*The Annals of Mathematical Statistics*
1972, Vol. 43, No. 1, 230-235

# SOME PROBABILITY INEQUALITIES RELATED TO THE LAW OF LARGE NUMBERS[1]

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Let $S_1, S_2, \cdots, S_n$ be integrable random variables (rv). Upper bounds of the Hájek-Rényi type are presented for $P(\max_{1\leq k\leq n} \phi_k S_k \geqq \varepsilon \mid \mathscr{G})$ where $\phi_1 \geqq \cdots \geqq \phi_n > 0$ are rv, $\varepsilon > 0$ and $\mathscr{G}$ is a $\sigma$-field. The theorems place no further assumptions on the $S_k$'s; some, in fact, do not even require the integrability. It is shown, however, that if the $S_k$'s are partial sums of independent rv or if $S_1, S_2, \cdots, S_n$ forms a submartingale, then some well-known inequalities follow as consequences of these theorems.

**1. Introduction.** Let $S_1, S_2, \cdots, S_n$ be random variables (rv). Let $\phi_1, \phi_2, \cdots, \phi_n$ be rv such that $\phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_n > 0$. This paper is concerned with finding upper bounds for $P[\max_{1\leqq k\leqq n} \phi_k S_k \geqq \varepsilon]$, where $\varepsilon > 0$.

If $S_1, S_2, \cdots, S_n$ is a submartingale relative to the sigma-fields $\mathscr{F}_k$ generated by $S_1, S_2, \cdots, S_k$, $k \leqq n$ and $c_1 \geqq c_2 \cdots \geqq c_n > 0$ constants, then a theorem of Chow (1960) shows that, for any $\varepsilon > 0$,

$$P[\max_{1\leqq k\leqq n} c_k S_k \geqq \varepsilon] \leqq c_n ES_n^+ + \sum_{k=1}^{n-1} (c_k - c_{k+1}) ES_k^+ - c_n \int_{[\max_{k\leqq n} c_k S_k < \varepsilon]} S_n^+ .$$

This result generalizes the well-known theorem of Hájek and Rényi (1955) which assumes that $S_k$ is the sum of independent rv which are centered at expectations. (Professor Chow has subsequently observed that his inequality remains true when $c_k$ is replaced by a $\mathscr{F}_{k-1}$-measurable rv $\phi_k$ where $\phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_n > 0$ a.e.).

In Section 2 we will prove similar inequalities where only certain integrability requirements are placed on $S_1, S_2, \cdots, S_n$. The technique to be used in the proofs follows a pattern similar to the one used by Kounias and Weng (1969) whose inequalities will be generalized in Section 2 (see Corollary 3). Another approach to obtaining inequalities of the desired type is demonstrated by Csörgö (1968, Inequality3 ); one applies a martingale analog of the Hájek-Rényi inequality to the martingale $((S_j - E(S_j \mid \mathscr{F}_{j-1})), \mathscr{F}_j, j \leqq n)$. In Section 3 some inequalities for non-integrable rv will be presented.

**2. An inequality for integrable rv.** Assume that $(\Omega, \mathscr{F}, P)$ is a probability space. If $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_n \subset \mathscr{F}$ are sigma-fields and $X_k$ is an $\mathscr{F}_k$-measurable rv, we write $(X_k, \mathscr{F}_k, 1 \leqq k \leqq n)$. Let $I_A$ denote the indicator function of the event $A$. As usual, $X^+ = \max(X, 0)$.

Received December 22, 1970.

[1] This work was supported by the National Research Council of Canada.

The main result of the paper follows.

THEOREM 1. *Let* $S_1, S_2, \cdots, S_n$ *be any* rv. *For* $1 \leqq k \leqq n$, *let* $\phi_k$ *be a positive* rv, *and assume* $\phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_n$ a.e. *For any* $\varepsilon > 0$, *define the stopping rule*

$$\begin{aligned} t &= \inf k \leqq n \quad \text{such that} \quad \phi_k S_k \geqq \varepsilon \\ &= n+1 \quad \text{if} \quad \phi_i S_i < \varepsilon \quad \text{for all} \quad i \leqq n\,. \end{aligned}$$

*Moreover, let* $A = [\max_{1 \leqq k \leqq n} \phi_k S_k \geqq \varepsilon] = [t \leqq n]$, *and define*

$$\begin{aligned} Z = \phi_n S_n &+ \textstyle\sum_{k=1}^{n-1} (\phi_k - \phi_{k+1}) S_k^+ \\ &+ \textstyle\sum_{k=1}^{n-1} I_{[t=k]}(\phi_k S_k - \phi_n S_n - \sum_{k=1}^{n-1} (\phi_i - \phi_{i+1}) S_i^+)\,. \end{aligned}$$

*Then,*

(i) $\varepsilon I_A \leqq Z I_A$; *and*

(ii) *for any sigma-field* $\mathscr{G}$ *such that* $E(\phi_k S_k^+ / \mathscr{G}) < \infty$ a.e., $1 \leqq k \leqq n$,

$$\varepsilon P(A \mid \mathscr{G}) \leqq E(\phi_1 S_1^+ I_A \mid \mathscr{G}) + \textstyle\sum_{k=2}^{n} E(\phi_k (S_k^+ - S_{k-1}^+) I_{[k \leqq t \leqq n]} \mid \mathscr{G}) \text{ a.e.}$$

PROOF. Note that $Z = \phi_1 S_1 = \phi_1 S_1^+$ on $[t = 1]$ and, on $[t = j]$ where $1 < j \leqq n$,

$$Z = \phi_j S_j^+ + \textstyle\sum_{k=1}^{j-1} (\phi_k - \phi_{k+1}) S_k^+ = \phi_1 S_1^+ + \sum_{k=2}^{j} \phi_k (S_k^+ - S_{k-1}^+)\,.$$

More compactly,

$$Z I_A = \phi_1 S_1^+ I_A + \textstyle\sum_{k=2}^{n} \phi_k (S_k^+ - S_{k-1}^+) I_{[k \leqq t \leqq n]}\,. \tag{1}$$

Because of the monotonicity of the $\phi_k$'s, it follows that $Z \geqq \phi_j S_j^+ \geqq \varepsilon$ on the event $[t = j]$ if $j \leqq n$. Hence $Z \geqq \varepsilon$ on the event $A$; this implies (i).

Taking conditional expectations on both sides of (i) and using (1) we arrive at (ii), and the proof is complete.

Now let us investigate some consequences of Theorem 1.

COROLLARY 1. *Let* $S_k$, $\phi_k$, $t$ *and* $A$ *be as defined in Theorem* 1. *Suppose that* $S_k \geqq 0$ a.e. *and that* $E(\phi_k S_k)^r < \infty$ *for all* $1 \leqq k \leqq n$ *and some* $r > 0$. *Then, for any* $\varepsilon > 0$,

$$\begin{aligned} \varepsilon^r PA &\leqq E(\phi_1^r S_1^r I_A) + \textstyle\sum_{k=2}^{n} E\{\phi_k^r (S_k^r - S_{k-1}^r) I_{[k \leqq t \leqq n]}\} \\ &= E(\phi_n^r S_n^r I_A) + \textstyle\sum_{k=1}^{n-1} E\{(\phi_n^r - \phi_{k+1}^r) S_k^r\} \\ &\quad + \textstyle\sum_{k=1}^{n-1} E\{I_{[t=k]}(\phi_k^r S_k^r - \phi_n^r S_n^r - \sum_{i=k}^{n-1} (\phi_i^r - \phi_{i+1}^r) S_i^r\}\,. \end{aligned}$$

PROOF. Note that $\varepsilon^r PA = \varepsilon^r P[\max_{1 \leqq k \leqq n} \phi_k^r S_k^r \geqq \varepsilon^r]$. The result follows by applying Theorem 1 (ii) to the rv $\{S_k^r\}$ $1 \leqq k \leqq n$, with $\mathscr{G} = \{\phi, \Omega\}$.

Suppose that $\phi_k = 1$ a.e. $(1 \leqq k \leqq n)$ in Theorem 1. In this case it is evident from (1) that $Z = S_t = S_t^+$ on $[t \leqq n]$. Hence, in view of Theorem 1 (i), we have the following result.

COROLLARY 2. *Let* $S_1, S_2, \cdots, S_n$ *be integrable* rv. *For* $\varepsilon > 0$, *let* $t = \inf k \leqq n$

*such that* $S_k \geqq \varepsilon$, $= n+1$ *if* $S_i < \varepsilon$ *for all* $i \leqq n$. *Then, for any* $\sigma$*-field* $\mathscr{G}$,

$$\varepsilon P(\max_{1\leqq k\leqq n} S_k \geqq \varepsilon \mid \mathscr{G}) \leqq E(S_t I_{[t\leqq n]} \mid \mathscr{G}) \text{ a.e.}$$

*In particular,* $\varepsilon P[\max_{1\leqq k\leqq n} S_k \geqq \varepsilon] \leqq E(S_t I_{[t\leqq n]})$.

REMARK. *If* $(S_k, \mathscr{F}_k, 1 \leqq k \leqq n)$ is a submartingale, it is an easy matter to demonstrate that

$$ES_t I_{[t\leqq n]} \leqq ES_n I_{[t\leqq n]} .$$

Thus the well-known result of Doob (1953, page 314) is a special case ot Corollary 2.

COROLLARY 3. *Let* $S_1, S_2, \cdots, S_n$ *and* $\phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_n > 0$ *be* rv. *Suppose that, for some* $r > 0$ *and for some sigma-field* $\mathscr{G}$, $E(\phi_k^r(S_k^+)^r \mid \mathscr{G}) < \infty$ a.e. *for each* $k \leqq n$. *Define* $X_1 = S_1$ *and* $X_k = S_k - S_{k-1}$ *for* $1 < k \leqq n$. *Then, for any* $\varepsilon > 0$ *and any positive integer* $m < n$,

(i) *if* $r \leqq 1$,

$$\varepsilon^r P(\max_{m\leqq k\leqq n} \phi_k S_k \geqq \varepsilon \mid \mathscr{G}) \leqq E(\phi_m^r \textstyle\sum_{k=1}^m (X_k^+)^r \mid \mathscr{G}) + \sum_{k=m+1}^n E(\phi_k^r (X_k^+)^r \mid \mathscr{G}) ;$$

(ii) *if* $r \geqq 1$,

$$\varepsilon^r P(\max_{m\leqq k\leqq n} \phi_k S_k \geqq \varepsilon \mid \mathscr{G}) \leqq [\textstyle\sum_{k=1}^m E^{r^{-1}}(\phi_m^r (X_k^+)^r \mid \mathscr{G}) + \sum_{k=m+1}^n E^{r^{-1}}(\phi_k^r (X_k^+)^r \mid \mathscr{G})]^r .$$

PROOF. For brevity, define the events

$$A = [\max_{m\leqq k\leqq n} \phi_k S_k \geqq \varepsilon] = [\max_{m\leqq k\leqq n} \phi_k S_k^+ \geqq \varepsilon] , \qquad \text{and}$$
$$E_k = [\max_{m\leqq i\leqq k} \phi_i S_i < \varepsilon] \qquad \text{where} \quad m \leqq k \leqq n .$$

Then, by Theorem 1 (ii),

$$\varepsilon^r P(A \mid \mathscr{G}) \leqq E(\phi_m^r (S_m^+)^r \mid \mathscr{G}) + \textstyle\sum_{k=m+1}^n E(\phi_k^r \{(S_k^+)^r - (S_{k-1}^+)^r\} I_{AE_{k-1}} \mid \mathscr{G}) . \tag{2}$$

If $r \leqq 1$, then, for any $k \geqq 1$, $(S_k^+)^r \leqq (S_{k-1}^+)^r + (X_k^+)^r \leqq \sum_{j=1}^k (X_k^+)^r$ by the $C_r$-inequality so that (i) follows from (2). Now suppose $r \geqq 1$. Define $Y = \phi_m S_m^+ + \sum_{k=m+1}^n \phi_k (S_k^+ - S_{k-1}^+) I_{E_{k-1}}$.

By rearranging the terms it is easily shown that $Y \geqq 0$ and $Y \geqq \varepsilon$ on $A$. Furthermore, since $S_m^+ \leqq S_{m-1}^+ + X_m^+ \leqq \sum_{k=1}^m X_k^+$,

$$Y \leqq \phi_m \textstyle\sum_{k=1}^m X_k^+ + \sum_{k=m+1}^n \phi_k X_k^+ .$$

Hence

$$\varepsilon^r P(A \mid \mathscr{G}) \leqq E(Y^r \mid \mathscr{G}) ,$$
$$\leqq E((\phi_m \textstyle\sum_{k=1}^m X_k^+ + \sum_{k=m+1}^n \phi_k X_k^+)^r \mid \mathscr{G}) .$$

(ii) follows when Minkowski's inequality for conditional expectations (see Loève (1963, page 348)) is applied twice.

REMARK. Corollary 3 generalizes the results of Kounias and Weng, who proved these inequalities for constant rv's $\phi_k$, $\mathscr{G} = \{\phi, \Omega\}$ and the nonnegative stochastic process $\{|S_k|\}$.

This corollary also contains the Hájek-Rényi inequality (since that result is also a special case of the Kounias-Weng theorems).

COROLLARY 4. *Let $(S_k, \mathscr{F}_k, 1 \leqq k \leqq n)$ be a submartingale. Let $\phi_k$ be a positive $\mathscr{F}_{k-1}$-measurable* rv *with $\phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_n$, and $E(\phi_k S_k^+) < \infty$ $1 \leqq k \leqq n$. Let $A = [\max_{1 \leqq k \leqq n} \phi_k S_k \geqq \varepsilon]$ for $\varepsilon > 0$ and let $\mathscr{G} \subset \mathscr{F}_1$ be a $\sigma$-field. Then*

$$\varepsilon P(A \mid \mathscr{G}) \leqq E(\phi_n S_n^+ I_A \mid \mathscr{G}) + \sum_{k=1}^{n-1} E((\phi_k - \phi_{k+1}) S_k^+ I_A \mid \mathscr{G}) .$$

PROOF. Since $A = [\max_{1 \leqq k \leqq n} \phi_k S_k^+ \geqq \varepsilon]$ and $(S_k^+, \mathscr{F}_k, k \geqq 1)$ is a submartingale (see Loève (1963) page 391), there is no harm in assuming $S_k \geqq 0$ a.e. Now

$$\phi_k S_k - \phi_n S_n - \sum_{i=k}^{n-1} (\phi_i - \phi_{i+1}) S_i = -\sum_{i=k+1}^{n} \phi_i (S_i - S_{i+1}) .$$

But

$$E\left(\sum_{i=k+1}^{n} \phi_i (S_i - S_{i-1}) \mid \mathscr{F}_k\right) = E\left(\sum_{i=k+1}^{n} \phi_i E(S_i - S_{i-1} \mid \mathscr{F}_{i-1}) \mid \mathscr{F}_k\right)$$
$$\geqq 0 .$$

So, defining $t$ as in Theorem 1,

$$\begin{aligned} &E\left(\sum_{k=1}^{n-1} I_{[t=k]} (\phi_k S_k - \phi_n S_n - \sum_{i=k}^{n-1} (\phi_i - \phi_{i+1}) S_i) \mid \mathscr{G}\right) \\ &\quad = E\left(\sum_{k=1}^{n-1} I_{[t=k]} E(\phi_k S_k - \phi_n S_n - \sum_{k=1}^{n-1} (\phi_i - \phi_{i+1}) S_i \mid \mathscr{F}_k) \mid \mathscr{G}\right) \\ &\quad \leqq 0 . \end{aligned}$$

So the result follows from Theorem 1 (ii).

REMARK. Taking $\mathscr{G} = \{\phi, \Omega\}$ in Corollary 4 provides a slightly sharper inequality than that of Chow (1960). Chow's result has been proved independently by Csörgö (1968, Inequality 1), and using yet another approach, by Whittle (1969). Some inequalities of the same type for the submartingale case have been obtained by Burkholder (1966, Theorems 6 and 8).

**3. Inequalities for non-integrable** rv. The following theorem gives inequalities, analogous to those in Theorem 1, for rv which may not be integrable. The theorem is a modification and generalization of some work of Heyde (1968).

THEOREM 2. *Let $S_1, S_2, \cdots, S_n$ be any* rv. *Let $\phi_1, \phi_2, \cdots, \phi_n$ be* rv *such that $\phi_1 \geqq \cdots \geqq \phi_n > 0$. Assume $T_1, T_2, \cdots, T_n$ are any* rv *satisfying $E(\phi_k T_k^+)^r < \infty$ for some $r > 0$ and all $k \leqq n$. Define $X_1 = S_1$, $Y_1 = T_1$, and $X_k = S_k - S_{k-1}$, $Y_k = T_k - T_{k-1}$ for $1 < k \leqq n$. For any $\varepsilon > 0$ and $0 < \eta < 1$, let $I_k$ be the indicator function of the event*

$$[\max_{1 \leqq j < k} \phi_j T_j < \varepsilon(1 - \eta), \phi_k T_k \geqq \varepsilon(1 - \eta)] .$$

*Define the events* $A = [\max_{1 \leq k \leq n} \phi_k S_k \geqq \varepsilon]$,

$$B = [\max_{1 \leq k \leq n} \phi_k T_k \geqq \varepsilon(1-\eta)], \qquad \textit{and the } \mathrm{rv}$$

$$U = \phi_n{}^r T_n{}^r + \sum_{k=1}^{n-1} (\phi_k{}^r - \phi_{k+1}^r)(T_k{}^+)^r + \sum_{k=1}^{n-1} I_k(\phi_k{}^r T_k{}^r - \phi_n{}^r T_n{}^r - \sum_{i=k}^{n-1} (\phi_i{}^r - \phi_{i+1}^r)(T_i{}^+)^r) .$$

*Then*

$$I_A \leqq (1-\eta)^{-r}\varepsilon^{-r} U I_B + I_{[\phi_1(X_1 - Y_1) \geqq \eta\varepsilon]} + \sum_{k=2}^n I_{[X_k \neq Y_k]} .$$

*In particular*,

$$PA \leqq (1-\eta)^{-r}\varepsilon^{-r} E(UI_B) + P[\phi_1(X_1 - Y_1) \geqq \eta\varepsilon] + \sum_{k=2}^n P[X_k \neq Y_k] .$$

PROOF. Let $C = [\max_{k \leq n} \phi_k(S_k - T_k) \geqq \eta\varepsilon]$. By a method similar to that used in Theorem 1, it is easily shown that $U \geqq (1-\eta)^r\varepsilon^r$ on $B$. Also, it is easily proved that

$$C \subset [\phi_1(X_1 - Y_1) \geqq \eta\varepsilon] \cup \bigcup_{k=2}^n I_{[X_k \neq Y_k]} .$$

Since $A \subset B \cup C$, the theorem follows.

REMARK. Theorem 1 is contained in Theorem 2 since one can take $S_k = T_k$ and let $\eta \downarrow 0$. Furthermore, if $Y_1, Y_2, \cdots, Y_n$ are independent, $EY_n{}^2 < \infty$, and the $\phi_k$'s are constant, then the extension of the Hájek-Rényi inequality obtained by Tomkins (1971) results.

Among the consequences of Theorem 2 are inequalities analogous to Corollaries 1, 2, 3, and 4. In particular, the next two corollaries extend the results of Chow.

COROLLARY 5. *Let* $S_1, S_2, \cdots, S_n$ *be any* rv *and define* $X_1 = S_1$, $X_k = S_k - S_{k-1}$, $k > 1$. *Let* $(T_k = \sum_{j=1}^k Y_j, \mathscr{F}_k, 1 \leq k \leq n)$ *be a nonnegative submartingale. And let* $\phi_k$ *be a positive* $\mathscr{F}_{k-1}$*-measurable* rv $(1 \leqq k \leqq n, \mathscr{F}_0 = \{\phi, \Omega\})$ *such that* $\phi_1 \geqq \phi_2 \geqq \cdots \geqq \phi_n$, *and* $E(\phi_k{}^r T_k{}^r) < \infty$ *for some* $r \geqq 1$, *and all* $1 \geqq k \geqq n$. *Then for any* $\varepsilon > 0$ *and* $0 < \eta < 1$,

$$P[\max_{1 \leq k \leq n} \phi_k S_k \geqq \varepsilon] \leqq \varepsilon^{-r}(1-\eta)^{-r}\{E(\phi_n{}^r T_n{}^r) + \sum_{k=1}^{n-1} E((\phi_k{}^r - \phi_{k+1}^r)T_k{}^r) + P[\phi_1(X_1 - Y_1) \geqq \eta\varepsilon] + \sum_{k=2}^n P[X_k \neq Y_k] .$$

PROOF. Defining $U$ as in Theorem 2, it is easily shown that $U \geqq 0$ a.e. Then, using an argument similar to that in the proof of Corollary 4, one shows that the expection of the second summation in the definition of $U$ is nonpositive. (It should be noted that $(T_k{}^r, \mathscr{F}_k, 1 \leqq k \leqq n)$ is a submartingale for all $r \geqq 1$).

COROLLARY 6. *Let* $(S_k = \sum_{j=1}^k X_j, \mathscr{F}_k, k \geqq 1)$ *be any stochastic process. Let* $c_1, c_2, \cdots$ *be any non-increasing sequence of reals with* $\lim c_n = 0$. *Let* $(T_k = \sum_{j=1}^k Y_j, \mathscr{F}_k, k \geqq 1)$ *be any nonnegative submartingale such that* $ET_k{}^r < \infty$

$(k \geqq 1)$ *for some* $r \geqq 1$. *If* $\sum_{k=1}^{\infty} P[X_k \neq Y_k] < \infty$ *and* $\sum_{k=2}^{\infty} c_k{}^r E(T_k{}^r - T_{k-1}^r) < \infty$, *then* $\lim_{n\to\infty} c_n S_n = 0$ a.e.

PROOF. By Kronecker's lemma, $c_m{}^r ET_m{}^r \to 0$ as $m \to \infty$.

Also, by the Borel-Cantelli lemma, $P[X_k \neq Y_k \text{ i.o.}] = 0$, so

$$\lim_{m\to\infty} c_m(S_m - T_m) = 0 \text{ a.e.}$$

But $(T_k{}^r, \mathscr{F}_k, k \geqq 1)$ is submartingale, so by Corollary 5, for any $\varepsilon > 0$, $0 < \eta < 1$ and any integer $m \geqq 1$,

$$\begin{aligned} P[\sup_{k\geqq m} c_k S_k \geqq \varepsilon] &\leqq (1-\eta)^{-r}\varepsilon^{-r}[c_m{}^r ET_m{}^r + \textstyle\sum_{k=m+1}^{\infty} c_k{}^r E(T_k{}^r - T_{k-1}^r)] \\ &\quad + P[c_m(S_m - T_m) \geqq \eta\varepsilon] + \textstyle\sum_{k=m+1}^{\infty} P[X_k \neq Y_k] \\ &\to_{m\to\infty} 0 . \end{aligned}$$

**4. Acknowledgment.** The author is grateful to referee and the editor for their helpful suggestions.

## REFERENCES


[1] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[2] CHOW, Y. S. (1960). A martingale inequality and the law of large numbers. *Proc. Amer. Math. Soc.* **11** 107–111.

[3] CSÖRGÖ, MIKLÓS (1968). On the strong law of large numbers and the central limit theorem for Martingales. *Trans. Amer. Math. Soc.* **131** 259–275.

[4] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[5] HÁJEK, J. and RÉNYI, A. (1955). Generalization of an inequality of Kolmogorov. *Acta Math. Acad. Sci. Hungar.* **6** 281–283.

[6] HEYDE, C. C. (1968). An extension of the Hâjek-Rényi inequality for the case without moment Conditions. *J. Appl. Prob.* **5** 481–483.

[7] KOUNIAS, EUSTRATIOS G. and WENG, TENG-SHAW (1969). An inequality and almost sure convergence. *Ann. Math. Statist.* **40** 1091–1093.

[8] LOÈVE, M. (1963). *Probability Theory,* 3rd. ed. Van Nostrand, Princeton.

[9] TOMKINS, R. J. (1971). On an inequality of Heyde. *J. Appl. Prob.* **8** 428–429.

[10] WHITTLE, P. (1969). Refinements of Kolmogorov's inequality. *Theor. Probability. Appl.* **14** 310–311.