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PROCESSES OBTAINABLE FROM BROWNIAN MOTION BY MEANS OF A RANDOM TIME CHANGE[1,2]

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1. Introduction. Our terminology throughout this paper will in general be that of [8]. The triple $(\Omega, \mathscr{A}, P)$ will denote our fixed fundamental probability space. A random variable will be an $\mathscr{A}$-measurable real-valued function. Throughout this paper we will assume that the Brownian motion we deal with is standard Brownian motion with *all* sample paths continuous and unbounded in both directions. If $\{X(t): t \in [a, b]\}$ is a collection of random variables, then $\sigma\{X(t): t \in [a, b]\}$ will denote the smallest sub-sigma field of $\mathscr{A}$ for which each $X(t)$, $t \in [a, b]$, is measurable. Furthermore if X is a random variable and $A \in \mathscr{A}$ then $[X \leqq s]$ will denote the event $\{\omega \in \Omega: X(\omega) \leqq s\}$ and I_A will denote the indicator of the event A.

The problem of finding what processes are random time changes of Brownian motion has been studied extensively (in the case of martingales) by K. E. Dambis in [2], by L. E. Dubins and G. Schwarz in [4]. In [4] L. E. Dubins and G. Schwarz showed that every continuous martingale can be transformed into standard Brownian motion by means of a random time change. In this paper we prove that if $\{X(t): t \in [0, +\infty)\}$ is a Brownian motion process and if $\{Y(\alpha): \alpha \in I\}$ is a stochastic process with sufficiently nice properties then $\{Y(\alpha): \alpha \in I\}$ can be obtained from $\{X(t): t \in [0, +\infty)\}$ by means of a random time change (see Definition 2.1 below). Furthermore for certain processes $\{Y(\alpha): \alpha \in [0, +\infty)\}$, the collection of stopping times we construct, "almost" has independent increments. The main results of this paper are Theorem 2.2, Theorem 2.4, Theorem 2.5, Theorem 2.6 and Corollary 2.7.

2. Main results.

DEFINITION 2.1. Let $I \subset [0, +\infty)$ and let $\{X(t): t \in [0, +\infty)\}$ and $\{Y(\alpha): \alpha \in I\}$ be stochastic processes defined on $(\Omega, \mathscr{A}, P)$. Then we say that $\{Y(\alpha): \alpha \in I\}$ can be obtained from $\{X(t): t \in [0, +\infty)\}$ by means of a random time change if and only if there exists a collection of random variables $\{T_\alpha: \alpha \in I\}$ defined on $(\Omega, \mathscr{A}, P)$ satisfying the following requirements:

$$\text{for each } \alpha \in I, \quad T_\alpha \geqq 0, \tag{2.1}$$

$$\text{for each } \omega \in \Omega, \quad T_\alpha(\omega) \text{ is non-decreasing in } \alpha, \tag{2.2}$$

$$\text{for each } \alpha \in I, [T_\alpha \leqq s] \in \sigma\{X(t): t \in [0, s]\} \text{ for every } s \in [0, +\infty), \quad \text{and} \tag{2.3}$$

$$\text{for each } \alpha \in I, \quad X(T_\alpha) = Y(\alpha) \text{ a.s.} \tag{2.4}$$

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In this paper we try to find out what type of processes can be obtained from a Brownian motion $\{X(t): t\in[0, +\infty)\}$ by means of a random time change. Let us first make the following very simple observation. Suppose $\{X(t): t\in[0, +\infty)\}$ is a Brownian motion process and $\{Y(\alpha): \alpha\in[0, +\infty)\}$ is a stochastic process such that for each $\alpha\in[0, +\infty)$, $\sigma\{Y(\alpha)\}\subset\sigma\{X(t): t\in[0, \alpha]\}$. For each $\alpha\in[0, +\infty)$, define T_α by $T_\alpha(\omega) = \inf\{t \geqq \alpha: X(t, \omega) = Y(\alpha, \omega)\}$. It is easy to show that each T_α is a stopping time for $\{X(t): t\in[0, +\infty)\}$ (that is, (2.3) is satisfied). Furthermore for each $\alpha\in[0, +\infty)$, $X(T_\alpha) = Y(\alpha)$ since $\{X(t): t\in[0, +\infty)\}$ has continuous sample paths. Hence if the process $\{Y(\alpha): \alpha\in[0, +\infty)\}$ is such that (2.2) holds for $\{T_\alpha: \alpha\in[0, +\infty)\}$ then $\{Y(\alpha): \alpha\in[0, +\infty)\}$ can be obtained from $\{X(t): t\in[0, +\infty)\}$ by means of a random time change. For example, define $Y(\alpha)$ by

$$\begin{aligned} Y(\alpha) &= X(\alpha) && \text{if } \alpha\in[0,1] \\ &= (\sup\nolimits_{t\in[0,\alpha]} X(t))I_A + (\inf\nolimits_{t\in[0,\alpha]} X(t))I_{A^c} && \text{if } \alpha > 1 \end{aligned}$$

where $A\in\sigma\{X(t): t\in[0, 1]\}$. Then $\{Y(\alpha): \alpha\in[0, +\infty)\}$ can be obtained from $\{X(t): t\in[0, +\infty)\}$ by means of a random time change.

Suppose $\{X(t): t\in[0, +\infty)\}$ is a Brownian motion process. The construction of a stopping time T so that $X(T)$ has the same distribution as a given random variable Y has been the subject of much discussion. In [7], D. H. Root showed that if $\sigma^2(Y) < +\infty$ and $E(Y) = 0$ then there is a stopping time T such that $\mathscr{L}(Y) = \mathscr{L}(X(T))$ and $E(T) = \sigma^2(Y)$. A second method of defining a stopping time T with finite expectation such that $X(T)$ and Y are equal in law has been given by L. Dubins in [3]. In view of these results the following theorem is of interest. However, the stopping times which we construct all have infinite expectations.

THEOREM 2.2. *Let $\{X(t): t\in[0, +\infty)\}$ be a Brownian motion process and let $\{Y(k): k = 0, 1, 2, \cdots\}$ be any stochastic process such that for each integer $k \geqq 0$ there exists a real number $c_k \geqq 0$ with $\sigma\{Y(k)\}\subset\sigma\{X(t): t\in[0, c_k]\}$. Then the process $\{Y(k): k = 0, 1, 2, \cdots\}$ can be obtained from $\{X(t): t\in[0, +\infty)\}$ by means of a random time change.*

PROOF. Define T_0 by

$$T_0(\omega) = \inf\{t \geqq c_0: X(t, \omega) = Y(0, \omega)\},$$

and for $k = 1, 2, \cdots$, define T_k by

$$T_k(\omega) = \inf\{t \geqq \max(c_k, T_{k-1}(\omega)): X(t, \omega) = Y(k, \omega)\}.$$

Then for each k, $T_k \geqq 0$ and $X(T_k) = Y(k)$ everywhere since $\{X(t): t\in[0, +\infty)\}$ has continuous sample paths. Furthermore for fixed $\omega\in\Omega$, $T_k(\omega) \leqq T_{k+1}(\omega)$. Hence in order to prove the theorem it suffices to prove that for each nonnegative integer k,

$$[T_k \leqq s]\in\sigma\{X(t): t\in[0, s]\} \quad \text{for every} \quad s\in[0, +\infty). \tag{2.5}$$

We prove this by induction on k. Let $s \in [0, +\infty)$. If $s < c_0$ then $[T_0 \leqq s] = \phi \in \sigma\{X(t): t \in [0, s]\}$ since $T_0 \geqq c_0$. Suppose that $c_0 \leqq s$. Now

$$[T_0 \leqq s] = \{\omega : \inf\{t \geqq c_0 : X(t,\omega) - Y(0,\omega) = 0\} \leqq s\}.$$

Let $A = [T_0 \leqq s] \cap [X(c_0) - Y(0) \leqq 0]$, and let $B = [T_0 \leqq s] \cap [X(c_0) - Y(0) > 0]$. Rewriting A and B we see that

$$A = \{\omega : \sup_{t \in [c_0, s]}(X(t,\omega) - Y(0,\omega)) \geqq 0\} \cap [X(c_0) - Y(0) \leqq 0],$$

and

$$B = \{\omega : \inf_{t \in [c_0, s]}(X(t,\omega) - Y(0,\omega)) \leqq 0\} \cap [X(c_0) - Y(0) > 0].$$

Now by hypothesis $\sigma\{Y(0)\} \subset \sigma\{X(t): t \in [0, c_0]\}$. Hence $[T_0 \leqq s] = A \cup B \in \sigma\{X(t): t \in [0, s]\}$ and therefore T_0 satisfies (2.5). Let k be a nonnegative integer and assume that T_k satisfies (2.5). Let $s \in [0, +\infty)$. If $s < c_{k+1}$ then $[T_{k+1} \leqq s] = \phi \in \sigma\{X(t): t \in [0, s]\}$ since $T_{k+1} \geqq c_{k+1}$. Suppose now that $c_{k+1} \leqq s$. Let

$$A = \{\omega : \inf\{t \geqq c_{k+1} : X(t,\omega) - Y(k+1,\omega) = 0\} \leqq s\},$$

and let

$$B = \{\omega : \inf\{t \geqq T_k(\omega) : X(t,\omega) - Y(k+1,\omega) = 0\} \leqq s\}.$$

Then $[T_{k+1} \leqq s] = A \cap B$. Moreover

$$\begin{aligned} A &= (A \cap [X(c_{k+1}) - Y(k+1) \leqq 0]) \cup (A \cap [X(c_{k+1}) - Y(k+1) \geqq 0]) \\ &= (\{\omega : \sup_{t \in [c_{k+1}, s]}(X(t,\omega) - Y(k+1,\omega)) \geqq 0\} \cap [X(c_{k+1}) - Y(k+1) \leqq 0]) \\ &\quad \cup (\{\omega : \inf_{t \in [c_{k+1}, s]}(X(t,\omega) - Y(k+1,\omega)) \leqq 0\} \cap [X(c_{k+1}) - Y(k+1) > 0]). \end{aligned}$$

Using the fact that $\sigma\{Y(k+1)\} \subset \sigma\{X(t): t \in [0, c_{k+1}]\}$, we see that $A \in \sigma\{X(t): t \in [0, s]\}$. Letting Q denote the rational numbers, B can be written as follows;

$$\begin{aligned} B &= \bigcap_{n=1}^{\infty} \{\omega: \text{ for some } t \in [T_k(\omega), s] \cap Q, |X(t,\omega) - Y(k+1,\omega)| \leqq 1/n\} \\ &= \bigcap_{n=1}^{\infty} \{\omega: \text{ for some } t \in [0, s] \cap Q, |X(t,\omega) - Y(k+1,\omega)| \leqq 1/n \\ &\qquad\qquad \text{and } T_k(\omega) \leqq t\} \\ &= \bigcap_{n=1}^{\infty} [\bigcup_{t \in [0,s] \cap Q}([|X(t) - Y(k+1)| \leqq 1/n] \cap [T_k \leqq t])]. \end{aligned}$$

By our induction assumption, T_k satisfies (2.5). Also by hypothesis $\sigma\{Y(k+1)\} \subset \sigma\{X(t): t \in [0, c_{k+1}]\}$ and $c_{k+1} \leqq s$. Hence we see that $B \in \sigma\{X(t): t \in [0, s]\}$. Therefore $[T_{k+1} \leqq s] = A \cap B \in \sigma\{X(t): t \in [0, s]\}$. □

Before beginning the proof of the major theorem of this paper, we state the following known result.

LEMMA 2.3. *Let $\{X(t): t \in [0, +\infty)\}$ be a Brownian Motion process and let $R_1, \cdots, R_n$ be independent random variables such that $\sigma\{X(t): t \in [0, +\infty)\}$ and $\sigma\{R_1, \cdots, R_n\}$ are independent sigma fields. Define $\mathscr{S}_1, \cdots, \mathscr{S}_n$ as follows;*

$$\mathscr{S}_1 = \inf\{t \geqq 0 : X(t) = R_1\}, \qquad \textit{and}$$

for $k = 2, \cdots, n$,

$$\mathscr{S}_k = \inf\{t \geqq 0 : X(t + \textstyle\sum_{i=1}^{k-1} \mathscr{S}_i) - X(\sum_{i=1}^{k-1} \mathscr{S}_i) = R_k\}.$$

Then $\mathscr{S}_1, \cdots, \mathscr{S}_n$ are independent random variables and for $k = 1, \cdots, n$, the process $\{X(t+\sum_{i=1}^k \mathscr{S}_i) - X(\sum_{i=1}^k \mathscr{S}_i) : t \in [0, +\infty)\}$ is a Brownian motion process independent of $\mathscr{S}_k$. Also for $k = 1, \cdots, n-1$, $\sigma\{X(t+\sum_{i=1}^k \mathscr{S}_i) - X(\sum_{i=1}^k \mathscr{S}_i) : t \in [0, +\infty)\}$ is independent of $\sigma\{R_{k+1}\}$. Furthermore if $R_1, \cdots, R_n$ are identically distributed then $\mathscr{S}_1, \cdots, \mathscr{S}_n$ are identically distributed.

PROOF. The proof follows from the strong Markov property for Brownian motion. See ([1] Theorem 12.42, page 269) or [6]. If $R_1, \cdots, R_n$ are identically distributed then the fact that $\mathscr{S}_1, \cdots, \mathscr{S}_n$ are identically distributed follows from ([5] Theorem 1, page 1605).

THEOREM 2.4. *Let $\{X(t) : t \in [0, +\infty)\}$ be a Brownian motion process and let $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ be a stochastic process such that the following requirements are satisfied:*

(2.6) $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ *has left continuous sample paths and independent increments with* $Y(0) = 0 = \lim_{n \to +\infty} Y(1/2^n)$ *on* Ω,

(2.7) *there exists a real number* $c \geqq 0$ *such that for every* $\alpha \in [0, +\infty)$, $\sigma\{Y(\alpha)\} \subset \sigma\{X(t) : t \in [0, \alpha + c]\}$,

(2.8) T *is a positive real number and the sample paths of* $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ *are of bounded variation on* $[0, T]$.

Then the process $\{Y(\alpha) : \alpha \in [0, T]\}$ can be obtained from $\{X(t) : t \in [0, +\infty)\}$ by means of a random time change.

PROOF. Let $\lambda = T + c$. Define the process $\{W(t) : t \in [0, +\infty)\}$ by $W(t) = X(t+\lambda) - X(\lambda)$ for $t \in [0, +\infty)$, and define the process $\{Z(t) : t \in [0, +\infty)\}$ by $Z(t) = Y(t) - X(\lambda)$ for $t \in [0, +\infty)$. Let $\alpha \in (0, T]$ and let n be any positive integer such that $1 \leqq [2^n \alpha]$ where $[2^n \alpha]$ denotes the largest integer $\leqq 2^n \alpha$. For such an α and n define $\bar{T}(\alpha, n, k, \cdot)$ for $k = 1, \cdots, [2^n \alpha]$ as follows:

$$\bar{T}(\alpha, n, 1, \omega) = \inf\{t \geqq 0 : W(t, \omega) = Z(1/2^n, \omega)\},$$

and for $k = 2, \cdots, [2^n \alpha]$,

$$\bar{T}(\alpha, n, k, \omega) = \inf\{t \geqq 0 : W(t + \textstyle\sum_{i=1}^{k-1} \bar{T}(\alpha, n, i, \omega), \omega) = Z(k/2^n, \omega)\}.$$

Define $\bar{\bar{T}}(\alpha, n, \cdot)$ and $T(\alpha, n, \cdot)$ by

$$\bar{\bar{T}}(\alpha, n, \omega) = \textstyle\sum_{k=1}^{[2^n \alpha]} \bar{T}(\alpha, n, k, \omega),$$

and

$$T(\alpha, n, \omega) = \bar{\bar{T}}(\alpha, n, \omega) + \lambda.$$

Clearly $T(\alpha, n, \omega) \leqq T(\beta, n, \omega)$ for $0 < \alpha \leqq \beta \leqq T$ since

$$\bar{T}(\alpha, n, i, \omega) = \bar{T}(\beta, n, i, \omega) \quad \text{for} \quad i = 1, \cdots, [2^n \alpha] \quad \text{and} \quad \omega \in \Omega. \tag{2.9}$$

CLAIM 1. Let $\alpha \in (0, T]$ and let n be any positive integer such that $1 \leqq [2^n \alpha]$. Then for each $\omega \in \Omega$, $T(\alpha, n, \omega) \leqq T(\alpha, n+1, \omega)$.

PROOF OF CLAIM 1. In order to prove the claim it suffices to show that $\tilde{T}(\alpha, n, \omega) \leqq \tilde{T}(\alpha, n+1, \omega)$ for each $\omega \in \Omega$. By definition

$$\tilde{T}(\alpha, n, \omega) = \textstyle\sum_{k=1}^{[2^n\alpha]} \tilde{T}(\alpha, n, k, \omega) \quad \text{and}$$

$$\tilde{T}(\alpha, n+1, \omega) = \textstyle\sum_{k=1}^{[2^{n+1}\alpha]} \tilde{T}(\alpha, n+1, k, \omega).$$

Notice first of all that $2[2^n\alpha] \leqq [2^{n+1}\alpha]$ and so the sum defining $\tilde{T}(\alpha, n+1, \omega)$ contains at least twice as many terms as the sum defining $\tilde{T}(\alpha, n, \omega)$. Now

$$\begin{aligned}\tilde{T}(\alpha, n+1, 2, \omega) &= \inf\left\{t \geqq 0 : W(t+\tilde{T}(\alpha, n+1, 1, \omega), \omega) = Z\left(\frac{2}{2^{n+1}}, \omega\right)\right\} \\ &= \inf\left\{t \geqq \tilde{T}(\alpha, n+1, 1, \omega) : W(t, \omega) = Z\left(\frac{2}{2^{n+1}}, \omega\right)\right\} \\ &\quad - \tilde{T}(\alpha, n+1, 1, \omega).\end{aligned}$$

Hence

$$\textstyle\sum_{k=1}^{2} \tilde{T}(\alpha, n+1, k, \omega) = \inf\{t \geqq \tilde{T}(\alpha, n+1, 1, \omega) : W(t, \omega) = Z(1/2^n, \omega)\},$$

and by comparing this with the definition of $\tilde{T}(\alpha, n, 1, \omega)$, one can see that $\tilde{T}(\alpha, n, 1, \omega) \leqq \sum_{k=1}^{2} \tilde{T}(\alpha, n+1, k, \omega)$. Let i be an integer such that $1 \leqq i < [2^n\alpha]$ and assume that $\sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega) \leqq \sum_{k=1}^{2i} \tilde{T}(\alpha, n+1, k, \omega)$. By definition

$$\begin{aligned}&\tilde{T}(\alpha, n, i+1, \omega) \\ &\quad = \inf\left\{t \geqq 0 : W\left(t + \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega), \omega\right) = Z\left(\frac{i+1}{2^n}, \omega\right)\right\} \\ &\quad = \inf\left\{t \geqq \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega) : W(t, \omega) = Z\left(\frac{i+1}{2^n}, \omega\right)\right\} - \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega).\end{aligned}$$

Therefore

$$\sum_{k=1}^{i+1} \tilde{T}(\alpha, n, k, \omega) = \inf\left\{t \geqq \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega) : W(t, \omega) = Z\left(\frac{i+1}{2^n}, \omega\right)\right\}. \tag{2.10}$$

In the same manner one obtains

$$\begin{aligned}&\sum_{k=1}^{2(i+1)} \tilde{T}(\alpha, n+1, k, \omega) \\ &\quad = \inf\left\{t \geqq \sum_{k=1}^{2i+1} \tilde{T}(\alpha, n+1, k, \omega) : W(t, \omega) = Z\left(\frac{2(i+1)}{2^{n+1}}, \omega\right)\right\}.\end{aligned} \tag{2.11}$$

By our induction assumption $\sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega) \leqq \sum_{k=1}^{2i+1} \tilde{T}(\alpha, n+1, k, \omega)$ since $\tilde{T}(\alpha, n+1, 2i+1, \omega) \geqq 0$, so by comparing (2.10) and (2.11) we see that $\sum_{k=1}^{i+1} \tilde{T}(\alpha, n, k, \omega) \leqq \sum_{k=1}^{2(i+1)} \tilde{T}(\alpha, n+1, k, \omega)$. This completes the proof of Claim 1.

CLAIM 2. Let $\alpha \in (0, T]$ and let n be any positive integer such that $1 \leqq [2^n\alpha]$. Then for $k = 1, \cdots, [2^n\alpha]$ and $\omega \in \Omega$,

$$X\left(\lambda + \sum_{i=1}^{k} \tilde{T}(\alpha, n, i, \omega), \omega\right) = Y\left(\frac{k}{2^n}, \omega\right).$$

PROOF OF CLAIM 2. $\{W(t): t\in[0,+\infty)\}$ has continuous sample paths so by the definition of $\tilde{T}(\alpha,n,k,\omega)$,

$$W\left(\sum_{i=1}^{k}\tilde{T}(\alpha,n,i,\omega),\omega\right)=Z\left(\frac{k}{2^n},\omega\right)$$

for $k=1,\cdots,[2^n\alpha]$ and $\omega\in\Omega$. Hence

$$X\left(\lambda+\sum_{i=1}^{k}\tilde{T}(\alpha,n,i,\omega),\omega\right)-X(\lambda,\omega)=Y\left(\frac{k}{2^n},\omega\right)-X(\lambda,\omega)$$

for $k=1,\cdots,[2^n\alpha]$ and $\omega\in\Omega$. This proves Claim 2.

CLAIM 3. Let $\alpha\in(0,T]$ and let n be any positive integer such that $1\leqq[2^n\alpha]$. Then $T(\alpha,n,\cdot)$ is a stopping time for $\{X(t): t\in[0,+\infty)\}$.

PROOF OF CLAIM 3. By definition

$$T(\alpha,n,\cdot)=\lambda+\tilde{T}(\alpha,n,\cdot)=\lambda+\textstyle\sum_{k=1}^{[2^n\alpha]}\tilde{T}(\alpha,n,k,\cdot).$$

We shall prove by induction that $\lambda+\sum_{k=1}^{[2^n\alpha]}\tilde{T}(\alpha,n,k,\cdot)$ is a stopping time for $\{X(t): t\in[0,+\infty)\}$. For any real number $s\geqq 0$, set $\mathscr{F}_s=\sigma\{X(t): t\in[0,s]\}$. If $0\leqq s<\lambda$ then $[\lambda+\tilde{T}(\alpha,n,1,\cdot)\leqq s]=\phi\in\mathscr{F}_s$. Assume that $\lambda\leqq s<+\infty$. Now

$$\begin{aligned}\tilde{T}(\alpha,n,1,\cdot)&=\inf\left\{t\geqq 0: W(t)=Z\left(\frac{1}{2^n}\right)\right\}\\&=\inf\left\{t\geqq 0: X(t+\lambda)=Y\left(\frac{1}{2^n}\right)\right\}\\&=\inf\left\{t\geqq \lambda: X(t)=Y\left(\frac{1}{2^n}\right)\right\}-\lambda.\end{aligned}$$

Therefore

$$\begin{aligned}&[\lambda+\tilde{T}(\alpha,n,1,\cdot)\leqq s]\\&=\left\{\omega:\inf\left\{t\geqq\lambda: X(t,\omega)=Y\left(\frac{1}{2^n},\omega\right)\right\}\leqq s\right\}\\&=\left(\left\{\omega:\inf\left\{t\geqq\lambda: X(t,\omega)-Y\left(\frac{1}{2^n},\omega\right)=0\right\}\leqq s\right\}\cap\left[X(\lambda)-Y\left(\frac{1}{2^n}\right)\leqq 0\right]\right)\\&\qquad\cup\left(\left\{\omega:\inf\left\{t\geqq\lambda: X(t,\omega)-Y\left(\frac{1}{2^n},\omega\right)=0\right\}\leqq s\right\}\cap\left[X(\lambda)-Y\left(\frac{1}{2^n}\right)>0\right]\right)\\&=\left(\left\{\omega:\sup_{t\in[\lambda,s]}\left(X(t,\omega)-Y\left(\frac{1}{2^n},\omega\right)\right)\geqq 0\right\}\cap\left[X(\lambda)-Y\left(\frac{1}{2^n}\right)\leqq 0\right]\right)\\&\qquad\cup\left(\left\{\omega:\inf_{t\in[\lambda,s]}\left(X(t,\omega)-Y\left(\frac{1}{2^n},\omega\right)\right)\leqq 0\right\}\cap\left[X(\lambda)-Y\left(\frac{1}{2^n}\right)>0\right]\right).\end{aligned}$$

But $1/2^n \leqq \alpha \leqq T \leqq \lambda \leqq s$ since $1 \leqq [2^n\alpha]$, and so $\sigma\{Y(1/2^n)\} \subset \mathscr{F}_s$. Thus $[\lambda + \tilde{T}(\alpha, n, 1, \cdot) \leqq s] \in \mathscr{F}_s$ and therefore $\lambda + \tilde{T}(\alpha, n, 1, \cdot)$ is a stopping time for $\{X(t): t \in [0, +\infty)\}$. Let i be any integer such that $1 \leqq i < [2^n\alpha]$ and assume that $\lambda + \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \cdot)$ is a stopping time for $\{X(t): t \in [0, +\infty)\}$. If $0 \leqq s < \lambda$ then $[\lambda + \sum_{k=1}^{i+1} \tilde{T}(\alpha, n, k, \cdot) \leqq s] = \phi \in \mathscr{F}_s$. Suppose that $\lambda \leqq s < +\infty$. Now

$$\begin{aligned}
&\tilde{T}(\alpha, n, i+1, \omega) \\
&= \inf\left\{t \geqq 0: W\left(t + \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega), \omega\right) = Z\left(\frac{i+1}{2^n}, \omega\right)\right\} \\
&= \inf\left\{t \geqq 0: X\left(t + \lambda + \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega), \omega\right) = Y\left(\frac{i+1}{2^n}, \omega\right)\right\} \\
&= \inf\left\{t \geqq \lambda + \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega): X(t, \omega) = Y\left(\frac{i+1}{2^n}, \omega\right)\right\} - \left(\lambda + \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega)\right).
\end{aligned}$$

For convenience set $f(\omega) = \lambda + \sum_{k=1}^{i} \tilde{T}(\alpha, n, k, \omega)$ for each $\omega \in \Omega$, and let Q denote the rational numbers. Then

$$\begin{aligned}
&\left[\lambda + \sum_{k=1}^{i+1} \tilde{T}(\alpha, n, k, \cdot) \leqq s\right] \\
&= \left\{\omega: \inf\left\{t \geqq f(\omega): X(t, \omega) = Y\left(\frac{i+1}{2^n}, \omega\right)\right\} \leqq s\right\} \\
&= \bigcap_{j=1}^{\infty} \left\{\omega: \text{ for some } t \in [f(\omega), s] \cap Q, \left|X(t, \omega) - Y\left(\frac{i+1}{2^n}, \omega\right)\right| \leqq \frac{1}{j}\right\} \\
&= \bigcap_{j=1}^{\infty} \left\{\omega: \text{ for some } t \in [0, s] \cap Q, \left|X(t, \omega) - Y\left(\frac{i+1}{2^n}, \omega\right)\right| \leqq \frac{1}{j} \text{ and } f(\omega) \leqq t\right\} \\
&= \bigcap_{j=1}^{\infty} \left[\bigcup_{t \in [0,s] \cap Q} \left(\left[\left|X(t) - Y\left(\frac{i+1}{2^n}\right)\right| \leqq \frac{1}{j}\right] \cap [f \leqq t]\right)\right].
\end{aligned}$$

By our induction assumption, f is a stopping time for $\{X(t): t \in [0, +\infty)\}$. Moreover $\sigma\{Y((i+1)/2^n)\} \in \mathscr{F}_s$ since $\alpha \leqq T \leqq \lambda \leqq s$ and $1 \leqq i < [2^n\alpha]$. Hence from the above we see that $[\lambda + \sum_{k=1}^{i+1} \tilde{T}(\alpha, n, k, \cdot) \leqq s] \in \mathscr{F}_s$. Thus by induction $T(\alpha, n, \cdot)$ is a stopping time for $\{X(t): t \in [0, +\infty)\}$ which proves Claim 3.

From Claim 1, if $\alpha \in (0, T]$ and $\omega \in \Omega$ then $\lim_{n \to +\infty} T(\alpha, n, \omega)$ exist in the extended real numbers. In the following claim we prove this limit is finite almost surely.

CLAIM 4. Let $A = [\lim_{n \to +\infty} T(T, n, \cdot) < +\infty]$. Then $P(A) = 1$ and $\lim_{n \to +\infty} T(\alpha, n, \omega) < +\infty$ for any $\alpha \in (0, T]$ and $\omega \in A$.

PROOF OF CLAIM 4. By (2.9), $T(\alpha, n, \omega) \leqq T(T, n, \omega)$ for any $\alpha \in (0, T]$ and $\omega \in \Omega$. Also $A = [\lim_{n \to \infty} \tilde{T}(T, n, \cdot) < +\infty]$ and so it suffices to show that

$$P[\lim_{n \to +\infty} \tilde{T}(T, n, \cdot) < +\infty] = 1. \tag{2.12}$$

For each integer n such that $1 \leqq [2^n T]$ define B_n by

$$B_n = \sum_{k=1}^{[2^n T]} \left| Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k-1}{2^n}\right) \right| = \sum_{k=1}^{[2^n T]} \left| Y\left(\frac{k}{2^n}\right) - Y\left(\frac{k-1}{2^n}\right) \right|,$$

and let $B = \lim_{n \to +\infty} B_n$. By condition (2.8), $B(\omega) < +\infty$ for each $\omega \in \Omega$. Let n be any positive integer with $2 \leqq [2^n T]$ and define $T^*(n, k, \cdot)$ for $k = 1, \cdots, [2^n T]$ by

$$T^*(n, 1, \omega) = \inf\left\{ t \geqq 0 : W(t, \omega) = \left| Z\left(\frac{1}{2^n}, \omega\right) - Z(0, \omega) \right| \right\}$$

and for $k = 2, \cdots, [2^n T]$,

$$T^*(n, k, \omega)$$
$$= \inf\left\{ t \geqq 0 : W\left(t + \sum_{i=1}^{k-1} T^*(n, i, \omega), \omega\right) = \sum_{i=1}^{k} \left| Z\left(\frac{k}{2^n}, \omega\right) - Z\left(\frac{k-1}{2^n}, \omega\right) \right| \right\}.$$

Since $\{W(t) : t \in [0, +\infty)\}$ has continuous sample paths,

$$W\left(\sum_{i=1}^{k} T^*(n, i, \cdot)\right) = \sum_{i=1}^{k} \left| Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k-1}{2^n}\right) \right| \quad \text{on } \Omega$$

and

$$W\left(\sum_{i=1}^{k} \tilde{T}(T, n, i, \cdot)\right) = Z\left(\frac{k}{2^n}\right) \quad \text{on } \Omega$$

for $1 \leqq k \leqq [2^n T]$. Therefore for $1 < k \leqq [2^n T]$,

$$T^*(n, k, \cdot) = \inf\left\{ t \geqq 0 : W\left(t + \sum_{i=1}^{k-1} T^*(n, i, \cdot)\right) - W\left(\sum_{i=1}^{k-1} T^*(n, i, \cdot)\right) = \left| Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k-1}{2^n}\right) \right| \right\}, \tag{2.13}$$

and

$$\tilde{T}(T, n, k, \cdot) = \inf\left\{ t \geqq 0 : W\left(t + \sum_{i=1}^{k-1} \tilde{T}(T, n, i, \cdot)\right) - W\left(\sum_{i=1}^{k-1} \tilde{T}(T, n, i, \cdot)\right) = Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k-1}{2^n}\right) \right\}. \tag{2.14}$$

By (2.7), $\sigma\{Z(k/2^n) - Z((k-1)/2^n)\} \subset \sigma\{X(t) : t \in [0, \lambda]\}$ for $k = 1, \cdots, [2^n T]$. Hence since $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ and $\{X(t) : t \in [0, +\infty)\}$ have independent increments we have that $Z(1/2^n) - Z(0)$, $Z(2/2^n) - Z(1/2^n), \cdots, Z([2^n T]/2^n) - Z(([2^n T] - 1)/2^n)$ are independent random variables with $\sigma\{Z(k/2^n) - Z((k-1)/2^n) : k = 1, \cdots, [2^n T]\}$ independent of $\sigma\{W(t) : t \in [0, +\infty)\}$. Lemma 2.3 and (2.13) now imply,

$$T^*(n, 1, \cdot), T^*(n, 2, \cdot), \cdots, T^*(n, [2^n T], \cdot) \tag{2.15}$$

are independent random variables.

In the proof of Claim 3 it was shown that $\lambda + \tilde{T}(T, n, 1, \cdot)$ is a stopping time for $\{X(t): t \in [0, +\infty)\}$ and so by ([1] Theorem 12.42, page 269), $\{W(t+\tilde{T}(T, n, 1, \cdot)) - W(\tilde{T}(T, n, 1, \cdot)): t \in [0, \infty)\}$ is a Brownian motion independent of $\sigma\{X(t): t \in [0, \lambda]\}$. Therefore by (2.14), Lemma 2.3, and [5, Theorem 1, page 1605],

$$\tilde{T}(T,n,2,\cdot),\cdots,\tilde{T}(T,n,[2^nT],\cdot) \tag{2.16}$$

are independent random variables such that for

$$k = 2,\cdots,[2^nT], \qquad \mathscr{L}(\tilde{T}(T,n,k,\cdot)) = \mathscr{L}(T^*(n,k,\cdot)).$$

Let n_0 be the smallest integer such that $1 \leqq [2^{n_0}T]$. For any integer $n \geqq n_0$, define H_n by

$$H_n(\omega) = \inf\{t \geqq 0: W(t,\omega) = B_n(\omega)\} \qquad \text{for} \quad \omega \in \Omega,$$

and define H by

$$H(\omega) = \inf\{t \geqq 0: W(t,\omega) = B(\omega)\} \qquad \text{for} \quad \omega \in \Omega.$$

Since

$$\sum_{k=1}^{i} \left| Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k-1}{2^n}\right) \right| \leqq \sum_{k=1}^{i+1} \left| Z\left(\frac{k}{2^n}\right) - Z\left(\frac{k-1}{2^n}\right) \right|,$$

it is clear that for each $n \geqq n_0$,

$$H_n = \textstyle\sum_{k=1}^{[2^nT]} T^*(n,k,\cdot) \qquad \text{on} \quad \Omega. \tag{2.17}$$

Since $B_n \to B < +\infty$ as $n \to +\infty$ and since $\sigma\{B_n: n = n_0, n_0+1, n_0+2, \cdots\} \subset \sigma\{X(t): t \in [0, \lambda]\}$ which is independent of $\sigma\{W(t): t \in [0, +\infty)\}$, it follows from ([5] Corollary 1A, page 1605), that

$$\mathscr{L}(H_n) \to \mathscr{L}(H) \qquad \text{as} \quad n \to +\infty. \tag{2.18}$$

By Claim 1, $\tilde{T}(T, n, \cdot) \leqq \tilde{T}(T, n+1, \cdot)$ and so for any real number γ, $[\lim_{n\to+\infty} \tilde{T}(T,n,\cdot) > \gamma] = \bigcup_{n=1}^{\infty} [\tilde{T}(T,n,\cdot) > \gamma]$.

Thus

$$\begin{aligned}
&P[\lim_{n\to+\infty} \tilde{T}(T,n,\cdot) > 2\gamma] \\
&\quad = \lim_{n\to+\infty} P[\tilde{T}(T,n,\cdot) > 2\gamma] \\
&\quad = \lim_{n\to+\infty} P[\textstyle\sum_{k=1}^{[2^nT]} \tilde{T}(T,n,k,\cdot) > 2\gamma] \\
&\quad \leqq \limsup_{n\to+\infty} (P[\tilde{T}(T,n,1,\cdot) > \gamma] + P[\textstyle\sum_{k=2}^{[2^nT]} \tilde{T}(T,n,k,\cdot) > \gamma]) \\
&\quad \leqq \limsup_{n\to+\infty} P[\tilde{T}(T,n,1,\cdot) > \gamma] \\
&\qquad\qquad + \limsup_{n\to+\infty} P[\textstyle\sum_{k=2}^{[2^nT]} \tilde{T}(T,n,k,\cdot) > \gamma],
\end{aligned} \tag{2.19}$$

for any real number γ. Now $Z(1/2^n) = Y(1/2^n) - X(\lambda) \to -X(\lambda)$ as $n \to +\infty$ since $Y(1/2^n) \to 0$. Furthermore for all n sufficiently large, $\sigma\{Y(1/2^n) - X(\lambda)\} \subset \sigma\{X(t):$

$t \in [0, \lambda]\}$. Hence letting T be defined by $T(\omega) = \inf\{t \geqq 0: W(t) = -X(\lambda)\}$, we get from ([5] Corollary 1A, page 1605) that $\mathscr{L}(\tilde{T}(T, n, 1, \cdot)) \to \mathscr{L}(T)$ as $n \to +\infty$. Also by (2.15) and (2.16), for any real number γ,

$$\begin{aligned} P[\textstyle\sum_{k=2}^{[2^nT]} \tilde{T}(T, n, k, \cdot) > \gamma] &= P[\textstyle\sum_{k=2}^{[2^nT]} T^*(n, k, \cdot) > \gamma] \\ &\leqq P[\textstyle\sum_{k=1}^{[2^nT]} T^*(n, k, \cdot) > \gamma] \\ &= P[H_n > \gamma]. \end{aligned}$$

Statements (2.18) and (2.19) now imply that

$$P[\lim_{n\to\infty} \tilde{T}(T, n, \cdot) > 2\gamma] \leqq P[T > \gamma] + P[H > \gamma]$$

for any real number γ belonging to the continuity sets of $\mathscr{L}(T)$ and $\mathscr{L}(H)$. Therefore since T and H are finite it follows that $P[\lim_{n\to+\infty} \tilde{T}(T, n, \cdot) = +\infty] = 0$. This completes the proof of Claim 4.

Define the stochastic process $\{T_\alpha: \alpha \in [0, T]\}$ as follows; $T_0 = 0$, and for $\alpha \in (0, T]$, $T_\alpha = \lim_{n\to\infty} T(\alpha, n, \cdot)$. By Claim 4, $T_\alpha < +\infty$ a.s. Also for any real number $s \geqq 0$, $[T_\alpha \leqq s] = \bigcap_{n=1}^{\infty} [T(\alpha, n, \cdot) \leqq s]$ by Claim 1. Claim 3 now implies,

(2.20) each T_α is a stopping time for $\{X(t): t \in [0, +\infty)\}$.

Claim 5. Let $\alpha \in [0, T]$. Then $X(T_\alpha) = Y(\alpha)$ a.s.

Proof of Claim 5. If $\alpha = 0$, $X(T_\alpha) = 0 = Y(0)$. Suppose that $\alpha \in (0, T]$. By Claim 4,

$$T_\alpha = \lim_{n\to+\infty} T(\alpha, n, \cdot) = \lim_{n\to+\infty} \tilde{T}(\alpha, n, \cdot) + \lambda < +\infty \qquad \text{a.s.},$$

and $\{X(t): t \in [0, +\infty)\}$ has continuous sample paths. Hence since $\{Y(\alpha): \alpha \in [0, +\infty)\}$ has left continuous sample paths,

$$\begin{aligned} X(T_\alpha) &= \lim_{n\to+\infty} X(\tilde{T}(\alpha, n, \cdot) + \lambda) \\ &= \lim_{n\to+\infty} Y\left(\frac{[2^n\alpha]}{2^n}\right) \qquad \text{by Claim 2} \\ &= Y(\alpha) \qquad \text{a.s.} \end{aligned}$$

This completes the proof of Claim 5.

We have now shown that

(i) for each $\alpha \in [0, T]$, T_α is a stopping time for $\{X(t): t \in [0, +\infty)\}$ (from (2.20)),
(ii) for each $\alpha \in [0, T]$, $X(T_\alpha) = Y(\alpha)$ a.s. (Claim 5),
(iii) for fixed ω, $T_\alpha(\omega)$ is non-decreasing in α (from (2.9)), and
(iv) for each $\alpha \in [0, T]$, $T_\alpha \geqq 0$ everywhere and finite almost surely (from Claim 4). □

THEOREM 2.5. *Let* $\{X(t): t \in [0, +\infty)\}$ *be a Brownian motion process and let* $\{Y(\alpha): \alpha \in [0, +\infty)\}$ *be a stochastic process such that the following requirements are satisfied*:

(2.21) $$\{Y(\alpha): \alpha \in [0, +\infty)\}$$

has left continuous sample paths and independent increments with

$$Y(0) = 0 = \lim_{n \to +\infty} Y\left(\frac{1}{2^n}\right) \quad on \quad \Omega,$$

(2.22) *there exists a real number* $c \geqq 0$ *such that for every* $\alpha \in [0, +\infty)$, $\sigma\{Y(\alpha)\} \subset \sigma\{X(t): t \in [0, \alpha + c]\}$,

(2.23) *for every positive integer* n, *the sample paths of* $\{Y(\alpha): \alpha \in [0, +\infty)\}$ *are of bounded variation on* $[0, n]$.

Then the process $\{Y(\alpha): \alpha \in [0, +\infty)\}$ *can be obtained from* $\{X(t): t \in [0, +\infty)\}$ *by the means of a random time change.*

PROOF. Let n be a fixed positive integer and let $\alpha \in (0, n]$. Set $\lambda_n = n + c$. For each integer i such that $1 \leqq [2^i \alpha]$, define $\tilde{T}^{(n)}(\alpha, i, k, \cdot)$ for $k = 1, \cdots, [2^i \alpha]$ as follows, $\tilde{T}^{(n)}(\alpha, i, 1, \cdot) = \inf\{t \geqq 0: X(t + \lambda_n) = Y(1/2^i)\}$ and for $k = 2, \cdots, [2^i \alpha]$,

$$\tilde{T}^{(n)}(\alpha, i, k, \cdot) = \inf\{t \geqq 0: X(t + \lambda_n + \textstyle\sum_{j=1}^{k-1} \tilde{T}^{(n)}(\alpha, i, j, \cdot)) = Y(k/2^i)\}.$$

For each positive integer n, define the process $\{T_\alpha^{(n)}: \alpha \in [0, n]\}$ as follows, $T_0^{(n)} = 0$ and for $\alpha \in (0, n]$,

$$T_\alpha^{(n)} = \lambda_n + \lim_{i \to +\infty} \textstyle\sum_{k=1}^{[2^i \alpha]} \tilde{T}^{(n)}(\alpha, i, k, \cdot).$$

It was shown in Theorem 2.4 that for each integer $n \geqq 1$, $\{T_\alpha^{(n)}: \alpha \in [0, n]\}$ satisfies the following requirements:

(i) for each $\alpha \in [0, n]$, $T_\alpha^{(n)}$ is a stopping time for $\{X(t): t \in [0, +\infty)\}$.
(ii) for each $\alpha \in [0, n]$, $X(T_\alpha^{(n)}) = Y(\alpha)$ a.s.,
(iii) for fixed ω, $T_\alpha^{(n)}(\omega)$ is non-decreasing in α, for $\alpha \in [0, n]$, and
(iv) for each $\alpha \in [0, n]$, $T_\alpha^{(n)} \geqq 0$ everywhere and finite almost surely.

Define the process $\{T_\alpha: \alpha \in [0, +\infty)\}$ by $T_\alpha = T_\alpha^{(1)}$ for $\alpha \in [0, 1)$, in general if n is a positive integer, define $T_\alpha = T_\alpha^{(n)}$ for $\alpha \in [n-1, n)$. Then for each $\alpha \in [0, +\infty)$, T_α is a stopping time for $\{X(t): t \in [0, +\infty)\}$, $X(T_\alpha) = Y(\alpha)$ a.s., and $T_\alpha \geqq 0$ everywhere and finite almost surely. Hence in order to prove the Theorem it suffices to show

(2.24) for fixed ω, $T_\alpha(\omega)$ is non-decreasing in α.

Now for each positive integer n and each $\omega \in \Omega$, $T_\alpha(\omega)$ is non-decreasing in α for α ranging in the interval $[n-1, n)$ since $T_\alpha^{(n)}(\omega)$ is non-decreasing in α for α ranging in $[0, n]$. Moreover for any $\alpha \in [n-1, n)$, $T_\alpha = T_\alpha^{(n)} \leqq T_n^{(n)}$. Also $T_n = T_n^{(n+1)}$. Therefore in order to prove (2.24) we need only prove

$$T_n^{(n)} \leqq T_n^{(n+1)} \qquad \text{for } n = 1, 2, 3, \cdots. \tag{2.25}$$

CLAIM 1. Let n and i be fixed positive integers. Then

$$\lambda_n + \textstyle\sum_{k=1}^{2^i n} \mathcal{T}^{(n)}(n, i, k, \cdot) \leqq \lambda_{n+1} + \sum_{k=1}^{2^i n} \mathcal{T}^{(n+1)}(n, i, k, \cdot).$$

PROOF OF CLAIM 1 by induction.

Now

$$\mathcal{T}^{(n)}(n, i, 1, \cdot) = \inf\left\{t \geqq 0 : X(t+\lambda_n) = Y\left(\frac{1}{2^i}\right)\right\}$$
$$= \inf\left\{t \geqq \lambda_n : X(t) = Y\left(\frac{1}{2^i}\right)\right\} - \lambda_n.$$

Therefore

$$\lambda_n + \mathcal{T}^{(n)}(n, i, 1, \cdot) = \inf\left\{t \geqq \lambda_n : X(t) = Y\left(\frac{1}{2^i}\right)\right\}.$$

In a similar manner,

$$\lambda_{n+1} + \mathcal{T}^{(n+1)}(n, i, 1, \cdot) = \inf\left\{t \geqq \lambda_{n+1} : X(t) = Y\left(\frac{1}{2^i}\right)\right\}.$$

Hence clearly $\lambda_n + \mathcal{T}^{(n)}(n, i, 1, \cdot) \leqq \lambda_{n+1} + \mathcal{T}^{(n+1)}(n, i, 1, \cdot)$ since $\lambda_n = n + c \leqq (n+1) + c = \lambda_{n+1}$. Now let $1 \leqq j < 2^i n$ and assume that

$$\lambda_n + \textstyle\sum_{k=1}^{j} \mathcal{T}^{(n)}(n, i, k, \cdot) \leqq \lambda_{n+1} + \sum_{k=1}^{j} \mathcal{T}^{(n+1)}(n, i, k, \cdot).$$

By definition

$$\mathcal{T}^{(n)}(n, i, j+1, \cdot)$$
$$= \inf\left\{t \geqq 0 : X\left(t + \lambda_n + \sum_{k=1}^{j} \mathcal{T}^{(n)}(n, i, k, \cdot)\right) = Y\left(\frac{j+1}{2^i}\right)\right\}$$
$$= \inf\left\{t \geqq \lambda_n + \sum_{k=1}^{j} \mathcal{T}^{(n)}(n, i, k, \cdot) : X(t) = Y\left(\frac{j+1}{2^i}\right)\right\} - \left(\lambda_n + \sum_{k=1}^{j} \mathcal{T}^{(n)}(n, i, k, \cdot)\right),$$

and so

$$\lambda_n + \textstyle\sum_{k=1}^{j+1} \mathcal{T}^{(n)}(n, i, k, \cdot) \tag{2.26}$$
$$= \inf\{t \geqq \lambda_n + \textstyle\sum_{k=1}^{j} \mathcal{T}^{(n)}(n, i, k, \cdot) : X(t) = Y(j+1/2^i)\}.$$

Likewise

$$\lambda_{n+1} + \textstyle\sum_{k=1}^{j+1} \mathcal{T}^{(n+1)}(n, i, k, \cdot) \tag{2.27}$$
$$= \inf\{t \geqq \lambda_{n+1} + \textstyle\sum_{k=1}^{j} \mathcal{T}^{(n+1)}(n, i, k, \cdot) : X(t) = Y(j+1/2^i)\}.$$

Using our induction assumption and comparing (2.26) and (2.27) we see that

$$\lambda_n+\sum_{k=1}^{j+1} \tilde{T}^{(n)}(n,i,k,\cdot) \leqq \lambda_{n+1}+\sum_{k=1}^{j+1} \tilde{T}^{(n+1)}(n,i,k,\cdot).$$

This completes the proof of Claim 1.

From the definitions of $T_n^{(n)}$ and $T_n^{(n+1)}$ and from Claim 1, it follows that statement (2.25) is true. □

The following two Theorems give another interesting property of the random variables T_α constructed in the proofs of Theorem 2.4 and Theorem 2.5.

THEOREM 2.6. *Let* $\{X(t): t \in [0, +\infty)\}$ *and* $\{Y(\alpha): \alpha \in [0, +\infty)\}$ *be as in Theorem* 2.4. *Then the collection of stopping times* $\{T_\alpha: \alpha \in [0, T]\}$ *constructed in the proof of Theorem* 2.4 *are such that the stochastic process* $\{T_\alpha: \alpha \in (0, T]\}$ *has independent increments.*

PROOF. Since $\{W(t): t \in [0, +\infty)\}$ has continuous sample paths we see that for $2 \leqq k \leqq [2^n\alpha]$,

$$\tilde{T}(\alpha,n,k,\cdot)=\inf\left\{t \geqq 0: W\left(t+\sum_{i=1}^{k-1} \tilde{T}(\alpha,n,i,\cdot)\right)-W\left(\sum_{i=1}^{k-1} \tilde{T}(\alpha,n,i,\cdot)\right) = Z\left(\frac{k}{2^n}\right)-Z\left(\frac{k-1}{2^n}\right)\right\}. \tag{2.28}$$

CLAIM 1. Let $\alpha \in (0, T]$ and let n be any positive integer such that $2 \leqq [2^n\alpha]$. Then the random variables $\tilde{T}(\alpha, n, 2, \cdot), \cdots, \tilde{T}(\alpha, n, [2^n\alpha], \cdot)$ are independent random variables.

PROOF OF CLAIM 1. For each $t \in [0, +\infty)$,

$$W(t+\tilde{T}(\alpha,n,1,\cdot))-W(\tilde{T}(\alpha,n,1,\cdot)) = X(t+\lambda+\tilde{T}(\alpha,n,1,\cdot))-X(\lambda+\tilde{T}(\alpha,n,1,\cdot)).$$

Also in the proof of Claim 3 of Theorem 2.4 it was shown that $\lambda+\tilde{T}(\alpha, n, 1, \cdot)$ is a stopping time for $\{X(t): t \in [0, +\infty)\}$. Hence by ([1] Theorem 12.42, page 269), $\{W(t+\tilde{T}(\alpha, n, 1, \cdot))-W(\tilde{T}(\alpha, n, 1, \cdot)): t \in [0, +\infty)\}$ is a Brownian motion independent of $\sigma\{X(t): t \in [0, \lambda]\}$, and also by hypothesis $Z(2/2^n)-Z(1/2^n)$, $Z(3/2^n)-Z(2/2^n), \cdots, Z([2^n\alpha]/2^n)-Z(([2^n\alpha]-1)/2^n)$ are independent random variables which are independent of $\{W(t+\tilde{T}(\alpha, n, 1, \cdot))-W(\tilde{T}(\alpha, n, 1, \cdot)): t \in [0, +\infty)\}$. Therefore by (2.28) and Lemma 2.3 it follows that $\tilde{T}(\alpha, n, 2, \cdot), \cdots, \tilde{T}(\alpha, n, [2^n\alpha], \cdot)$ are independent random variables. This proves Claim 1.

Let $0 < \alpha < \beta \leqq T$. Then

$$T_\beta - T_\alpha = \lim_{n\to+\infty} \left[\sum_{k=1}^{[2^n\beta]} \tilde{T}(\beta,n,k,\cdot)-\sum_{k=1}^{[2^n\alpha]} \tilde{T}(\alpha,n,k,\cdot)\right].$$

Using (2.9) in Theorem 2.4 repeatedly we obtain

$$\begin{aligned} T_\beta - T_\alpha &= \lim_{n\to+\infty} \sum_{k=[2^n\alpha]+1}^{[2^n\beta]} \tilde{T}(\beta,n,k,\cdot) \\ &= \lim_{n\to+\infty} \sum_{k=[2^n\alpha]+1}^{[2^n\beta]} \tilde{T}(T,n,k,\cdot) \end{aligned} \tag{2.29}$$

for any $0 < \alpha < \beta \leqq T$. Suppose now that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_j \leqq T$. Then by (2.29),

$$T_{\alpha_{i+1}} - T_{\alpha_i} = \lim_{n \to +\infty} \sum_{k=[2^n\alpha_i]+1}^{[2^n\alpha_{i+1}]} \tilde{T}(T, n, k, \cdot) \tag{2.30}$$

for $i = 1, \cdots, j-1$. The fact that $T_{\alpha_2} - T_{\alpha_1}, \cdots, T_{\alpha_j} - T_{\alpha_{j-1}}$ are independent random variables now follows from Claim 1 and (2.30). □

COROLLARY 2.7. *Let* $\{X(t) : t \in [0, +\infty)\}$ *and* $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ *be as in Theorem* 2.5. *Then the random variables* $\{T_\alpha : \alpha \in [0, +\infty)\}$ *constructed in the proof of Theorem* 2.5 *have the following properties*:

(2.31) *the stochastic process* $\{T_\alpha : \alpha \in (0, 1)\}$ *has independent increments, and*

(2.32) *for each integer* $n \geqq 1$, *the stochastic process* $\{T_\alpha : \alpha \in [n, n+1)\}$ *has independent increments.*

PROOF. The proof follows directly from Theorem 2.6 and by the nature of the way the process $\{T_\alpha : \alpha \in [0, +\infty)\}$ was constructed in Theorem 2.5. □

Let $\{x_k\}_{k=1}^{+\infty}$ be any strictly increasing sequence of positive real numbers converging to $+\infty$. Notice that by slightly modifying the construction of the process $\{T_\alpha : \alpha \in [0, +\infty)\}$ in Theorem 2.5, (2.31) and (2.32) could be replaced by

(2.31′) the stochastic process $\{T_\alpha : \alpha \in (0, x_1)\}$ has independent increments, and

(2.32′) for each integer $n \geqq 1$, the stochastic process $\{T_\alpha : \alpha \in [x_n, x_{n+1})\}$ has independent increments.

Notice also that by a slight modification in the proof of Theorem 2.4, hypotheses (2.6) and (2.21) could be replaced by

(2.6′) $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ has *right* continuous sample paths and independent increments with $Y(0) = 0$.

We conclude this work by generating some examples of processes $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ which satisfy the hypothesis of Theorem 2.5.

Let $\{X(t) : t \in [0, +\infty)\}$ be a Brownian motion process, let $\{f_k : k = 1, 2, 3, \cdots\}$ be a collection of Borel measurable functions, let $\{t_k\}_{k=0}^{\infty}$ be any strictly increasing sequence of nonnegative real numbers such that for some constant $c \geqq 0$, $t_k \leqq k + c$ for $k = 0, 1, 2, \cdots$. Finally for any real number α let $\langle \alpha \rangle$ denote the largest integer strictly less than α. Define the stochastic process $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ as follows: $Y(\alpha) = 0$ for $\alpha \in [0, 1]$ and $Y(\alpha) = \sum_{k=1}^{\langle \alpha \rangle} f_k(X(t_k) - X(t_{k-1}))$ for $\alpha > 1$. Since $t_k \leqq k + c$ for $k = 0, 1, \cdots$, it follows that

$$\sigma\{Y(\alpha)\} \subset \sigma\{X(t) : t \in [0, \alpha + c]\}$$

for each $\alpha \in [0, +\infty)$. Hence (2.22) holds. Clearly (2.23) holds and $\{Y(\alpha) : \alpha \in [0, +\infty)\}$ has sample paths which are left continuous and right continuous at 0 with $Y(0) = 0$. Furthermore if $0 \leqq \alpha_1 \leqq 1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n$ then

$$Y(\alpha_2) - Y(\alpha_1) = Y(\alpha_2) = \sum_{k=1}^{\langle \alpha_2 \rangle} f_k(X(t_k) - X(t_{k-1}))$$

and for $i = 2, \cdots, n$,

$$Y(\alpha_{i+1}) - Y(\alpha_i) = \sum_{k=\langle\alpha_i\rangle+1}^{\langle\alpha_{i+1}\rangle} f_k(X(t_k) - X(t_{k-1})) \qquad \text{if} \quad \langle\alpha_i\rangle < \langle\alpha_{i+1}\rangle;$$
$$= 0 \qquad \text{otherwise.}$$

Hence we see that $\{Y(\alpha): \alpha \in [0, +\infty)\}$ has independent increments since $\{X(t): t \in [0, +\infty)\}$ has independent increments. Therefore $\{Y(\alpha): \alpha \in [0, +\infty)\}$ satisfies the hypothesis of Theorem 2.5.

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