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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR k-DIMENSIONAL RENEWAL THEORY[1]

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1. Introduction. Let $\{\mathbf{X}_n, n \geqq 1\}$ be a sequence of random vectors in $\mathbb{R}^k$ defined on some probability triple $(\Omega, \mathscr{F}, P)$ and set $\mathbf{S}_n = \sum_{i=1}^n \mathbf{X}_i$ for $n \geqq 1$, and $\mathbf{S}_0 = \mathbf{0}$. Let $h:\mathbb{R}^k \to [0, \infty)$ be a function with continuous first partial derivatives, such that $h(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$, $\mathbf{x} \in \mathbb{R}^k$; assume furthermore that h is homogeneous of degree one (i.e., for all $\mathbf{x} \in \mathbb{R}^k$, $\lambda \geqq 0$, $h(\lambda\mathbf{x}) = \lambda h(\mathbf{x})$). We define the associated point process $\{M(t): t \geqq 0\}$ by $M(t) = \min\{n \geqq 1 : h(\mathbf{S}_n) > t\}$, where $M(t) = \infty$ if no such n exists.

The main result of this paper is a functional central limit theorem (invariance principle) for the process $\{M(t): t \geqq 0\}$. Section 2 is devoted to two preliminary lemmas and the theorem is proved in Section 3.

The ordinary central limit theorem for $\{M(t): t \geqq 0\}$ was given by Farrell [4]. Bickel and Yahav [1] discuss renewal theory for which h is any norm giving the Euclidean topology in $\mathbb{R}^k$. Related material on k-dimensional renewal theory may be found in Farrell [3] and Stam [5].

Our analysis shall be carried out in $D[0, 1]$, the space of right continuous functions on $[0, 1]$ having left limits and endowed with the Skorohod metric d. For an account of the weak convergence of probability measures on $D[0, 1]$ the reader is referred to the book by Billingsley (1968). We shall use $\Rightarrow$ to denote weak convergence of probability measures. When stochastic processes or ordinary random variables appear in such an expression we mean the measures induced by these functions. Let $C[0, 1] \equiv C$ denote the space of continuous functions on $[0, 1]$ and ρ the uniform metric on C and D; $C^k \equiv C^k[0, 1]$ and $D^k \equiv D^k[0, 1]$ will denote the product spaces of k copies of C and D respectively, with the appropriate product topologies.

2. Preliminaries. Let $\boldsymbol{\mu} \in \mathbb{R}^k$, $\boldsymbol{\mu} \neq \mathbf{0}$ and define the random functions Y_n, H_n in D^k and D induced by the sequence of partial sums $\{\mathbf{S}_n, n \geqq 1\}$ as follows

$$Y_n(t) = [\mathbf{S}_{[nt]} - nt\boldsymbol{\mu}]/n^{\frac{1}{2}}$$
$$H_n(t) = [h(\mathbf{S}_{[nt]}) - nth(\boldsymbol{\mu})]/n^{\frac{1}{2}}.$$

Let $\cdot$ denote the ordinary scalar product in $\mathbb{R}^k$ and $\nabla h = (\partial h/\partial x_1, \cdots, \partial h/\partial x_k)$. Note that ∇h is a homogeneous function of degree 0, in particular $\nabla h(t\boldsymbol{\mu}) = \nabla h(\boldsymbol{\mu})$ for all $t \in [0, 1]$.

LEMMA 1. *If $Y_n \Rightarrow \xi$ in D^k then $H_n \Rightarrow \nabla h(\boldsymbol{\mu}) \cdot \xi$ in D, where $\nabla h(\boldsymbol{\mu}) \cdot \xi$ is the scalar product of the process ξ and the constant vector $\nabla h(\boldsymbol{\mu})$.*

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PROOF. For each $\omega \in \Omega$ we have

$$H_n(t) = [h(n\{Y_n(t)/n^{\frac{1}{2}} + t\boldsymbol{\mu}\}) - nth(\boldsymbol{\mu})]/n^{\frac{1}{2}}$$
$$= [nh(\{Y_n(t)/n^{\frac{1}{2}} + t\boldsymbol{\mu}\}) - nth(\boldsymbol{\mu})]/n^{\frac{1}{2}} \quad \text{by the homogeneity of } h.$$

Then, by Taylor's theorem, we obtain

$$H_n(t) = n^{\frac{1}{2}}[h(t\boldsymbol{\mu}) + \nabla h(t\boldsymbol{\mu} + \theta_n(t)Y_n(t)/n^{\frac{1}{2}}) \cdot Y_n(t)/n^{\frac{1}{2}} - h(t\boldsymbol{\mu})],$$

where $0 \leqq \theta_n(t) \leqq 1$. This reduces to

$$H_n(t) = \nabla h(t\boldsymbol{\mu} + \theta_n(t)Y_n(t)/n^{\frac{1}{2}}) \cdot Y_n(t).$$

Because $Y_n \Rightarrow \xi$ we get $\nabla h(\boldsymbol{\mu}) \cdot Y_n \Rightarrow \nabla h(\boldsymbol{\mu}) \cdot \xi$ by the continuous mapping theorem; so by Theorem 4.1 of [2] to complete the proof of the lemma it is sufficient to show that $\rho(H_n, \nabla h(\boldsymbol{\mu}) \cdot Y_n) \Rightarrow 0$ as $n \to \infty$ since this implies that $d(H_n, \nabla h(\boldsymbol{\mu}) \cdot Y_n) \Rightarrow 0$. Observe that from the above

$$\rho(H_n, \nabla h(\boldsymbol{\mu}) \cdot Y_n) = \sup_{0 \leqq t \leqq 1} |H_n(t) - \nabla h(\boldsymbol{\mu}) \cdot Y_n(t)|$$
$$= \sup_{0 \leqq t \leqq 1} |(f_n(t) - \nabla h(\boldsymbol{\mu})) \cdot Y_n(t)|,$$

where $f_n(t) = \nabla h(t\boldsymbol{\mu} + \theta_n(t)Y_n(t)/n^{\frac{1}{2}})$. Let $||\cdot||$ be the usual Euclidean norm on $\mathbb{R}^k$ and define Q a compact subset of $\mathbb{R}^k$ as $Q = \{\mathbf{x} : ||x - t\boldsymbol{\mu}|| \leqq K, \text{ for some } t \in [0, 1]\}$ where K is some large positive constant. Since ∇h is uniformly continuous on Q, given $\varepsilon > 0$ there exists $\delta > 0$ such that for $||\mathbf{x}|| < \delta$ we have $\sup_{0 \leqq t \leqq 1} ||\nabla h(t\boldsymbol{\mu} + \mathbf{x}) - \nabla h(t\boldsymbol{\mu})|| \leqq \varepsilon^{\frac{1}{2}}$. Hence there is a sequence of positive real numbers, $\{a_n : n \geqq 1\}$, with $a_n \to 0$ as $n \to \infty$ such that for $||\mathbf{x}|| < \delta n^{-\frac{1}{4}}$ we have $\sup_{0 \leqq t \leqq 1} ||\nabla h(t\boldsymbol{\mu} + \mathbf{x}) - \nabla h(t\boldsymbol{\mu})|| \leqq \varepsilon^{\frac{1}{2}} a_n$. Now we define A_n, B_n, $C_n \in \mathscr{F}$ as

$$A_n = \{\rho(H_n, \nabla h(\boldsymbol{\mu}) \cdot Y_n) \leqq \varepsilon\}$$
$$B_n = \{\sup_{0 \leqq t \leqq 1} ||n^{-\frac{1}{4}} Y_n(t)|| < \delta\}$$
$$C_n = \{\sup_{0 \leqq t \leqq 1} ||a_n Y_n(t)|| \leqq \varepsilon^{\frac{1}{2}}\}.$$

Because $Y_n \Rightarrow \xi$ it follows that $n^{-\frac{1}{4}} Y_n \Rightarrow 0$ and $a_n Y_n \Rightarrow 0$ as $n \to \infty$, hence $P(B_n) \to 1$ and $P(C_n) \to 1$ as $n \to \infty$. Also from the above

$$B_n \subset \{\sup_{0 \leqq t \leqq 1} ||f_n(t) - \nabla h(\boldsymbol{\mu})|| \leqq \varepsilon^{\frac{1}{2}} a_n\}$$

and so $B_n \cap C_n \subset A_n$, which implies that $P(A_n) \to 1$ and hence $\rho(H_n, \nabla h(\boldsymbol{\mu}) \cdot Y_n) \Rightarrow 0$, completing the proof.

Next we define random functions T_n in D by

$$T_n(t) = [h(S_{M(nt)}) - M(nt)h(\boldsymbol{\mu})]/n^{\frac{1}{2}}$$

and choose a constant $c > 0$ such that $ch(\boldsymbol{\mu}) > 1$.

LEMMA 2. *If* $Y_n \Rightarrow \xi$ *and* $P\{\xi \in C^k\} = 1$ *then* $T_n \Rightarrow h(\boldsymbol{\mu})^{-\frac{1}{2}}(\nabla h(\boldsymbol{\mu}) \cdot \xi)$.

PROOF. The proof follows closely that of Theorem 17.3 in Billingsley (1968) and involves a random change of time in the functions H_n. We first show that

(2.1) $$\sup_{0 \leqq v \leqq u} |M(v) - v/h(\boldsymbol{\mu})|/u \Rightarrow 0$$

as $u \to \infty$.

Since from Lemma 1 $H_n \Rightarrow \nabla h(\boldsymbol{\mu}) \cdot \xi$ we have

(2.2) $$\sup_{0 \leqq t \leqq s} |h(S_{[t]}) - th(\boldsymbol{\mu})|/s \Rightarrow 0 \qquad \text{as} \quad s \to \infty.$$

But $M(v) > t$ implies $h(S_{[t]}) \leqq v$ and hence $\sup_{0 \leqq v \leqq u}(M(v) - v/h(\boldsymbol{\mu}))/u > \varepsilon$ implies

(2.3) $$\sup_{0 \leqq t \leqq u(\varepsilon + h(\boldsymbol{\mu})^{-1})} |h(S_{[t]}) - th(\boldsymbol{\mu})| \geqq h(\boldsymbol{\mu})u\varepsilon,$$

furthermore $M(v) < t$ implies there exists an s, $0 \leqq s \leqq t$ with $h(S_{[s]}) > v$ and hence for $\varepsilon < h(\boldsymbol{\mu})^{-1}$

(2.4) $$\inf_{0 \leqq v \leqq u}(M(v) - v/h(\boldsymbol{\mu}))/u < -\varepsilon \qquad \text{implies}$$
$$\sup_{0 \leqq t \leqq u(h(\boldsymbol{\mu})^{-1} - \varepsilon)} |h(S_{[t]}) - th(\boldsymbol{\mu})| \geqq h(\boldsymbol{\mu})u\varepsilon.$$

By (2.2) the probabilities of (2.3) and (2.4) go to 0 as $u \to \infty$ which proves (2.1). Now define random functions Φ_n in $D[0, 1]$ by

$$\Phi_n(t) = \begin{cases} M(nt)/cn & \text{if} \quad M(nt)/cn \leqq 1, \\ 1 & \text{otherwise;} \end{cases}$$

and define Φ by $\Phi(t) = t/ch(\boldsymbol{\mu})$, $0 \leqq t \leqq 1$. Then $\Phi \in C \cap D_0$, where D_0 consists of those functions ϕ of D which satisfy $0 \leqq \phi(t) \leqq 1$ for all $t \in [0, 1]$ and are non-decreasing. We will use the result of Billingsley ((1968) page 145), that if x_n, ϕ_n are random functions in D and D_0 respectively, $(x_n, \phi_n) \Rightarrow (x, \phi)$ in D^2 and $P\{x \in C\} = P\{\phi \in C\} = 1$ then $x_n \circ \phi_n \Rightarrow x \circ \phi$, where $\circ$ denotes the composition of functions.

We have $\Phi_n \Rightarrow \Phi$ from (2.1), $H_{cn} \Rightarrow \nabla h(\boldsymbol{\mu}) \cdot \xi$ where $H_{cn}(t) = [h(\mathbf{S}_{[cnt]}) - cnth(\boldsymbol{\mu})]/(cn)^{\frac{1}{2}}$. Also $P\{\Phi \in C\} = P\{\nabla h(\boldsymbol{\mu}) \cdot \xi \in C\} = 1$ so by Theorem 4.4 of Billingsley (1968) and the remarks above $H_{cn} \circ \Phi_n \Rightarrow \nabla h(\boldsymbol{\mu}) \cdot (\xi \circ \Phi)$; however, $H_{cn} \circ \Phi_n = T_n/c^{\frac{1}{2}}$, if $M(nt)/cn \leqq 1$. Since $P\{M(n)/cn \leqq 1\} \to 1$ we have that $T_n \Rightarrow c^{\frac{1}{2}} \nabla h(\boldsymbol{\mu}) \cdot (\xi \circ \Phi)$. Finally, $Y_n \Rightarrow \xi$ implies $Y_{ch(\boldsymbol{\mu})n} \Rightarrow \xi$ and hence $Y_n \equiv (ch(\boldsymbol{\mu}))^{\frac{1}{2}} Y_{ch(\boldsymbol{\mu})n} \circ \Phi \Rightarrow (ch(\boldsymbol{\mu}))^{\frac{1}{2}} \xi \circ \Phi$, which shows that ξ has the same distribution as $(ch(\boldsymbol{\mu}))^{\frac{1}{2}} \xi \circ \Phi$, so the result follows.

3. The main result. Define random functions M_n in D by

$$M_n(t) = [M(nt) - nt/h(\boldsymbol{\mu})]/n^{\frac{1}{2}};$$

we can now prove our main result.

THEOREM. *Under the conditions of Lemma* 2

$$M_n \Rightarrow -h(\boldsymbol{\mu})^{-\frac{3}{2}}(\nabla h(\boldsymbol{\mu}) \cdot \xi).$$

PROOF.

$$\begin{aligned} T_n(t) &\geqq [nt - M(nt)h(\boldsymbol{\mu})]/n^{\frac{1}{2}} \\ &\geqq [h(\mathbf{S}_{M(nt)-1}) - M(nt)h(\boldsymbol{\mu})]/n^{\frac{1}{2}} \\ &= T_n(t) - [h(\mathbf{S}_{M(nt)}) - h(\mathbf{S}_{M(nt)-1})]/n^{\frac{1}{2}}. \end{aligned}$$

By Lemma 2 $T_n \Rightarrow h(\boldsymbol{\mu})^{-\frac{1}{2}}(\nabla h(\boldsymbol{\mu}) \cdot \xi)$, so to complete the proof it is sufficient to show that $\sup_{0 \leqq t \leqq 1} |h(\mathbf{S}_{M(nt)}) - h(\mathbf{S}_{M(nt)-1})|/n^{\frac{1}{2}} \Rightarrow 0$ as $n \to \infty$. By the homogeneity of h and Taylor's theorem,

$$\begin{aligned}
&[h(\mathbf{S}_{M(nt)}) - h(\mathbf{S}_{M(nt)-1})]/n^{\frac{1}{2}} \\
&\quad = [nh(\{\mathbf{S}_{M(nt)-1} + \mathbf{X}_{M(nt)}\}/n) - nh(\mathbf{S}_{M(nt)-1}/n)]/n^{\frac{1}{2}} \\
&\quad = n^{-\frac{1}{2}}\mathbf{X}_{M(nt)} \cdot \nabla h(\{\mathbf{S}_{M(nt)-1} + \psi_n(t)\mathbf{X}_{M(nt)}\}/n),
\end{aligned}$$

where $0 \leqq \psi_n(t) \leqq 1$. Because of the problem of the measurability of $\psi_n(t)$ we first show that $\sup_{0 \leqq t \leqq 1} |n^{-\frac{1}{2}}\mathbf{X}_{M(nt)} \cdot \nabla h(\mathbf{S}_{M(nt)}/n)| \Rightarrow 0$. Using the Schwarz inequality we obtain

$$\begin{aligned}
&\sup_{0 \leqq t \leqq 1} |n^{-\frac{1}{2}}\mathbf{X}_{M(nt)} \cdot \nabla h(\mathbf{S}_{M(nt)}/n)| \\
&\quad \leqq \sup_{0 \leqq t \leqq 1} ||n^{-\frac{1}{2}}\mathbf{X}_{M(nt)}|| \sup_{0 \leqq t \leqq 1} ||\nabla h(\mathbf{S}_{M(nt)}/n)||.
\end{aligned}$$

If Φ_n and Φ are as defined in the proof of Lemma 2 then with the same random change of time argument we have that $Y_{cn} \circ \Phi_n \Rightarrow \xi \circ \Phi$, where $Y_{cn} \circ \Phi_n(t) = [\mathbf{S}_{M(nt)} - M(nt)\boldsymbol{\mu}]/(cn)^{\frac{1}{2}}$. Define the functional $g: D^k \to \mathbb{R}$ by $g(x) = \sup_{0 \leqq t \leqq 1} ||x(t) - x(t-)||$, g is measurable and continuous at $x \in C^k$. Since $P\{\xi \circ \Phi \in C^k\} = 1$, applying the continuous mapping theorem (Billingsley (1968) Theorem 5.1) we get $g(Y_{cn} \circ \Phi_n) \Rightarrow g(\xi \circ \Phi) = 0$ which implies that $\sup_{0 \leqq t \leqq 1} ||n^{-\frac{1}{2}}\mathbf{X}_{M(nt)}|| \Rightarrow 0$. Let $f \in D^k$ be given by $f(t) = t\boldsymbol{\mu}$ for $t \in [0, 1]$, then since $Y_n \Rightarrow \xi$ we have $S_{[n\cdot]}/n \Rightarrow f$ and by a random change of time $\mathbf{S}_{M(n\cdot)}/n \Rightarrow h(\boldsymbol{\mu})^{-1}f$.

Now using the continuous mapping theorem once more we have

$$\sup_{0 \leqq t \leqq 1} ||\nabla h(\mathbf{S}_{M(nt)}/n)|| \Rightarrow \sup_{0 \leqq t \leqq 1} ||\nabla h(h(\boldsymbol{\mu})^{-1}f(t))||,$$

a constant. Finally, using an argument similar to that in the proof of Lemma 1 it is easily shown that

$$\sup_{0 \leqq t \leqq 1} |n^{-\frac{1}{2}}\mathbf{X}_{M(nt)} . (\nabla h(\{\mathbf{S}_{M(nt)-1} + \psi_n(t)\mathbf{X}_{M(nt)}\}/n) - \nabla h(\mathbf{S}_{M(nt)}/n))| \Rightarrow 0,$$

which completes the proof.

Notice that when the $\{\mathbf{X}_n, n \geqq 1\}$ are independent, identically distributed random vectors with $E\mathbf{X}_1 = \boldsymbol{\mu}$ and positive definite covariance matrix $\boldsymbol{\Sigma}$, then ξ is a k-dimensional Brownian motion (with dependent components); but in this case $-\xi$ has the same distribution as ξ so the conclusion of the theorem may be replaced by

$$M_n \Rightarrow h(\boldsymbol{\mu})^{-\frac{1}{2}}(\nabla h(\boldsymbol{\mu}) \cdot \xi).$$

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