The Annals of Mathematical Statistics
1972, Vol. 43, No. 4, 1363-1368

OPTIMAL STOPPING FOR PARTIAL SUMS

BY D. A. DARLING,[1] T. LIGGETT[2]
AND H. M. TAYLOR[3]

University of California, Irvine
University of California, Los Angeles
and Cornell University

We determine $\sup E[r(S_T)]$, where S_n is a sequence of partial sums of independent identically distributed random variables, for two reward functions: $r(x) = x^+$ and $r(x) = (e^x - 1)^+$. The supremum is taken over all stop rules T. We give conditions under which the optimal expected return is finite. Under these conditions, optimal stopping times exist, and we determine them.

The problem has an interpretation in an action timing problem in finance.

1. Introduction and summary. Let $X, X_1, X_2, \cdots$ be independent and identically distributed random variables having partial sums $S_n = X_1 + \cdots + X_n$, where $S_0 = 0$. Let T be a stopping time relative to (S_n), let β be a "discount factor" satisfying $0 \leqq \beta \leqq 1$, and let r be a nonnegative reward function. For two reward functions, we solve the problem of finding a stopping time T^* that maximizes the expected discounted reward

$$E[\beta^T r(x + S_T)] \,. \tag{1.1}$$

Dubins and Teicher [5] considered this problem with $r(z) = z$, assuming $\beta < 1$. Here we solve this problem for two other cases, using an entirely different approach. Our first case is $r(z) = z^+$, and $\beta = 1$ under the moment conditions $E[X] = -\mu < 0$ and $E[X^2] = \sigma^2 + \mu^2 < \infty$. We show the optimal rule is to stop at the first n, if any, for which $x + S_n \geqq E[M]$ where $M = \max\{S_0, S_1, \cdots\}$. Under this optimal stopping time T^*, the optimal expected reward is

$$E[(x + S_{T^*})^+] = E[(x + M - E[M])^+] \,. \tag{1.2}$$

Our second case has $r(z) = (e^z - 1)^+$ under the assumption $\beta E[e^X] < 1$. Here it is optimal to stop at the first n, if any, for which $x + S_n \geqq \ln E[e^M]$ and $M = \max\{S_j : 0 \leqq j < \tau\}$, τ being independent of $\{X_i\}$ and having the geometric distribution $P[\tau > k] = \beta^k$, $k \geqq 0$.

The optimal expected discounted return is

$$E[\beta^{T^*}(\exp\{x + S_{T^*}\} - 1)^+] = E[(e^{x+M} - E[e^M])^+]/E[e^M] \,.$$

We allow the possibility of never stopping and when this occurs, no reward

Received July 2, 1971; revised January 1972.

[1] Supported by Public Health Service Research Grant No. G.M. 10525(08) National Institutes of Health.

[2] Partially supported by United States Air Force Contract AF-AFOSR-69-1781.

[3] Partially supported by the National Science Foundation under Grant GK 21460.

is received. Thus we interpret $E[r(x + S_T)]$ as $\int_{T<\infty} r(x + S_T)\, dP$, where P is the basic underlying probability measure. This convention was introduced by Dynkin [6].

Since one would not wish to stop at a position having a cost, under the convention, the same stopping problem results if a possibly negative r is replaced by its positive part. Thus, it is no loss in generality to assume r nonnegative, which we have done to ensure the existence of the expectation in (1.1). (The alternative is to restrict attention to stopping times for which (1.1) is defined.)

A model in mathematical economics formulated by Samuelson [10] motivated one of us (Taylor) to study these optimal stopping problems. In these models, $x + S_n$ (respectively, $\exp\{x + S_n\}$ in our second case) represents the market price of an asset on day n. Consider an option whose owner has the privilege of purchasing this asset at a normalized price of zero (respectively, one) and then reselling at the market price. Our stopping problem corresponds to the owner's problem of choosing the optimal time to exercise his option. Note that in this context, never stopping has the natural interpretation of never exercising the option. To say that the daily price changes $X_1, X_2, \cdots$ are independent and identically distributed corresponds to the classical random walk model for market prices that was first suggested by Bachelier [1]. A collection of more recent studies on the random walk hypothesis is contained in Cootner [3].

2. Proof of optimality. The proofs in both cases result from the following lemma. A quite general lemma with a similar content appears in Dubins and Savage [4]. See Blackwell [2] for an early version.

LEMMA. *Let $X, X_1, X_2, \cdots$ be independent and identically distributed random variables, let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$. Let r and f be nonnegative functions and β a constant satisfying* $0 \leqq \beta \leqq 1$. *If for all x,*

$$f(x) \geqq r(x) \tag{2.1}$$

and

$$f(x) \geqq E[\beta f(x + X)]\,, \tag{2.2}$$

then

$$f(x) \geqq E[\beta^T r(x + S_T)] \tag{2.3}$$

for all x and stopping times T.

PROOF. Write $T(n) = \min\{T, n\}$, where T is an arbitrary stopping time. Inequality (2.2) implies that $(\beta^n f(x + S_n);\, n \geqq 0)$ is a supermartingale which leads to

$$\begin{aligned} f(x) &\geqq E[\beta^{T(n)} f(x + S_{T(n)})] \\ &\geqq E[\beta^{T(n)} r(x + S_{T(n)})] \\ &\geqq E[\beta^T r(x + S_T);\, T \leqq n]\,, \end{aligned}$$

where for any event A, $E[\cdot\,; A] = \int_A \cdot\, dP$. The last two inequalities result from

(2.1) and the nonnegativity of r, respectively. Finally by monotone convergence, the last expectation increases to $E[\beta^T r(x+S_T)]$ as n increases indefinitely, which concludes the proof. □

We apply this to the arithmetic problem, using for f the function

$$f(x) = E[(x + M - E[M])^+] .$$

Kiefer and Wolfowitz [8] have shown that $E[M] < \infty$ whenever $E[X^2] < \infty$ so that f is well defined. Use Jensen's inequality to see that

$$f(x) = E[(x + M - E[M])^+] \geqq x^+ = r(x) .$$

Since M has the same distribution as $(X+M)^+$ we have

$$\begin{aligned} f(x) &= E[(x + M - E[M])^+] \\ &= E[(x + (X+M)^+ - E[M])^+] \\ &\geqq E[(x + X + M - E[M])^+] \\ &= E\{E[(x + X + M - E[M])^+ \mid X]\} \\ &= E[f(x+X)] . \end{aligned}$$

It now follows from our lemma that

$$f(x) \geqq E[(x+S_T)^+] \tag{2.4}$$

for all stopping times T. To complete the proof, we need only show that

$$T^* = \min\{n \geqq 0 : x + S_n \geqq E[M]\}$$

achieves equality in (2.4). But

$$\begin{aligned} E[x + M; T^* < \infty] &= E[M - S_{T^*} + x + S_{T^*}; T^* < \infty] \\ &= E[M - S_{T^*}; T^* < \infty] + E[x + S_{T^*}; T^* < \infty] . \end{aligned}$$

Since the process is spatially homogeneous and strong Markov,

$$E[M - S_{T^*}; T^* < \infty] = E[M]P[T^* < \infty] .$$

Thus

$$\begin{aligned} E[x + S_{T^*}; T^* < \infty] &= E[x + M; T^* < \infty] - E[M]P[T^* < \infty] \\ &= E[(x + M - E[M])^+] . \end{aligned}$$

Since T^* achieves an upper bound on expected rewards, T^* is optimal.

It is of interest to compare the optimal expected reward in this case with the best one could do if one could "look into the future" and observe the whole sequence (S_n) before deciding when to stop. With perfect information about the future, one would of course choose to stop at that point at which the maximum M is attained. So, for example, if $x = 0$ the optimal expected return would be $E[M]$. The somewhat surprising result is that

$$e^{-1} \leqq \frac{E[(M - E[M])^+]}{E[M]} \leqq 1 \tag{2.5}$$

where the bounds are independent of the distribution of X. (We exclude the case $M \equiv 0$, which is of no interest in this context.) Furthermore, both bounds are best possible.

Since $M \geqq 0$, the upper bound is clear. To obtain the more interesting lower bound, we introduce the variables $T_0 \equiv 0$ and

$$T_{k+1} = \min\{n > T_k : S_n > S_{T_k}\}$$

where the minimum of an empty set is taken to be infinity. Let

$$N = \max\{k : T_k < \infty\} .$$

Then N has a geometric distribution with parameter

$$\alpha = P[N = 0] = P[T_1 = \infty] .$$

Note that $\alpha > 0$ since $S_n \to -\infty$ and that $\alpha < 1$ since $M \not\equiv 0$. M can then be written in the form

$$M = \sum_{k=1}^{N} (S_{T_k} - S_{T_{k-1}}) .$$

Using the strong Markov property of the process (S_n), we conclude that

$$E[M \mid N] = CN$$

for some constant C. Taking the expected value of both sides of this relation yields

$$C = E[M]/E[N] .$$

Now we use Jensen's inequality for conditional expectations to obtain

$$\begin{aligned} E[(M - E[M])^+] &= E[E\{(M - E[M])^+ \mid N\}] \\ &\geqq E[E[M - E[M] \mid N]]^+ \\ &= E\left[N \frac{E[M]}{E[N]} - E[M]\right]^+ . \end{aligned}$$

So,

$$\frac{E[(M - E[M])^+]}{E[M]} \geqq \frac{E[(N - E[N])^+]}{E[N]} .$$

It is interesting to note that $N = M$ if X takes on only the values $+1$ and -1, and in this case, $\alpha = 1 - P[X = 1]/P[X = -1]$. Thus we have reduced the proof of the lower bound in (2.5) to this particularly simple case, where explicit computations are possible.

Carrying out these computations, we see that since N is geometrically distributed with parameter α,

$$R(x) = E[(N - x)^+]/E[N]$$

is obtained for $x \geqq 0$ by linearly interpolating the values

$$R(n) = (1 - \alpha)^n$$

for nonnegative integers n. Since $E[N] = (1 - \alpha)/\alpha$, $E[(N - E[N])^+]/E[N]$ is a monotone increasing function of α which approaches 1 as $\alpha \uparrow 1$ and e^{-1} as

$\alpha \downarrow 0$. This establishes the lower bound in (2.5) and shows that both bounds are tight.

For the geometric stock option problem we use for f the function

$$f(x) = E[(e^{x+M} - E[e^M])^+]/E[e^M]$$

where now $M = \max\{S_j;\, 0 \leqq j < \tau\}$ and τ, independent of $\{X_i\}$, has the geometric distribution $P[\tau > k] = \beta^k$, for $k = 0, 1, \cdots$. (Note that this is consistent with our previous definition for M, provided we interpret $\beta = 1$ as meaning $\tau = \infty$.) To show that f is well defined we need to show that $E[e^M] < \infty$, which the following theorem, of independent interest, does.

THEOREM. *Let $T = \min\{n : S_n > 0\}$. The following are equivalent:*

(a) $\beta E[e^X] < 1$,
(b) $E[\beta^T \exp\{S_T\}] < 1$,
(c) $E[e^M] < \infty$.

PROOF. (b) $\Leftrightarrow$ (c): $E[e^M] = \sum_{n=0}^{\infty} \alpha(1-\alpha)^n\{E[\exp\{S_T\} \mid T < \tau]\}^n$ where $\alpha = P[T \geqq \tau]$. So $E[e^M] < \infty$ if and only if $(1-\alpha)E[\exp\{S_T\} \mid T < \tau] < 1$. But $(1-\alpha)E[\exp\{S_T\} \mid T < \tau] = E[\beta^T \exp\{S_T\}]$.

(c) $\Rightarrow$ (a): $\{\beta E[e^X]\}^n = E[\exp\{S_n\};\, \tau > n] \leqq E[e^M] < \infty$ for all n. Hence $\beta E[e^X] \leqq 1$. Suppose $\beta E[e^X] = 1$. Then $\beta^n \exp\{S_n\}$ forms a uniformly integrable martingale which converges to zero on the event $\{T = \infty\}$. Hence $E[\beta^T \exp\{S_T\}] = 1$. Since (b) is equivalent to (c) this implies $E[e^M] = \infty$, a contradiction. Hence $\beta E[e^X] < 1$.

(a) $\Rightarrow$ (b): $\{E[e^X]\}^{-n} \exp\{S_n\}$ is a nonnegative martingale. By Fatou's Lemma

$$E[\{E[e^X]\}^{-T} \exp\{S_T\}] \leqq 1 .$$

Since $T \geqq 1$, from (a) $\beta^T E[e^X]^T \leqq \beta E[e^X]$ and

$$\{\beta E[e^X]\}^{-1} E[\beta^T \exp\{S_T\}] \leqq E[\{\beta E[e^X]\}^{-T} \beta^T \exp\{S_T\}] \leqq 1$$

or

$$E[\beta^T \exp\{S_T\}] \leqq \beta E[e^X] < 1$$

which concludes the proof. □

Again, using Jensen's inequality, it is easily seen that $f(x) \geqq r(x) = (e^x - 1)^+$ for all x. Also M has the same distribution as $(X + IM)^+$ where $I = 0$ with probability β and $I = 1$ with probability $1 - \beta$, and is independent of $X, X_1, X_2, \cdots$. Hence,

$$\begin{aligned} f(x) &= E[(\exp(x + X + IM)^+ - E[e^M])^+]/E[e^M] \\ &\geqq E\{E[\exp(x + X + IM) - E[e^M]]^+ \mid X\}/E[e^M] \\ &\geqq \beta E[f(x + X)] . \end{aligned}$$

We conclude

$$f(x) \geqq E[\beta^T(\exp\{x + S_T\} - 1)^+]$$

for all stopping times, T.

To complete our argument we will show

$$f(x) = E[\beta^{T^*}(\exp\{x + S_{T^*}\} - 1)^+]$$

for $T^* = \min\{n \geqq 0 : x + S_n \geqq A\}$ where $A = \ln E[e^M]$. Again

$$\begin{aligned} E[e^{x+M}; x + M \geqq A] &= E[\exp\{M - S_{T^*}\}\exp\{x + S_{T^*}\}; x + M \geqq A] \\ &= E[e^M] \times E[\exp\{x + S_{T^*}\}; x + M \geqq A]\,. \end{aligned}$$

Hence

$$\begin{aligned} &E[(e^{x+M} - E[e^M]); x + M \geqq A] \\ &\quad = E[e^M] \times E[(\exp\{x + S_{T^*}\} - 1); x + M \geqq A]\,, \end{aligned}$$

and

$$E[(\exp\{x + S_{T^*}\} - 1); x + M \geqq A] = E[(e^{x+M} - E[e^M])^+]/E[e^M]\,.$$

To complete the proof note

$$\begin{aligned} E[(\exp\{x + S_{T^*}\} - 1); x + M \geqq A] &= E[(\exp\{x + S_{T^*}\} - 1); T^* > \tau] \\ &= E[\beta^{T^*}(\exp\{x + S_{T^*}\} - 1)]\,. \end{aligned}$$

Using the theorem on page 573 of Feller [7] it is easy to see

$$\begin{aligned} A = \ln E[e^M] &= \ln (1 - \beta) \textstyle\sum_0^\infty \beta^n E[\exp\{M_n\}] \\ &= \textstyle\sum_1^\infty \frac{\beta^n}{n} E[\exp\{S_n^+\} - 1]\,. \end{aligned}$$

REFERENCES

[1] BACHELIER, L. (1900). Théorie de la spéculation. *Annals de l'École Normale Supérieure, Ser. 3* **17** 21–86; [3] is English translation.

[2] BLACKWELL, D. (1954). On optimal systems. *Ann. Math. Statist.* **25** 394–397.

[3] COOTNER, P. H., ed. (1964). *The Random Character of Stock Market Prices.* M.I.T. Press.

[4] DUBINS, L. E. and SAVAGE, L. J. (1965). *How to Gamble if You Must.* McGraw-Hill, New York.

[5] DUBINS, L. E. and TEICHER, H. (1967). Optimal stopping when the future is discounted. *Ann. Math. Statist.* **38** 601–605.

[6] DYNKIN, E. B. (1963). Optimal selection of a stopping time for a Markov process. *Dokl. Akad. Nauk USSR* **150** 238–240; (English trans. *Soviet Math.* **4** 627–629).

[7] FELLER, W. (1966). *An Introduction to Probability Theory and its Applications* **2**. Wiley, New York.

[8] KIEFER, J. and WOLFOWITZ, J. (1956). On the characteristics of the general queuing process, with application to random walk. *Ann. Math. Statist.* **27** 147–161.

[9] SAMUEL, E. (1967). On the existence of the expectation of randomly stopped sums. *J. Appl. Probability* **4** 197–200.

[10] SAMUELSON, P. A. (with an Appendix by H. P. McKean) (1965). Rational theory of warrant pricing. *Indust. Manag. Review* **6** 13–31.