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PROBABILISTIC ANALYSIS OF A CAPACITATED VEHICLE ROUTING PROBLEM II[1]

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A fleet of vehicles located at a common depot must serve customers located throughout the plane. Without loss of generality, the depot will be located at the origin. Each vehicle must start at the depot, travel in turn to each customer its serves and go back to the depot. Each vehicle can serve at most k customers. The objective is to minimize the total distance traveled by the fleet. In our model, the customers $X_1, \dots, X_n$ are independent and uniformly distributed over the unit disc. If $R'(X_1, \dots, X_n)$ denotes the optimal solution with these customer locations, we show that with overwhelming probability we have

$$\left| R'(X_1, \dots, X_n) - \frac{2}{k} \sum_{i \le n} \|X_i\| - \xi \sqrt{n} \right| \le K(n \log n)^{1/3},$$

where ξ and K are constants that depend on k only.

1. Introduction. A general routing problem is as follows. A fleet of vehicles, starting at a common depot, must serve a set of customers with demands. Without loss of generality, the depot will be located at the origin. Each vehicle has unit capacity, and the sum of the demands of the customers it will serve must not exceed 1. The vehicles must start at the depot, travel to the customers they serve and go back to the depot. The objective is to minimize the cost (= total distance traveled). In a previous model [Rhee (1993a)], we studied the situation where both customers' locations and demands are random, with special focus on the case where demand is uniform over $[0, 1]$. In the present paper, we focus on the case where the demands are fixed and equal to $1/k$ ($k \ge 2$). (This integer k is fixed once and for all.) This leads to dramatically different results. Throughout the paper, $R'(x_1, \dots, x_n)$ will denote the optimal solution of the routing problem with customer locations $x_1, \dots, x_n$ (each of them having demand $1/k$). We denote by $\|x\|$ the distance from x to the origin. Our main result is as follows.

THEOREM 1.1. *There exist constants ξ, K depending on k only with the following property. Consider r.v.'s $X_1, \dots, X_n$ that are independent and uni-*

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formly distributed over the unit disc. Then

$$P\left(\left|R'(X_1,\dots,X_n) - \frac{2}{k}\sum_{i\le n}\|X_i\| - \xi\sqrt{n}\right| \ge K(n\log n)^{1/3}\right) \le \exp\left(-\frac{n^{1/3}}{K(\log n)^{2/3}}\right). \tag{1.1}$$

It must be pointed out that this result captures second-order effects. The term $\sum_{i\le n}\|X_i\|$ is asymptotically $N(na, b\sqrt{n})$ for some constants a, b (by the central limit theorem).

Thus

$$\frac{R'(X_1,\dots,X_n) - na}{\sqrt{n}} \sim \xi + Y, \tag{1.2}$$

where Y is asymptotically $N(0, b)$ and where $\sim$ means up to effects of order $n^{-1/6}(\log n)^{1/3}$. It would be of interest to determine the value of ξ. The proof will show that $\xi = \pi\alpha$, where α is a constant related to a simpler problem. In particular, when $k = 2$, α is the constant that arises in the "minimum matching problem," where one computes the minimum possible sum of the distances between a pair of points when the points—except one when n is odd—are matched in pairs. The factor π simply arises as the area of the unit disc.

There is nothing magical about the term $(n \log n)^{1/3}$. This is simply an artifact of the method of proof. It seems actually perfectly conceivable that

$$R'(X_1,\dots,X_n) - \frac{2}{k}\sum_{i\le n}\|X_i\| - \xi\sqrt{n}$$

is of order 1 with probability close to 1. Proving such a result seems, however, to be quite beyond the range of present day techniques.

We would also like to mention that we have chosen the hypothesis that $X_1,\dots,X_n$ are distributed uniformly in the unit disc for symmetry reasons. Standard modifications of our proof would show that (1.2) still holds up to effects of $o(1)$ for considerably more general distributions.

We now discuss the methods of the paper. A first obvious idea is that the main contribution to the total distance traveled is the "radical collection cost" $(2/k)\sum_{i\le n}\|x_i\|$. Thus, it is natural to consider the functional

$$S'(x_1,\dots,x_n) = R'(x_1,\dots,x_n) - \frac{2}{k}\sum_{i\le n}\|x_i\|.$$

Observe that S' depends only on the points, not on their order.

The quantity $S'(X_1,\dots,X_n)$ will be studied via subadditivity methods. There are, however, two reasons that this study is not routine and does not fit in the scheme of Steele (1991). The first is that it is not monotonic; that is, that adding one customer may decrease the value of S' rather than increase it. The second is that, since the definition of S' singles out the depot, the functional S' is neither translation invariant nor does it satisfy the homogeneity conditions of Steele (1991). There are two main tools for the proof. The first tool is an inequality that shows that the value of S' (or, more accurately, of a slightly different functional) on a set F is nearly equal to the sum of the values on the subsets of certain partitions of F. This tool serves as a substitute for subadditivity. The proof of this inequality is done in Section 2. The second tool is the introduction of a functional S_∞ that expresses what would happen (after renormalization) if the depot were located "at infinity." There are two ideas in doing this. First, S_∞ will well approximate S' when the diameter of the set of customers is small compared to their distance to the origin. Second, the new functional no longer singles out the depot, so it is translation invariant and satisfies homogeneity conditions that make it amenable to the usual subadditivity methods. Developing the properties of S_∞ is the purpose of Section 3. Finally the various tools come together in Section 4 to prove Theorem 1.1. In the main argument, the unit disc is partitioned in little pieces A_j that resemble squares. The inequality of Section 2 shows that the cost $S'(X_1, \dots, X_n)$ of the total set of customers is nearly the sum of the costs corresponding to the little pieces A_j, and for each such piece, the cost can be well approximated using S_∞. Moreover, this shows that $S'(X_1, \dots, X_n)$ is nearly equal to a sum of independent random variables, and the concentration of $S'(X_1, \dots, X_n)$ around its mean follows. (It should be pointed out that changing one single customer location could conceivably create a big variation in S', so martingales do not seem to be of use here.) This approach to concentration, that is apparently new, has already been applied with great success to other problems [Rhee (1993c)].

2. Inequalities. While the overall plan of attack is rather simple, its implementation runs into a number of technical difficulties. These are genuinely low order hurdles, which nonetheless must be addressed in order to make the proofs valid. One of these problems is better faced from the beginning, since delaying would only force us to repeat steps. We have found that is is more convenient to study a slightly different optimization problem first. We have no intuitive explanation to offer as to why the change is beneficial; only trial and error convinced us that the $\varepsilon - \delta$ proofs are better written with the new formulation. In this new formulation, we require that there be at most one vehicle that serves less than k customers. The cost of a schedule is then computed as the sum of the distances traveled by the vehicles that serve k customers, plus pk^{-1} times the distance traveled by the one vehicle that may serve $p \le k$ customers. We denote by $R(x_1, \dots, x_n)$ the minimum possible cost (in the reformulated problem) to serve $x_1, \dots, x_n$, and we set

$$S(x_1, \dots, x_n) = R(x_1, \dots, x_n) - \frac{2}{k} \sum_{i \le n} \|x_i\|.$$

This might be the place to point out that for $k = 2$, $S(x_1, \dots, x_n)$ is exactly the cost of the minimal matching of $x_1, \dots, x_n$. In that case, the proof of Theorem 1.1 can be considerably simplified, and much sharper results are available [Rhee (1993b)].

LEMMA 2.1. *Assume that a vehicle serves customers* $x_1, \dots, x_p$ *(in that order). Set*

$$a = \sum_{1 \le i < p} \|x_{i+1} - x_i\| \ge \max_{i, j \le p} \|x_i - x_j\|.$$

Then the distance R *traveled by this vehicle satisfies*

$$R \ge \frac{2}{p} \sum_{1 \le i \le p} \|x_i\| + \frac{2a}{p} \ge \frac{2}{p} \sum_{i \le p} \|x_i\|. \tag{2.1}$$

PROOF. We observe that, for all $1 < j < p$, we have

$$\|x_j\| \le \|x_1\| + \sum_{1 \le i \le j-1} \|x_{i+1} - x_i\|,$$

$$\|x_j\| \le \|x_p\| + \sum_{j \le i < p} \|x_i - x_{i+1}\|$$

so that

$$2\|x_j\| \le \|x_1\| + \|x_p\| + a = R$$

so that

$$2 \sum_{1 \le j \le p} \|x_j\| \le p\|x_1\| + p\|x_p\| + (p - 2)a$$

$$= pR - 2a.$$

We should also point out that the weaker inequality $R \ge (2/p)\sum_{i \le p}\|x_i\|$ is obvious, since $R \ge 2\max_{i \le p}\|x_i\|$. □

Throughout the paper, $T(x_1, \dots, x_n)$ will denote the length of the shortest tour through points $x_1, \dots, x_n$ of the plane.

For simplicity, we will say that a vehicle is *complete* if it serves k customers, and *incomplete* otherwise. Beside the minor restriction that there be at most one incomplete vehicle, (a sharper form of) the following lemma is proved as Theorem 2.2 of Haimovich, Rinnooy Kan and Stougie (1988). The simple argument is reproduced for the convenience of the reader.

LEMMA 2.2. *Consider points* $x_1, \dots, x_n$. *Then*

$$0 \le S(x_1, \dots, x_n) \le 3T(x_1, \dots, x_n). \tag{2.2}$$

PROOF. (a) By (2.1), if a vehicle serves points $y_1, \dots, y_p$, the distance it travels is at least $(2/p)\sum_{l \le p}\|y_l\|$. Thus the contribution of the vehicle that serves them has a total cost of at least $(2/k)\sum_{l \le p}\|y_l\|$. The left side inequality of (2.2) follows by summation over all vehicles.

(b) We can assume that $x_1, \dots, x_n$ are numbered in such a way that visiting them in that order produces a shortest tour. Consider the following strategy. For $(k+1)p \le n$, the pth vehicle serves $x_{kp+1}, \dots, x_{(k+1)p}$ in that order, for a total travel distance of

$$\|x_{kp+1}\| + \|x_{(k+1)p}\| + U_p,$$

where

$$U_p = \sum_{kp+1 \le i < (k+1)p} \|x_{i+1} - x_i\|.$$

We observe that, obviously, for $kp + 1 \le i, j \le (k+1)p$ we have

$$\|x_i\| \ge \|x_j\| - U_p$$

so that

$$\|x_{kp+1}\| + \|x_{(k+1)p}\| \le \frac{2}{k} \sum_{kp+1 \le i \le (k+1)p} \|x_i\| + 2U_p.$$

Hence the pth vehicle travels a distance at most

$$\frac{2}{k} \sum_{kp+1 \le i \le (k+1)p} \|x_i\| + 3U_p.$$

When n/k is not an integer, the last vehicle will serve items $x_{qk+1}, \dots, x_n$, where $q = [n/k]$. The preceding argument shows that it will travel a distance

$$\frac{2}{n - qk} \sum_{qk < i \le n} \|x_i\| + 3 \sum_{qk < i < n} \|x_{i+1} - x_i\|.$$

After weighting this last contribution by $(n - qk)/k$ and summing the corresponding contributions, we obtain the result. □

We will often combine (2.2) with the fact [Steele (1990)] that for a subset G of the plane, of diameter D, the length T of the shortest tour through G satisfies

$$T \le KD\sqrt{\operatorname{card} G}. \tag{2.3}$$

For a subset F of the plane, we use the notation $R(F) = R(x_1, \dots, x_n)$ if $F = \{x_1, \dots, x_n\}$, and a similar notation for S. Throughout the paper, K will denote a constant depending on k only, not necessarily the same at each occurrence.

PROPOSITION 2.3. *Consider two finite subsets F, G of the plane, and let D be the diameter of $F \cup G$, that is,*

$$D = \sup\{\|x - y\|;\, x, y \in F \cup G\}.$$

Then

$$|S(F \cup G) - S(F)| \le KD\sqrt{\operatorname{card} G}. \tag{2.4}$$

COMMENT. If we change just one customer, conceivably the change of value of $S(F)$ can be as large as KD. Iterating this estimate shows that changing p customers cannot create a variation of $S(F)$ larger than KDp. Proposition 2.3 makes a crucial improvement of this trivial estimate, replacing p by $\sqrt{p}$.

PROOF. We observe first that there is no loss of generality to assume that F and G are disjoint.

Step 1. We bound $R(F \cup G)$ knowing $R(F)$. Consider an optimal solution to the problem of serving the customers in F. At most one vehicle is incomplete; we denote by $y_1, \dots, y_r$ the customers it serves. As is shown by Lemma 2.2, the contribution of this vehicle to $R(F)$ is at least $(2/k)\sum_{i \le r}\|y_i\|$. Thus, the total length traveled by the other vehicles is at most $R(F) - (2/k)\sum_{i \le r}\|y_i\|$. This means that we can serve the customers of $F \setminus \{y_1, \dots, y_r\}$ with vehicles that are complete and that travel at most a total of $R(F) - (2/k)\sum_{i \le r}\|y_i\|$. Now, by Lemma 2.2, the customers of $G \cup \{y_1, \dots, y_r\}$ can be served with vehicles for a total travel distance (in such a way that at most one vehicle is incomplete)

$$R(G \cup \{y_1, \dots, y_r\}) \le \frac{2}{k} \sum_{i \le r} \|y_i\| + \frac{2}{k} \sum_{z \in G} \|z\| + 3T,$$

where T is the length of a tour through $G \cup \{y_1, \dots, y_r\}$.

From (2.3), we have

$$T \le KD\Big(\operatorname{card}(G \cup \{y_1, \dots, y_r\})^{1/2}\Big).$$

Thus, we have shown how to serve the customers of $F \cup G$ with a total cost at most

$$\left(R(F) - \frac{2}{k} \sum_{i \le r} \|y_i\|\right) + \left(\frac{2}{k} \sum_{i \le r} \|y_i\| + \frac{2}{k} \sum_{z \in G} \|z\| + KD\sqrt{k + \operatorname{card} G}\right),$$

with at most one incomplete vehicle. Thus, since we can assume $\operatorname{card} G \ge 1$,

$$\begin{aligned} R(F \cup G) &\le R(F) + \frac{2}{k} \sum_{z \in G} \|z\| + KD\sqrt{k + \operatorname{card} G} \\ &\le R(F) + \frac{2}{k} \sum_{z \in G} \|z\| + KD\sqrt{\operatorname{card} G}. \end{aligned}$$

COMMENT. The reader has observed that some effort is devoted to ensure that the routing constructed has at most one incomplete vehicle. This feature will be common in all the proofs of this section.

Step 2. We bound $R(F)$ knowing $R(F \cup G)$. Consider an optimal tour through $F \cup G$. There is at most one incomplete vehicle. Consider the class V of vehicles that consists of this vehicle, as well as of all the vehicles that serve at least one customer of G. Then V contains at most $1 + \operatorname{card} G$ vehicles.

Consider the set F' of customers of F that are served by a vehicle of V. Thus, V serves the customers of $F' \cup G$. It follows from Lemma 2.1 that the contribution of the vehicles of V to $R(F \cup G)$ is at least $(2/k)\sum_{x \in F' \cup G}\|x\|$. Thus, we can serve the customers in $(F \cup G) \setminus (F' \cup G) = F \setminus F'$ with a cost of at most

$$R(F \cup G) - \frac{2}{k} \sum_{x \in F' \cup G} \|x\|$$

and with vehicles that are all full. Now, by Lemma 2.2 and (2.3) we have

$$R(F') \leq \frac{2}{k} \sum_{x \in F'} \|x\| + K\sqrt{\operatorname{card} F'}\,.$$

Thus

$$R(F) \leq R(F \cup G) - \frac{2}{k} \sum_{x \in G} \|x\| + K\sqrt{\operatorname{card} F'}\,.$$

This completes the proof since $\operatorname{card} F' \leq k(1 + \operatorname{card} G) \leq K \operatorname{card} G$. □

We come now to the cornerstone inequality. This inequality relates the cost of serving customers in a domain with the sum of the costs of serving customers in "regular" subdomains. The notion of regularity we need is formalized as follows.

DEFINITION 2.4. For a number $L > 0$, we say that a domain A of the plane has property $H(L)$ if the following occurs. Consider any number $\beta > 0$ and consider the domain

$$I_\beta(A) = \{x \in A; \exists\, y \notin A, d(x, y) \leq \beta\}.$$

Consider n points $x_1, \dots, x_n$ in $I_\beta(A)$. Then

$$T(x_1, \dots, x_n) \leq L + \sqrt{L\beta n}\,. \tag{2.5}$$

COMMENTS. (1) The use of this definition is that a square A (and a moderate distortion of a square) satisfies property $H(L)$, where L is proportional to the perimeter P of A. To see it, we mimic the usual proof that $T(x_1, \dots, x_n) \leq K\sqrt{n}$ when $x_1, \dots, x_n$ belong to the unit square. We divide $I_\beta(A)$ into small squares of side of order $\min(\sqrt{P\beta/n}\,, \beta)$. The tour visits points inside each square in an arbitrary order (creating a contribution of at most $\sqrt{2P\beta n}$), and moves from one square to the next in a spiral pattern (see Figure 1) thereby creating a contribution of order P (length of one turn) times $1 + \beta\sqrt{n/P\beta}$ (number of turns)

(2) A main feature of (2.5) is that for β small, the quantity $L\beta$ is much smaller than the area of A. For these small values of β and large n (these

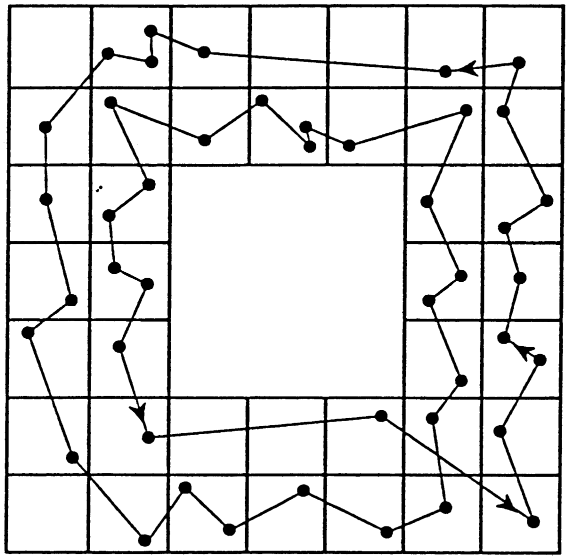

FIG. 1.

will be the values we will use later), (2.5) improves upon the estimate

$$T(x_1, \dots, x_n) \leq \tfrac{3}{2} \text{ perimeter of } A + \sqrt{2n \text{ area}(A)}\,.$$

of Haimovich, Rinnooy Kan and Stougie (1988).

Before we start the main body of work, we prove one more easy fact.

LEMMA 2.5. *Consider disjoint sets* $F_1, \dots, F_s$. *Then*

$$R\Big(\bigcup_{i \leq s} F_i\Big) \leq \sum_{i \leq s} R(F_i) + KD\sqrt{s}\,,$$

where D *is the diameter of* $\bigcup_{i \leq s} F_i$.

PROOF. For each $i \leq s$, consider a routing of cost $R(F_i)$. Each of these tours might use at most one vehicle that is incomplete. Denote by G_i the customers this vehicle serves. The contribution of this vehicle to $R(F_i)$ is, by Lemma 2.1, at least $(2/k)\sum_{x \in G_i} \|x\|$. Thus, we can serve all the customers in $\bigcup_{i \leq s} F_i \setminus G$ (where $G = \bigcup_{i \leq s} G_i$) with vehicles that are all complete and at a cost at most

$$\sum_{i \leq s} R(F_i) - \frac{2}{k} \sum_{x \in G} \|x\|.$$

To complete the proof, we use from Lemma 2.2 the fact that

$$R(G) \leq \frac{2}{k} \sum_{x \in G} \|x\| + KD\sqrt{\operatorname{card} G}\,. \qquad \square$$

THEOREM 2.6. *Consider a domain A and suppose that we have a partition $A = \bigcup_{l \le s} A_l$, where each domain A_l satisfies condition $H(L_l)$ for a certain number L_l. Set $D_0 = \max_{l \le s} \operatorname{diam}(A_l)$. Then, for each subset F of A, with* card $F = n$, *we have*

$$\begin{aligned} \sum_{l \le s} R(F \cap A_l) &\le R(F) + K\left(L \log n + \sqrt{L D_0 s \log n} + D\sqrt{s}\right) \\ &\le R(F) + K\left(L \log n + D_0 s + D\sqrt{s}\right), \end{aligned} \tag{2.6}$$

where D is the diameter of A and where $L = \sum_{l \le n} L_l$.

PROOF. We fix an optimal routing that serves the points of F, that will be referred to as "the optimal routing of F." We have to construct routings of the sets $F \cap A_l$ that satisfy (2.6). For x in F, we denote by $C(x)$ the collection of points of F that are served by the same vehicle as x. If this vehicle is complete and if all the points of $C(x)$ belong to a given set A_l, the points of $C(x)$ will be served by the same vehicle in the routing of $F \cap A_l$ that we construct.

We have to decide how to serve the other points.

The main idea is articulated in two phases. First, in an optimal routing, vehicles have a tendency to serve points close to each other. Thus, most of the points x of A_l and such that $C(x)$ is not included in A_l will be close to the boundary of A_l. On the other hand, combining Lemma 2.2 and (2.5) shows that the closer the points are to the boundary of A_l, the more efficiently they can be served. Thereby, it is natural to proceed to a "stratification" of these points as a function of their distance to the boundary of A_l; this distance of a point x of A_l to the boundary of A_l is appropriately measured by the largest q for which $x \in I_{2^{-q}}(A_l)$. A careful choice of the various parameters is necessary in order to obtain the relatively sharp bound (with only a $\log n$ parasitic term) (2.6) provides. Weaker bounds (involving error terms that are powers of n) are considerably easier to prove.

Consider the smallest integer q_0 such that $D_0 \le 2^{-q_0}$ and the smallest integer q_1 with $2^{-q_1} n \le 2^{-q_0}$. For $q_0 \le q \le q_1$, we set

$$B_q = \bigcup_{l \le s} I_{2^{-q}}(A_l).$$

We set $B_{q_1+1} = \varnothing$. For $q_0 \le q \le q_1$, we consider the set F_q of points that have the property that $C(x)$ is contained in B_q, but is not contained in B_{q+1}, and, moreover, that $C(x)$ meets at least two different sets A_l. We observe that the sets F_q are disjoint. Also, if $x \in F_q$, then since $C(x)$ is not contained in B_{q+1}, we can find $y \in C(x)$ and $l \le s$ such that $y \in A_l$, $y \notin I_{2^{-q-1}}(A_l)$. Since $C(x)$ meets at least two different sets A_l, we can find $z \in A_m \cap C(x)$, $m \ne l$. By definition of $I_{2^{-q-1}}(A_l)$, we have $\|y - z\| \ge 2^{-q-1}$. Thus

(2.7) $C(x)$ contains two points that are at least a distance 2^{-q-1} apart.

In the optimal routing of F, there is at most one vehicle that is incomplete. Denote by G the set of customers this vehicle serves (so that card $G < k$). If we use (2.7), Lemma 2.1 and summation over $q_0 \le q \le q_1$, we see that the vehicles that serve $F' = G \cup \bigcup_{q_0 \le q \le q_1} F_q$ in the optimal routing of F must travel a distance at least

$$\frac{2}{k} \sum_{x \in F'} \|x\| + \frac{1}{k} \sum_{q_0 \le q < q_1} 2^{-q} \operatorname{card} F_q. \tag{2.8}$$

This fact will be precious in bounding the cardinality of the sets F_q. Indeed, since the quantity (2.8) is bounded by the cost of an optimal routing of F, the coefficient 2^{-q} in front of card F_q implies the (intuitive) fact that card F_q decreases fast as q decreases.

We now fix $l \le s$, and we show how to serve the customers of $A_l \cap F' = A_l \cap (G \cup \bigcup_q F_q)$. There we will take advantage of the fact that the customers close to the boundary of A_l can be served more efficiently.

First, we consider the customers of

$$Q'(1) = \left(F_{q_1} \cup G\right) \cap I_{2^{-q_1}}(A_l).$$

We set $m'(1) = \operatorname{card} Q'(1)$ and we set $m(1) = k[m'(1)/k]$. We consider a subset $Q(1)$ of $Q'(1)$ with card $Q(1) = m(1)$.

Since D_l satisfies property $H(L_l)$, we see by Lemma 2.2 that we can serve the customers of $Q(1)$, with complete vehicles, with a cost at most

$$\frac{2}{k} \sum_{x \in Q(1)} \|x\| + 3\left(L_l + \sqrt{2^{-q_1} L_l m(1)}\right).$$

In the second stage, we consider the customers of

$$Q'(2) = (Q'(1) \setminus Q(1)) \cup \left[\left(F_{q_1 - 1} \cup G\right) \cap I_{2^{-q_1+1}}(A_l)\right].$$

We set $m'(2) = \operatorname{card} Q'(2)$. We observe that $m'(2) \le k + \operatorname{card}(F_{q_1-1} \cap A_l)$. We set $m(2) = k[m'(2)/k]$ and we consider a subset $Q(2)$ of $Q'(2)$ with card $Q'(2) = m(2)$. By Lemma 2.2 again, these customers can be served with complete vehicles and a cost of at most

$$\frac{2}{k} \sum_{x \in Q(2)} \|x\| + 3\left(L_l + \sqrt{2^{-q_1+1} L_l m(2)}\right).$$

Continuing in this fashion and by summation, we can serve all the customers of $U_l = A_l \cap (G \cup \bigcup_q F_q)$ with a cost of at most

$$\begin{aligned} &\frac{2}{k} \sum_{x \in U_l} \|x\| + 3\left((q_1 - q_0) L_l + \sqrt{L_l} \sum_{q_0 \le q \le q_1} \sqrt{2^{-q} m(q)}\right) \\ &\le \frac{2}{k} \sum_{x \in U_l} \|x\| \\ &\quad + 3\left((q_1 - q_0) L_l + \sqrt{(q_1 - q_0) L_l} \sqrt{\sum_{q_0 \le q \le q_1} 2^{-q} m(q)}\right) \end{aligned} \tag{2.9}$$

by the Cauchy–Schwarz inequality. We now sum these inequalities over $l \le s$, writing now $m_l(q)$ rather than $m(q)$ to indicate the dependence on l. We remember that $m_l(q) \le k + \operatorname{card}(F_q \cap D_l)$, and we apply the Cauchy–Schwarz inequality again. We find that we have succeeded to serve all the customers of $F' = G \cup \bigcup_q F_q$ with a cost at most

$$(2.10)\quad \frac{2}{k}\sum_{x\in F'}\|x\| + 3\left((q_1-q_0)L + \sqrt{(q_1-q_0)L}\sqrt{2ks2^{-q_0} + \sum_{q_0\le q\le q_1} 2^{-q}\operatorname{card} F_q}\right).$$

To make use of this bound, we must have a bound on $Z = \sum_{q_0\le q<q_1} 2^{-q}\operatorname{card} F_q$. Our task is to extract such a bound for (2.10) itself. (Observe that $2^{-q_1}\operatorname{card} F_{q_1} \le 2^{-q_1}n \le 2^{-q_0}$). We recall that (2.8) shows that the cost in the optimal routing serving the customers of F' is at least

$$(2.11)\quad \frac{2}{k}\sum_{x\in F'}\|x\| + \frac{Z}{k}.$$

It is not possible to serve the customers of F' with a cost of less than (2.11), with at most one incomplete vehicle, since otherwise the routing with which the proof started would not be optimal. We can serve the customers of F' with a cost (2.10), with, however, possibly s vehicles that are incomplete (one for each subdomain). The proof of Lemma 2.5 shows that this routing can be modified to obtain a routing that serves the customers of F' with at most one incomplete vehicle and a cost at most

$$(2.12)\quad \frac{2}{k}\sum_{x\in F'}\|x\| + 3\left((q_1-q_0)L + \sqrt{(q_1-q_0)L(3ks2^{-q_0}+Z)}\right) + KD\sqrt{s}.$$

We now know that (2.11) is less than (2.12). Using the inequalities $\sqrt{A+B} \le \sqrt{A} + \sqrt{B}$ and $3\sqrt{aZ} \le (9/2)ak + Z/(2k)$, we get, using the fact that $2^{-q_0} \le 2D_0$,

$$\begin{aligned} Z &\le K\left((q_1-q_0)L + \sqrt{(q_1-q_0)Lks2^{-q_0}} + D\sqrt{s}\right) \\ &\le K\left((q_1-q_0)L + s2^{-q_0} + D\sqrt{s}\right) \\ &\le K\left((q_1-q_0)L + sD_0 + D\sqrt{s}\right). \end{aligned}$$

Plugging this bound into (2.10) and recalling that $q_1 - q_0 \le K\log n$ by definition of q_1 yields the result. □

3. Away from the depot. In this section we study the case where the central depot is located far away from the customers. Given a subset F of $\mathbb{R}^2$, $M, a \in \mathbb{R}$, we set

$$R_{M,a}(F) = R(F + (M,a)),$$

where

$$F + (M, a) = \{(u + M, v + a) \colon (u, v) \in F\}.$$

We set

$$S_{M,a}(F) = R_{M,a}(F) - \frac{2}{k} \sum_{x \in F} \| x + (M, a) \|.$$

In other words, the depot is now located at $(-M, -a)$ rather than 0. We observe that, by Lemma 2.2, we have

$$0 \leq S_{M,a}(F) \leq K(\operatorname{card} F)^{1/2}. \tag{3.1}$$

We set

$$S_\infty(F) = \lim_{M \to \infty} S_{M,a}(F). \tag{3.2}$$

First of all, we show that the limit exists and is independent of a. Using the inequality $\sqrt{s^2 + t^2} \leq s + 2t^2/s$, for $s, t \geq 0$, we see that for any point $x = (u, v)$, we have for $M > \|x\|$,

$$0 \leq \| x + (M, a) \| - (u + M) \leq \frac{1}{2} \frac{(v + a)^2}{u + M} \leq \frac{1}{2} \frac{(\|x\| + a)^2}{M - \|x\|}. \tag{3.3}$$

Consider now points $x_1 = (u_1, v_1), \dots, x_p = (u_p, v_p)$. Set

$$f(M, a) = \| x_1 + (M, a) \| + \| x_p + (M, a) \| - \frac{2}{p} \sum_{i \leq p} \| x_i + (M, a) \|.$$

Set $A = \max_{i \leq p} \|x_i\|$. By (3.3) we get, for $M > A$,

$$\left| f(M, a) - \left(u_1 + u_p - \frac{2}{p} \sum_{i \leq p} u_i \right) \right| \leq 2 \frac{(A + a)^2}{M - A}.$$

Thus

$$|f(M, a) - f(M', a')| \leq 2 \frac{(A + a)^2}{M - A} + 2 \frac{(A + a')^2}{M' - A}. \tag{3.4}$$

Suppose now that we number for convenience the vehicles of our fleet. Consider an optimal routing when the depot is at (M, a). Consider the routing [with the depot now at (M', a')] where each vehicle serves the same customers in the same order. This routing, combined with (3.4), shows that

$$S_{M',a}(F) \leq S_{M,a}(F) + 2 \operatorname{card} F \left(\frac{(B + a)^2}{M - B} + \frac{(B + a')^2}{M' - B} \right), \tag{3.5}$$

where $B = \max\{\|x\|, \|x\| \in F\}$. Also, we can exchange (M, a) and (M', a') in (3.5). The proof that the limit exists in (3.2) is then easily concluded.

It should be pointed out that the value of $S_\infty(F)$ can be reformulated as the minimal cost of a routing problem, where the cost of a vehicle that serves

$x_1 = (u_1, v_1), \dots, x_p = (u_p, v_p)$ (in that order) is

$$u_1 + u_p - \frac{2}{p} \sum_{i \le q} u_i + \sum_{1 \le i < p} \|x_{i+1} - x_i\|$$

and where at most one vehicle serves less than k customers. In particular, when $k = 2$, $S_\infty(F)$ is (at least when the number of customers is even) equal to the minimum cost of a "simple matching." When $k \ge 2$, $S_\infty(F)$ is related to (but different from) the minimum cost of a decomposition of F into a union of "k-chains."

Consider a positive measure ν on $\mathbb{R}^2$. We will say that a random subset F of $\mathbb{R}^2$ is generated by a Poisson point process of intensity measure ν if for any two disjoint subsets A, B of $\mathbb{R}^2$, the random sets $F \cap A$ and $F \cap B$ are independent, and if card $F \cap A$ is a Poisson r.v. of expectation $\nu(A)$. In many cases we will have $d\nu = \lambda\, du\, dv$ for some $\lambda > 0$. We will then simply say that the Poisson point process is of constant intensity λ.

We now assume that F is a random subset of $[0,1]^2$ that is generated by a Poisson point process of constant intensity λ. (Why we are now interested in the unit square rather than the unit disc will become apparent in the beginning of Section 4.)

We set $\theta(\lambda) = ES_\infty(F)$. The main result is as follows.

THEOREM 3.1. *There exists a universal constant α such that for $\lambda \ge 2$ we have*

$$\left| \frac{\theta(\lambda)}{\sqrt{\lambda}} - \alpha \right| \le K \frac{\log \lambda}{\sqrt{\lambda}}. \tag{3.6}$$

The main idea in the introduction of S_∞ is that this functional is now translation invariant and has the appropriate homogeneity (= scaling) properties. Thus the proof can follow (a quantitative version of) the usual arguments of Steele (1981).

PROOF.

Step 1. The proof will rely on subadditivity. Denote by $(C_i)_{i \le p^2}$ the partition of $[0,1]^2$ into squares of side p^{-1}. It follows from Lemma 2.5, Theorem 2.6 and the definition of S_∞ that

$$\left| S_\infty(F) - \sum_{i \le p^2} S_\infty(F \cap C_i) \right| \le K(p + p\log(1 + \operatorname{card} F)).$$

By concavity of the log function, and since $E \operatorname{card} F = \lambda$, we have

$$\left| \theta(\lambda) - \sum_{i \le p^2} ES_\infty(F \cap C_i) \right| \le Kp\log(1 + \lambda). \tag{3.7}$$

Step 2. Fix $i \le p^2$ and consider the affine transformation U that sends C_i to $[0, 1]^2$. By scaling, it should be clear from the existence of the limit in (3.2) that

$$S_\infty(F \cap C_i) = \frac{1}{p} S_\infty(U(F \cap C_i)).$$

Since $U(F \cap C_i)$ is a Poisson process with intensity λ / p^2, we have

$$ES_\infty(F \cap C_i) = \frac{1}{p}\theta\left(\frac{\lambda}{p^2}\right)$$

so that, from (3.7),

$$\left|\theta(\lambda) - p\theta\left(\frac{\lambda}{p^2}\right)\right| \le Kp\log(1+\lambda).$$

Replacing λ by λp^2 and setting $f(\lambda) = \theta(\lambda)/\sqrt{\lambda}$, we get

$$|f(\lambda p^2) - f(\lambda)| \le K\frac{\log(1+\lambda p^2)}{\sqrt{\lambda}}. \tag{3.8}$$

Step 3. Consider $r, s \ge 0$. Then, using (3.8) for $4^r 9^s \lambda$ instead of λ and $p = 2$, we obtain, for $\lambda \ge 2$,

$$|f(4^{r+1}9^s\lambda) - f(4^r 9^s \lambda)| \le K\frac{1}{2^r 3^s \sqrt{\lambda}}(r + s + \log \lambda). \tag{3.9}$$

The same inequality holds if the term $4^{r+1}9^s$ is replaced by $4^r 9^{s+1}$ (taking now $p = 3$). Since

$$\sum_{r' \ge r,\, s' \ge s} (r' + s')2^{-r'}3^{-s'} \le K(r+s)2^{-r}3^{-s},$$

it follows from (3.9) that for $r' \ge r$, $s' \ge s$, we have

$$|f(4^{r'}9^{s'}\lambda) - f(4^r 9^s \lambda)| \le \frac{K}{2^r 3^s \sqrt{\lambda}}(r + s + \log \lambda).$$

This shows that

$$h(\lambda) = \lim_{\min(r,s)\to\infty} f(4^r 9^s \lambda)$$

exists and that

$$|h(\lambda) - f(\lambda)| \le K\frac{\log \lambda}{\sqrt{\lambda}}.$$

Step 4. To conclude the proof, it suffices to show that $h(\lambda)$ does not depend on λ [we then set $\alpha = h(\lambda)$].

In this step, we prove the inequality

$$|\theta(\lambda) - \theta(\mu)| \le K\sqrt{|\lambda - \mu|}, \tag{3.10}$$

which is an important technical step toward this goal. There is no loss of generality to assume $\lambda \le \mu$.

By Lemma 2.2 and an obvious limit argument, we see that for two subsets F, G of $[0,1]^2$ we have

$$|S_\infty(F \cup G) - S_\infty(F)| \le K\sqrt{\operatorname{card} G}\,. \tag{3.11}$$

We also observe that if F and G, respectively, are generated by two independent Poisson point process with respective constant intensities λ and $\mu - \lambda$, then $F \cup G$ is generated by a Poisson point process of constant intensity μ. Then (3.10) follows from (3.11) by taking expectations.

Step 5. Since the ratio $\log 2/\log 3$ is irrational, the numbers $a\log 2 + b\log 3$ $(a, b \in \mathbb{Z})$ are dense in R. Thus, given λ, μ, $\varepsilon > 0$, we can find $a, b \in \mathbb{Z}$ such that

$$\left|\frac{\lambda}{\mu} - 4^a 9^b\right| \le \varepsilon\frac{\lambda}{\mu}. \tag{3.12}$$

For $r, s \in \mathbb{N}$, we have

$$(1-\varepsilon)\lambda 4^r 9^s \le \mu 4^{a+r} 9^{b+s} \le (1+\varepsilon)\lambda 4^r 9^s.$$

From (3.10) we observe that

$$|\theta(\mu 4^{a+r}9^{b+s}) - \theta(\lambda 4^r 9^s)| \le K\sqrt{\varepsilon\lambda}\,2^r 3^s$$

so that

$$\left|f(\mu 4^{a+r}9^{b+s})\sqrt{\frac{\mu}{\lambda}}\,2^a 3^b - f(\lambda 4^r 9^s)\right| \le K\sqrt{\varepsilon}\,.$$

Letting $r, s \to \infty$ gives

$$\left|h(\mu)\sqrt{\frac{\mu}{\lambda}}\,2^a 3^b - h(\lambda)\right| \le K\sqrt{\varepsilon}\,.$$

In view of (3.12) and since ε is arbitrary, this shows that $h(\lambda) = h(\mu)$, and concludes the proof. □

We will need a result comparable to Theorem 3.1, but for nonuniform Poisson point processes. Rather than proving a very general result, we focus on the form we will need later on.

THEOREM 3.2. *Consider a function h on $[0,1]^2$ and $0 \le a \le 1$, and assume that*

$$1 \le h(x) \le 2, \tag{3.13}$$

$$\forall\, x, y \in [0,1]^2, \qquad |h(x) - h(y)| \le a\|x - y\|. \tag{3.14}$$

Consider $\mu \ge 1$ and a subset G of $[0,1]^2$ that is generated by a Poisson point process with intensity measure $\mu h(u,v)\,du\,dv$. (Thus, for a Borel subset A of $[0,1]^2$, $\operatorname{card}(G \cap A)$ is a Poisson r.v. of expectation $\mu\int\int_A h(u,v)\,du\,dv$.) Then

$$\left|ES_\infty(G) - \alpha\sqrt{\mu}\int\int_{[0,1]}\sqrt{h(u,v)}\,du\,dv\right| \le K\left(1 + \log\mu + (a\mu\log\mu)^{1/3}\right). \tag{3.15}$$

PROOF.

Step 1. Consider the squares $(C_i)_{i \le p^2}$ as in the proof of Theorem 3.1. Thus we have, mimicking the proof of (3.7),

$$\left| ES_\infty(G) - \sum_{i \le p^2} ES_\infty(G \cap C_i) \right| \le Kp \log(1 + \mu). \tag{3.16}$$

Step 2. Denote by b_i the minimum of h over $C_{i'}$. Proceeding as in Step 2 of Theorem 3.1, we see that, in distribution,

$$S_\infty(G \cap C_i) = \frac{1}{p} S_\infty(G'), \tag{3.17}$$

where G' is a Poisson point process on $[0,1]^2$ with intensity measure $p^{-2}\mu h_i(u,v)\,du\,dv$, where $b_i \le h_i(u,v) \le b_i + \sqrt{2}\,a/p$.

Step 3. Consider two independent Poisson point processes G_1, G_2 with respective intensity measures $p^{-2}\mu b_i\,du\,dv$ and $p^{-2}\mu(h_i(u,v) - b_i)\,du\,dv$. Then $G_1 \cup G_2$ is distributed like G'. Using Proposition 2.3 and taking expectations, we have

$$|ES_\infty(G') - ES_\infty(G_1)| \le K\sqrt{a\mu p^{-3}}. \tag{3.18}$$

We observe that, for $\lambda \le 2$, we have $|\theta(\lambda) - \alpha\sqrt{\lambda}| \le K$. Combining with Theorem 3.1, we have

$$\left| ES_\infty(G_1) - \frac{\alpha}{p}\sqrt{\mu b_i} \right| \le K(1 + \log \mu). \tag{3.19}$$

Thus from (3.17)–(3.19), we have

$$\left| ES_\infty(G \cap C_i) - \frac{\alpha}{p^2}\sqrt{\mu b_i} \right| \le \frac{K}{p}(1 + \log \mu) + K\frac{\sqrt{a\mu}}{p^{5/2}}.$$

By (3.16) we get

$$\left| ES_\infty(G) - \frac{\alpha\sqrt{\mu}}{p^2} \sum_{i \le p^2} \sqrt{b_i} \right| \le Kp(1 + \log \mu) + K\frac{\sqrt{a\mu}}{\sqrt{p}}.$$

Since for $(u,v) \in C_i$, we have, using (3.13),

$$\sqrt{b_i} \le \sqrt{h(u,v)} \le \sqrt{b_i} + \frac{a}{p},$$

we have

$$\left| \frac{1}{p^2} \sum_{i \le p^2} \sqrt{b_i} - \iint_{[0,1]^2} \sqrt{h(u,v)}\,du\,dv \right| \le \frac{a}{p} \le \frac{\sqrt{a\mu}}{\sqrt{p}}.$$

The result then follows by taking p of order $\max(1, a^{1/3}\mu^{1/3}(\log \mu)^{-2/3})$. □

4. Proof of Theorem 1.1. Before the proof starts, we still need to develop one basic tool, that is, how to approximate $S(F)$ when the diameter of F is small compared to the distance of F from the origin.

Given a number $0 < a < 1$, we consider the map g_a from $[0,1]^2$ to $\mathbb{R}^2$ that sends the point with coordinates (u, v) to the point with *polar* coordinates $1 + au, av$. The purpose of this map is to transfer results previously obtained for the unit square to the disc. We start with an elementary fact that shows that, for a small, $a^{-1}g_a$ is almost an isometry.

LEMMA 4.1. *For x, x' in $[0,1]^2$, we have*

$$a(1 - Ka)\|x - x'\| \le \|g_a(x) - g_a(x')\| \le a(1 + Ka)\|x - x'\|.$$

PROOF. Let $x = (u, v)$ and $x' = (u', v')$. Thus

$$\begin{aligned}\|g_a(x) - g_a(x')\|^2 &= (1 + au)^2 + (1 + au')^2 \\ &\quad - 2(1 + au)(1 + au')\cos a(v - v') \\ &= a^2(u - u')^2 + 2(1 + au)(1 + au')(1 - \cos a(v - v')).\end{aligned}$$

For $|t| \le 1$, we have

$$\left|1 - \cos t + \frac{t^2}{2}\right| \le |Kt|^3$$

so that

$$\frac{a^2(v - v')^2}{2}(1 - Ka) \le 1 - \cos a(v - v') \le \frac{a^2(v - v')^2}{2}(1 + Ka).$$

The result follows easily. □

In view of Lemma 4.1, it is intuitive that $S(g_a(F))$ should be very close to $aS_\infty(F)$.

PROPOSITION 4.2. *There exists $a_0 > 0$ such that whenever $a \le a_0$, for any set $F \subset [0,1]^2$, we have*

$$|S(g_a(F)) - aS_\infty(F)| \le Ka^2\sqrt{\operatorname{card} F}.$$

PROOF.

Step 1. Consider $M > 2$. Consider points $x_1, \dots, x_p$ of F and their images $y_1, \dots, y_p$ under g_a. We compare the quantities

$$\begin{aligned}A = {}&\|x_1 + (M,0)\| + \|x_p + (M,0)\| + \sum_{1 \le i < p} \|x_{i+1} - x_i\| \\ &- \frac{2}{p}\sum_{i \le p} \|x_i + (M,0)\|\end{aligned}$$

and

$$B = \|y_1\| + \|y_p\| + \sum_{1 \le i < p} \|y_{i+1} - y_i\| - \frac{2}{p} \sum_{i \le p} \|y_i\|.$$

We set $x_i = (u_i, v_i)$. Since $\|y_i\| = 1 + au_i$, we have

$$B = a\left(u_1 + u_p - \frac{2}{p} \sum_{i \le p} u_i\right) + \sum_{1 \le i < p} \|y_{i+1} - y_i\|. \tag{4.1}$$

We recall that by (3.3) we have

$$0 \le \| x_i + (M,0) \| - (u_i + M) \le \frac{1}{M-2}.$$

Thus we get, by Lemma 4.1,

$$|aA - B| \le \frac{pa}{M-2} + \sum_{1 \le i < p} |a\|x_{i+1} - x_i\| - \|y_{i+1} - y_i\||$$

$$\le \frac{pa}{M-2} + Ka^2 \sum_{1 \le i < p} \|x_{i+1} - x_i\|.$$

Using Lemma 2.1, we get

$$|aA - B| \le \frac{pa}{M-2} + Kpa^2 A. \tag{4.2}$$

Step 2. We prove that

$$S(g_a(F)) \le aS_\infty(F) + Ka^2\sqrt{\text{card } F}. \tag{4.3}$$

Consider a routing of the customers [with the depot at $(M,0)$] with cost $R(F + (M,0))$. To this routing corresponds a routing of $g_a(F)$ where the same vehicles serve the image by g_a of the same customers in the same order. Setting $n = \text{card } F$, summation of the inequality (4.2) over the vehicles in this routing shows that

$$S(g_a(F)) \le aC + Kpa^2 C + \frac{na}{M-2},$$

where

$$C = R(F + (M,0)) - \frac{2}{p} \sum_{x \in F} \| x + (M,0) \|.$$

By Lemma 2.2, $C \le K\sqrt{n}$, so that

$$S(g_a(F)) \le aC + Kpa^2\sqrt{n} + \frac{na}{M-2}.$$

Letting $M \to \infty$ and using the definition of S_∞ yields (4.3).

Step 3. To control $S_\infty(F)$ knowing $S(g_a(F))$, we proceed in a similar way, noting that the term a^2A in (4.2) can be replaced by aB. □

The following is pretty close to Theorem 1.1.

THEOREM 4.3. *Set $\xi = \pi\alpha$. Assume that the subset F of the unit disc is generated by a Poisson point process with constant intensity λ. Then*

$$P\left(\left|S(F) - \xi\sqrt{\lambda}\right| \geq K(\lambda \log \lambda)^{1/3}\right) \leq K \exp\left(-\frac{\lambda^{1/3}}{K(\log \lambda)^{2/3}}\right).$$

PROOF. The overall program is as follows. We will decompose the unit disc into the union of small pieces A_i. The value of $S(F)$ is related to $\Sigma S(F \cap A_i)$ through Lemma 2.5 and Theorem 2.6 Each piece A_i (but one) looks rather like a little square, and $S(F \cap A_i)$ can be related to S_∞ of a new set via Proposition 4.2. The expectation of this latter quantity has been studied through (3.15). This scheme allows one to write $S(F)$ as an approximate sum of independent random variables with known expectation, and some control on their magnitude, from which the inequality of Theorem 4.3 will follow by standard methods.

Step 1. We define a suitable partition of the unit disc. Consider two integers r, r', to be determined later. We consider the domains $(A_i)_{0 \leq i \leq rr'}$, given as follows. The domain A_0 is the disc of center zero and radius $(1 + 2\pi r^{-1})^{-r'}$. The domains $(A_i)_{1 \leq i \leq rr'}$ are an enumeration of the domains $(W_{l,m})$ for $0 \leq l \leq r'$, $0 \leq m < r$, where, in polar coordinates,

$$W_{l,m} = \left\{(\rho, \theta); (1 + 2\pi r^{-1})^{-l-1} < \rho \leq (1 + 2\pi r^{-1})^{l}; \frac{2\pi m}{r} \leq \theta < \frac{2\pi(m+1)}{r}\right\}.$$

It should be clear that $W_{l,m}$ satisfies condition $H(L_{l,m})$ of Definition 2.4, where $L_{l,m}$ is bounded by a constant times the length of the boundary of $W_{l,m}$, that is,

$$L_{l,m} \leq \frac{K}{r}(1 + 2\pi r^{-1})^{l}.$$

Thus

$$\sum_{l,m} L_{l,m} \leq Kr.$$

Step 2. We combine Lemma 2.5 and Theorem 2.6. We see that, provided

$$(1 + 2\pi r^{-1})^{-r'} \leq r^{-1}, \tag{4.4}$$

the number D_0 of Theorem 2.6 satisfies $D_0 \leq r^{-1}$. Also, we have $s = rr' + 1 \leq 2rr'$.

Thus

$$\left|S(F) - \sum_{0 \leq i \leq rr'} S(F \cap A_i)\right| \leq K\left(r \log \operatorname{card} F + \sqrt{rr' \log \operatorname{card} F}\right). \tag{4.5}$$
$$\leq K(r \log \operatorname{card} F + r').$$

We set $\eta_i = S(F \cap A_i)$. The r.v.'s η_i are independent, so that, within a perturbation of smaller order, $S(F)$ is expressed as a sum of independent r.v.'s.

Step 3. We study the expectation of the r.v. η_i. Set $a = 2\pi/r$. If $A_i = W_{l,m}$, it should be clear by homogeneity that η_i is distributed like $(1+a)^{-l-1}\eta$, where $\eta = S(G)$, G being the subset of the domain $g_a([0,1]^2)$ generated by a Poisson point process with constant intensity $\lambda' = \lambda(1+a)^{-2l-2}$. Consider the set $G' = g_a^{-1}(G)$. By Proposition 4.2, we have

$$|S(G) - aS_\infty(G')| \le Ka^2\sqrt{\operatorname{card} G}$$

so that, taking expectations, and since the area of $g_a([0,1]^2)$ is $\le Ka^2$,

$$|ES(G) - aES_\infty(G')| \le Ka^3\sqrt{\lambda}(1+a)^{-l-1}. \tag{4.6}$$

To study $S_\infty(G')$, one is tempted to use Theorem 3.1. However, one must be cautious, since G' is generated by a Poisson point process with *non*uniform intensity. The intensity measure of G' is the inverse image by g_a on $[0,1]^2$ of the measure $\lambda(1+a)^{-2l-2}\,du\,dv$ on $g_a([0,1]^2)$, that is, the measure

$$\lambda(1+a)^{-2l-2}a^2(1+au)\,du\,dv.$$

In distribution, we can write $G' = G_1 \cup G_2$, where G_1 is generated by a Poisson point process with constant intensity

$$\lambda(1+a)^{-2l-2}a^2\,du\,dv$$

on $[0,1]^2$, while G_2 has intensity

$$\lambda(1+a)^{-2l-2}a^3u\,du\,dv.$$

By Proposition 2.3, we have

$$|S_\infty(G') - S_\infty(G_1)| \le K\sqrt{\operatorname{card} G_2}$$

so that, taking expectations,

$$|ES_\infty(G') - ES_\infty(G_1)| \le K\big(\lambda(1+a)^{-2l-2}a^3\big)^{1/2}. \tag{4.7}$$

By Theorem 3.2, we have

$$\begin{aligned}\left|ES_\infty(G_1) - \alpha a\sqrt{\lambda}(1+a)^{-l-1}\right| &\le K\log\big(\lambda(1+a)^{-2l-2}a^2\big)\\ &\le K\log\lambda.\end{aligned} \tag{4.8}$$

Since $E\eta_i = (1+a)^{-l-1}ES(G)$, we get, by combining the previous inequalities (4.6), (4.7) and (4.8), that

$$\begin{aligned}&\left|E\eta_i - \alpha a^2\sqrt{\lambda}(1+a)^{-2l-2}\right|\\ &\qquad\le Ka(1+a)^{-l-1}\log\lambda + K\sqrt{\lambda}\,a^{5/2}(1+a)^{-2l-2}.\end{aligned} \tag{4.9}$$

We recall that this holds for $1 \le i \le rr'$, $A_i = W_{l,m}$, $a = 2\pi/r$. We now sum these inequalities, for $0 \le l < r'$, $0 \le m < r$. Since $\sum_{l\ge 1}(1+a)^{-l} \le K/a \le$

Kr, we get

$$\left| \sum_{1 \le i \le rr'} E\eta_i - 2\pi\alpha\sqrt{\lambda}\left(a \sum_{0 \le l < r'} (1+a)^{-2l-2} \right) \right| \le Kr \log \lambda + K\sqrt{a\lambda}.$$

Now we have, under (4.4),

$$\left| a \sum_{0 \le l < r'} (1+a)^{-2l-2} - \tfrac{1}{2} \right| \le Ka$$

so that

$$\left| \sum_{1 \le i \le rr'} E\eta_i - \pi\alpha\sqrt{\lambda} \right| \le Kr \log \lambda + K\sqrt{a\lambda}.$$

From Lemma 2.2, we see that under (4.4) this remains true if the sum for $i \ge 1$ is replaced by the sum for $i \ge 0$. Combining with (4.5), we get

$$\text{(4.10)} \quad \begin{aligned} |S(F) - \pi\alpha\sqrt{\lambda}| \le & \left| \sum_{0 \le i \le rr'} (\eta_i - E\eta_i) \right| \\ & + K\left(r \log \operatorname{card} F + r' + r \log \lambda + K\sqrt{\lambda/r} \right). \end{aligned}$$

Step 4. We choose r of order $\lambda^{1/3}(\log \lambda)^{-2/3}$, $r' = Kr \log \lambda$, so that (4.4) holds and (4.10) becomes

$$\text{(4.11)} \quad \begin{aligned} |S(F) - \pi\alpha\sqrt{\lambda}| \le & \left| \sum_{0 \le i \le rr'} (\eta_i - E\eta_i) \right| \\ & + K\lambda^{1/3}(\log \lambda)^{-2/3} \log \operatorname{card} F + (\lambda \log \lambda)^{1/3}. \end{aligned}$$

Since $\operatorname{card} F$ is a Poisson r.v. of expectation $2\pi\lambda$, we have $\operatorname{card} F \le K\lambda$ with probability $1 - Ke^{-\lambda}$. Thus, to prove Theorem 4.3, it suffices to prove that

$$\text{(4.12)} \quad P\left(\left| \sum_{0 \le i \le rr'} \eta_i - E\eta_i \right| \ge K(\lambda \log \lambda)^{1/3} \right) \le \exp\left(-\frac{\lambda^{1/3}}{K(\log \lambda)^{2/3}} \right).$$

It follows from Lemma 2.2 that

$$0 \le \eta_i \le K \operatorname{diam} A_i \sqrt{\operatorname{card}(F \cap A_i)}$$

and hence $E\eta_i \le K(\operatorname{diam} A_i)^2\sqrt{\lambda}$. Setting $\xi_i = \eta_i - E\eta_i$, we have

$$\text{(4.13)} \quad |\xi_i| \le K(\operatorname{diam} A_i)^2\sqrt{\lambda} + K \operatorname{diam} A_i \sqrt{\operatorname{card}(F \cap A_i)}.$$

Consider a Bernoulli sequence $(\varepsilon_i)_{0 \le i \le rr'}$ [i.e., $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$] that is independent of the sequence ξ_i. Then it follows from Giné and Zinn (1984), Lemma 2.7, that to prove (4.12), one can replace ξ_i by $\varepsilon_i \xi_i$. Using the subgaussian inequality [see, e.g., Ledoux and Talagrand (1991), (4.1), page 90], we get that for all $B > 0$ we have

$$P\left(\left|\sum_{0\le i\le rr'} \varepsilon_i \xi_i\right| \ge (\lambda \log \lambda)^{1/3}\right) \le 2\exp\left(-\frac{(\lambda\log\lambda)^{2/3}}{2B}\right) + P\left(\sum_{0\le i\le rr'} \xi_i^2 \ge B\right) \tag{4.14}$$

and we proceed to bound the last term.

For $m < r'$, consider the set I_m of indexes i such that $A_i = W_{l,m}$ for some l, $0 \le l \le r$. The union of the sets A_i for $i \in I_m$ is the annulus

$$D_l = \left\{x; (1+a)^{-l-1} \le \|x\| \le (1+a)^{-l}\right\}.$$

By (4.13), using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we have

$$\sum_{i\in I_l} \xi_i^2 \le K\lambda a^3(1+a)^{-3l} + Ka^2(1+a)^{-2l}\operatorname{card}(F\cap D_l).$$

We recall that if Y is a Poisson r.v. with expectation μ, we have $E\exp tY = \exp\mu(e^t - 1)$. In particular, $E\exp Y \le e^{2\mu}$, so that if $u \ge 4\mu$, $P(Y \ge u) \le \exp(-u + 2\mu) \le \exp(-u/2)$. Since the r.v. $\operatorname{card}(F \cap D_l)$ is Poisson with expectation λ area $D_l \le 2\pi\lambda a$, we have

$$P(\operatorname{card}(F\cap D_l) \ge K\lambda a) \le \exp(-\lambda a)$$

so that, with the same probability, we have

$$\sum_{i\in I_l^2} \xi_i^2 \le K\lambda a^3(1+a)^{-2l}.$$

It follows that with probability less than or equal to $r'\exp(-\lambda a)$, we have

$$\sum_{1\le i\le rr'} \xi_i^2 \le K\lambda a^2.$$

The reader will check that under (4.4) the term for $i = 0$ has a smaller order contribution. Taking $B = K\lambda a^2$ in (4.14) finishes the proof, since $B \le K\lambda r^{-2} \le K\lambda^{1/3}(\log\lambda)^{4/3}$. □

COROLLARY 4.4. *Consider r.v.'s $(X_i)_{i\le n}$ that are uniformly distributed over the unit disc. Then, for some number ξ independent of n, we have*

$$P\left(\left|S(X_1,\dots,X_n) - \xi\sqrt{n}\right| \ge K(n\log n)^{1/3}\right) \le K\exp\left(-\frac{n^{1/3}}{K(\log n)^{2/3}}\right).$$

PROOF. In Theorem 4.3, we take $\lambda = n$. We have $P(\operatorname{card} F = n) \ge 1/K\sqrt{n}$ and, conditionally on this event, F is distributed like $\{X_1,\dots,X_n\}$. □

To complete the proof of Theorem 1.1, it suffices to prove the following proposition.

PROPOSITION 4.5. *For a subset F of the unit disc, we have*

$$|R(F) - R'(F)| \le K.$$

PROOF. In the definition of R, we do not require that at most one vehicle is incomplete and we do not give a weight to that vehicle. Thus, for any subset F of the unit disc, we have $R'(F) \le R(F) + k$.

Consider a routing with cost $R'(F)$. Denote by G the subset of F that consists of the customers that are served by a vehicle that is not complete. If an incomplete vehicle serves a set H of customers, it travels at least

$$2\max_{x\in H}\|x\| \ge \frac{2}{\operatorname{card} H}\sum_{x\in H}\|x\| \ge \frac{2}{k-1}\sum_{x\in H}\|x\|.$$

Thus, the incomplete vehicles travel at least $(2/(k-1))\Sigma_{x\in G}\|x\|$. By optimality of the routing, we have

$$\frac{2}{k-1}\sum_{x\in G}\|x\| \le R'(G). \tag{4.15}$$

On the other hand, by Lemma 2.2,

$$R(G) \le \frac{2}{k}\sum_{x\in G}\|x\| + 3T(G), \tag{4.16}$$

where $T(G)$ is the length of the shortest tour through G. Since $R'(G) \le R(G) + k$, we have

$$R'(G) \le \frac{2}{k}\sum_{x\in G}\|x\| + 3T(G) + k.$$

Combining with (4.15), we see that

$$\frac{2}{k(k-1)}\sum_{x\in G}\|x\| \le k + 3T(G). \tag{4.17}$$

For $p \ge 1$, let us denote by N_p the number of points in G that satisfy $2^{-p} < \|x\| \le 2^{-p+1}$. Thus

$$\sum_{x\in G}\|x\| \ge \sum_{p\ge 1} 2^{-p}N_p.$$

On the other hand, it is simple to see that $T(G) \le K\Sigma_{p\ge 1}2^{-p}\sqrt{N_p} + K$. Thus by (4.17) we have

$$\sum_{p\ge 1} 2^{-p}N_p \le K\left(\sum_{p\ge 1} 2^{-p}\sqrt{N_p} + 1\right).$$

Consider $m \geq 1$, to be determined later, and let $I = \{p \geq 1;\ N_p \geq m\}$. Thus

$$\begin{aligned}\sqrt{m} \sum_{p \in I} 2^{-p} \sqrt{N_p} &\leq \sum_{p \geq 1} 2^{-p} N_p \leq K\Bigg(\sum_{p \geq 1} 2^{-p} \sqrt{N_p} + 1\Bigg) \\ &\leq K\Bigg(\sum_{p \in I} 2^{-p} \sqrt{N_p} + \sum_{p \notin I} 2^{-p} \sqrt{N_p} + 1\Bigg) \\ &\leq K\Bigg(1 + \sqrt{m} + \sum_{p \in I} 2^{-p} \sqrt{N_p}\Bigg).\end{aligned}$$

Taking $m = 4K^2$ gives $\sum_{p \in I} 2^{-p}\sqrt{N_p} \leq K$, so that $\sum_{p \geq 1} 2^{-p} N_p \leq K$. Going back to (4.16) gives $R(G) \leq K$. As the vehicles that serve the other customers are complete, we have $R(F) \leq R(G) + R'(F \setminus G) \leq R'(F) + K$. $\square$

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