The Annals of Mathematical Statistics
1971, Vol. 42, No. 1, 410–412

SUPERPOSITIONS AND THE CONTINUITY OF INFINITELY DIVISIBLE DISTRIBUTIONS

BY BARTHEL W. HUFF

The University of Georgia

1. Basic properties of infinitely divisible distributions. Let $F_Y(x) = P[Y \leqq x]$ be an infinitely divisible probability distribution function. Its characteristic function $f_Y(u) = Ee^{iuY}$ has a representation of the form

$$f_Y(u) = \exp\left\{i\gamma_Y u - \frac{{\sigma_Y}^2 u^2}{2} + \int_{-\infty}^{\infty}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) dM_Y(x)\right\},$$

where $\gamma_Y, {\sigma_Y}^2$, and M_Y are the Lévy parameters uniquely associated with the given distribution. The Lévy spectral function M_Y is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$, is asymptotically zero $(M(-\infty) = 0 = M(\infty))$, and satisfies the integrability condition

$$\int_{-1}^{-0} + \int_{+0}^{+1} x^2 \, dM_Y(x) < \infty.$$

In [2], P. Hartman and A. Wintner proved that a necessary and sufficient condition that F_Y be continuous is that $\int_{-\infty}^{\infty} dM_Y(x) = \infty$ or ${\sigma_Y}^2 > 0$. Howard G. Tucker in [4] and M. Fisz and V. S. Varadarajan in [1] have shown that a sufficient condition for F_Y to be absolutely continuous is that $\int_{-\infty}^{\infty} dM_{ac}(x) = \infty$, where M_{ac} denotes the absolutely continuous component of M_Y.

We shall show that the distribution functions associated with certain non-negative infinitely divisible random variables are continuous if and only if they are continuous at their first rise.

2. Superposition of Brownian motion onto an infinitely divisible random variable. Let $\{W(t)/t \in [0, \infty)\}$ be a standard Brownian motion; i.e. a separable differential process with sample paths that are almost surely continuous and such that $\mathscr{L}(W(t)) = \mathscr{N}(0, t)$. Let X be a nonnegative infinitely divisible random variable that is independent of $\{W(t)\}$. The Lévy spectral function M_X associated with X vanishes on the negative half-axis and we assume that it satisfies the stronger integrability condition

$$\int_{+0}^{+1} x \, dM_X(x) < \infty. \tag{1}$$

Consequently, the characteristic function of X can be written in the form

$$f_X(u) = \exp\{i\gamma_X u + \int_0^{\infty} (e^{iux} - 1) \, dM_X(x)\},$$

where $\gamma_X \geqq 0$. The superposition $Y = W(X)$ also has an infinitely divisible distribution. We have shown in [3] (more general results appear in [3] and [5]) that $Y = W(X)$ has Lévy parameters $\gamma_Y = 0$, ${\sigma_Y}^2 = \gamma_X$, and

$$\begin{aligned} M_Y(x) &= \int_{-\infty}^{x} \int_0^{\infty} (2\pi t)^{-\frac{1}{2}} \exp\{-y^2/2t\} \, dM_X(t) \, dy, \qquad & x < 0 \\ &= -M_Y(-x), & x > 0. \end{aligned}$$

Received March 2, 1970.

We now note that M_Y is symmetric, absolutely continuous, and $M_Y(+0) = \frac{1}{2}M_X(+0)$. Thus $\int_{-\infty}^{\infty} dM_Y(x) = \infty$ if and only if $\int_0^{\infty} dM_X(x) = \infty$ and we have proved

THEOREM 1. *Let X be a nonnegative infinitely divisible random variable that is independent of $\{W(t)\}$ and has a Lévy spectral function satisfying* (1). *Then the following statements are equivalent*:

(i) *$W(X)$ has an absolutely continuous distribution.*
(ii) *$W(X)$ has a continuous distribution.*
(iii) *X has a continuous distribution or $\gamma_X > 0$.*

Note that (iii) would imply that the distribution of X is continuous at the origin.

3. Superposition of Brownian motion onto a nonnegative random variable. Let X be any nonnegative random variable that is independent of $\{W(t)\}$. The superposition $W(X)$ has characteristic function

$$f_{W(X)}(u) = \int_0^{\infty} e^{-\frac{1}{2}tu^2}\, dF_X(t).$$

The Helly–Bray Theorem implies that if $\langle X_n \rangle$ is a sequence of random variables independent of $\{W(t)\}$ and such that $X_n \to_{\mathscr{L}} X$, then $W(X_n) \to_{\mathscr{L}} W(X)$. We now find the distribution function of such a superposition.

THEOREM 2. *Let X be a nonnegative random variable that is independent of $\{W(t)\}$. Then*

$$\begin{aligned} F_{W(X)}(x) &= P[X=0] + \int_0^{\infty}\int_{-\infty}^{x} (2\pi y)^{-\frac{1}{2}} e^{-t^2/2y}\, dt\, dF_X(y), \qquad & x > 0 \\ &= \int_0^{\infty}\int_{-\infty}^{x} (2\pi y)^{-\frac{1}{2}} e^{-t^2/2y}\, dt\, dF_X(y), & x < 0. \end{aligned}$$

PROOF. It suffices to outline the proof for $x > 0$. Set

$$X_n = 0 \cdot I_{[X=0]} + \sum_{k=1}^{\infty} \frac{k}{2^n} \cdot I_{[(k-1)/2^n < X \leqq k/2^n]}.$$

Then

$$\begin{aligned} F_{W(X_n)}(x) &= P[W(X_n) \leqq x] \\ &= P[X=0] + \textstyle\sum_{k=1}^{\infty} P[W(k/2^n) \leqq x] \cdot P[X_n = k/2^n] \\ &= P[X=0] + \textstyle\sum_{k=1}^{\infty} \int_{-\infty}^{x} (2\pi k/2^n)^{-\frac{1}{2}} e^{-t^2/2k/2^n}\, dt \cdot P[X_n = k/2^n] \\ &= P[X=0] + \int_0^{\infty}\int_{-\infty}^{x} (2\pi y)^{-\frac{1}{2}} e^{-t^2/2y}\, dt\, dF_{X_n}(y). \end{aligned}$$

Since $X_n \downarrow X$, an application of the Helly–Bray Theorem to the last expression completes our proof.

Since the Lebesgue Dominated Convergence Theorem allows us to take limits inside, we immediately have

COROLLARY 1. *Let X be a nonnegative random variable independent of $\{W(t)\}$. Then $F_{W(X)}(x)$ is continuous at all $x \neq 0$. $F_{W(X)}(x)$ is continuous at $x = 0$ if and only if $P[X=0] = 0$,* i.e., *F_X takes no jump at the origin.*

Now suppose X^* is a nonnegative infinitely divisible random variable with Lévy spectral function satisfying (1). We assume we have a copy that is independent of $\{W(t)\}$. Let $\gamma = \inf\{x: F_{X^*}(x) > 0\}$ be the first rise of the distribution of X^*. Set $X = X^* - \gamma$. We note that $\gamma_X = 0$, that X has continuous distribution if and only if X^* has continuous distribution, and that $F_X(x)$ is continuous at the origin if and only if $F_{X^*}(x)$ is continuous at $x = \gamma$. Applying Theorem 1, we see that X is continuous if and only if $W(X)$ is continuous. Corollary 1 implies that the latter condition is equivalent to $F_X(x)$ being continuous at the origin. This completes the proof of

COROLLARY 2. *Let X be any nonnegative infinitely divisible random variable with Lévy spectral function satisfying* (1). *Then the distribution of X is continuous if and only if it is continuous at its first rise.*

REFERENCES

[1] FISZ, M. and VARADARAJAN, V. S. (1963). A condition for absolute continuity of infinitely divisible distributions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **1** 335–339.

[2] HARTMAN, P. and WINTNER, A. (1942). On the infinitesimal generators of integral convolutions. *Amer. J. Math.* **64** 273–298.

[3] HUFF, BARTHEL W. (1969). The strict subordination of differential processes. *Sankhyā Ser. A* **31** 403–412.

[4] TUCKER, H. G. (1962). Absolute continuity of infinitely divisible distributions. *Pacific J. Math.* **12** 1125–1129.

[5] ZOLOTAREV, V. M. (1958). Distribution of the superposition of infinitely divisible processes. *Theor. Probability Appl.* **3** 185–188.