The Annals of Mathematical Statistics
1971, Vol. 42, No. 5, 1748–1751

ON AN INEQUALITY FOR ORDER STATISTICS

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The problem of finding a Chebyshev type inequality for random variables with unknown or non-existent variance was considered by Z. W. Birnbaum (1970). In this present paper, a statistic, T, similar to, but simpler than Birnbaum's, is considered. The statistic is independent of location and scale parameters for families of bell-shaped distributions and so may be considered to be a competitor to Student's t. An inequality establishing an upper bound for $P(|T| > \lambda)$ is proved. This bound is considerably smaller than the corresponding bound found by Birnbaum. Finally, an improvement of the latter is offered.

1. Introduction and summary. For a sample of size $2n+1$ from a "bell-shaped" distribution, Z. W. Birnbaum (1970) has proposed a studentized Chebyshev type inequality. Denoting by $X_{(j)}$, the jth order statistic, he establishes that

$$P(|S| > \lambda) \leqq \binom{2n+1}{n-r}\binom{2r}{r}[\lambda(\lambda-1)]^{-r}2^{-(n+r)} \qquad \text{for } \lambda > 1, \tag{1.1}$$

where $S = (V-\mu)/(W-U)$, μ is the median of the distribution, $U = X_{(n+1-r)}$, $V = X_{(n+1)}$ and $W = X_{(n+1+r)}$. In this paper, a statistic similar to S is considered and an inequality similar to (1.1) is developed. The inequality gives a considerably smaller upper bound for the probability of large values of the statistic. Finally, an improvement of Birnbaum's inequality is provided.

2. Notation and definitions. Let $X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(2n+1)}$ be the order statistics for a sample of size $2n+1$ of a random variable X. We consider the two order statistics

$$U = X_{(n+1-r)}, \qquad W = X_{(n+1+r)} \tag{2.1}$$

for some integer r, $1 \leqq r \leqq n$. We shall use $W-U$, the interquantile range between two sample quantiles and $\frac{1}{2}(U+W)$, a generalized midrange, to form the statistic

$$T = \frac{\frac{1}{2}(U+W)-\mu}{W-U}. \tag{2.2}$$

Comparing T and S, given by (1.1), we see that they have the same denominator and in each case the numerator is the difference between μ and an estimate of μ. Thus, T and S are statistics of the same form.

We shall assume that X has a bell-shaped probability density function $f(x)$, that is

$$f(\mu-x) = f(\mu+x) \qquad \text{for } x \geqq 0 \text{ and} \tag{2.3}$$

$$f(\mu+x) \quad \text{is nonincreasing} \qquad \text{for } x \geqq 0. \tag{2.4}$$

Received January 11, 1971.

Let $F(x)$ be the cumulative distribution of X and denote by $\mathscr{F}_B$ the family of all distributions having densities satisfying (2.3) and (2.4).

3. The inequality. If X has a probability distribution in $\mathscr{F}_B$, then

$$P(|T| > \lambda) \leqq \binom{n}{r}(\lambda - \tfrac{1}{2})^{-2r}2^{-2r} \qquad \text{for } \lambda > \tfrac{1}{2}. \tag{3.1}$$

PROOF. Clearly T is independent of location and scale parameters, remaining unchanged under linear transformations of X. Consequently, we can assume without loss of generality that

$$\mu = 0. \tag{3.2}$$

The density function $f(x)$ is assumed to be symmetric about zero and to be a nonincreasing function of $|x|$. Because the corresponding cumulative distribution function, $F(x)$, is concave for $x \geqq 0$, convex for $x \leqq 0$ and $F(0) = \frac{1}{2}$, it follows that for $0 < m \leqq 1$, $w \neq 0$,

$$\frac{F(w) - F(mw)}{w - mw} \leqq \frac{F(mw) - \frac{1}{2}}{mw} \leqq \frac{F(w) - \frac{1}{2}}{mw}. \tag{3.3}$$

The joint density of U and W is

$$g(u, w) = K(n, r)F^{n-r}(u)[F(w) - F(u)]^{2r-1}[1 - F(w)]^{n-r} \cdot f(u)f(w) \tag{3.4}$$

for $u < w$ and zero otherwise,

where

$$K(n, r) = (2n+1)!/[(n-r)!]^2(2r-1)!. \tag{3.5}$$

By symmetry, we may write

$$P(|T| > \lambda) = 2P(T > \lambda) = 2\int\int_{T>\lambda} g(u, w)\, du\, dw. \tag{3.6}$$

For $\lambda > \frac{1}{2}$,

$$P(T > \lambda) = P\left(\frac{W + U}{W - U} > 2\lambda\right) = P(U > mW), \tag{3.7}$$

where

$$m = \frac{2\lambda - 1}{2\lambda + 1} \quad \text{and} \quad 0 < m < 1. \tag{3.8}$$

Combining (3.4) through (3.8), we have

$$P(|T| > \lambda) = 2K(n, r)\int_{w=0}^{\infty}\int_{u=mw}^{w} F^{n-r}(u)[F(w) - F(u)]^{2r-1} \cdot [1 - F(w)]^{n-r} f(u)f(w)\, du\, dw. \tag{3.9}$$

Noting that $F(u)$ is nondecreasing, we obtain the bound for the inner integral

$$(3.10)\qquad \begin{aligned} &\int_{mw}^{w} F^{n-r}(u)[F(w)-F(u)]^{2r-1}f(u)\,du \\ &\leqq F^{n-r}(w)\int_{mw}^{w}[F(w)-F(u)]^{2r-1}f(u)\,du \\ &= (2r)^{-1}F^{n-r}(w)[F(w)-F(mw)]^{2r}, \end{aligned}$$

which yields by direct substitution,

$$(3.11)\quad P(|T| > \lambda) \leqq r^{-1}K(n,r)\int_0^\infty F^{n-r}(w)[F(w)-F(mw)]^{2r}\cdot[1-F(w)]^{n-r}f(w)dw.$$

Multiplying through in (3.3) by $w(1-m)$ and substituting into (3.11) we obtain

$$(3.12)\qquad \begin{aligned} P(|T| > \lambda) &\leqq \frac{K(n,r)}{r}\left(\frac{1-m}{m}\right)^{2r}\int_{\frac{1}{2}}^{1} z^{n-r}(1-z)^{n-r}(z-\tfrac{1}{2})^{2r}\,dz \\ &= \frac{K(n,r)}{2r}\left(\frac{1-m}{m}\right)^{2r}\int_{0}^{1} z^{n-r}(1-z)^{n-r}(z-\tfrac{1}{2})^{2r}\,dz \\ &= \frac{K(n,r)}{2r}\left(\frac{1-m}{m}\right)^{2r}\cdot\frac{(2r)!(n-r)!n!}{2^{2r}r!(2n+1)!}. \end{aligned}$$

The latter step may be verified as follows: Let

$$(3.13)\qquad J(a,b) = \int_0^1 z^a(1-z)^a(z-\tfrac{1}{2})^b\,dz,$$

then integration by parts yields

$$(3.14)\qquad J(a,b) = \frac{b-1}{2(a+1)}J(a+1,b-2),$$

and the result follows by iteration. Since $(1-m)/m = (\lambda-\frac{1}{2})^{-1}$, combining (3.5) and (3.12), one obtains (3.1).

4. Conclusions and remarks. We note that for $\lambda > \frac{1}{2}$, $P(|T| > \lambda) \leqq P(|T| > \frac{1}{2})$. However,

$$(4.1)\qquad \begin{aligned} P(|T| > \tfrac{1}{2}) = P\left(\left|\frac{W+U}{W-U}\right| > 1\right) &= P(U, W > 0) + P(U, W < 0) \\ &= P(U > 0) + P(W < 0) \end{aligned}$$

since $U < W$. By symmetry

$$(4.2)\qquad P(|T| > \tfrac{1}{2}) = 2P(U > 0) = 2\sum_{j=n-r+1}^{2n+1} b(j; 2n+1, \tfrac{1}{2})$$

where $b(j;n,p)$ is the usual binomial probability function. Denoting this last probability by $B_{\frac{1}{2}}$ and noting that $B_{\frac{1}{2}}$ is conveniently tabulated for small n and approximated for large n, we may make a slight improvement in (3.1) writing

$$(4.3)\qquad P(|T| > \lambda) \leqq \min\left(B_{\frac{1}{2}}, \tbinom{n}{r}(2\lambda-1)^{-2r}\right) \qquad \text{for } \lambda \geqq \tfrac{1}{2}.$$

An improvement over Birnbaum's result can be achieved by replacing his upper bound for $J(n-r, 2r)$ (page 431) by the exact value of the integral to yield

$$P(|S| > \lambda) \leqq 2^{-2r}\binom{n}{r}\binom{2r}{r}[\lambda(\lambda-1)]^{-r} = \binom{n}{r}\binom{2r}{r}[(2\lambda-1)^2-1]^{-r}, \qquad \lambda > 1. \tag{4.4}$$

Calling this new bound $B(S)$ and the bound given in (3.1) $B(T)$, we see that

$$B(T)/B(S) = [1-(2\lambda-1)^{-2}]^r/\binom{2r}{r}, \qquad \lambda > 1. \tag{4.5}$$

Since for both inequalities spread is measured in units of $W-U$, it is noteworthy that $B(T)$ is considerably smaller than $B(S)$ for all values of λ at which the two are comparable. In addition, the bound $B(T)$ is valid for a greater range of values of λ.

REFERENCE

BIRNBAUM, Z. W. (1970). On a statistic similar to Student's *t*. *Nonparametric Techniques in Statistical Inference*, ed. M. L. Puri. Cambridge Univ. Press 427–433.