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PRODUCTS OF TWO POLYKAYS WHEN ONE HAS WEIGHT 5[1]

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1. Introduction and summary. Fisher [5] introduced a combinatorial method to obtain sampling cumulants of k-statistics as linear functions of cumulants of an infinite parent population. Kendall [6] systematized Fisher's combinatorial method by providing rules for the same and their proofs. Tukey [11] considered the sample statistic $k_{rs\cdots}$ in order to simplify the presentation of sampling moment formulae of the k-statistics when samples are drawn from a finite population. These $k_{rs\cdots}$, termed generalized k-statistics by Abdel-Aty [1] and polykays[2] by Tukey [12], were in fact considered earlier by Dressel [3] for the seminvariant case ($r, s, \cdots \neq 1$). Wishart [13] modified the combinatorial method to obtain products of k-statistics as linear combinations of polykays, obtained products of polykays by algebraic manipulation, and applied these to the case of a finite population. He provided formulae for products of k-statistics through weight 8, and of polykays through weight 6. These have appeared again in David, Kendall and Barton ([2], 196–200, Table 2.3). Schaeffer and Dwyer [8] provided formulae for products of seminvariant polykays through weight 8. Tracy [9] supplied formulae for all products of polykays of weight 7.

Dwyer and Tracy [4] modified and extended the combinatorial method to obtain products of two polykays. They presented general formulae resulting from this method for products $k(P)k(Q)$, where $k(P) = k_P = k_{p_1\cdots p_\pi}$ is a polykay having any weight and weight$(Q) \leqq 4$. Such formulae may be looked upon as rules of multiplication of a polykay by another of weight up to 4. It is the purpose of this paper to extend these results to the case of weight$(Q) = 5$. The formulae are presented together for compactness in a tabular form in Table 1, each column of which reads a formula for some Q. Checks indicated in Section 4 are applied more easily in the tabular presentation. Illustrations showing the use of the formulae appear in Tables 2 and 3.

2. Notation. The formulae feature the notation and concepts of Dwyer and Tracy [4], to which reference might be made for a more detailed presentation of the notation and basic theory. Thus $P_i|2$ denotes a 2-part partition of p_i and has $C(P_i|2)$ for its combinatorial coefficient. The collection of polykays having the same pattern function and combinatorial coefficient is indicated by a symbolic addition $\oplus$.

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[2] Kendall and Stuart [7] call these multiple k-statistics or l-statistics and remark [7, p. 304] that "the l-statistics are called 'polykays' by some American authors, but we feel that there are limits to linguistic miscegenation which should not be exceeded." David, Kendall and Barton [2], however, call them polykays.

3. Use of Table 1. This table presents formulae for each of the seven cases $k(P)k(Q)$ where weight$(Q) = 5$. The seven products appear at the head of the columns. The linear expansion for each appears in that column. There are 74 terms possible in such expansions, which are listed and numbered. However, not every term appears in a given expansion. If there is no entry in a column against a term, it denotes that that term does not appear in the expansion, i.e. the corresponding coefficient is zero. The coefficient of a term in the expansion of a product is obtained by dividing the corresponding entry by the entry in the "Divisor" column. Terms are arranged in blocks according to a common divisor.

As an illustration, one would read

$$(1) \qquad k(P)k(5) = k(P5) + k(P\oplus 5)/n - 5k(P\oplus 41)/n - 10k(P\oplus 32)/n^{(2)} + \cdots .$$

The Σ's in terms number 37 through 74 indicate summations over all possible partitions of the indicated order. By the pattern rule [4], all terms represented under a given Σ have the same coefficient.

4. Checks. The formulae have been checked by various methods. A useful expression for checking all the seven formulae simultaneously is [4]

$$(2) \qquad k(P)k^5(1) = \sum \binom{5}{u} C(U) \sum C(T) k(P\oplus U, T)/n^{5-\tau}$$

where $0 \leqq u \leqq 5$, $t = 5-u$, τ is the order of T. The detailed expansion of $k(P)k^5(1)$, as obtained from (2), appears in the last column of Table 1. (Multiplication by n^5 saves writing denominators). Also $k^5(1)$ may be expressed as a linear function of polykays of weight 5, as in [13], so that

$$(3) \qquad k(P)k^5(1) = k(P)k(5)/n^4 + 5k(P)k(41)/n^3 + 10k(P)k(32)/n^3 + 10k(P)k(31^2)/n^2 + 15k(P)k(2^21)/n^2 + 10k(P)k(21^3)/n + k(P)k(1^5)$$

as shown in the last row of the table. A check for the coefficients of a particular term appearing in the various formulae is thus obtained by verifying that the entries in that row multiplied by the corresponding ones in the last row add up to the product of the last two entries in the row. Thus, for term number 35, $k(P\oplus 1^4, 1)$,

$$(4) \qquad -6(5n^2) + 4(10n^3) + 1(15n^3) - 3(10n^4) + 5(n^5) = n^{(4)}(5n).$$

Another check for Q involving unit parts ([4], page 1185) has also been applied.

5. Applications and illustration. The general formulae appearing in Table 1 may be used to obtain linear expansions for specific products by specifying P. To consider a simple illustration, we choose P of low weight, say $P = 21$, and consider the product $k(21)k(5)$. Here $P_i = 2$, $P_j = 1$ and many terms in the expansion of $k(P)k(5)$ do not appear because of the small number of parts involved. The terms which appear are listed in Table 2.

TABLE 1

Formulae for $k(P)k(Q)$ with weight $(Q)=5$

	Term	$k(P)k(5)$	$k(P)k(41)$	$k(P)k(32)$
1	$k(P5)$	1		
2	$k(P41)$		1	
3	$k(P32)$			1
4	$k(P31^2)$			
5	$k(P2^21)$			
6	$k(P21^3)$			
7	$k(P1^5)$			
8	$k(P\oplus 5)$	1		
9	$k(P\oplus 4, 1)$		1	
10	$k(P\oplus 3, 2)$			1
11	$k(P\oplus 2, 3)$			1
12	$k(P\oplus 1, 4)$		1	
13	$k(P\oplus 3, 1^2)$			
14	$k(P\oplus 2, 21)$			
15	$k(P\oplus 2, 1^3)$			
16	$k(P\oplus 1, 31)$			
17	$k(P\oplus 1, 2^2)$			
18	$k(P\oplus 1, 21^2)$			
19	$k(P\oplus 1, 1^4)$			
20	$k(P\oplus 41)$	-5	1	
21	$k(P\oplus 32)$	-10		1
22	$k(P\oplus 31, 1)$		-4	
23	$k(P\oplus 2^2, 1)$		-3	
24	$k(P\oplus 21, 2)$			-3
25	$k(P\oplus 1^2, 3)$			-1
26	$k(P\oplus 21, 1^2)$			
27	$k(P\oplus 1^2, 21)$			
28	$k(P\oplus 1^2, 1^3)$			
29	$k(P\oplus 31^2)$	20	-4	-1
30	$k(P\oplus 2^21)$	30	-3	-3
31	$k(P\oplus 21^2, 1)$		12	
32	$k(P\oplus 1^3, 2)$			2
33	$k(P\oplus 1^3, 1^2)$			
34	$k(P\oplus 21^3)$	-60	12	5
35	$k(P\oplus 1^4, 1)$		-6	
36	$k(P\oplus 1^5)$	24	-6	-2

$k(P)k(31^2)$	$k(P)k(2^21)$	$k(P)k(21^3)$	$k(P)k(1^5)$	Divisor	$n^5k(P)k^5(1)$
				1	n
				1	$5n^2$
				1	$10n^2$
1				1	$10n^3$
	1			1	$15n^3$
		1		1	$10n^4$
			1	1	n^5
				n	1
				n	$5n$
				n	$10n$
				n	$10n$
				n	$5n$
1				n	$10n^2$
	2			n	$30n^2$
		1		n	$10n^3$
2				n	$20n^2$
	1			n	$15n^2$
		3		n	$30n^3$
			5	n	$5n^4$
				$n^{(2)}$	5
				$n^{(2)}$	10
2				$n^{(2)}$	$20n$
	1			$n^{(2)}$	$15n$
	2			$n^{(2)}$	$30n$
1				$n^{(2)}$	$10n$
-3		3		$n^{(2)}$	$30n^2$
	-2	3		$n^{(2)}$	$30n^2$
		-1	10	$n^{(2)}$	$10n^3$
1				$n^{(3)}$	10
	1			$n^{(3)}$	15
-6	-2	3		$n^{(3)}$	$30n$
	-2	1		$n^{(3)}$	$10n$
2		-3	10	$n^{(3)}$	$10n^2$
-3	-2	1		$n^{(4)}$	10
4	1	-3	5	$n^{(4)}$	$5n$
2	1	-1	1	$n^{(5)}$	1

TABLE 1—*continued*

	Term	$k(P)k(5)$	$k(P)k(41)$	$k(P)k(32)$
37	$\sum C(P_i\,\|\,2)k(P{:}P_i\,\|\,2\oplus 41)$	$5n$	-1	
38	$\sum C(P_i\,\|\,2)k(P{:}P_i\,\|\,2\oplus 32)$	$10n$		-1
39	$\sum C(P_i\,\|\,2)k(P1{:}P_i\,\|\,2\oplus 31)$		$4n$	
40	$\sum C(P_i\,\|\,2)k(P1{:}P_i\,\|\,2\oplus 2^2)$		$3n$	
41	$\sum C(P_i\,\|\,2)k(P2{:}P_i\,\|\,2\oplus 21)$			$3n$
42	$\sum C(P_i\,\|\,2)k(P3{:}P_i\,\|\,2\oplus 1^2)$			n
43	$\sum C(P_i\,\|\,2)k(P1^2{:}P_i\,\|\,2\oplus 21)$			
44	$\sum C(P_i\,\|\,2)k(P21{:}P_i\,\|\,2\oplus 1^2)$			
45	$\sum C(P_i\,\|\,2)k(P1^3{:}P_i\,\|\,2\oplus 1^2)$			
46	$\sum C(P_i\,\|\,3)k(P{:}P_i\,\|\,3\oplus 31^2)$	$10n^2$	$-4n$	$-n$
47	$\sum C(P_i\,\|\,3)k(P{:}P_i\,\|\,3\oplus 2^21)$	$15n^2$	$-3n$	$-3n$
48	$\sum C(P_i\,\|\,2)k(P\oplus 1{:}P_i\,\|\,2\oplus 31)$	$-20n$	$4n$	2
49	$\sum C(P_i\,\|\,2)k(P\oplus 1{:}P_i\,\|\,2\oplus 2^2)$	$-15n$	$3(n-1)$	3
50	$\sum C(P_i\,\|\,2)k(P\oplus 2{:}P_i\,\|\,2\oplus 21)$	$-30n$	6	$3n$
51	$\sum C(P_i\,\|\,2)k(P\oplus 3{:}P_i\,\|\,2\oplus 1^2)$	$-10n$	4	$n-1$
52	$\sum C(P_i\,\|\,3)k(P1{:}P_i\,\|\,3\oplus 21^2)$		$6n^2$	
53	$\sum C(P_i\,\|\,3)k(P2{:}P_i\,\|\,3\oplus 1^3)$			n^2
54	$\sum C(P_i\,\|\,3)k(P1^2{:}P_i\,\|\,3\oplus 1^3)$			
55	$\sum C(P_i\,\|\,2)k(P\oplus 2, 1{:}P_i\,\|\,2\oplus 1^2)$		$-6n$	
56	$\sum C(P_i\,\|\,2)k(P\oplus 1, 1{:}P_i\,\|\,2\oplus 21)$		$-12n$	
57	$\sum C(P_i\,\|\,2)k(P\oplus 1, 2{:}P_i\,\|\,2\oplus 1^2)$			$-3n$
58	$\sum C(P_i\,\|\,2)k(P\oplus 1, 1^2{:}P_i\,\|\,2\oplus 1^2)$			
59	$\sum C(P_i\,\|\,4)k(P{:}P_i\,\|\,4\oplus 21^3)$	$10n^2(n+1)$	$-6n(n+1)$	$-2n(2n-1)$
60	$\sum C(P_i\,\|\,3)k(P\oplus 1{:}P_i\,\|\,3\oplus 21^2)$	$-30n(n+1)$	$6(n^2+3)$	$6(2n-1)$
61	$\sum C(P_i\,\|\,3)k(P\oplus 2{:}P_i\,\|\,3\oplus 1^3)$	$-10n(n+1)$	$6(n+1)$	$(n+2)(n-1)$
62	$\sum C(P_i\,\|\,3)k(P\oplus 1, 1{:}P_i\,\|\,3\oplus 1^3)$		$-4n(n+1)$	
63	$\sum C(P_i\,\|\,2)k(P\oplus 21{:}P_i\,\|\,2\oplus 1^2)$	$60n$	$-6(n+3)$	$-3(2n-1)$
64	$\sum C(P_i\,\|\,2)k(P\oplus 1^2{:}P_i\,\|\,2\oplus 21)$	$60n$	$-12n$	$-3(n+2)$
65	$\sum C(P_i\,\|\,4)k(P1{:}P_i\,\|\,4\oplus 1^4)$		$n^2(n+1)$	
66	$\sum C(P_i\,\|\,2)k(P\oplus 1^2, 1{:}P_i\,\|\,2\oplus 1^2)$		$12n$	
67	$\sum C(P_i\,\|\,2)C(P_j\,\|\,2)k(P{:}P_i\,\|\,2\oplus 21{:}P_j\,\|\,2\oplus 1^2)$	$-10n(n-1)$	$6(n-1)$	n^2-2n+2
68	$\sum C(P_i\,\|\,5)k(P{:}P_i\,\|\,5\oplus 1^5)$	$n^3(n+5)$	$-n^2(n+5)$	$-n^2(n-1)$
69	$\sum C(P_i\,\|\,4)k(P\oplus 1{:}P_i\,\|\,4\oplus 1^4)$	$-5n^2(n+5)$	$n(n^2+2n+21)$	$5n(n-1)$
70	$\sum C(P_i\,\|\,3)k(P\oplus 1^2{:}P_i\,\|\,3\oplus 1^3)$	$20n(n+2)$	$-4(n^2+2n+6)$	$-(n^2+8n-8)$
71	$\sum C(P_i\,\|\,2)k(P\oplus 1^3{:}P_i\,\|\,2\oplus 1^2)$	$-60n$	$12(n+1)$	$5n$
72	$\sum C(P_i\,\|\,3)C(P_j\,\|\,2)k(P{:}P_i\,\|\,3\oplus 1^3{:}P_j\,\|\,2\oplus 1^2)$	$-n^2(n-1)$	$n(n-1)$	$n(n^2-2n+2)/10$
73	$\sum C(P_i\,\|\,2)C(P_j\,\|\,2)k(P1{:}P_i\,\|\,2\oplus 1^2{:}P_j\,\|\,2\oplus 1^2)$		$-n(n-1)(n-4)$	
74	$\sum C(P_i\,\|\,2)C(P_j\,\|\,2)k(P\oplus 1{:}P_i\,\|\,2\oplus 1^2{:}P_j\,\|\,2\oplus 1^2)$	$10n(n-1)$	$-(n+6)(n-1)$	$-(n^2-2n+2)$
	$n^5k(P)k^5(1)$	n	$5n^2$	$10n^2$

$k(P)k(31^2)$	$k(P)k(2^21)$	$k(P)k(21^3)$	$k(P)k(1^5)$	Divisor	$n^5k(P)k^5(1)$
				$n^{(2)}$	0
				$n^{(2)}$	0
-2				$n^{(2)}$	0
	-1			$n^{(2)}$	0
	-2			$n^{(2)}$	0
-1				$n^{(2)}$	0
$3n$		-3		$n^{(2)}$	0
	$2n$	-3		$n^{(2)}$	0
		n	-10	$n^{(2)}$	0
2				$n^{(3)}$	0
	2			$n^{(3)}$	0
-2				$n^{(3)}$	0
	-1			$n^{(3)}$	0
	-2			$n^{(3)}$	0
-1				$n^{(3)}$	0
$-6n$	$-2n$	6		$n^{(3)}$	0
	$-2n$	2		$n^{(3)}$	0
n^2		$-3n$	20	$n^{(3)}$	0
6	$2(n-1)$	-3		$n^{(3)}$	0
$6n$	4	-6		$n^{(3)}$	0
	$2(n+1)$	-3		$n^{(3)}$	0
$-3n$		$3(n+1)$	-30	$n^{(3)}$	0
$6n$	$4n$	-6		$n^{(4)}$	0
$-6n$	$-2(n+3)$	6		$n^{(4)}$	0
-6	$-2(n-1)$	2		$n^{(4)}$	0
$2(n^2+n+4)$	$4(n-1)$	$-6(n+1)$	40	$n^{(4)}$	0
9	$2n$	-3		$n^{(4)}$	0
$3n$	6	-3		$n^{(4)}$	0
$-2n(n+1)$	$-n(n-1)$	$6n$	-30	$n^{(4)}$	0
$-6(n+1)$	$-2n$	$3(n+3)$	-30	$n^{(4)}$	0
$-n$	$-2n/3$	1		$n^{(4)}$	0
$2n(n+2)$	$2n(n-1)$	$-6n$	24	$n^{(5)}$	0
$-2(n^2+2n+6)$	$-(n^2+5n-6)$	$6(n+1)$	-30	$n^{(5)}$	0
$n^2+2n+16$	$2(3n-2)$	$-(3n+8)$	20	$n^{(5)}$	0
$-(3n+8)$	$-2(n+1)$	$n+6$	-10	$n^{(5)}$	0
$-(n^2+8n-8)/10$	$-(n^2-2n+2)/5$	$n/2$	-2	$n^{(5)}$	0
$2(n^2-5n+4)$	$(n^2-3n+3)(n-4)/3$	$-n(n-4)$	$5(n-4)$	$n^{(5)}$	0
$3n-2$	$(n^2-n+3)/3$	$-(n+1)$	5	$n^{(5)}$	0
$10n^3$	$15n^3$	$10n^4$	n^5		Check

TABLE 2
Formula for k(P)k(5), with P = 21

No.	Term	Interpretation	Coefficient
1	$k(215)$	$k(521)$	1
8	$k(21\oplus 5)$	$k(71)+k(62)$	$1/n$
20	$k(21\oplus 41)$	$k(62)+k(53)$	$-5/n^{(2)}$
21	$k(21\oplus 32)$	$k(53)+k(44)$	$-10/n^{(2)}$
37	$\sum C(2\mid 2)k(21:2\mid 2\oplus 41)$	$2k(521)$	$5/(n-1)$
38	$\sum C(2\mid 2)k(21:2\mid 2\oplus 32)$	$2k(431)$	$10/(n-1)$
48	$\sum C(21\mid 2)k(21\oplus 1:2\mid 2\oplus 31)$	$2k(422)$	$-20/(n-1)^{(2)}$
49	$\sum C(2\mid 2)k(21\oplus 1:2\mid 2\oplus 2^2)$	$2k(332)$	$-15/(n-1)^{(2)}$
50	$\sum C(2\mid 2)k(21\oplus 2:2\mid 2\oplus 21)$	$2k(332)$	$-30/(n-1)^{(2)}$
51	$\sum C(2\mid 2)k(21\oplus 3:2\mid 2\oplus 1^2)$	$2k(422)$	$-10/(n-1)^{(2)}$

TABLE 3
Formula for k(P)k(21), with P = 5

Term	Interpretation	Coefficient
$k(521)$	$k(521)$	1
$k(5\oplus 2, 1)$	$k(71)$	$1/n$
$k(5\oplus 1, 2)$	$k(62)$	$1/n$
$\sum C(5\mid 2)k(5:5\mid 2\oplus 21)$	$5[k(62)+k(53)]+10[k(53)+k(44)]$	$-1/n^{(2)}$
$\sum C(5\mid 2)k(51:5\mid 2\oplus 11)$	$2[5k(521)+10k(431)]$	$1/(n-1)$
$\sum C(5\mid 3)k(5:5\mid 3\oplus 111)$	$6[10k(422)+15k(332)]$	$-1/(n-1)^{(2)}$

The linear expansion for $k(21)k(5)$ is obtained as the sum of the products of the entries in the last two columns. On collecting terms, we get

$$\begin{aligned} k(21)k(5) &= k(71)/n+(n-6)k(62)/n^{(2)}-15k(53)/n^{(2)}-10k(44)/n^{(2)} \\ &\quad +(n+9)k(521)/(n-1)+20k(431)/(n-1)-60k(422)/(n-1)^{(2)} \\ &\quad -90k(332)/(n-1)^{(2)}. \end{aligned} \tag{5}$$

It may be noted that the last six terms in Table 2 call for partitioning 2 into two parts and the only possibility is 1, 1. These, when added to the two parts of 5 in all possible ways, produce the coefficient 2 in the third column.

The product $k(21)k(5)$ could be considered alternatively, and perhaps more simply, as $k(P)k(21)$ with $P=5$. Using the expansion for $k(P)k(21)$ in [4], and ignoring terms which do not appear with a one-part P, we get Table 3.

Collecting terms again leads to (5). Generally, to expand $k(P)k(Q)$ with minimum effort, one should choose P to be the one with larger weight. Thus the formula for $k(P)k(5)$ is used to greater advantage when weight $(P) \geqq 5$.

It may be remarked that the formula (5) obtained above for $k(21)k(5)$ is not yet available in literature as [2], [13] give formulae of weight 8 for products of k-statistics only and [8] covers products of seminvariant polykays. This example thus illustrates how the general formulae of this paper may be used to obtain formulae for specific products, in extension of those given earlier [2], [8], [9], [13]. Some of the results here are in fact used in obtaining products of weight 9 [10].

We feel, however, that the main point in obtaining general formulae of the type presented in Table 1 is not as much to be able to obtain specific ones from them as it is to set up general multiplication rules applicable to any $k(P)$, providing an analysis of the coefficients obtained in specific products, and thus to be able to classify a multitude of specific products formulae into certain types.

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