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ON THE NONEXISTENCE OF A THREE SERIES CONDITION FOR SERIES OF NONINDEPENDENT RANDOM VARIABLES[1]

BY DAVID GILAT[2]

University of California, Berkeley

As is well known (see, for example, page 148 of [1]), if the series of conditional means μ_n and the series of conditional variances $\sigma_n{}^2$ of a series of uniformly bounded, real-valued random variables X_n converge almost surely, then so does $\sum X_n$. In a lecture, Aryeh Dvoretzky presented an example of a convergent series $\sum X_n$ of uniformly bounded variables for which $\sum \mu_n$ and $\sum \sigma_n{}^2$ diverge almost surely.

Of course μ_n and $\sigma_n{}^2$ refer to the conditional expectation and conditional variance of X_n given $X_1, \cdots, X_{n-1}$, except that μ_1 and $\sigma_1{}^2$ are unconditional.

For Dvoretzky's example, let $\{S_n\}$ be the sequence of independent random variables with

$$P\{S_n = n^{-\frac{1}{2}}\} = \tfrac{1}{2} = P\{S_n = -n^{-\frac{1}{2}}\} \qquad \text{for} \quad n \geqq 1,$$

let $S_0 \equiv 0$, and let $X_n = S_n - S_{n-1}$ for $n \geqq 1$.

It is natural to ask: *Is there some condition on the* μ_n *and* $\sigma_n{}^2$ *of a series of uniformly bounded random variables* $\sum X_n$, *which is necessary and sufficient for* $\sum X_n$ *to converge almost surely*?

The purpose of this note is to answer this query in the negative and in a rather strong sense.

Let $Y_n = S_n + S_{n-1}$, where the S_n are the same as were used to define X_n, and verify these two facts:

(i) $\sum Y_n$ diverges almost surely;

(ii) The vector-valued sequence $(\mu_n, \sigma_n{}^2)$ associated with the Y_n-process has the same finite dimensional distributions as does the $(\mu_n, \sigma_n{}^2)$-sequence associated with the X_n-process.

Since, in view of (ii), the Y_n-process has essentially the same first and second conditional moments as does the X_n-process, and yet $\sum Y_n$ diverges whereas $\sum X_n$ converges, the answer to the query above is clearly "no".

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REFERENCE

[1] NEVEU, J. (1965). *Mathematical Foundations of the Calculus of Probability*. Holden-Day, San Francisco.

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[2] Presently at Columbia University.

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