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A CONSISTENT ESTIMATION OF KERNEL FUNCTIONS IN THE MULTIPLE WIENER INTEGRALS

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1. Introduction. It is well known from the results of Itô [4] that any L^2-functional F of Brownian motion with $E[F] = 0$ has an orthogonal representation, i.e., $F = \sum_{p=1}^{\infty} I_p(f_p)$ where $I_p(f_p)$ is the pth degree multiple Wiener integral. The estimation of the kernel function f_p $(p = 1, 2, \cdots)$ is often required in solving various problems in nonlinear analysis and involves tremendous computations in the usual L^2-norm approximation. As a direct application of a result of Isaccson [3], we give here a consistent estimator for the kernel function f_p of L^2-functional of the form $F = \sum_{p=1}^{\infty} I_p(f_p)$.

2. Notations and preliminaries. Let $\{X(t, \omega)\}_{t \in [0,T]}$ be standard Brownian motion defined on a probability space $(\Omega, \mathscr{F}, P)$. Let B_t (for $t \in [0, T]$) denote the σ-field generated by sets of the form

$$E = \{\omega; [X(s_1, \omega), \cdots, X(s_n, \omega)] \in B^n\} \tag{2.1}$$

where $s_1, s_2, \cdots, s_n \in [0, t]$ and B^n is an n-dimensional Borel set. Let $\mathbf{B}_t$ denote the completion of B_t under P.

We shall write $L^2(\mathbf{B}_t)$ for $L^2(\Omega, \mathbf{B}_t, P)$, the Hilbert space of $\mathbf{B}_t$-measurable, real-valued functions square integrable with respect to P. We assume that $L^2(B_T)$ is separable. Let $L_t(X)$ denote the closed subspace of $L^2(\mathbf{B}_t)$ spanned by all finite linear combinations of the form $\sum_{i=1}^{n} c_i X(s_i, \omega)$ where the c_i's are real constants, and $s_1, s_2, \cdots, s_n \in [0, t]$.

We refer to Itô [4] for its definition and the various properties of the multiple Wiener integral;

$$I_p(f_p; t) = \int_0^t \cdots \int_0^t f_p(s_1, s_2, \cdots, s_p)\, dX(s_1)\, dX(s_2) \cdots dX(s_p) \tag{2.2}$$

for $f_p \in L^2([0, T]^p)$, where $L^2([0, T]^p)$ is the Hilbert space of Lebesgue square integrable functions on $[0, T]^p$. Denote $I_p(f_p) = I_p(f_p; T)$. The following results are due to Itô [4]. Any $F \in L^2(\mathbf{B}_T)$ can be expressed in the form:

$$F = \sum_{p=1}^{\infty} I_p(f_p) = \sum_{p=1}^{\infty} I_p(\tilde{f}_p)\,, \tag{2.3}$$

furthermore if

$$\sum_{p=1}^{\infty} I_p(f_p) = F = \sum_{p=1}^{\infty} I_p(g_p)\,, \tag{2.4}$$

then

$$\tilde{f}_p = \tilde{g}_p\,,$$

where $\tilde{f}_p(s_1, \cdots, s_p) = 1/(p!) \sum_{(\pi)} f_p(s_{\pi_1}, \cdots, s_{\pi_p})$, $(\pi) = (\pi_1, \cdots, \pi_p)$ running over all permutation of $(1, 2, \cdots, p)$.

When a sequence of random variables Y_n converges to a random variable Y in probability, we shall write $P \lim_{n\to\infty} Y_n = Y$.

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We quote the following definition and theorem from Isaacson [3]:

DEFINITION. A real-valued process $\phi(s, \omega)$ is in $M_1(X)$ if:

(i) $\phi(s, \omega)$ is adapted to $\{\mathbf{B}_s\}$
(ii) $\phi(s, \omega)$ is measurable on $([0, T] \times \Omega, \sigma([0, T]) \times \mathscr{F})$
(iii) $\int_0^t E[\phi(s, \omega)]^2 ds < \infty$ for all $t \in [0, T]$.

THEOREM [3]. *Let $\phi(s, \omega)$ be in $M_1(X)$ and $Y(t, \omega) = \int_0^t \phi(s, \omega)\, dX(s, \omega)$. Then*

$$P \lim_{\Delta t \to 0} \Delta Y(t, \omega)/\Delta X(t, \omega) = \phi(t, \omega) \qquad \text{a.a.} \quad t \in [0, T] \tag{2.5}$$

where

$$\Delta Y(t, \omega) = Y(t + \Delta t, \omega) - Y(t, \omega)$$
$$\Delta X(t, \omega) = X(t + \Delta t, \omega) - X(t, \omega) \qquad \textit{for} \quad \Delta t > 0 .$$

3. A consistent estimator. We shall give a sequence of lemmas before stating our theorem.

LEMMA 1. *Let $0 < s < t \leqq T$. Then*

$$E^s[I_p(f_p; t)] \equiv E[I_p(f_p; t) \mid \mathbf{B}_s] = I_p(f_p; s) \tag{3.1}$$

and

$$\begin{aligned} &E^s[\textstyle\int_0^t \cdots \int_0^t f_p(t_1, \cdots, t_{p-q}, s_{p-q+1}, \cdots, s_p)\, dX(s_{p-q+1}) \cdots dX(s_p)] \\ &\quad = \textstyle\int_0^s \cdots \int_0^s f_p(t_1, \cdots, t_{p-q}, s_{p-q+1}, \cdots, s_p)\, dX(s_{p-q+1}) \cdots dX(s_p) \end{aligned} \tag{3.2}$$

for each $(t_1, \cdots, t_{p-q}) \in [0, T]^{p-q}$.

PROOF. If f_p is a special elementary function (see [4]), this lemma is easily verified by the definition of the multiple Wiener integral. In the general case we can show it by approximating f_p with a special elementary function and making use of the properties of the multiple Wiener integral. This completes the proof.

For any $F \in L^2(\mathbf{B}_t)$, the projection of F into $L_t(X)$ is denoted by P_t, i.e., $P_t F = \mathrm{Proj}_{L_t(X)} F$. We consider P_t as the projection operator from $L^2(\mathbf{B}_t)$ to $L_t(X)$.

LEMMA 2. *Let $F \in L^2(\mathbf{B}_t)$ with $F = \sum_{p=1}^{\infty} I_p(f_p; t)$. Then*

$$P_t F = I_1(f_1; t) . \tag{3.3}$$

The lemma follows easily from the orthogonality of the multiple Wiener integrals of different degrees and the fact that the linear space $L_t(X)$ is characterized by the ordinary Wiener integrals, i.e., $L_t(X) = \{I_1(f); f \in L^2([0, t])\}$ (see [5]).

We remark that any Z in $L_t(X)$ can be expressed in terms of the complete orthonormal system in $L_t(X)$.

From [4], we can express $I_p(f_p; t)$ as:

$$I_p(f_p; t) = p \textstyle\int_0^t \phi(s_1, \omega)\, dX(s_1, \omega) , \tag{3.4}$$

where

$$\begin{aligned} \phi(s_1, \omega) &= (p-1)! \textstyle\int_0^{s_1} (\cdots [\int_0^{s_{p-1}} f_p(s_1, \cdots, s_p)\, dX(s_p)] \cdots)\, dX(s_2) \\ &= \textstyle\int_0^{s_1} \cdots \int_0^{s_1} f_p(s_1, s_2, \cdots, s_p)\, dX(s_2) \cdots dX(s_p) . \end{aligned} \tag{3.5}$$

Denote

$$D_t I_p(f_p; t) = P \lim_{\Delta t \to 0} \Delta I_p(f_p; t)/\Delta X(t, \omega) \tag{3.6}$$

where

$$\Delta I_p(f_p; t) = I_p(f_p; t + \Delta t) - I_p(f_p; t) .$$

LEMMA 3.

$$D_t I_p(f_p; t) = p \int_0^t \cdots \int_0^t f(t, s_2, \cdots, s_p)\, dX(s_2) \cdots dX(s_p) \tag{3.7}$$

for almost all $t \in [0, T]$, *and*

$$\begin{aligned} &D_t\{\int_0^t \cdots \int_0^t f_p(t_1, \cdots, t_{p-q}, s_{p-q+1}, \cdots, s_p)\, dX(s_{p-q+1}) \cdots dX(s_p)\} \\ &\quad = q \int_0^t \cdots \int_0^t f_p(t_1, \cdots, t_{p-q}, t, s_{p-q+2}, \cdots, s_p)\, dX(s_{p-q+2}) \cdots dX(s_p) \end{aligned} \tag{3.8}$$

for almost all $t \in [0, T]$ *and each* $(t_1, \cdots, t_{p-q}) \in [0, T]^{p-q}$.

PROOF. We first show (3.7), and (3.8) then follows similarly. We note that $\phi(s, \omega)$ given in (3.5) belong to $M_1(X)$, since

(i) $\phi(s, \omega)$ is adapted to $\{\mathbf{B}_s\}$ if f_p is a special elementary function and in the general case, approximating f_p by special elementary functions, we can have that $\phi(s, \omega)$ is adapted to $\{\mathbf{B}_s\}$.

(ii) $\sigma([0, T]) \times \mathscr{F}$-measurability of $\phi(s, \omega)$ follows easily from Theorem 2.1 of Doob [2] page 430, and

(iii) $\int_0^t E[\phi(s, \omega)]^2\, ds = \int_0^t \{E[\int_0^s \cdots \int_0^s f_p(s, s_2, \cdots, s_p)\, dX(s_2) \cdots dX(s_p)]^2\, ds \leqq (p-1)!/p \int_0^t \cdots \int_0^t \tilde{f}_p^2(s_1, \cdots, s_p)\, ds_1 \cdots ds_p < \infty$.

Now we obtain from Theorem [3] that

$$\begin{aligned} D_t I_p(f_p; t) &= D_t\{p \int_0^t \phi(s, \omega)\, dX(s, \omega)\} \\ &= p\phi(t, \omega) \\ &= p \int_0^t \cdots \int_0^t f_p(t, s_2, \cdots, s_p)\, dX(s_2) \cdots dX(s_p) \end{aligned} \tag{3.9}$$

for almost all $t \in [0, T]$. This completes the proof.

Since given $F = \sum I_p(f_p)$, it is only possible to estimate f_p uniquely up to a symmetric function according to (2.4), we assume that f_p $(p = 2, 3, \cdots, n)$ is a symmetric function, i.e., $f_p = \tilde{f}_p$ in the following theorem.

The product of two operators is understood as successive operation and the following notations are adopted:

$$\begin{aligned} G_{t_1}F &= D_{t_1}E^{t_1}(I - P_T)F \\ &= D_{t_1}E^{t_1}F - D_{t_1}E^{t_1}P_T F \\ G_{t_j} &= D_{t_j}E^{t_j}(I - P_{t_{j-1}})F \qquad \text{for } j = 2, 3, \cdots, p , \end{aligned} \tag{3.10}$$

where I stands for the identity operator,

$$G^*_{t_j}F = D_{t_j}E^{t_j}F \qquad \text{for } j = 1, 2, \cdots, p . \tag{3.11}$$

THEOREM 1. *Let* $F \in L^2(\mathbf{B}_T)$ *with* $E[F] = 0$ *and* $F = \sum_{p=1}^{\infty} I_p(f_p)$. *Then* $1/(p!) \times G^*_{t_p}P_{t_p}G_{t_{p-1}}G_{t_{p-2}} \cdots G_{t_2}G_{t_1}F$ *is a consistent estimator of* $f_p(t_1, \cdots, t_p)$, *for almost all* $(t_1, \cdots, t_p) \in [0, T]^p$ *for* $p = 2, 3, \cdots, n$.

PROOF. Let $0 < t_p < t_{p-1} < \cdots < t_2 < t_1 \leqq T$, and $F = \sum_{q=1}^{p} I_q(f_q) + F^*$, where $F^* = \sum_{q=p+1}^{\infty} I_q(f_q)$.

According to Clark [1] (Lemma 1, Theorem 2) and (3.4), we can write

$$F^* = \sum_{q=p+1}^{\infty} I_q(f_q) = \int_0^T \phi_1(s, \omega)\, dX(s, \omega)\,,$$

where

$$\phi_1(s, \omega) = \sum_{q=p+1}^{\infty} q \int_0^s \cdots \int_0^s f_q(s, s_2, \cdots, s_q)\, dX(s_2) \cdots dX(s_q)\,.$$

The above series converges in the sense of $L^2(\mathbf{B}_T)$ norm for a.a. $s \in [0, T]$. By the dominated convergence theorem,

$$\begin{aligned} E[\phi_1(s, \omega)]^2 &= \sum_{q=p+1}^{\infty} q^2(q-1)! \int_0^s \cdots \int_0^s f_q^2(s, s_2, \cdots, s_q)\, ds_2 \cdots ds_q \\ &= \sum_{q=p+1}^{\infty} (q!)^2 \int_0^s \left[\int^{s_2} \cdots \left[\int_0^{s_{q-1}} f_q^2(s, s_2, \cdots, s_q)\, ds_q\right] \cdots\right] ds_2 \end{aligned}$$

and

$$\begin{aligned} \int_0^T E[\phi_1(s, \omega)]^2\, ds &= \sum_{q=p+1}^{\infty} q! \int_0^T \cdots \int_0^T f_q^2(s_1, \cdots, s_q)\, ds_1 \cdots ds_q \\ &\leqq \|F\|^2_{L^2(\mathbf{B}_T)} < \infty\,. \end{aligned}$$

We can obtain that $\phi_1(s, \omega) \in M_1(X)$ and $\phi_1(s, \omega) \in L^2(\mathbf{B}_T)$ for a.a. $s \in [0, T]$. Now

$$P_T F = I_1(f_1) \quad \text{and} \quad (I - P_T)F = \sum_{q=2}^{p} I_q(f_q) + F^* \qquad \text{(by Lemma 2)}\,,$$

$$E^{t_1}(I - P_T)F = \sum_{q=2}^{p} I_q(f_q; t_1) + E^{t_1}F^* \qquad \text{(by Lemma 1)}\,.$$

and

$$D_{t_1} E^{t_1}(I - P_T)F \equiv G_{t_1}F = \sum_{q=1}^{p-1} (q+1) I_q(f_{q+1}; t_1) + G_{t_1}^* F^*$$
$$\text{for a.a.} \quad t_1 \in [0, T] \quad \text{(by Lemma 3)}\,.$$

By identifying $G_{t_1}^* F^* = \phi_1(t_1, \omega)$, we get $G_{t_1}F \in L^2(\mathbf{B}_T)$ for a.a. $t_1 \in [0, T]$. Therefore,

$$P_{t_1} G_{t_1} F = 2I_1(f_2; t_1)\,, \qquad \text{and}$$
$$(I - P_{t_1}) G_{t_1} F = \sum_{q=2}^{p-1} (q+1) I_q(f_{q+1}; t_1) + G_{t_1}^* F^*\,.$$

Again using the results [1] and (3.4), we get

$$G_{t_1}^* F^* = \int_0^{t_1} \phi_2(s, \omega)\, dX(s, \omega)\,,$$

where

$$\phi_2(s, \omega) = \sum_{q=p+1}^{\infty} q(q-1) \int_0^s \cdots \int_0^s f_q(t_1, s, s_3, \cdots, s_q)\, dX(s_3) \cdots dX(s_q)\,,$$

and $\phi_2(s, \omega)$ are in $M_1(X)$ and $L^2(\mathbf{B}_{t_1})$ for a.a. $(t_1, s) \in [0, T]^2$. Therefore,

$$G_{t_2} G_{t_1} F = \sum_{q=1}^{p-2} (q+2)(q+1) I_q^*(f_{q+2}; t_2) + G_{t_2}^* G_{t_1}^* F^*$$

for almost all $(t_1, t_2) \in [0, T]^2$, where

$$I_q^*(f_{q+2}; t_2) = \int_0^{t_2} \cdots \int_0^{t_2} f_{q+2}(t_1, t_2, s_3, \cdots, s_{q+2})\, dX(s_3) \cdots dX(s_{q+2})\,.$$

After repeating $(p-1)$ steps of the above operations, we obtain

$$G_{t_{p-1}} G_{t_{p-2}} \cdots G_{t_2} G_{t_1} F = p! \int_0^{t_{p-1}} f_p(t_1, t_2, \cdots, t_{p-1}, s_p)\, dX(s_p) + G_{t_{p-1}}^* \cdots G_{t_1}^* F^*\,,$$

where $G_{t_{p-1}}^* \cdots G_{t_1}^* F^*$ is the sum of multiple Wiener integrals of degree greater than 1. Hence

$$P_{t_{p-1}}G_{t_{p-1}} \cdots G_{t_1}F = p! \int_0^{t_{p-1}} f_p(t_1, \cdots, s_p)\, dX(s_p) ,$$
$$E^{t_p}P_{t_{p-1}}G_{t_{p-1}} \cdots G_{t_1}F = p! \int_0^{t_p} f_p(t_1, \cdots, t_{p-1}, s_p)\, dX(s_p) ,$$

and

$$D_{t_p}E^{t_p}P_{t_{p-1}}G_{t_{p-1}} \cdots G_{t_1}F = p!\, f_p(t_1, \cdots, t_p)$$

for almost all $(t_1, \cdots, t_p) \in [0, T]^p$. This completes the proof of the theorem.

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