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APPROACHABILITY IN A TWO-PERSON GAME

BY TIEN-FANG HOU

Bell Telephone Laboratories, Incorporated

1. Introduction. Let $M = ||M(i, j)||$ be an $r \times s$ matrix whose elements $M(i, j)$ are probability distributions with finite $E||\cdot||^\alpha$, $||\cdot||$ is the Euclidean norm and $\alpha > 1$, in a Euclidean k-space $\mathscr{E}^k$. We associate with M a game between two players, I and II, with the following infinite sequence of engagements: At the nth engagement, $n = 1, 2, \cdots$, player I selects $i = 1, \cdots, r$ with probability $p_n(1), \cdots, p_n(r)$, $\sum_{i=1}^r p_n(i) = 1$, and player II selects $j = 1, \cdots, s$ with probability $q_n(1), \cdots, q_n(s)$, $\sum_{j=1}^s q_n(j) = 1$. Each selection is made without either player knowing the choice of the other player. Having chosen i and j, payoff $Y_n \in \mathscr{E}^k$ is then determined according to the distribution $M(i, j)$. The point Y_n and probabilities

$$(1.1) \qquad p_n = (p_n(1), \cdots, p_n(r)) \quad \text{and} \quad q_n = (q_n(1), \cdots, q_n(s))$$

are announced to both players after each engagement. We call p_n player I's move and q_n player II's move.

A strategy for player I is a sequence of functions $f = \{f_n\}$, $n = 0, 1, 2, \cdots$, where f_n is defined on the $3n$-tuples $(p_1, q_1, Y_1; \cdots; p_n, q_n, Y_n)$ with value p_{n+1} in

$$(1.2) \qquad P = \{p = (p(1), \cdots, p(r)) : \textstyle\sum_1^r p(i) = 1 \quad \text{and} \quad p(i) \geqq 0\},$$

and $p_1 = f_0$ is simply a point of P. For player II, a strategy $g = \{g_n\}$ is defined similarly, except that

$$(1.3) \qquad g_n(p_1, q_1, Y_1; \cdots; p_n, q_n, Y_n) = q_{n+1} \in Q \quad \text{and} \quad q_1 = g_0 \in Q,$$

where

$$(1.4) \qquad Q = \{q = (q(1), \cdots, q(s)) : \textstyle\sum_1^s q(j) = 1 \quad \text{and} \quad q(j) \geqq 0\}.$$

For a given M, each strategy pair f, g determines a sequence of random variables $Y_1, Y_2, \cdots$ (vector payoffs) in $\mathscr{E}^k$.

Our objective here is to investigate the controllability of the center of gravity of the actual payoffs $\bar{Y}_n = \sum_1^n Y_m/n$ in a long series of plays.

We denote the Euclidean distance between $\bar{Y}_n$ and a nonempty set S in k-space by $\delta(\bar{Y}_n, S)$. For a given M, the set S is said to be approachable (see [1] and [2]) by I in M, if there exists an f^* for I such that, for every g,

$$(1.5) \qquad \delta(\bar{Y}_n, S) \to 0 \quad \text{a.s.,}$$

where $Y_1, Y_2, \cdots$ are the payoffs determined by f^*, g. The set S is excludable by

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I in M, if there exists an f^* for I and a k-space set S' with $\delta(S, S') > 0$ such that, for every g,

$$\delta(\bar{Y}_n, S') \to 0 \quad \text{a.s.} \tag{1.6}$$

For player II, approachability and excludability are defined similarly with g^* instead of f^* and f instead of g.

Since approachability and excludability for a set S are the same as approachability and excludability for the closure of S, we may assume that S is closed. Each superset of an approachable set is approachable, each subset of an excludable set is excludable, and no set is approachable by one player and also excludable by the other player. Moreover, any condition for approachability implies a condition for excludability by their definitions; hence we may focus our attention only on approachability.

In Section 2, we introduce the strong law of large numbers for two-person games with stochastic vector payoffs (Theorem 1). In Section 3, we present a necessary and sufficient condition for approachability (Theorem 3). In Section 4, we prove that the class of approachable (excludable) sets for a given M depends only on the matrix of mean values of M (Theorem 5).

Blackwell [1] proposed Theorem 3 and Theorem 5 as unsolved problems. In [2] Blackwell assumes that the $M(i,j)$ are probability distributions over a closed bounded convex set of k-space and gives a sufficient condition for approachability. Under Blackwell's assumption, we presented these results and introduced a simple proof to show that Blackwell's condition was sufficient in our unpublished works [4], [5], [6]. In this paper our proofs cover a more general case. We assume that the $M(i,j)$ are probability distribution with finite $E\|\cdot\|^\alpha$, for some $\alpha > 1$, in k-space.

Examples are given in Section 5.

2. The strong law of large numbers.

THEOREM 1. *Let* $M = \|M(i,j)\|$ *be defined as in Section* 1; *let* $\omega_n = E(Y_n \mid p_1, q_1, Y_1, p_2, q_2, \cdots, Y_{n-1}, p_n, q_n)$, $n = 1, 2, \cdots$, *denote the conditional expectation of* Y_n *given* $(p_1, q_1, Y_1, p_2, q_2, \cdots, Y_{n-1}, p_n, q_n)$; *and let* $\bar{\omega}_n = \sum_1^n \omega_m/n$. *Then*

$$\delta(\bar{Y}_n, \bar{\omega}_n) \to 0 \quad \text{a.s.} \tag{2.1}$$

PROOF. For each $n \geqq 1$, let

$$X_n = Y_n - \omega_n \tag{2.2}$$

and let $\mathscr{F}_{n-1}$ be the σ-field generated by $(p_1, q_1, Y_1, p_2, q_2, \cdots, Y_{n-1}, p_n, q_n)$, then

$$\omega_n = E(Y_n \mid \mathscr{F}_{n-1}) \tag{2.3}$$

and $\{\sum_1^n X_m/m, \mathscr{F}_n, n \geqq 1\}$ is a martingale.

Let σ^α be the upper bound of $E\|\cdot\|^\alpha$ of the distributions $M(i,j)$, $\alpha > 1$. Then we have

$$\sum_1^\infty E\|X_n\|^\alpha/n^\theta \leqq \sum_1^\infty \sigma^\alpha/n^\theta < \infty \text{ with } \theta=\min(\alpha,2) \quad \text{and} \tag{2.4}$$

$$\sum_1^n X_m/m \quad \text{converges a.s.} \tag{2.5}$$

by the martingale convergence theorem [3]. Moreover

$$\{\sum_1^n X_m\}/n = \bar{Y}_n - \bar{\omega}_n \quad \text{converges to the zero vector a.s.} \tag{2.6}$$

by the Kronecker lemma [8]. □

The statement (2.6) is, in a sense, a particular form of the strong law of large numbers for stochastic games.

Moreover, this theorem is also true for the weak approachability and weak excludability [7].

3. Necessary and sufficient condition for approachability. *Let* $\bar{M} = \|\bar{M}(i,j)\|$ be the $r \times s$ matrix whose elements $\bar{M}(i,j)$ are the mean values of distributions $M(i,j)$ defined in Section 1, Ω be the convex hull of the $r \times s$ elements of $\bar{M}$,

$$\omega(p,q) = \sum_{i=1}^r \sum_{j=1}^s p(i)\bar{M}(i,j)q(j) \qquad \text{for} \quad p \in P \quad \text{and} \quad q \in Q, \tag{3.1}$$

and ω_n, $n \geqq 1$, be defined as in Theorem 1. Then Ω is a closed bounded (compact) convex subset of $\mathscr{E}^k$ with diameter $K < \infty$,

$$\omega_n = E(Y_n \mid p_n, q_n) = \sum_{i=1}^r \sum_{j=1}^s p_n(i)\bar{M}(i,j)q_n(j) = \omega(p_n, q_n) \in \Omega, \tag{3.2}$$

and $\bar{\omega}_n \in \Omega$. Let

$$R(p) = \{\omega(p,q) : q \in Q\} \qquad \text{for} \quad p \in P, \tag{3.3}$$

$$T(q) = \{\omega(p,q) : p \in P\} \qquad \text{for} \quad q \in Q, \tag{3.4}$$

and ω^l, $1 \leqq l \leqq k$, be the lth coordinate of a point

$$\omega = (\omega^1, \cdots, \omega^k) \in \mathscr{E}^k. \tag{3.5}$$

Then $R(p)$ ($T(q)$) is the range of the conditional expected payoff of one engagement given that I selects p (II selects q).

According to Theorem 1, we may focus our attention on $\bar{\omega}_n$ (instead of $\bar{Y}_n$) for the investigation of approachability, and we may use the following sufficient statements: For a given M, a nonempty set S in k-space is said to be approachable by I in M, if there exists an f^* for I such that, for every g, $\delta(\bar{\omega}_n, S) \to 0$ a.s. S is not approachable by I if there exist a g^* for II and a $\Delta > 0$ such that for every f, $\delta(\bar{\omega}_n, S) \geqq \Delta$ infinitely often a.s. Since $\bar{\omega}_n \in \Omega$ for all $n \geqq 1$, we have that S is approachable by one player if and only if $\bar{S} \cap \Omega$ is approachable by him (by Theorem 1), where $\bar{S}$ represents the closure of S. Hence we may assume that S is a closed subset of Ω.

DEFINITION 1. A nonempty subset B of Ω is said to be an *insufficient subset* of a closed set $S \subset \Omega$ for player I, if there exist an open set $U(B)$ in $\mathscr{E}^k$ and a number $\Delta > 0$ such that

(i) $S \cap U(B) = B$ and

(ii) for each integer $n > 10K/\Delta$ and $u \in U(B)$, there exists a strategy g^* for II such that for every strategy f for I

$$\text{Prob}\left\{ \left(\frac{n}{N}u+\left(1-\frac{n}{N}\right)\bar{\omega}_{N-n}, S\right) \geqq \Delta \right. \tag{3.6}$$

$$\left. \text{for some integer } N \geqq n \,\middle|\, f, g^* \right\} = 1.$$

An insufficient subset B of S for I has the following implication. If after n plays of the game $\bar{\omega}_n \in U(B)$, $n > 10K/\Delta$, then there is a sequence of moves $q_{n+1}, q_{n+2}, \cdots$ for II such that, for any strategy of I we have with probability one that there is a random integer $N \geqq n$ such that $\delta(\bar{\omega}_N, S) \geqq \Delta$.

$$\left(\text{Note: } \bar{\omega}_N = \frac{n}{N}\bar{\omega}_n+\left(1-\frac{n}{N}\right)\left(\frac{1}{N-n}\sum_{n+1}^{N}\omega_j\right).\right)$$

$n > 10K/\Delta$ is used such that the distance $\delta(\bar{\omega}_n, \bar{\omega}_{n-1}) = \delta(\omega_n, \bar{\omega}_{n-1})/n \leqq K/n$ is much smaller than Δ and this restriction depends only on Δ.

We write $\Delta(B)$ for the supremum of all values of Δ for which the above conditions are satisfied.

DEFINITION 2. Let $\mathscr{B}$ denote the collection of all insufficient subsets B of S,

$$B^* = \bigcup_{B\in\mathscr{B}} B,\ U^* = \bigcup_{B\in\mathscr{B}} U(B), \quad \text{and} \quad S' = S \sim B^* = \{\omega\colon \omega \in S \ \text{ and } \ \omega \notin B^*\}, \tag{3.7}$$

then $B^* = S \bigcap U^*$, $S' = S \sim U^*$, and S' is a closed subset of Ω. S' is called the *sufficient subset* of S.

Let

$$\mathscr{B}_k = \{B\colon B \in \mathscr{B} \ \text{ and } \ \Delta(B)/2 \geqq 1/k\}, \qquad B_k^* = \bigcup_{B\in\mathscr{B}_k} B, \quad \text{and} \quad U_k^* = \bigcup_{B\in\mathscr{B}_k} U(B) \tag{3.8}$$

for $k \geqq 1$, then

$$B_k^* = S \bigcap U_k^*, \ \bigcup_{k=1}^{\infty} B_k^* = B^* \quad \text{and} \quad \bigcup_{k=1}^{\infty} U_k^* = U. \tag{3.9}$$

THEOREM 2. *S' contains no insufficient subset.*

We may assume that $S' \neq \varnothing$. The proof of this theorem is established through two lemmas as follows.

LEMMA 1. *For every $\Delta' > 0$, there exists a $\theta > 0$ such that $\omega \in \Omega \sim U^*$ and $\delta(\omega, S') \geqq \Delta'$ imply $\delta(\omega, S) \geqq \theta$.*

PROOF. Suppose that this lemma is not true. Then there exists a sequence of points $\{\omega_i: i \geqq 1\}$ in $\Omega \sim U^*$ such that $\delta(\omega_i, S') \geqq \Delta'$ for each i and $\delta(\omega_i, S) \to 0$ as $i \to \infty$. Let s_i be the closest point in S to ω_i for each i, then $\delta(\omega_i, s_i) \to 0$ as $i \to \infty$ and $\{s_i: i \geqq 1\}$ is a sequence of points in the compact set S. Hence there exist an infinite subsequence $\{s_{i_j}: j \geqq 1\}$ of $\{s_i\}$ and a point $s \in S$ such that

$$s_{i_j} \to s \quad \text{and} \quad \delta(\omega_{i_j}, s) \to 0 \qquad \text{as } j \to \infty. \tag{3.10}$$

$\delta(\omega_{i_j}, S') \geqq \Delta'$ for all j imply $s \notin S'$, therefore $s \in B^* = S \cap U^*$. But $\delta(s, \Omega \sim U^*) > 0$ for $s \in B^*$, which contradicts (3.10). □

LEMMA 2.

$$\delta_k = \sup_{\omega \in \Omega \cap [U^* \sim U_k^*]} \delta(\omega, \Omega \sim U^*) \to 0 \qquad as \quad k \to \infty, \tag{3.11}$$

where $\delta_k = 0$ *if* $\Omega \cap [U^* \sim U_k^*] = \varnothing$.

PROOF. $U_k \subset U_{k+1}$ implies $\delta_k \geqq \delta_{k+1}$ for each $k \geqq 1$. Hence we may assume that $\Omega \cap [U^* \sim U_k^*] \neq \varnothing$ for each k.

Suppose this lemma is not true. Then there exist a $\Delta > 0$ and a sequence $\{\omega_k: \omega_k \in \Omega \cap [U^* \sim U_k^*]$ and $k \geqq 1\}$ such that $\delta(\omega_k, \Omega \sim U^*) \geqq \Delta$ for each k. The compactness of Ω implies that there exist an infinite subsequence $\{\omega_{k_i}: i \geqq 1\}$ of $\{\omega_k\}$ and $\omega' \in \Omega$ such that $\omega_{k_i} \to \omega'$ as $i \to \infty$ and $\delta(\omega', \Omega \sim U^*) \geqq \Delta$. Hence $\omega' \in \Omega \cap U^*$ and $\omega' \in \Omega \cap U_{k'}^*$ for some finite positive integer k'. But this contradicts the assumption of the existence of $\omega_k \in \Omega \cap [U^* \sim U_k^*]$ for each $k \geqq 1$. □

Lemma 2 implies the following corollary.

COROLLARY 1. *For each* $\varepsilon > 0$ *there exists a positive integer* $j(\varepsilon)$ *such that*

$$\sup_{\omega \in \Omega \cap [U^* \sim U_k^*]} \delta(\omega, \Omega \sim U^*) \leqq \varepsilon \qquad for\ all \quad k \geqq j(\varepsilon). \tag{3.12}$$

PROOF OF THEOREM 2. Suppose S' does contain an insufficient subset B', $U(B')$ is the associated open set and Δ' is the associated positive number defined in Definition 1. If $\bar{\omega}_n \in U(B')$ for some sufficiently large n, then there exists a sequence of moves $q_{n+1}, q_{n+2}, \cdots$ for II such that, for any strategy of I there is an integer $N \geqq n$ with probability 1 such that

$$\delta(\bar{\omega}_N, S') \geqq \Delta'. \tag{3.13}$$

If $\bar{\omega}_N \in \Omega \sim U^*$, then

$$\delta(\bar{\omega}_N, S) \geqq \theta > 0 \tag{3.14}$$

by Lemma 1; if $\bar{\omega}_N \in \Omega \cap U_{j(\varepsilon)}^*$ for some positive integer $j(\varepsilon)$ with $0 < \varepsilon \leqq \Delta'/2$ defined in Corollary 1, then there exists a sequence of moves $q_{N+1}, q_{N+2}, \cdots$ for II such that, for any strategy of I there is an integer $N' \geqq N$ with probability 1 such that

$$\delta(\bar{\omega}_{N'}, S) \geqq 1/j(\varepsilon) \tag{3.15}$$

by the definition of $U^*_{j(\varepsilon)}$; and if $\bar{\omega}_N \in \Omega \cap [U^* \sim U^*_{j(\varepsilon)}]$, then $\delta(\bar{\omega}_N, \Omega \sim U^*) \leqq \varepsilon$ by Corollary 1. In the latest case, let u be the closest point in $\Omega \sim U^*$ to $\bar{\omega}_N$, then $\delta(\bar{\omega}_N, u) \leqq \varepsilon$, $\delta(u, S') \geqq \Delta' - \varepsilon \geqq \Delta'/2$, and $\delta(u, S) \geqq \theta'$ for some $\theta' > 0$ by Lemma 1. ε can be chosen so small that $\theta' - \varepsilon = \theta'' > 0$. Hence

$$\delta(\bar{\omega}_N, S) \geqq \theta' - \varepsilon \geqq \theta''. \tag{3.16}$$

Therefore, if $\bar{\omega}_n \in U(B')$ for some $n > 10K/\Delta''$, $\Delta'' = \min\{\Delta', \theta, \theta'', 1/j(\varepsilon)\} > 0$, then there exists a sequence of moves $q_{n+1}, q_{n+2}, \cdots$ for II such that, for any strategy of I there is an integer $N' \geqq n$ with probability 1 such that $\delta(\bar{\omega}_{N'}, S) \geqq \Delta''$. Thus $S \cap U(B') \supset B'$ is an insufficient subset of S, which contradicts the definition of S'. ∎

THEOREM 3. *A set $S \subset \mathscr{E}^k$ is approachable by player* I *if and only if S', the sufficient subset of $\bar{S} \cap \Omega$, is nonempty.*

Without loss of generality, we may assume that S is a closed subset of Ω by Theorem 1, that is $\bar{S} \cap \Omega = S$.

PROOF OF NECESSITY. Let $S' = \varnothing$, then $\{U(B): B \in \mathscr{B}\}$ is an open covering of the compact set S. Hence this open covering contains a finite covering $\{U(B_i)$: $B_i \in \mathscr{B}$ and $1 \leqq i \leqq j < \infty\}$ of S. Let $\Delta = \min\{\Delta(B_i)\colon 1 \leqq i \leqq j\}$ and $\Delta' = \delta(S, \mathscr{E}^k \sim \bigcup_{i=1}^{j} U(B_i))$, then $\Delta'' = \min(\Delta/2, \Delta') > 0$. Based on Definition 1, there exists a strategy g^* for II such that, for every strategy of I,

$$\delta(\bar{\omega}_n, S) \geqq \Delta'' \quad \text{i.o. a.s.} \tag{3.17}$$

Where g^* can be constructed as follows: Let q_n be arbitrary if $n \leqq 10K/\Delta''$ or $\delta(\bar{\omega}_{n-1}, S) \geqq \Delta''$. Let N_1 be the first $n > 10K/\Delta''$ with $\delta(\bar{\omega}_n, S) < \Delta''$, then $\bar{\omega}_n \in U(B_i)$ for some $1 \leqq i \leqq j$; let $q_{N_1+1}, q_{N_1+2}, \cdots, q_{N_2}$ be the associated moves in the definition of the insufficient subset B_i of S such that N_2 is the first $n \geqq N_1$ with $\delta(\bar{\omega}_n, S) \geqq \Delta''$. For $n > N_2, q_n$ can be constructed in a similar way. (3.17) implies that S is not approachable by I. ∎

The proof of the sufficiency is established through Theorem 4, Lemma 3 and Lemma 4.

THEOREM 4. *A closed nonempty set $S \subset \mathscr{E}^k$ is approachable by player* I *if the following condition is satisfied: For each k-space point $v \notin S$, let u be the closest point in S to v and H be the hyperplane of k-space through u and perpendicular to the segment $\overline{vu}$; then there exists a move $p^* \in P$ such that H separates v from $R(p^*)$.*

PROOF. Let K be the diameter of Ω and $\langle u, v\rangle$ represent the inner product of the vectors u and v. Let player II use any strategy and let player I use the following strategy: Let p_1 be arbitrary and let $p_n, n > 1$, be an arbitrary element of P if $\bar{\omega}_{n-1} \in S$. If $\bar{\omega}_{n-1} \notin S$, $n > 1$, let u_{n-1} be the closest point in S to $\bar{\omega}_{n-1}$, H_{n-1} be the hyperplane through u_{n-1} and perpendicular to the segment $\overline{\bar{\omega}_{n-1}u_{n-1}}$, $\lambda_{n-1} \in P$ such that H_{n-1} separates $\bar{\omega}_{n-1}$ from $R(\lambda_{n-1})$, and $p_n = \lambda_{n-1}$. For

$\bar{\omega}_{n-1} \notin S$, we have $\omega_n \in R(p_n) = R(\lambda_{n-1})$, and $\langle \bar{\omega}_{n-1} - u_{n-1}, \omega_n - u_{n-1} \rangle \leqq 0$. Since $\omega_n, \bar{\omega}_{n-1} \in \Omega$ and $\delta(\bar{\omega}_n, \bar{\omega}_{n-1}) = \delta(\omega_n, \bar{\omega}_{n-1})/n \leqq K/n$, we have

$$\begin{aligned} \delta^2(\bar{\omega}_n, S) &\leqq \delta^2(\bar{\omega}_n, u_{n-1}) \\ &= \delta^2(\bar{\omega}_{n-1}, u_{n-1}) + 2\langle \bar{\omega}_{n-1} - u_{n-1}, \bar{\omega}_n - \bar{\omega}_{n-1} \rangle + \delta^2(\bar{\omega}_n, \bar{\omega}_{n-1}) \\ &\leqq (1-2/n)\delta^2(\bar{\omega}_{n-1}, S) + K^2/n^2 \qquad \text{for} \quad n \geqq 2. \end{aligned}$$

If we assume that $\delta^2(\bar{\omega}_{n-1}, S) \leqq K^2/(n-1)$, then we have

$$\delta^2(\bar{\omega}_n, S) \leqq K^2/n. \tag{3.18}$$

If $\bar{\omega}_{n-1} \in S$, then

$$\delta^2(\bar{\omega}_n, S) \leqq \delta^2(\bar{\omega}_n, \bar{\omega}_{n-1}) \leqq K^2/n^2 \leqq K^2/n.$$

Since we must have $S \cap \Omega \neq \emptyset$ to satisfy the given condition, (3.18) is true for $n = 1$. Hence (3.18) is true for all $n \geqq 1$ by induction. Thus $\delta(\bar{\omega}_n, S) \to 0$ and $\delta(\bar{Y}_n, S) \to 0$ a.s. □

LEMMA 3. *For a given k-space hyperplane* $H[\sum_1^k \alpha^l \omega^l + \alpha^0 = 0$ *for some finite real* α^l *and* $\alpha^0]$, *let*

$$A = \{\omega : \omega \in \mathscr{E}^k \text{ and } \textstyle\sum_1^k \alpha^l \omega^l + \alpha^0 < 0\} \quad \text{and} \quad A^c = \mathscr{E}^k \sim A.$$

If $R(p) \not\subset A^c$ *for all* $p \in P$, *then there exists a* $q^* \in Q$ *such that* $T(q^*) \subset A$.

PROOF. $R(p) \not\subset A^c$, $p \in P$, implies that there exists a $q \in Q$ such that $\omega(p, q) \in A$. Without loss of generality, we may assume that H is $\omega^1 = 0$ by a linear mapping. Under this assumption, we may rewrite the hypothesis as follows: For every $p \in P$, there exists a $q \in Q$ such that $\omega^1(p, q) < 0$, where $\omega^1(p, q)$ is the first coordinate of $\omega(p, q)$.

Let $\bar{M}^* = \|\bar{M}^1(i,j)\|$ be the matrix of finite real numbers whose elements $\bar{M}^1(i,j)$ are the first coordinates of $\bar{M}(i,j)$. Then there exists a finite real number $\mathscr{V}$, the value of $\bar{M}^*$, such that by the von Neumann Minimax Theorem [9],

$$\begin{aligned} \max_{p \in P} \min_{q \in Q} \textstyle\sum_{i=1}^r \sum_{j=1}^s p(i)\bar{M}^1(i,j)q(j) &= \mathscr{V} \\ &= \min_{q \in Q} \max_{p \in P} \textstyle\sum_{i=1}^r \sum_{j=1}^s p(i)\bar{M}^1(i,j)q(j). \end{aligned} \tag{3.19}$$

Since

$$\min_{q \in Q} \textstyle\sum_{i=1}^r \sum_{j=1}^s p(i)\bar{M}^1(i,j)q(j) < 0 \qquad \text{for each} \quad p \in P$$

by the hypothesis, and since P and Q are closed bounded convex subsets of r- and s-space respectively, we have $\mathscr{V} < 0$. Hence, there exists a $q^* \in Q$ such that

$$\max_{p \in P} \textstyle\sum_{i=1}^r \sum_{j=1}^s p(i)\bar{M}^1(i,j)q^*(j) < 0,$$

that is, $\omega^1 < 0$ for all $\omega \in T(q^*)$. Thus $T(q^*) \subset A$ and the proof is completed. □

For a special case, $r = s = k = 2$, a direct proof of Lemma 3 without the use of the von Neumann Minimax Theorem was given by the writer [5].

LEMMA 4. *Let u be the closest point in a closed set $S \subset \Omega$ to $v \in [\mathscr{E}^k \sim S]$. Moreover, for some real α^l and α^0, let $\sum_1^k \alpha^l\omega^l + \alpha^0 = 0$ be the hyperplane H through u and perpendicular to the segment $\overline{vu}$ such that $\sum_1^k \alpha^l v^l + \alpha^0 < 0$* (i.e., $v \in A$). *If there is a $q^* \in Q$ such that $T(q^*) \subset A$, then $u \in B$ for some insufficient subset B of S.*

PROOF. Let $V = \{\omega : \omega \in \mathscr{E}^k$ and $\delta(v, \omega) < \delta(v, u)\}$ and $V^* = \{\omega : \omega \in \mathscr{E}^k$ and $\delta(v, \omega) = \delta(v, u)\}$, then $S \cap V = \varnothing$ by the hypothesis.

$T(q^*) = \{\omega(p, q^*) : p \in P\}$ is the convex hull of the r points

$$t_i = \sum_{j=1}^s \bar{M}(i, j)q^*(j), \; i = 1, \cdots, r.$$

For each $t_i \notin V$, let L_i be the tangent line of V^* from t_i *such* that $\delta(u, L_i) \leqq \delta(u, L_i')$ for all $L_i' \in \{L_i' : L_i'$ is a tangent line of V^* from $t_i\}$. Let L_l be one of these L_i such that

$$\delta^* = \delta(u, L_l) \leqq \delta(u, L_i) \quad \text{for all} \quad 1 \leqq i \leqq r \quad \text{and} \quad t_i \notin V$$

and let $\delta^* = \delta(u, v)$ if $t_i \in V$ for all $1 \leqq i \leqq r$.

Since $T(q^*)$, $V \subset A$, and $\delta(u, v) > 0$, we have $\delta^* > 0$ and $U = \{\omega : \omega \in \mathscr{E}^k$ and $\delta(u, \omega) < \delta^*/2\}$ is a nonempty open set in k-space including u.

If $\bar{\omega}_n \in U$ for any $n \geqq 1$ and $q_m = q^*$ for $m > n$, then for $m > n$, $\bar{\omega}_m$ will move toward $T(q^*)$ through V, where V is an open set disjoint from S. Hence there exists a $\Delta > 0$ such that for each $n > 10K/\Delta$ and $\bar{\omega}_n \in U$

$$\text{Prob}\,\{\delta(\bar{\omega}_N, S) \geqq \Delta \quad \text{for some integer} \quad N \geqq n \,|\, f, q_m = q^* \text{ for } m > n\} = 1$$

for every strategy f for I.

We conclude that $B = U \cap S$ with $u \in B$ is an insufficient subset of S by Definition 1. □

PROOF OF SUFFICIENCY OF THEOREM 3.

Let $u \in S' \neq \varnothing$, $v \in [\mathscr{E}^k \sim S']$, $H[\sum_1^k \alpha^l\omega^l + \alpha^0 = 0]$ be defined as in Theorem 4 such that $v \in A = \{\omega : \omega \in \mathscr{E}^k$ and $\sum_1^k \alpha^l\omega^l + \alpha^0 < 0\}$. Since S' contains no insufficient subset, we must have $T(q) \not\subset A$ for all $q \in Q$ by Lemma 4. $T(q) \not\subset A$ for all $q \in Q$ implies that we cannot have $R(p) \not\subset A^c$ for all $p \in P$ by Lemma 3. Hence, there exists a $p^* \in P$ such that $R(p^*) \subset A^c$. So that H separates v from $R(p^*)$. Thus S' is approachable by player I by Theorem 4. □

If we replace M by M', the transpose of M, then we have a necessary and sufficient condition for approachability for player II.

4. Approachability for games in $\bar{M}$.

THEOREM 5. *For a given M, the class of approachable* (*excludable*) *sets for player* I (II) *depends only on $\bar{M}$.*

PROOF. For each $n \geqq 1$, the move p_{n+1} of player I in the proof of sufficiency of Theorem 3 and the move q_{n+1} of player II in the proof of necessity of the same theorem can be written as functions of $(\omega_1, \cdots, \omega_n)$ only, where

$$\omega_n = \sum_{i=1}^r \sum_{j=1}^s p_n(i)\bar{M}(i, j)q_n(j).$$

Hence the condition described in Theorem 3 is also the necessary and sufficient condition for approachability in $\bar{M}$. Thus a set $S \subset \mathscr{E}^k$ is approachable by I (II) in M if and only if S is approachable by I (II) in $\bar{M}$. □

5. Examples.

5.1. Suppose $\bar{M}$ is given as

$$\bar{M} = \begin{Vmatrix} (1, 1) & (0, 1) \\ (1, 0) & (0, 0) \end{Vmatrix}.$$

Let $\mathscr{H}$ be the collection of all subsets $H = \{(x, h(x)) \colon 0 \leqq x \leqq 1\}$, where $h(x)$ is continuous on $0 \leqq x \leqq 1$ with the following conditions:

(i) $0 \leqq h(x) \leqq 1$,

(ii) $\max\{-h(x)/(1-x), [h(x)-1]/x\} \leqq [h(x')-h(x)]/(x'-x) \leqq \min\{[1-h(x)]/(1-x), h(x)/x\}$

for $0 < x < x' \leqq 1$ and $0 \leqq x' < x < 1$,

(iii) $-h(0) \leqq [h(x')-h(0)]/x' \leqq 1-h(0)$ for $0 < x' \leqq 1$,

(iv) $h(1)-1 \leqq [h(x')-h(1)]/(x'-1) \leqq h(1)$ for $0 \leqq x' < 1$.

That is, no chord of h, when extended, intersects either the line segment joining (0, 1) and (1, 1) or the line segment joining (0, 0) and (1, 0) except the four end points. Then we have the following results:

1. If the closure of a set S in 2-space contains an $H \in \mathscr{H}$, then S is approachable by player I. An approachable strategy with respect to H is described in the proof of Theorem 4.

2. If we define a minimal closed approachable set by player I to be a closed set which is approachable by him and intersects each $T(q)$, $q \in Q$, in at most one point, then the collection $\mathscr{H}$ and the collection of minimal closed approachable sets for player I are equivalent.

3. An approachable set S which does not contain any $\mathscr{H}$-set is shown in Figure 1, where $S = \bigcup_{k=1}^{8} S_i$, S_1 is the line segment joining (0, 1/6) and (1/7, 2/7), S_2 is the convex hull of the three points (1/7, 2/7), (1/3, 4/9), and (1/4, 1/2), S_3 is the line segment joining (1/4, 1/2) and (1/2, 2/3), S_4 is the line segment joining (1/3, 4/9) and (1/2, 1/3), S_5 is the line segment joining (1/2, 2/3) and (2/3, 5/9), S_6 is the line segment joining (1/2, 1/3) and (3/4, 1/2), S_7 is the convex hull of the three points (2/3, 5/9), (3/4, 1/2), and (6/7, 5/7), and S_8 is the line segment joining (6/7, 5/7) and (1, 5/6). None of the extensions of the line segments forming the upper (lower) envelope of S intersect the line segment joining $\bar{M}(2, 2)$ and $\bar{M}(2, 1)$ [$\bar{M}(1, 2)$ and $\bar{M}(1, 1)$] except the two end points. Hence S is approachable by I by Theorem 4.

5.2. Suppose $\bar{M}$ is given as

$$\bar{M} = \begin{Vmatrix} (1, 1) & (0, 0) \\ (1, 0) & (0, 0) \end{Vmatrix},$$

then a closed set in 2-space is approachable by player I if and only if it contains an $R(p)$ for some $p \in P$.

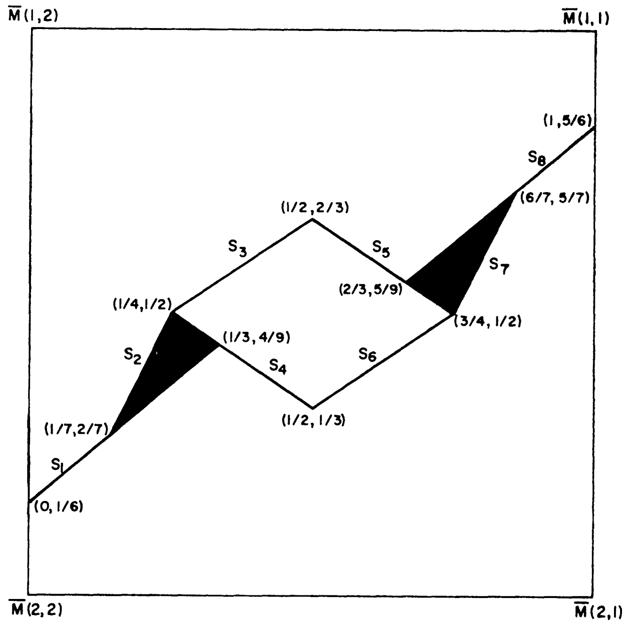

FIG. 1. $S = \bigcup_{k=1}^{8} S_k$ is a closed approachable set (by player I) but contains no H of $\mathscr{H}$.

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