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ON THE ASYMPTOTIC EFFICIENCY OF MEDIAN UNBIASED ESTIMATES

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1. Summary. It is shown that, for sequences of median unbiased estimates, the asymptotic efficiency never exceeds Fisher's bound. In other words: assumptions on the convergence of the distributions of the estimates are superfluous in this case.

2. The main results. Let $(X, \mathscr{A})$ be a measurable space and $P_\vartheta | \mathscr{A}, \vartheta \in \Theta$, a parametrized family of p-measures (= probability measures). This paper is exclusively concerned with the case $\Theta \subset \mathbb{R}$ (the set of real numbers). We shall assume that $P_\vartheta | \mathscr{A}, \vartheta \in \Theta$, is dominated by some σ-finite measure $\mu | \mathscr{A}$. For each $\vartheta \in \Theta$ let $h(\cdot, \vartheta)$ be a density of $P_\vartheta | \mathscr{A}$ with respect to $\mu | \mathscr{A}$. By a sequence of estimates, say $(T_n)_{n \in \mathbb{N}}$, we mean a sequence of $\mathscr{A}^n$-measurable maps $T_n: X^n \to \Theta$. For notational convenience we shall consider T_n as a map defined on $X^{\mathbb{N}}$ (with $T_n((x_\nu)_{\nu \in \mathbb{N}})$ depending on $x_1, \cdots, x_n$ only) and $\mathscr{A}^n$ as σ-algebra of subsets of $X^{\mathbb{N}}$ (namely of cylinders with the base in X^n).

Up to recently, comparisons of efficiency were confined to asymptotically normal sequences of estimates, where the asymptotic concentration of the estimate is determined by the "asymptotic variance." One of the most important results in this area is that of LeCam (1953, 1958), C. R. Rao (1963), Bahadur (1964) and Schmetterer (1966) that (under suitable regularity conditions on the family of densities) for every sequence of estimates which is asymptotically distributed according to $N(\vartheta, n^{-\frac{1}{2}}\sigma(\vartheta))$, the relation $\sigma(\vartheta) \geqq I(\vartheta)^{-\frac{1}{2}}$ holds for Lebesgue-a.a. $\vartheta \in \Theta$, where

$$I(\vartheta) \equiv \int \left(\frac{\partial \log h(x, \vartheta)}{\partial \vartheta}\right)^2 P_\vartheta(dx).$$

Written in a slightly different way, this means that for every sequence $(T_n)_{n \in \mathbb{N}}$ of estimates for which the sequence of distributions of $n^{\frac{1}{2}}(T_n - \vartheta)$ converges weakly to a normal distribution with mean zero, we have for Lebesgue-a.a. $\vartheta \in \Theta$ and all $t', t'' > 0$

$$\lim_{n \in \mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t'n^{-\frac{1}{2}} < T_n(\mathbf{x}) < \vartheta + t''n^{-\frac{1}{2}}\} \leqq \Phi(t'' I(\vartheta)^{\frac{1}{2}}) - \Phi(-t' I(\vartheta)^{\frac{1}{2}}),$$

where $\Phi(t) \equiv \int_{-\infty}^{t} (2\pi)^{-\frac{1}{2}} \exp[-\frac{1}{2}u^2]\, du$.

In a number of papers Wolfowitz stressed the arbitrariness of restricting the comparison to asymptotically normal estimates. Under suitable regularity conditions on the family of densities and under the assumption that the sequence of distribution functions of $n^{\frac{1}{2}}(T_n - \vartheta)$ converges uniformly in both arguments,

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Wolfowitz (1965) has shown that—with the possible exception of a countable subset of Θ—

$$\lim_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t''n^{-\frac{1}{2}} + w(\vartheta)n^{-\frac{1}{2}} \leqq T_n(\mathbf{x}) \leqq \vartheta + t''n^{-\frac{1}{2}} + W(\vartheta)n^{-\frac{1}{2}}\}$$
$$\leqq \Phi(t''I(\vartheta)^{\frac{1}{2}}) - \Phi(-t''I(\vartheta)^{\frac{1}{2}}).$$

The functions w and W are—with the possible exception of a set which is the countable union of nowhere dense sets—the infimum and supremum of the set of medians of the limiting measure (which, in general, may depend on ϑ). Possible extensions of this result for $\Theta \subset \mathbb{R}^k$ were investigated by Kaufmann (1966). Schmetterer (1966) improved this result, mainly by substituting the assumption of uniform convergence by the (weaker) assumption of continuous convergence and by relaxing the regularity conditions.

With the same justification with which Wolfowitz questioned the asymptotic normality assumption for the sequence of estimates one could question his assumption of weak uniform convergence: Why should a statistician confine himself to estimates for which the sequence of distributions of $n^{\frac{1}{2}}(T_n - \vartheta)$ converges at all? A sequence of estimates $(T_n)_{n\in\mathbb{N}}$ for which $(P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t'n^{-\frac{1}{2}} < T_n(\mathbf{x}) < \vartheta + t''n^{-\frac{1}{2}}\})_{n\in\mathbb{N}}$ does not converge would nevertheless be preferred to a sequence of estimates, say $(T_n^*)_{n\in\mathbb{N}}$, for which $(P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t'n^{-\frac{1}{2}} < T_n^*(\mathbf{x}) < \vartheta + t''n^{-\frac{1}{2}}\})_{n\in\mathbb{N}}$ converges if for all $\vartheta \in \Theta$, $t', t'' > 0$

$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t'n^{-\frac{1}{2}} < T_n(\mathbf{x}) < \vartheta + t''n^{-\frac{1}{2}}\}$$
$$\geqq \lim_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t'n^{-\frac{1}{2}} < T_n^*(\mathbf{x}) < \vartheta + t''n^{-\frac{1}{2}}\}.$$

It therefore seems useful to look for other conditions which are justifiable from the operational point of view. One such condition is median unbiasedness:

DEFINITION. The estimate T_n is median unbiased for the parameter ϑ if

$$P_\vartheta^{\mathbb{N}}\{\mathbf{x}: T_n(\mathbf{x}) \geqq \vartheta\} \geqq \tfrac{1}{2} \quad \text{and} \quad P_\vartheta^{\mathbb{N}}\{\mathbf{x}: T_n(\mathbf{x}) \leqq \vartheta\} \geqq \tfrac{1}{2} \quad \text{for all} \quad \vartheta \in \Theta.$$

To obtain an example of a sequence of median unbiased estimates $(T_n)_{n\in\mathbb{N}}$ for which the distributions of $n^{\frac{1}{2}}(T_n - \vartheta)$ are not convergent, consider the family of normal distributions $N(\vartheta, 1)$, $\vartheta \in \mathbb{R}$, and let

$$T_n(\mathbf{x}) \equiv \frac{1}{m(n)} \sum_{i=1}^{m(n)} x_i, \qquad n \in \mathbb{N}.$$

$n^{\frac{1}{2}}(T_n - \vartheta)$ is distributed according to $N(0, (n/m(n))^{\frac{1}{2}})$. If we choose $m(n) = 2^{[\log_2 n]}$ (where $[a]$ denotes the largest integer smaller or equal to a), then the sequence $(n/m(n))$, $n \in \mathbb{N}$, oscillates in the interval $[1, 2)$. Therefore the sequence $N(0, (n/m(n))^{\frac{1}{2}})$, $n \in \mathbb{N}$, does not converge.

The results of this paper are obtained under the following

Regularity conditions:

(i) *For every* $\vartheta_0 \in \Theta$ *there exists an* $\mathscr{A}$*-measurable function* $M(\cdot, \vartheta_0)$ *with* $P_{\vartheta_0}(M(\cdot, \vartheta_0)^2) < \infty$ *and a number* $r(\vartheta_0) > 0$ *such that* $|\vartheta' - \vartheta_0| \leqq r(\vartheta_0)$ *and*

$|\vartheta''-\vartheta_0| \leqq r(\vartheta_0)$ *imply*

$$\left|\frac{h(x,\vartheta')}{h(x,\vartheta'')}-1\right| \leqq M(x,\vartheta_0)\,|\vartheta'-\vartheta''| \quad \textit{for all} \quad x \in X;$$

(ii) $\vartheta \to \frac{\partial}{\partial\vartheta} h(x,\vartheta)$ *is continuous on* Θ *for* μ-a.a. $x \in X$;

(iii) $P_\vartheta\left(\left(\frac{\partial}{\partial\vartheta}\log h(\cdot,\vartheta)\right)^2\right) > 0$ *for all* $\vartheta \in \Theta$.

We remark that condition (i) has been used earlier by Daniels ((1961) page 152 Formula 2.2) as one of a set of conditions assuring the aymptotic efficiency of maximum likelihood estimates.

The regularity conditions used here refer to the first derivative of $\vartheta \to h(x,\vartheta)$ only, whereas the corresponding regularity conditions of Bahadur ((1964) pages 1545–6) assume that the second derivative is continuous and that

$$\int \frac{\partial^k}{\partial\vartheta^k} h(x,\vartheta)P_\vartheta(dx) = \frac{\partial^k}{\partial\vartheta^k}\int h(x,\vartheta)P_\vartheta(dx), \qquad k = 1, 2.$$

On the other hand: If the second derivative has the properties assumed by Bahadur, our assumption (i) may be replaced by Bahadur's assumption (iv) that for every $\vartheta_0 \in \Theta$ there exists an $\mathscr{A}$-measurable function $B(\cdot,\vartheta_0)$ with $P_{\vartheta_0}(B(\cdot,\vartheta_0)) < \infty$ and a number $r(\vartheta_0) > 0$ such that

$$|\vartheta-\vartheta_0| < r(\vartheta_0) \quad \text{implies} \quad \left|\frac{\partial^2}{\partial\vartheta^2}\log h(x,\vartheta)\right| \leqq B(x,\vartheta_0) \quad \text{for all } x \in X.$$

Hence, strictly speaking, the regularity conditions used here are incomparable with the regularity conditions of Bahadur.

The conditions stated by Schmetterer ((1966) page 305, Theorem 2.1 and page 308, Theorem 2.2) are weaker than our conditions. However, for his Lemma 2.2, attributed to Daniels (1961), a correct proof is not available.

We remark that the regularity conditions (i), (ii), (iii) are not sufficient to guarantee the asymptotic normality of the maximum likelihood estimates. For this reason, they must not be compared with the conditions of Wolfowitz ((1965) pages 252–3) which also include conditions for the asymptotic normality of maximum likelihood estimates.

THEOREM 1. *If* $\Theta \subset \mathbb{R}$ *is an open set and if the regularity conditions* (i), (ii), (iii) *are fulfilled, then we have for every sequence* $(T_n)_{n\in\mathbb{N}}$ *of median unbiased estimates*:

(1) $\limsup_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t'n^{-\frac{1}{2}} \leqq T_n(\mathbf{x}) \leqq \vartheta + t''n^{-\frac{1}{2}}\}$

$$\leqq \Phi(t''I(\vartheta)^{\frac{1}{2}}) - \Phi(-t'I(\vartheta)^{\frac{1}{2}}) \qquad \textit{for all} \quad t',t'' > 0 \quad \textit{and all} \quad \vartheta \in \Theta.$$

PROOF. Let $\vartheta \in \Theta$ and $t > 0$ be arbitrarily fixed. We have $\vartheta + tn^{-\frac{1}{2}} \in \Theta$ for all sufficiently large n. Median unbiasedness implies $P^{\mathbb{N}}_{\vartheta+tn^{-1/2}}\{\mathbf{x}\colon T_n(\mathbf{x}) \geqq \vartheta + tn^{-\frac{1}{2}}\} \geqq \frac{1}{2}$ for all sufficiently large n. As $\frac{1}{2} > \Phi(-\delta t I(\vartheta)^{\frac{1}{2}})$ for all $\delta > 0$, Lemma 1 (10) implies for all $\delta > 0$ and all sufficiently large $n \in \mathbb{N}$:

$$(2)\quad P^{\mathbb{N}}_{\vartheta+tn^{-1/2}}\{\mathbf{x}\colon T_n(\mathbf{x}) \geqq \vartheta + tn^{-\frac{1}{2}}\}$$

$$> P^{\mathbb{N}}_{\vartheta+tn^{-1/2}}\left\{\mathbf{x}\colon \frac{\sum_{i=1}^n \log h(x_i, \vartheta + tn^{-\frac{1}{2}}) - \sum_{i=1}^n \log h(x_i, \vartheta) + \frac{1}{2}t^2 I(\vartheta)}{tI(\vartheta)^{\frac{1}{2}}} \geqq (1+\delta)tI(\vartheta)^{\frac{1}{2}}\right\}.$$

Applying the fundamental lemma of Neyman and Pearson (Lehmann (1959) page 65, Theorem 1 (ii)) for

$$p_0 = \prod_{i=1}^n h(x_i, \vartheta), \qquad p_1 = \prod_{i=1}^n h(x_i, \vartheta + tn^{-\frac{1}{2}}),$$

$$\varphi = 1_{\{\mathbf{x}:[\sum_{i=1}^n \log h(x_i, \vartheta+tn^{-1/2}) - \sum_{i=1}^n \log h(x_i, \vartheta) + \frac{1}{2}t^2 I(\vartheta)]/tI(\vartheta)^{1/2} \geqq (1+\delta)tI(\vartheta)^{1/2}\}}$$

and
$$\varphi^* = 1_{\{\mathbf{x}\colon T_n(\mathbf{x}) \geqq \vartheta + tn^{-1/2}\}}$$

we obtain inequality (2) with $P_{\vartheta+tn^{-1/2}}$ replaced by P_ϑ.

Using Lemma 1 (9) we obtain

$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}\colon T_n(\mathbf{x}) \geqq \vartheta + tn^{-\frac{1}{2}}\} \geqq \Phi(-(1+\delta)tI(\vartheta)^{\frac{1}{2}}).$$

As the relation holds for all $\delta > 0$,

$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}\colon T_n(\mathbf{x}) \geqq \vartheta + tn^{-\frac{1}{2}}\} \geqq \Phi(-tI(\vartheta)^{\frac{1}{2}}).$$

The other inequality follows similarly.

Roughly speaking, Theorem 1 gives an upper bound for the asymptotic concentration of sequences of median unbiased estimates. Under suitable regularity conditions, this maximal asymptotic concentration is achieved by the sequence of maximum likelihood estimates. As maximum likelihood estimates are not median unbiased in general the question arises under what condition median unbiased estimates with maximal asymptotic concentration exist. In Pfanzagl (1971) conditions will be given under which a sequence of estimates can be adjusted in such a way that (i) each estimate of the sequence becomes median unbiased, (ii) the asymptotic behavior of the sequence remains unchanged. The results obtained in that paper imply in particular that median unbiased estimates with maximal asymptotic concentration exist for all exponential families fulfilling certain regularity conditions. In view of the fact that families with monotone likelihood ratios admit median unbiased estimates with strong optimum properties (see Lehmann (1959) page 83 and Pfanzagl (1970)), this is what one would expect intuitively.

If the distributions are symmetric, the maximum likelihood estimates are median unbiased without any adjustment: Let $X = \mathbb{R}$, $\mathscr{A} = \mathscr{B}$, $\Theta = \mathbb{R}$ and $h(x, \vartheta) \equiv f(x - \vartheta)$ with $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Under appropriate regularity conditions on f, the maximum likelihood estimate is uniquely determined and asymptotically distributed according to $N(\vartheta, n^{-\frac{1}{2}} I(\vartheta)^{-\frac{1}{2}})$. The symmetry of f implies that the maximum likelihood estimate $\hat{\vartheta}_n$ is symmetrically distributed about ϑ (which follows easily from $\prod_{i=1}^n h(2\vartheta - x_i, 2\vartheta - \hat{\vartheta}_n) = \prod_{i=1}^n h(x_i, \hat{\vartheta}_n)$) and therefore median unbiased.

Theorem 1 can be given in an equivalent formulation in terms of loss functions: For every $\vartheta \in \Theta$, let $L_\vartheta: \mathbb{R} \to \mathbb{R}$ be a measurable function which assumes its minimal value at 0 and is nondecreasing as the argument moves away from 0 in either direction:

$$(3) \quad t_2 < t_1 < 0 < t_1' < t_2' \quad \text{implies} \quad L_\vartheta(t_2) \geqq L_\vartheta(t_1) \geqq L_\vartheta(0) \leqq L_\vartheta(t_1') \leqq L_\vartheta(t_2').$$

COROLLARY 1. *If $\Theta \subset \mathbb{R}$ is an open set and if the regularity conditions* (i), (ii), (iii) *are fulfilled, then we have for every sequence $(T_n)_{n \in \mathbb{N}}$ of median unbiased estimates and any loss function with the property* (3) *stated above*:

$$(4) \quad \liminf_{n \in \mathbb{N}} \int L_\vartheta(n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta)) P_\vartheta^{\mathbb{N}}(d\mathbf{x}) \geqq \int L_\vartheta(u) \left(\frac{I(\vartheta)}{2\pi}\right)^{\frac{1}{2}} \exp[-\tfrac{1}{2}u^2 I(\vartheta)]\, du \qquad \textit{for all} \quad \vartheta \in \Theta.$$

PROOF. For $L_\vartheta(t) \equiv 1_{(-\infty, -t') \cup (t'', \infty)}(t)$, Theorem 1 implies

$$\begin{aligned} &\liminf_{n \in \mathbb{N}} \int L_\vartheta(n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta)) P_\vartheta^{\mathbb{N}}(d\mathbf{x}) \\ &\quad = \liminf_{n \in \mathbb{N}} (1 - P_\vartheta^{\mathbb{N}}\{\mathbf{x}: -t'' \leqq n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \leqq t''\} \\ &\quad \geqq 1 - \int_{-t' I(\vartheta)^{1/2}}^{t'' I(\vartheta)^{1/2}} (2\pi)^{-\frac{1}{2}} \exp[-\tfrac{1}{2}u^2]\, du \\ &\quad = \int L_\vartheta(u) \left(\frac{I(\vartheta)}{2\pi}\right)^{\frac{1}{2}} \exp[-\tfrac{1}{2}u^2 I(\vartheta)]\, du. \end{aligned}$$

As any loss function with the property (3) can be pointwise approximated by an increasing sequence of elementary functions $L_\vartheta^{(k)}$ of the type

$$a_0 + \sum_{i=1}^k a_i 1_{(-\infty, -t_i') \cup (t_i'', \infty)},$$

with $a_i > 0$ for $i = 1, \cdots, k$, the assertion follows easily.

Without median unbiasedness for all sample sizes, a similar conclusion can be obtained if the sequence of distributions of $n^{\frac{1}{2}}(T_n - \vartheta)$ converges to a limiting distribution with median 0:

THEOREM 2. *Assume that $\Theta \subset \mathbb{R}$ is an open set and that the regularity conditions* (i), (ii), (iii), *are fulfilled. Assume, furthermore, that the sequence of p-measures induced by $P_\vartheta^{\mathbb{N}}$ and the sequence $n^{\frac{1}{2}}(T_n - \vartheta)$ converges weakly to some p-measure, say Q_ϑ, with median* 0.

Then for all $t', t'' > 0$ and Lebesgue-a.a. $\vartheta \in \Theta$:

$$(5) \quad Q_\vartheta[-t', t''] \leqq \Phi(t'' I(\vartheta)^{\frac{1}{2}}) - \Phi(-t' I(\vartheta)^{\frac{1}{2}})$$

and therefore

(6) $$\limsup_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: \vartheta - t'n^{-\frac{1}{2}} \leqq T_n(\mathbf{x}) \leqq \vartheta + t''n^{-\frac{1}{2}}\} \leqq \Phi(t''I(\vartheta)^{\frac{1}{2}}) - \Phi(-t'I(\vartheta)^{\frac{1}{2}}).$$

PROOF. (i) First we shall show that the assumptions of Lemma 3 are fulfilled: Let $y < 0$ be arbitrary. Let $z \in [y, 0]$ be such that $Q_\vartheta\{z\} = 0$. Then

$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq y\} \geqq \lim_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq z\} = Q_\vartheta[z, \infty) \geqq Q_\vartheta[0, \infty) \geqq \tfrac{1}{2}.$$

The other inequality follows similarly.

(ii) Now we shall show that the assertion follows from the assertion of Lemma 3. Let $N_\vartheta \equiv \{r \in \mathbb{R}: Q_\vartheta\{r\} = 0\}$. Let $t > 0$ be given. Using Lemma 3 (with $t'' = s$) we obtain for Lebesgue-a.a. $\vartheta \in \Theta$ and all $s \in (t, \infty) \cap N_\vartheta$:

$$Q_\vartheta[s, \infty) = \lim_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq s\} \geqq \Phi(-sI(\vartheta)^{\frac{1}{2}}).$$

As Φ is continuous and N_ϑ is dense in $\mathbb{R}$, this implies $Q_\vartheta(t, \infty) \geqq \Phi(-tI(\vartheta)^{\frac{1}{2}})$.

Furthermore,

$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) > t\} \geqq \liminf_{n\in\mathbb{N}} P^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq s\} = Q_\vartheta[s, \infty).$$

As N_ϑ is dense in $\mathbb{R}$, this implies

$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) > t\} \geqq Q_\vartheta(t, \infty).$$

The other inequality follows similarly.

Theorem 2 generalizes a result of LeCam (1953) from normal to arbitrary limiting p-measures. It is furthermore closely related to the theorem of Wolfowitz cited above. It is more special than the theorem of Wolfowitz because the limiting p-measure is assumed to have median 0. This assumption is, however, most natural. If it were not fulfilled, then there would exist $\vartheta \in \Theta$ such that

$$\limsup_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: T_n(\mathbf{x}) \geqq \vartheta\} < \tfrac{1}{2} \quad \text{or} \quad \limsup_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: T_n(\mathbf{x}) \leqq \vartheta\} < \tfrac{1}{2}.$$

Beyond this difference, the assumption of uniform (Wolfowitz (1965)) or continuous (Schmetterer (1966)) convergence of distribution functions is eliminated. The price paid for this is the occurrence of an exceptional ϑ-set of Lebesgue-measure 0 on which the sequence of estimates may be "super-efficient". That the set of super-efficiency may, in fact, be uncountable, can be seen from the example given by LeCam ((1953) page 291).

COROLLARY 2. *Assume that $\Theta \subset \mathbb{R}$ is an open set and that the regularity conditions* (i), (ii), (iii), *are fulfilled. Assume, furthermore, that the sequence of p-measures induced by $P_\vartheta^{\mathbb{N}}$ and $n^{\frac{1}{2}}(T_n - \vartheta)$ converges weakly to some p-measure, say Q_ϑ, with* *median* 0. *Then for any loss function* L_ϑ *with property* (3) *and Lebesgue*-a.a. $\vartheta \in \Theta$:

$$(7) \qquad \int L_\vartheta(u) Q_\vartheta(du) \geqq \int L_\vartheta(u) \left(\frac{I(\vartheta)}{2\pi}\right)^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}u^2 I(\vartheta)\right] du$$

and

$$(8) \quad \liminf_{n\in\mathbb{N}} \int L_\vartheta(n^{\frac{1}{2}}(T_n(\mathbf{x})-\vartheta)) P_\vartheta^{\mathbb{N}}(d\mathbf{x}) \geqq \int L_\vartheta(u) \left(\frac{I(\vartheta)}{2\pi}\right)^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}u^2 I(\vartheta)\right] du.$$

PROOF. (7) and (8) follow similarly as in Corollary 1.

3. A few lemmas.

LEMMA 1. *Assume that regularity conditions* (i), (ii), (iii) *are fulfilled. Then for all* $t > 0$ *and all* $c \in \mathbb{R}$

$$(9) \quad \lim_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\left\{\mathbf{x}: \frac{\sum_{i=1}^n \log h(x_i, \vartheta + tn^{-\frac{1}{2}}) - \sum_{i=1}^n \log h(x_i, \vartheta) + \frac{1}{2}t^2 I(\vartheta)}{tI(\vartheta)^{\frac{1}{2}}} < c\right\} = \Phi(c)$$

$$(10) \quad \lim_{n\in\mathbb{N}} P_{\vartheta+tn^{-\frac{1}{2}}}^{\mathbb{N}}\left\{\mathbf{x}: \frac{\sum_{i=1}^n \log h(x_i, \vartheta + tn^{-\frac{1}{2}}) - \sum_{i=1}^n \log h(x_i, \vartheta) + \frac{1}{2}tI(\vartheta)}{tI(\vartheta)^{\frac{1}{2}}} < c\right\}$$

$$= \Phi(c - tI(\vartheta)^{\frac{1}{2}}).$$

PROOF. For all ϑ, ϑ' with $|\vartheta - \vartheta_0| \leqq r(\vartheta_0)$ and $|\vartheta' - \vartheta_0| \leqq r(\vartheta_0)$ we have

$$\left|\frac{h(x, \vartheta') - h(x, \vartheta)}{(\vartheta' - \vartheta) h(x, \vartheta)}\right| \leqq M(x, \vartheta_0).$$

For notational convenience let $l(x, \vartheta) \equiv (\partial/\partial\vartheta) \log h(x, \vartheta)$. As

$$l(x, \vartheta) = \lim_{\vartheta' \to \vartheta} \frac{h(x, \vartheta') - h(x, \vartheta)}{(\vartheta' - \vartheta) h(x, \vartheta)},$$

this implies $|l(x, \vartheta)| \leqq M(x, \vartheta_0)$ for $|\vartheta - \vartheta_0| < r(\vartheta_0)$ and $\int l(x, \vartheta_0) P_{\vartheta_0}(dx) = 0$. By the Mean Value Theorem, we have $\log h(x, \vartheta) - \log h(x, \vartheta_0) = l(x, \vartheta')(\vartheta - \vartheta_0)$ with $|\vartheta' - \vartheta_0| \leqq |\vartheta - \vartheta_0|$. Hence $|\log h(x, \vartheta) - \log h(x, \vartheta_0)| \leqq M(x, \vartheta_0)|\vartheta - \vartheta_0|$ for $|\vartheta - \vartheta_0| < r(\vartheta_0)$, whence

$$(11) \qquad \lim_{\vartheta \to \vartheta_0} \int \left(\frac{\log h(x, \vartheta) - \log h(x, \vartheta_0)}{\vartheta - \vartheta_0}\right)^2 P_{\vartheta_0}(dx) = \int (l(x, \vartheta_0))^2 P_{\vartheta_0}(dx).$$

Furthermore,

$$\frac{\int l(x, \vartheta_0) P_{\vartheta_0}(dx)}{\vartheta - \vartheta_0} = -\frac{\int l(x, \vartheta) P_\vartheta(dx) - \int l(x, \vartheta) P_{\vartheta_0}(dx)}{\vartheta - \vartheta_0}$$

$$= -\int \frac{h(x, \vartheta) - h(x, \vartheta_0)}{(\vartheta - \vartheta_0) h(x, \vartheta_0)} l(x, \vartheta) P_{\vartheta_0}(dx).$$

As

$$\left|\frac{h(x,\vartheta)-h(x,\vartheta_0)}{(\vartheta-\vartheta_0)h(x,\vartheta_0)}\, l(x,\vartheta)\right| \leqq M^2(x,\vartheta_0)$$

for $|\vartheta-\vartheta_0| < r(\vartheta_0)$ and $\lim_{\vartheta\to\vartheta_0} l(x,\vartheta) = l(x,\vartheta_0)$ for μ-a.a. $x\in X$, this implies

$$\lim_{\vartheta\to\vartheta_0}\frac{\int l(x,\vartheta)P_{\vartheta_0}(dx)}{\vartheta-\vartheta_0} = -\int (l(x,\vartheta_0))^2 P_{\vartheta_0}(dx).$$

Hence for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ such that $|\vartheta-\vartheta_0| < \delta(\varepsilon)$ implies

$$\left|\int l(x,\vartheta)P_{\vartheta_0}(dx)+(\vartheta-\vartheta_0)I(\vartheta_0)\right| \leqq |\vartheta-\vartheta_0|\varepsilon.$$

As $\int_{\vartheta_0}^{\vartheta} l(x,\vartheta')\,d\vartheta' = \log h(x,\vartheta)-\log h(x,\vartheta_0)$ for μ-a.a. $x\in X$, this implies

$$\left|\int(\log h(x,\vartheta)-\log h(x,\vartheta_0))P_{\vartheta_0}(dx)+\tfrac{1}{2}(\vartheta-\vartheta_0)^2 I(\vartheta_0)\right| \leqq \tfrac{1}{2}(\vartheta-\vartheta_0)^2\varepsilon,$$

whence

$$\lim_{\vartheta\to\vartheta_0}\frac{\int(\log h(x,\vartheta)-\log h(x,\vartheta_0))P_{\vartheta_0}(dx)}{(\vartheta-\vartheta_0)^2} = -\tfrac{1}{2}I(\vartheta_0). \tag{12}$$

This implies

$$\lim_{\vartheta\to\vartheta_0}\frac{\int(\log h(x,\vartheta)-\log h(x,\vartheta_0))P_{\vartheta_0}(dx)}{\vartheta-\vartheta_0} = 0 \tag{13}$$

Let $f_{ni}(\mathbf{x}) \equiv \log h(x_i,\vartheta_0+tn^{-\frac{1}{2}})-\log h(x_i,\vartheta_0)$. From (12), (13) and (11) we obtain, respectively:

$$\lim_{n\in\mathbb{N}}\sum_{i=1}^{n}\int f_{ni}(\mathbf{x})P_{\vartheta_0}^{\mathbb{N}}(d\mathbf{x}) = -\tfrac{1}{2}t^2 I(\vartheta_0),$$

$$\lim_{n\in\mathbb{N}}\sum_{i=1}^{n}\left(\int f_{ni}(\mathbf{x})P_{\vartheta_0}^{\mathbb{N}}(d\mathbf{x})\right)^2 = 0,$$

$$\lim_{n\in\mathbb{N}}\sum_{i=1}^{n}\int f_{ni}^2(\mathbf{x})P_{\vartheta_0}^{\mathbb{N}}(d\mathbf{x}) = t^2 I(\vartheta_0).$$

Hence

$$\lim_{n\in\mathbb{N}}\sum_{i=1}^{n}\int\left(f_{ni}(\mathbf{x})-\int f_{ni}(\mathbf{y})P_{\vartheta_0}^{\mathbb{N}}(d\mathbf{y})\right)^2 P_{\vartheta_0}^{\mathbb{N}}(d\mathbf{x}) = t^2 I(\vartheta_0).$$

By assumption, $tn^{-\frac{1}{2}} < r(\vartheta_0)$ implies

$$\frac{n^{\frac{1}{2}}}{t}|f_{ni}(\mathbf{x})| = \frac{|\log h(x_i,\vartheta_0+tn^{-\frac{1}{2}})-\log h(x_i,\vartheta_0)|}{|(\vartheta_0+tn^{-\frac{1}{2}})-\vartheta_0|} \leqq M(x_i,\vartheta_0).$$

Elementary computations show that this implies Lindeberg's condition (see Billingsley, page 42, Theorem 7.2). Hence (9) follows.

Relation (10) may be proved as follows:

Let

$$\psi_n(\mathbf{x},t) \equiv \frac{\sum_{i=1}^{n}\log h(x_i,\vartheta+tn^{-\frac{1}{2}})-\sum_{i=1}^{n}\log h(x_i,\vartheta)+\frac{1}{2}t^2 I(\vartheta)}{tI(\vartheta)^{\frac{1}{2}}}.$$

Then

$$\int_{\{\psi_n(\mathbf{x},t)<c\}} P^{\mathbb{N}}_{\vartheta+tn^{-1/2}}(d\mathbf{x})$$
$$= \int_{\{\psi_n(\mathbf{x},t)<c\}} \prod_{i=1}^{n} \frac{h(x_i, \vartheta+tn^{-\frac{1}{2}})}{h(x_i,\vartheta)} P_\vartheta^{\mathbb{N}}(d\mathbf{x})$$
$$= \int_{\{\psi_n(\mathbf{x},t)<c\}} \exp\left[\sum_{i=1}^n \log h(x_i, \vartheta+tn^{-\frac{1}{2}}) - \sum_{i=1}^n \log h(x_i,\vartheta)\right] P_\vartheta^{\mathbb{N}}(d\mathbf{x})$$
$$= \exp\left[-\tfrac{1}{2}t^2 I(\vartheta)\right] \int_{\{\psi_n(\mathbf{x},t)<c\}} \exp\left[tI(\vartheta)^{\frac{1}{2}}\psi_n(\mathbf{x},t)\right] P_\vartheta^{\mathbb{N}}(d\mathbf{x}).$$

By (9) this implies

$$\lim_{n\in\mathbb{N}} \int_{\{\psi_n(\mathbf{x},t)<c\}} P^{\mathbb{N}}_{\vartheta+tn^{-1/2}}(d\mathbf{x})$$
$$= \exp\left[-\tfrac{1}{2}t^2 I(\vartheta)\right](2\pi)^{-\frac{1}{2}} \int_{-\infty}^{c} \exp\left[v\cdot tI(\vartheta)^{\frac{1}{2}}\right]\exp\left[-\tfrac{1}{2}v^2\right] dv$$
$$= \Phi(c - tI(\vartheta)^{\frac{1}{2}}).$$

LEMMA 2. *If $F_n\colon \mathbb{R}\to\mathbb{R}$, $n\in\mathbb{N}$, is a sequence of $\mathscr{B}$-measurable functions such that $\liminf_{n\in\mathbb{N}} F_n(x) \geqq 0$ for all $x\in\mathbb{R}$, then for every sequence $(x_n)_{n\in\mathbb{N}}\to 0$:*

$$\limsup_{n\in\mathbb{N}} F_n(x+x_n) \geqq 0 \quad \text{for Lebesgue-a.a.} \quad x\in\mathbb{R}.$$

PROOF. Let $F_n^* = \inf(F_n, 0)$. We have $\lim_{n\in\mathbb{N}} F_n^*(x) = 0$ for all $x\in X$. According to Lemma 4 of Bahadur ((1964) page 1549) (see also Schmetterer ((1966) pages 303–304)) there exists a subsequence $\mathbb{N}_0\subset\mathbb{N}$ such that $\lim_{n\in\mathbb{N}_0} F_n^*(x+x_n) = 0$ for every sequence $(x_n)_{n\in\mathbb{N}}\to 0$ and Lebesgue-a.a. $x\in\mathbb{R}$.

As $F_n(x+x_n) \geqq F_n^*(x+x_n)$ for all $x\in\mathbb{R}$, $n\in\mathbb{N}$ we obtain

$$\limsup_{n\in\mathbb{N}} F_n(x+x_n) \geqq \limsup_{n\in\mathbb{N}} F_n^*(x+x_n) \geqq \lim_{n\in\mathbb{N}_0} F_n^*(x+x_n) = 0.$$

LEMMA 3. *Assume that $\Theta\subset\mathbb{R}$ is an open set and that regularity conditions* (i), (ii), (iii) *are fulfilled. Assume furthermore that*

$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x})-\vartheta) \geqq y\} \geqq \tfrac{1}{2} \qquad \textit{for all} \quad y<0$$
$$\liminf_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x})-\vartheta) \leqq y\} \geqq \tfrac{1}{2} \qquad \textit{for all} \quad y>0.$$

Then for all $t', t'' > 0$ and Lebesgue-a.a. $\vartheta\in\Theta$

$$\limsup_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x})-\vartheta) \geqq t''\} \geqq 1-\Phi(t''I(\vartheta)^{\frac{1}{2}})$$
$$\limsup_{n\in\mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x})-\vartheta) \leqq -t'\} \geqq \Phi(-t'I(\vartheta)^{\frac{1}{2}}).$$

PROOF. Let $F_n(y,\vartheta) \equiv P_\vartheta^{\mathbb{N}}\{\mathbf{x}: n^{\frac{1}{2}}(T_n(\mathbf{x})-\vartheta)\geqq y\}$. By assumption:

$$\liminf_{n\in\mathbb{N}} F_n(y,\vartheta) \geqq \tfrac{1}{2}.$$

By Lemma 2, applied for $F_n - \frac{1}{2}$ instead of F_n, for each pair $s>0$, $y<0$ there exists a Lebesgue-null set $M_{s,y}\in\mathscr{B}$ such that $\limsup_{n\in\mathbb{N}} F_n(y, \vartheta+sn^{-\frac{1}{2}}) \geqq \frac{1}{2}$ for $\vartheta\in M_{s,y}$. Let $M_0 \equiv \bigcup\{M_{s,y}\colon s\in(0,\infty)\cap\mathbb{Q},\ y\in(-\infty,0)\cap\mathbb{Q}\}$, where $\mathbb{Q}$ is the set of all rationals. We have $\lambda(M_0)=0$. Let $\vartheta\notin M_0$, $s\in\mathbb{Q}$ and $y<0$ be given and choose

$y_0 \in (y, 0) \cap \mathbb{Q}$. As $y \to F_n(y, \vartheta + sn^{-\frac{1}{2}})$ is nonincreasing, we obtain

$$\limsup_{n \in \mathbb{N}} F_n(y, \vartheta + sn^{-\frac{1}{2}}) \geqq \limsup_{n \in \mathbb{N}} F_n(y_0, \vartheta + sn^{-\frac{1}{2}}) \geqq \tfrac{1}{2}.$$

Hence, given $\vartheta \notin M_0$ and $s \in \mathbb{Q}$,

$$\limsup_{n \in \mathbb{N}} F_n(y, \vartheta + sn^{-\frac{1}{2}}) \geqq \tfrac{1}{2} \tag{14}$$

holds for all $y < 0$, not only for $y \in (-\infty, 0) \cap \mathbb{Q}$.

We have

$$\begin{aligned} P^{\mathbb{N}}_{\vartheta + sn^{-1/2}}\{\mathbf{x} : n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq t\} &= P^{\mathbb{N}}_{\vartheta + sn^{-1/2}}\{\mathbf{x} : n^{\frac{1}{2}}(T_n(\mathbf{x}) - (\vartheta + sn^{-\frac{1}{2}})) \geqq t - s\} \\ &= F_n(t - s, \vartheta + sn^{-\frac{1}{2}}). \end{aligned}$$

Let $t > 0$ be given. Using (14) we obtain for $\vartheta \notin M_0$ and $s \in (t, \infty) \cap \mathbb{Q}$:

$$\limsup_{n \in \mathbb{N}} P^{\mathbb{N}}_{\vartheta + sn^{-1/2}}\{\mathbf{x} : n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq t\} \geqq \tfrac{1}{2}.$$

Similarly as in the proof of Theorem 1 we conclude from this that

$$\limsup_{n \in \mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x} : n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq t\} \geqq \Phi(-sI(\vartheta)^{\frac{1}{2}}).$$

As $s \in (t, \infty) \cap \mathbb{Q}$ was arbitrary, continuity of Φ implies

$$\limsup_{n \in \mathbb{N}} P_\vartheta^{\mathbb{N}}\{\mathbf{x} : n^{\frac{1}{2}}(T_n(\mathbf{x}) - \vartheta) \geqq t\} \geqq \Phi(-tI(\vartheta)^{\frac{1}{2}}).$$

This is one of the inequalities constituting the assertion. The other one is proved by the dual argument.

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