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ON THE DISTRIBUTION OF THE SPHERICITY TEST CRITERION IN CLASSICAL AND COMPLEX NORMAL POPULATIONS HAVING UNKNOWN COVARIANCE MATRICES

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1. Introduction and summary. Let $\mathbf{x}:p\times 1$ be distributed $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are both unknown. Let $\mathbf{S}$ be the sum of product matrix of a sample of size N. To test the hypothesis of sphericity, namely, $H_0:\boldsymbol{\Sigma} = \sigma^2\mathbf{I}_p$, where $\sigma^2 > 0$ is unknown, against $H_1:\boldsymbol{\Sigma} \neq \sigma^2\mathbf{I}_p$, Mauchly [10] obtained the likelihood ratio test criterion for H_0 in the form $W = |\mathbf{S}|/[(\text{tr}\,\mathbf{S})/p]^p$. Thus the criterion W is a power of the ratio of the geometric mean and the arithmetic mean of the roots $\theta_1, \theta_2, \cdots, \theta_p$ of $|\mathbf{S}-\theta\mathbf{I}| = \mathbf{0}$ (see Anderson [1]). In the null case, Machly [10] gave the density of W for $p = 2$ and Consul [3], [4] for any p in terms of Meijer's G-function defined in the next section.

In this paper we have obtained the general moments of W both in real and complex cases for arbitrary covariance matrices, and also the corresponding distributions of W in terms of the G-function.

2. Some definitions and results. In this section we give a few definitions and some lemmas which are needed in the sequel. First we define Meijer's G-function by

$$(2.1)\qquad G_{p,q}^{m,n}\left(x\,\Big|\,{}^{a_1,\cdots,a_p}_{b_1,\cdots,b_q}\right) = (2\pi i)^{-1}\int_C \frac{\prod_{j=1}^m \Gamma(b_j-s)\prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s)\prod_{j=n+1}^p \Gamma(a_j-s)}\,x^s\,ds,$$

where an empty product is interpreted as unity and C is a curve separating the singularities of $\prod_{j=1}^m \Gamma(b_j-s)$ from those of $\prod_{j=1}^n \Gamma(1-a_j+s)$, $q \geqq 1$, $0 \leqq n \leqq p \leqq q$, $0 \leqq m \leqq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$.

The G-function of (2.1) can be expressed in terms of a finite number of generalised hypergeometric functions (see Pillai, Al-Ani and Jouris [11] and Luke [9]). Further, the hypergeometric function of matrix variates is defined by

$${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; \mathbf{S}, \mathbf{T}) = \sum_{k=0}^{\infty} \Sigma_\kappa \frac{(a_1)_\kappa \cdots (a_p)_\kappa C_\kappa(\mathbf{S})C_\kappa(\mathbf{T})}{(b_1)_\kappa \cdots (b_q)_\kappa C_\kappa(\mathbf{I}_m)k!},$$

where the zonal polynomials, $C_\kappa(\cdot)$, and $(\cdot)_\kappa$ are defined in [6].

LEMMA 2.1. *Let $\mathbf{T}:m\times m$ be an arbitrary complex symmetric matrix. Then*

$$(2.2)\qquad \int_{S>0} \exp\left(-\tfrac{1}{2}\,\text{tr}\,\mathbf{S}\right)|\mathbf{S}|^{t-\frac{1}{2}(m+1)}(\text{tr}\,\mathbf{S})^q C_\kappa(\mathbf{TS})\,d\mathbf{S} = \Gamma_m(t, \kappa)2^{tm+k+q}\Gamma(mt+k+q)C_\kappa(\mathbf{T})/\Gamma(mt+k),$$

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where $\Gamma_m(t,\kappa)$ *is defined in Constantine* [2], *and* $R(t) > \frac{1}{2}(m-1)$. (See Khatri [8]).

Similarly, in the complex case, we have the following lemma.

LEMMA 2.2. *Let* $\mathbf{T}$ *be as in Lemma* 2.1 *and* $\mathbf{S}: m\times m$ *be an Hermitian matrix. Then*

$$\int_{\bar{S}'=S>0} \exp(-\operatorname{tr}\mathbf{S})|\mathbf{S}|^{a-m}(\operatorname{tr}\mathbf{S})^j \tilde{C}_\kappa(\mathbf{TS})\,d\mathbf{S} = \tilde{\Gamma}_m(a,\kappa)\Gamma(am+k+j)\tilde{C}_\kappa(\mathbf{T})/\Gamma(am+k). \tag{2.3}$$

3. Distribution of W **in the real case.** Let $\mathbf{S}: p\times p$ be distributed as Wishart (n, p, Σ). Then the distribution of the latent roots $g_1, g_2, \cdots, g_p$ of $\mathbf{S}$ has been given in the form suggested by Pillai (see Pillai, Al-Ani and Jouris [11]),

$$k(p,n,\Sigma)|\mathbf{G}|^{\frac{1}{2}(n-p-1)}\exp(-\tfrac{1}{2}\operatorname{tr}\mathbf{G})\Pi_{i<j}(g_i-g_j)\,{}_0F_0(\mathbf{M},\mathbf{G}) \qquad \infty > g_1 \geqq g_2 \cdots \geqq g_p > 0 \tag{3.1}$$

where $\mathbf{M} = \frac{1}{2}(\mathbf{I}-\Sigma^{-1})$, $\mathbf{G} = \operatorname{diag}(g_1, g_2, \cdots, g_p)$ and

$$k(p,n,\Sigma) = |\Sigma|^{-\frac{1}{2}n}\pi^{\frac{1}{2}p^2}/\{2^{\frac{1}{2}pn}\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}p)\}.$$

Now the sphericity criterion, $W = |\mathbf{G}|/\{(\operatorname{tr}\mathbf{G})/p\}^p$ and the hth moment of W can be easily shown to be

$$E(W^h) = \frac{p^{ph}|\Sigma|^{-\frac{1}{2}n}}{\Gamma_p(\frac{1}{2}n)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\mathbf{M})}{k!}2^k\frac{\Gamma_p(\frac{1}{2}n+h,\kappa)\Gamma(\frac{1}{2}pn+k)}{\Gamma(\frac{1}{2}pn+ph+k)}. \tag{3.2}$$

Further we prove the following theorem.

THEOREM 3.1. *For any finite p, the* pdf of W is

$$f(w) = C(p,n,\Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{2^k C_\kappa(\mathbf{M})}{k!}p^{\frac{1}{2}-\frac{1}{2}pn-k}\Gamma(\tfrac{1}{2}pn+k) \cdot w^{\frac{1}{2}(n-p-1)}G_{p,p}^{p,0}\left(w\,\middle|\,{}^{a_1,\cdots,a_p}_{b_1,\cdots,b_p}\right), \tag{3.3}$$

where

$$C(p,n,\Sigma) = \pi^{\frac{1}{4}p(p-1)}|\Sigma|^{-\frac{1}{2}n}(2\pi)^{\frac{1}{2}(p-1)}/\Gamma_p(\tfrac{1}{2}n),$$

$$a_j = (k+j-1)/p+\tfrac{1}{2}(p-1);\ b_j = k_j+\tfrac{1}{2}(p-j).$$

Special case. For $p = 2$, (3.3) reduces to

$$f(w) = \frac{|\Sigma|^{-\frac{1}{2}n}}{2\Gamma(n-1)}w^{\frac{1}{2}(n-3)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\Gamma(n+k)}{k!}C_\kappa(\mathbf{M})w^{k_1+\frac{1}{2}} \cdot {}_2F_1(a_2-b_2, a_1-b_2; a_1+a_2-b_1-b_2, 1-w). \tag{3.4}$$

PROOF. Applying Gauss and Legendre's multiplication formula on

$$\Gamma[p(\tfrac{1}{2}n+h+\ k/p)]$$

we have from (3.2)

$$E(W^h) = C(p, n, \Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}[\{2^k C_\kappa(\mathbf{M})p^{\frac{1}{2}-\frac{1}{2}pn-k}\Gamma(\tfrac{1}{2}pn+k)\}/k!]$$
$$\prod_{j=1}^{p}[\Gamma\{\tfrac{1}{2}n+h+k_j-\tfrac{1}{2}(j-1)\}/\Gamma\{\tfrac{1}{2}n+((k+j-1)/p)+h\}].$$

Using inverse Mellin transform, the density of W has the form

$$f(w) = C(p, n, \Sigma)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{2^k C_\kappa(\mathbf{M})}{k!}p^{\frac{1}{2}-\frac{1}{2}pn-k}\Gamma(\tfrac{1}{2}pn+k)w^{\frac{1}{2}(n-p-1)} \cdot(2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty}w^{-r}\frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}dr, \tag{3.5}$$

where $r = \frac{1}{2}n+h-\frac{1}{2}(p-1)$, $b_j = k_j+\frac{1}{2}(p-j)$, $a_j = (k+j-1)/p+\frac{1}{2}(p-1)$. Noting that the integral in (3.5) is in the form of Meijer's G-function, we can write the density of W as in (3.3).

REMARK. Putting $\Sigma = \sigma^2\mathbf{I}$ in (3.3) and (3.4) we can easily deduce the result of Consul [3], [4], and Mauchly [10].

4. Distribution of W in the complex case. Let $\tilde{\mathbf{S}}: p\times p$ be distributed as a complex Wishart $(n, p, \tilde{\Sigma})$ (see [5], [7]). Then, as in the real case, the distribution of the latent roots $g_1, g_2, \cdots, g_p$ of $\tilde{\mathbf{S}}$ can be given in the form

$$k(p, n, \tilde{\Sigma})\,{}_0\tilde{F}_0(\tilde{\mathbf{M}}_1, \mathbf{G})\exp(-\operatorname{tr}\mathbf{G})|\mathbf{G}|^{n-p}\prod_{i<j}(g_i-g_j)^2\prod_{i=1}^{p}dg_i, \tag{4.1}$$

where $\tilde{\mathbf{M}}_1 = \mathbf{I}_p-\tilde{\Sigma}^{-1}$ and $k(p, n, \tilde{\Sigma}) = |\tilde{\Sigma}|^{-n}\pi^{p(p-1)}/\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(p)$.

Now the hth moment of W is

$$\frac{p^{ph}}{\tilde{\Gamma}_p(n)}|\Sigma|^{-n}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_\kappa(\tilde{\mathbf{M}}_1)}{k!}\Gamma(np+k)\tilde{\Gamma}_p(n+h, \kappa)/\Gamma(np+k+ph). \tag{4.2}$$

Further we have the following theorem.

THEOREM 4.1. *The density of W is*

$$f(w) = \frac{\pi^{\frac{1}{2}p(p-1)}|\Sigma|^{-n}(2\pi)^{\frac{1}{2}(p-1)}}{\tilde{\Gamma}_p(n)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\tilde{C}_\kappa(\tilde{\mathbf{M}}_1)}{k!}\Gamma(pn+k) \cdot p^{\frac{1}{2}-pn-k}w^{n-p}G_{p,p}^{p,0}\left(w\,\middle|\,{}^{a_1,\cdots,a_p}_{b_1,\cdots,b_p}\right),$$

where $a_j = (k/p)+(j-1)/p+(p-1)$, and $b_j = k_j-j+p$.

PROOF. The proof is similar to that of Theorem 3.1 and hence is omitted.

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