

## FAMILIES OF INFINITELY DIVISIBLE DISTRIBUTIONS CLOSED UNDER MIXING AND CONVOLUTION<sup>1</sup>

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Certain families of probability distribution functions maintain their infinite divisibility under repeated mixing and convolution. Examples on the continuum and lattice are given. The main tools used are Pólya's criteria and the properties of log-convexity and complete monotonicity. Some light is shed on the relationship between these two properties.

**0. Introduction and summary.** The concept of infinite divisibility [1], [5], [6] has a central role in probability theory. It has been shown by Goldie [2], and Steutel [7] that for many families of infinitely divisible distribution functions, the property of infinite divisibility is preserved under mixing. Surprisingly, this curious preservation of infinite divisibility can be shown to hold for certain families even when mixing and convolution are applied repeatedly. Specifically we will exhibit such sets of infinitely divisible probability distribution functions on both the lattice and continuum which are closed under mixing, convolution and weak convergence.

The main tools used are Pólya's criteria and the properties of log-convexity and complete monotonicity. Some light is shed on the relationship between these two properties.

In our discussion the following abbreviations and notation will be employed:

cdf: cumulative distribution function; e.g.  $F(x)$ ,  
 $G(x)$ ;  
pdf: probability density function; e.g.  $f(x)$ ,  $g(x)$ ;  
ch.f.: characteristic function; e.g.  $\phi(t)$ ,  $\gamma(t)$ .

We will frequently deal with symmetric distribution functions, for which the characteristic function is real and even. If further  $\phi(t)$  is completely monotonic, it will have (Feller [1] page 416) the representation

$$(0.1) \quad \phi(t) = \int_{-\infty}^{\infty} e^{itx} dF(x) = \int_0^{\infty} e^{-|t|x} d\mathfrak{F}(x).$$

It should be noted that  $F(x)$  and  $\mathfrak{F}(x)$  do not have a simple relationship, even though  $\mathfrak{F}(x)$  is itself a cdf.

### 1. Some structural properties.

**DEFINITION 1.0.** A cdf  $F(x)$  will be said to be a finite mixture of cdf's  $\{F_{\alpha_j}(x)\}$  if  $F(x) = \sum_{j=1}^N p_j F_{\alpha_j}(x)$  for some  $0 \leq p_j \leq 1$ ,  $\sum_{j=1}^N p_j = 1$ . It will be said to be

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Received January 11, 1971.

<sup>1</sup> Supported, in part, by the Office of Naval Research under contract NO0014-68-A-0091.

a general mixture or simply mixture if  $F(x) = \int_0^\infty F_\alpha(x) dG(\alpha)$  where  $G(\alpha)$  is a cdf. The same convention will be employed for mixtures of nonnegative functions.

We shall use the following lemmas for the proof of which we refer to Lukacs (1960).

LEMMA 1.1. *An infinitely divisible characteristic function has no real zeros.*

LEMMA 1.2. *An infinitely divisible ch.f. which is analytic has no zeros in the interior of its strip of regularity.*

LEMMA 1.3. *A ch.f. which is the limit of a sequence of infinitely divisible ch.f.'s is infinitely divisible.*

DEFINITION 1.1. Throughout this paper the symbol  $\mathcal{L}$  will denote any well-defined set of infinitely divisible ch.f.'s, closed under finite mixing and multiplication. Specifically, if  $\phi_1 \in \mathcal{L}$ ,  $\phi_2 \in \mathcal{L}$ , then  $p_1\phi_1 + p_2\phi_2 \in \mathcal{L}$ ,  $p_1 \geq 0$ ,  $p_2 \geq 0$ ,  $p_1 + p_2 = 1$ ; and  $\phi_1\phi_2 \in \mathcal{L}$ .

THEOREM 1.1. *If  $\phi \in \mathcal{L}$ , then  $\phi$  is real, positive and hence even.*

PROOF. If  $\phi$  is not real then, for some real  $t_0$ ,  $\arg \phi(t_0) \neq 0$ . As  $\phi(0) = 1$  and  $\phi$  is continuous, there is a  $t_1$  such that  $\arg \phi(t_1) = \pi/n$  for some integer  $n > 0$ . Therefore  $\{\phi(t_1)\}^n < 0$  and  $\{\phi(t_1)\}^{2n} > 0$ . It follows that for a suitable value of  $p$ , with  $0 < p < 1$ ,  $p\{\phi(t_1)\}^n + (1-p)\{\phi(t_1)\}^{2n} = 0$ . Hence, by Lemma 1.1,  $p\phi^n + (1-p)\phi^{2n}$  is not infinitely divisible and, by definition,  $\phi \notin \mathcal{L}$ . This proves that for all  $\phi \in \mathcal{L}$   $\phi$  must be real, and hence even. Positivity is required by Lemma 1.1.  $\square$

THEOREM 1.2. *If  $\phi \in \mathcal{L}$ , and  $\phi$  is not identically equal to 1, then  $\phi$  has no second moment.*

PROOF. If  $\phi$  is even with variance  $\sigma^2 \neq 0$ , then

$$(1.1) \quad \lim_{n \rightarrow \infty} \phi^n(t/n^{1/2}) = \phi(t) = e^{-\sigma^2 t^2/2}.$$

Suppose  $\phi(t) \in \mathcal{L}$ . Then by definition of  $\mathcal{L}$  the ch.f.

$$(1.2) \quad \beta(t) = \frac{1}{2}\{\phi^n(t) + \phi^{2n}(t)\}$$

is in  $\mathcal{L}$ , and therefore infinitely divisible for all  $n$ . Hence  $\beta(t/n^{1/2})$  is infinitely divisible for all  $n$ , so that by (1.1) and Lemma 1.3 the ch.f.  $\zeta(t) = \frac{1}{2}\{\phi(t) + \phi^2(t)\}$  is infinitely divisible. But  $\zeta(t) = 0$  for  $t^2 = 2\pi i/\sigma^2$ , which contradicts Lemma 1.2. Hence  $\phi(t)$  cannot be in  $\mathcal{L}$ .  $\square$

COROLLARY. *If  $\phi \in \mathcal{L}$  and  $\phi$  is not identically equal to one, then  $\phi$  is not analytic in any neighborhood of  $t = 0$ . Moreover  $\phi$  cannot have a convergence strip.*

PROOF. Analyticity in a neighborhood of zero implies the existence of a second moment. Also, evenness of  $\phi(t)$  implies symmetry of any convergence strip about the real axis, which would imply analyticity in a neighborhood of zero, and a second moment contradicting Theorem 1.2.  $\square$

An alternative proof of the fact that a ch.f.  $\phi \in \mathcal{L}$  cannot have a convergence strip is the following.

Let  $t = u + iv$ , where  $u$  and  $v$  are real, and let  $\phi(t) = \int_{-\infty}^{\infty} e^{itz} dF(x)$ . If  $\phi(t)$  has a convergence strip, there will be a set of values  $v$ , the "convergence interval," for which  $\phi(iv) = \int e^{-vz} dF(x) < \infty$ . For each such  $v$  the function

$$(1.3) \quad \phi_v(u) = \phi(u + iv)/\phi(iv) = \int e^{iuz} e^{-vz} dF(x) / \int e^{-vz} dF(x) = \int e^{iuz} dF_v(x)$$

will be a ch.f., and  $\phi_v(u)$  will not be even in  $u$  for  $v \neq 0$ . Hence, as in the proof of Theorem 1.1, for some  $n, u_1$ , and  $p$  we will have  $p\phi_v^n(u_1) + (1-p)\phi_v^{2n}(u_1) = 0$  and  $p\phi^n(u_1 + iv)/\phi^n(iv) + (1-p)\phi^{2n}(u_1 + iv)/\phi^{2n}(iv) = 0$ . Hence  $p_1\phi^n(u_1 + iv) + p_2\phi^{2n}(u_1 + iv) = 0$ ,  $p_1 > 0$ ,  $p_2 > 0$ ,  $p_1 + p_2 = 1$ . But  $\phi \in \mathcal{L}$  implies that  $\phi^* = p_1\phi^n + p_2\phi^{2n} \in \mathcal{L}$  and  $\phi^*$  is infinitely divisible. Moreover  $\phi^*$  and  $\phi$  have the same convergence interval and  $\phi^*(u_1 + iv) = 0$ . This contradicts Lemma 1.2.

The following related theorem may be of independent interest.

THEOREM 1.3. Let  $\phi(t) = \int e^{itz} dF(x)$  be any infinitely divisible ch.f. having a convergence strip  $S$ . For any real  $v$  in the convergence interval, the cdf

$$(1.4) \quad F_v(x) = \int_{-\infty}^x e^{-vy} dF(y) / \int_{-\infty}^{\infty} e^{-vy} dF(y)$$

will also be infinitely divisible.

PROOF. By Lemma 1.2,  $\phi(t) \neq 0$  in  $S$ , so that  $\phi(t) = \exp\{\psi(t)\}$  where  $\psi(t) = \int_0^t \{\phi'(w)/\phi(w)\} dw$  is analytic in  $S$ . Also  $\phi(t) = \{\phi_N(t)\}^N$  where  $\phi_N(t) = \exp\{N^{-1}\psi(t)\}$  is a ch.f. having the same convergence strip  $S$ . From (1.3) and (1.4)  $F_v(x)$  has the ch.f.  $\phi_v(u) = \phi(u + iv)/\phi(iv) = \{\phi_N(u + iv)/\phi_N(iv)\}^N = \{\phi_{vN}(u)\}^N$ , and  $\phi_{vN}(u) = \phi_N(u + iv)/\phi_N(iv)$  is itself a ch.f. Hence  $\phi_v(u)$  is infinitely divisible.  $\square$

## 2. Examples of classes $\mathcal{L}$ .

LEMMA 2.1. (Pólya's criterion). If a real function  $\phi$  is even, continuous, convex on  $(0, \infty)$ ,  $\phi(0) = 1$  and  $\phi(t) \rightarrow p$  ( $0 \leq p \leq 1$ ) as  $t \rightarrow \infty$ , then it is a ch.f.

PROOF. In [6] this lemma is proved under the additional condition that  $\phi(t) \rightarrow 0$  for  $t \rightarrow \infty$ . By considering  $p + (1-p)\phi$ , it is easily seen that this condition can be relaxed to  $\phi(t) \rightarrow p$  ( $0 \leq p \leq 1$ ).  $\square$

LEMMA 2.2.<sup>2</sup> If a ch.f.  $\phi$  is even positive and log-convex on  $(0, \infty)$ , then it is infinitely divisible.

<sup>2</sup> Suggested by R. A. Horn, On certain power series and sequences, *J. London Math. Soc.* (2nd series), Vol. 2, Part 1, 1970, p. 160-162.

PROOF. For every positive  $p$ ,  $\phi^p$  is log-convex and hence convex on  $(0, \infty)$ , i.e.  $\phi^p$  satisfies the conditions of Lemma 2.1. As  $\phi^p$  is a ch.f. for all  $p > 0$  it follows that  $\phi$  is infinitely divisible.  $\square$

LEMMA 2.3. *The family of log-convex functions is closed under mixing, multiplication and raising to a positive power. The limit of a sequence of log-convex functions is log-convex.* (For a proof, see e.g. Kingman [4]).

DEFINITION 2.1.  $\mathcal{L}_0$  is the class of all real (even), positive ch.f.'s that are log-convex on  $(0, \infty)$ .

THEOREM 2.1. *The class  $\mathcal{L}_0$  is closed under (a) mixing, (b) multiplication, (c) raising to a positive power, (d) scaling and (e) passage to the limit.*

Closure under (a) and (b) has been defined in Definition 1.1. By (c) we mean that if  $\phi \in \mathcal{L}_0$ , then  $\phi^p \in \mathcal{L}_0$  for all  $p > 0$ . By (d): if  $\phi(t) \in \mathcal{L}_0$ , then  $\phi(pt) \in \mathcal{L}_0$  for all  $p > 0$ . By (e): if  $\phi_n \in \mathcal{L}_0$ , and if  $\lim \phi_n = \phi_0$  is a ch.f., then  $\phi_0 \in \mathcal{L}_0$ .

PROOF. Only (a) is non-trivial, and this follows from Lemma 2.3.  $\square$

An important subclass of  $\mathcal{L}_0$  is the class of completely monotonic ch.f.'s.

DEFINITION 2.2.  $\mathcal{L}_1$  is the class of all real ch.f.'s that are completely monotonic on  $(0, \infty)$ .

THEOREM 2.2. *The class  $\mathcal{L}_1$  is a proper subclass of  $\mathcal{L}_0$ , and  $\mathcal{L}_1$  is closed under mixing, multiplication, raising to a positive integer power and passage to the limit.* (We will see in Section 4 that  $\mathcal{L}_1$  is *not* closed under raising to a positive power.)

PROOF. Any completely monotonic function  $\phi(t)$  is positive and has the representation (cf. (0.1))

$$(2.1) \quad \phi(t) = \int_0^\infty e^{-tx} d\mathfrak{F}(x), \quad t \geq 0,$$

where  $\mathfrak{F}(x)$  is non-decreasing. It follows from (2.1) and Lemma 2.3 that every completely monotonic function is log-convex on  $(0, \infty)$ . The converse is not true since log-convex functions need not be differentiable everywhere. Even log-convex functions which are analytic need not be completely monotonic as we will see. Consequently  $\mathcal{L}_1$  is a proper subclass of  $\mathcal{L}_0$ . Every  $\phi_1 \in \mathcal{L}_1$  is of the form (2.1) on  $(0, \infty)$ , where  $\mathfrak{F}(x)$  is a cdf on  $[0, \infty)$ . That the class  $\mathcal{L}_1$  is closed under the operations stated, now follows from well-known properties of Laplace-Stieltjes transforms.  $\square$

We see from (2.1) that the distributions corresponding to  $\mathcal{L}_1$  are all mixtures of symmetric Cauchy distributions with ch.f.'s  $\exp(-|t|x)$ .

DEFINITION 2.3.  $\mathcal{L}_\alpha$  is the class of all real (even) ch.f.'s that are completely monotonic functions of  $|t|^\alpha$  ( $0 < \alpha \leq 1$ ).

**THEOREM 2.3.** *The class  $\mathcal{L}_\alpha$  is closed under mixing, multiplication, raising to a positive integer power, scaling and passage to the limit. If  $\alpha < \beta$ , then  $\mathcal{L}_\alpha$  is a proper subset of  $\mathcal{L}_\beta$ .*

**PROOF.** If  $\phi_\alpha \in \mathcal{L}_\alpha$ , then by definition  $\phi_\alpha$  is of the form

$$(2.2) \quad \phi_\alpha(t) = \phi_1(|t|^\alpha),$$

with  $\phi_1 \in \mathcal{L}_1$ . As  $t^\alpha$  has a completely monotonic derivative on  $(0, \infty)$ , it follows (cf. [1] page 417) that  $\phi_\alpha$  is completely monotonic in  $(0, \infty)$  and hence infinitely divisible. The fact that  $\mathcal{L}_\alpha$  is closed as stated follows from (2.1) and (2.2). As, for  $\alpha < \beta$ ,  $\phi_1(|t|^\alpha)$  is completely monotonic in  $|t|^\beta$ , and  $\phi_1(|t|^\beta)$  need not be completely monotonic in  $|t|^\alpha$ , it follows that  $\mathcal{L}_\alpha$  is a proper subset of  $\mathcal{L}_\beta$  if  $\alpha < \beta$ .  $\square$

As for (2.1) the set  $\mathcal{L}_\alpha$  consists of ch.f.'s of the form  $\phi(t) = \int_0^\infty e^{-|t|^\alpha x} d\mathcal{G}(x)$ . Hence the corresponding distributions for  $\mathcal{L}_\alpha$  are mixtures of the symmetric stable distributions of index  $\alpha$ . (Instead of "index" some authors use "exponent," or "order.")

**REMARK.** Though every ch.f. which is even and log-convex on  $(0, \infty)$  is infinitely divisible and convex on  $(0, \infty)$  a ch.f. may be infinitely divisible and convex without being log-convex. An example of this is the ch.f.  $\phi(t)$ , of the form

$$(2.3) \quad \phi = e^{\lambda(\zeta-1)},$$

with

$$(2.4) \quad \zeta(t) = \frac{1}{2}(e^{-|t|} + e^{-t^2}).$$

Here  $\phi$  is the ch.f. of a compound Poisson distribution, and therefore infinitely divisible. One easily verifies that  $\phi$  is convex if  $\lambda$  is sufficiently large. However  $\zeta$  is not convex, and hence  $\phi$  is not log-convex.

### 3. An analogue for lattice distributions.

**LEMMA 3.1.** (Pólya): *A real function  $f$ , which is even, continuous, nonnegative, convex on  $(0, \pi)$ , periodic with period  $2\pi$  and such that  $f(0) = 1$  is the ch.f. of a distribution on the lattice of integers.*

**PROOF.** See [6] and the proof of Lemma 2.1.

**LEMMA 3.2.** *A ch.f. that is real, positive, log-convex on  $(0, \pi)$  and periodic with period  $2\pi$ , is infinitely divisible.*

**PROOF.** See proof of Lemma 2.2.

**DEFINITION 3.1.**  $\mathcal{L}_0^*$  is the class of ch.f.'s satisfying the conditions of Lemma 3.2.

**THEOREM 3.1.** *The class  $\mathcal{L}_0^*$  is closed under mixing, convolution and passage to the limit.*

**PROOF.** See proof of Theorem 3.

**COROLLARY.** *If  $\phi \in \mathcal{L}_0$ , and if  $\phi$  is integrable, then*

$$(3.1) \quad \phi^*(t) = \sum_{k=-\infty}^{\infty} \phi(t + 2k\pi) / \{ \sum_{k=-\infty}^{\infty} \phi(2k\pi) \}$$

*is a ch.f. in  $\mathcal{L}_0^*$ .*

**PROOF.** As  $\phi(t)$  is even, non-increasing on  $(0, \infty)$  and integrable, we have

$$(3.2) \quad \sum_{-\infty}^{\infty} \phi(2k\pi) = \phi(0) + 2 \sum_1^{\infty} \phi(2k\pi) \leq 1 + (2/2\pi) \int_0^{\infty} \phi(t) dt < \infty.$$

For  $0 \leq t \leq \pi$  the monotonicity of  $\phi(t)$  yields

$$(3.3) \quad \begin{aligned} \sum_{|k| \geq N} \phi(2k\pi + t) &= \sum_{k=N}^{\infty} \phi(2k\pi + t) + \sum_{k=-N}^{\infty} \phi(2k\pi - t) \\ &\leq \sum_{k=N}^{\infty} \phi(2k\pi) + \sum_{k=N-1}^{\infty} \phi(2k\pi) \leq 2 \sum_{k=N-1}^{\infty} \phi(2k\pi). \end{aligned}$$

It follows that  $\sum \phi(2k\pi + t)$  is uniformly convergent, and that  $\phi^*(t)$  is a ch.f. It is easily verified that  $\phi^*(t)$  satisfies the conditions of Lemma 3.2.  $\square$

Using the above Corollary we prove

**THEOREM 3.2.** *If  $f(x)$  is the pdf corresponding to an integrable ch.f.  $\phi \in \mathcal{L}_0$ , then*

$$(3.4) \quad p_k = f(k) / \sum_{-\infty}^{\infty} f(n) \quad (k = 0, \pm 1, \pm 2, \dots)$$

*is an infinitely divisible probability distribution on  $(0, \pm 1, \pm 2, \dots)$ .*

**PROOF.** From Poisson's summation formula (see e.g. [1] page 592) we have

$$(3.5) \quad \sum_{-\infty}^{\infty} \phi(t + 2k\pi) = \sum_{-\infty}^{\infty} f(k) e^{ikt}.$$

From the Corollary to Theorem 3.1 it follows that

$$(3.6) \quad \phi^*(t) = \sum_{-\infty}^{\infty} f(k) e^{ikt} / \sum_{-\infty}^{\infty} f(k)$$

is an infinitely divisible ch.f. This concludes the proof.  $\square$

As an example we take the Cauchy distribution with ch.f.  $\phi(t) = \exp(-|t|)$  and pdf  $f(x) = \pi^{-1}(1 + x^2)^{-1}$ . From (3.4) we obtain the infinitely divisible lattice distribution with

$$(3.7) \quad p_k = \frac{c}{1 + k^2} \quad (k = 0, \pm 1, \pm 2, \dots),$$

where (c.f. [1] page 594)  $c = \pi^{-1}(1 - e^{-2\pi})(1 + e^{-2\pi})^{-1}$ .

**4. Complete monotonicity, log-convexity, and infinite divisibility.** It has been stated above that a function may be log-convex, real analytic, and monotonic decreasing without being completely monotonic. We will demonstrate this by

example and show its interest for questions of infinite divisibility. We will need the following side-result.

**THEOREM 4.1.** *If the ch.f.  $\phi(t)$  is completely monotonic, and  $\phi^{1/n}(t)$  is itself a completely monotonic ch.f. for all  $n$  then  $\phi(t^2)$  is infinitely divisible.*

As proof we need only note that if  $\phi^{1/n}(t)$  is completely monotonic then

$$(4.1) \quad \phi^{1/n}(|t|) = \int_0^\infty e^{-|t|x} d\mathcal{F}_n(x).$$

Hence

$$(4.2) \quad \phi^{1/n}(|t|^2) = \int_0^\infty e^{-t^2x} d\mathcal{F}_n(x),$$

being a mixture of characteristic functions is itself a characteristic function, so that  $\phi(t^2)$  is infinitely divisible.  $\square$

Examples illustrating Theorem 4.1 are  $\phi(t) = \exp\{-|t|^\alpha\}$ ,  $0 \leq \alpha \leq 1$ , and  $\phi(t) = \{1 + |t|^\alpha\}^{-1}$ ,  $0 \leq \alpha \leq 1$ .

Let us now demonstrate that  $\phi(t)$  completely monotonic does not imply that  $\phi^{1/n}(t)$  is completely monotonic.  $\phi^{1/n}(t)$  is of course log-convex and real analytic in  $(0, \infty)$  since  $\phi(t)$  is. Consider the ch.f.  $\phi(t) = \frac{1}{2}(e^{-|t|} + e^{-|2t|})$  which is clearly completely monotonic. If  $\phi^{1/n}(t)$  were completely monotonic for *all*  $n$ , then from Theorem 4.1 it would follow that  $\phi(t^2) = \frac{1}{2}(e^{-t^2} + e^{-2t^2})$  is infinitely divisible. This contradicts Lemma 1.2. Hence for *some*  $n$ ,  $\phi^{1/n}(t)$  will not be completely monotonic.

It will next be shown, indeed, that for this example  $\phi^{1/n}(t)$  is not completely monotonic for *any*  $n$ .

**THEOREM 4.2.** *Let  $\phi(t)$  be a completely monotonic real ch.f. with representation*

$$(4.3) \quad \phi(t) = \int_0^\infty e^{-|t|x} d\mathcal{F}(x)$$

where  $\mathcal{F}(x)$  is a cdf. Then  $\phi^{1/n}(t)$  is completely monotonic if and only if  $\mathcal{F}(x)$  is the convolution of  $n$  equal distributions i.e.  $\mathcal{F}(x) = G_n^{*n}(x)$ .

**PROOF.** Suppose  $\phi^{1/n}(t)$  is completely monotonic. Then  $\phi^{1/n}(t)$  has the representation

$$(4.4) \quad \phi^{1/n}(t) = \int_0^\infty e^{-|t|x} d\mathcal{G}_n(x)$$

so that

$$(4.5) \quad \phi(t) = \left\{ \int_0^\infty e^{-|t|x} d\mathcal{G}_n(x) \right\}^n = \int_0^\infty e^{-|t|x} d\mathcal{G}_n^{*n}(x).$$

It is known however that two Laplace-Stieltjes transforms are equal if and only if their cdf's are equal (Feller [1] page 408). Hence from (4.3) and (4.5)  $\mathcal{F}(x) = G_n^{*n}(x)$ . The converse is also immediate from (4.5).  $\square$

It follows from Theorem 4.2 that the function

$$(4.6) \quad \phi_n(t) = \left\{ \frac{1}{2}(e^{-|t|} + e^{-|2t|}) \right\}^{1/n}$$

cannot be completely monotonic for any positive integral  $n$ , since the lattice distribution  $\mathfrak{F}(x)$  with mass  $\frac{1}{2}$  at  $x = 1$ , and  $x = 2$  is indecomposable (cf. Lukacs (1960) Theorem 5.1.1).

The same behavior is of interest for probability density functions. For example the probability density function  $f(x) = K(e^{-x} + e^{-2x})^{1/2}$  is log-convex and real analytic on  $(0, \infty)$  but is not completely monotonic. It has been shown by F. W. Steutel [8] that such log-convex density functions on  $(0, \infty)$  are infinitely divisible (see also [3]). The example shows that they need not be completely monotonic.

From Theorem 4.1 and Theorem 4.2 we find the following extension of Theorem 2.3 for the symmetric stable distributions.

**THEOREM 4.3.** *Let  $\phi(t)$  be any mixture of the symmetric stable ch.f.'s  $\exp\{-|t|^{2\alpha}x\}$ ,  $0 \leq 2\alpha \leq 2$  of the form*

$$(4.7) \quad \phi(t) = \int_0^\infty e^{-|t|^{2\alpha}x} dG(x)$$

*where  $G(x)$  is itself infinitely divisible. Then  $\phi(t)$  is infinitely divisible.*

**PROOF.** Writing  $\phi(t) = \phi(|t|^2)$  we have

$$(4.8) \quad \phi(t) = \int_0^\infty e^{-|t|^{2\alpha}x} dG(x).$$

By Theorem 4.2  $\phi^{1/n}(t)$  is completely monotonic as a function of  $t^\alpha$  with  $0 \leq \alpha \leq 1$ , and hence as a function of  $t$ . Hence, by Theorem 4.1,  $\phi(t^2) = \phi(t)$  is infinitely divisible.  $\square$

**5. Some unanswered questions.** In Section 1 we saw that all  $\phi \in \mathcal{L}$  are real and even, and that no non-degenerate  $\phi \in \mathcal{L}$  can have a finite second moment. We shall now show that no non-degenerate  $\phi \in \mathcal{L}_0$  can have a finite (absolute) first moment. In fact,  $\phi(t)$  cannot have a derivative at  $t = 0$ . For there will be a value  $t_1 > 0$ , at which  $\phi(t_1) < 1$ . The convexity of  $\phi(t)$  implies that

$$\frac{\phi(t) - 1}{t} \leq \frac{\phi(t_1) - 1}{t_1} = \phi_1 < 0, \quad 0 < t \leq t_1.$$

If  $\phi$  were differentiable at  $t = 0$ , one would have  $\phi'(0) \leq \phi_1 < 0$  violating the evenness of  $\phi(t)$ .

The following questions remain unanswered.

1. Can any  $\phi \in \mathcal{L}$  have a finite first moment?
2. Can the class  $\mathcal{L}_0$  be imbedded in a large class?

All mixtures of symmetric stable distributions of index  $\alpha$ , with  $0 \leq \alpha \leq 1$ , are infinitely divisible (cf. Theorem 2.3). For  $\alpha = 2$  (mixtures of normal distributions) this is not true. Therefore the following question is of interest.

3. Is there any  $\alpha > 1$  such that all mixtures of ch.f.'s of the form  $e^{-\lambda|t|^\alpha}$  are infinitely divisible?



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