x-h) = Wq(2x-2h-1)+Lq(x-h-1)$$

and similarly $q(x+h) = Wq(2x+2h-1)+Lq(x+h-1)$.
Hence

$$L[q(x-h-1)-q(x+h-1)]+W[q(2x-2h-1)-q(2x+2h-1)] \geqq K$$

for all $h > 0$.
Clearly, there must exist a discontinuity of $q(x)$ at $x-1$ of size K. By induction, there exists a discontinuity of size K at $x-j, j = 1, 2, \cdots$, which leads to contradiction, since $q(x)$ is identically 1 for $x \leqq 1$.

REMARK. If $p = 1$ with probability 1, then

$$q(x) = 1 \quad \text{for} \quad x < 1, \quad = 0 \quad \text{for} \quad x \geqq 1$$

so that $q(x)$ has a discontinuity at $x = 1$.

7. Stationary Markov betting strategies are optimal. For Model I, we have shown that $q(x)$ is continuous. For Model II, Breiman, Section 3 of [1], showed that $q(x)$ is lower semi-continuous (we note that the result of Theorem 5.1(A) is assumed in the argument). The remainder of this section uses only the lower semi-continuity of $q(x)$ and hence holds for both Models. With $\Phi(x,p) = \inf_b \phi(x,p,b)$ as defined in (3.3), and with the appropriate range for b understood, let $B(x,p) = \{b : \phi(x,p,b) = \Phi(x,p)\}$. We may choose $b^*(x,p) \in B(x,p)$ such that $b^*(x,p)$ is jointly Borel-measurable. (A proof of this statement can be based on Lemma 3.4).

Given this b^* we define a particular stationary Markov betting function by

$$b_j^* = b^* \quad \text{for all} \quad j.$$

Assuming the existence of such a b^*, Ferguson ([3], pages 803–804) showed

$$q(x) = q^{b^*}(x).$$

[His proof is in the context of a game with allowable bets of \$1 and \$2 only, but holds also for Models I and II with trivial modifications.]

Thus $\{b_j^*\}$ is a (not necessarily unique) optimal betting function.

8. Asymptotic forms for the ruin function and optimal betting function, for large fortunes. If $F(p)$ has no mass in the neighborhood of $p = 1$, then, as $x \to \infty$,

(i) the ruin function $q(x) \sim cr^x$, where c, r are constants, and $c > 1$, $0 < r < 1$ (r is specified in terms of $F(p)$, but no general formula for c has been found) [Breiman [1]].

(ii) the optimal (stationary) betting function, $b^*(x,p)$, $\sim b^0(p)$ as $x \to \infty$ where $b^0(p)$ is an uncomplicated expression involving p and r [Ferguson [3]].

What can we say if $F(p)$ is unrestricted (although we exclude the trivial case where it is concentrated at 1)? With r, $b^0(p)$ as above,

(A) $q(x) \geqq r^x$ all x, and as $x \to \infty \limsup [q(x)r^{-x}x] \leqq \exp[(\text{mass of } F \text{ at } p = 1)/r]$.

(B) Given $0 < \rho < 1, \eta > 0, \delta > 0, R > r$, there exists a stationary Markov betting strategy $b = b(x,p)$ and an x_0 such that

(a) $|q^b(x) - q(x)| < \delta R^x$ all x.

(b) $|b(x,p) - b^0(p)| < \eta$ all $x \geqq x_0$, and all $p \in [0, \rho]$.

Details are in Truelove [6].

For one specific case, stronger results have been obtained (David Cantor and Thomas S. Ferguson, paper in preparation). If $F(p)$ has mass concentrated at $p = 0, 1$, then (i) and (ii) hold and the constant c can be calculated from a formula.

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