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PROBABILISTIC ANALYSIS OF AN ALGORITHM IN THE THEORY OF MARKETS IN INDIVISIBLE GOODS

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A model of commodity trading consists of n traders, each bringing to the market his own individual good and each having his own preference for the goods on the market. The trade results in a so-called core allocation, that is, an exchange of goods which cannot be destabilized by a coalition of traders. Shapley and Scarf, who proposed the model, proved the existence of such an exchange by means of an algorithm invented by Gale. The algorithm determines sequentially a cyclic decomposition of the set of traders into trading groups with equally priced goods that satisfies the stability requirement. In this paper the work of the algorithm is studied under an assumption that the traders' individual preferences are independent and uniform. It is shown that the decreasing sequence of the market sizes has the same distribution as a Markov chain $\{\nu_i\}$ on integers in which the next state ν' is obtained from the current state ν by randomly mapping $[\nu]$ into $[\nu]$ and deleting all the cycles. The number of steps of the algorithm is proved to be asymptotically normal with mean and variance both of order $n^{1/2}$.

1. Introduction. Shapley and Scarf [17] discussed a model of commodity trading which can be summarized as follows: There are n traders and trader j brings to market an indivisible good γ_j, say. At the end of trading, trader j will depart with good $\gamma_{\tau(j)}$, where τ is a permutation on $[n] = \{1, 2, \dots, n\}$. The permutation τ is referred to as an allocation.

Each trader orders the goods γ_j, $j \in [n]$, in some order of preference. Let π_j denote the permutation on $[n]$ induced by the trader j's preferences. That is, trader j prefers $\gamma_{\pi_j(i)}$ to $\gamma_{\pi_j(i+1)}$ for $i \in [n-1]$.

The main question discussed by Shapley and Scarf was as to the existence of an allocation with the following property: There does not exist a nonempty set (coalition) of traders $S \subseteq [n]$ who can, by changing their choices, enforce an allocation in which each of them *strictly* improves on his own outcome. An allocation with these properties is said to be a *core allocation*. Shapley and Scarf showed that core allocations exist. They also describe a simple algorithm for computing a core allocation. The algorithm's invention is credited to David Gale.

Let t stand for a tentative allocation which will be amended throughout the algorithm. Initially $t(j) = \pi_j(1)$ for $j \in [n]$. In other words, each trader

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makes a bid for the good at the top of his/her preference list. The functional digraph (the digraph with vertex set $[n]$ and edges $(i, t(i))$ for $i \in [n]$) of t splits into disjoint (weak) components. Each component contains a unique directed cycle. We assign $\tau(j) = t(j)$ for each j which lies on a cycle. The vertices C of these cycles are then removed and this leaves a rooted forest. (The traders holding the goods C achieve the best possible results by trading among themselves, ignoring the rest of the traders and their goods.) If x is a root of this forest, then $t(x) \in C$, but this good has already been claimed. Therefore, $t(x)$ is redefined as the good most desired by x in $\overline{C} = [n] \setminus C$. Thus now t defines a function from $\overline{C}$ to $\overline{C}$. We repeat the process by removing the new cycles and so on, until all vertices have been removed and τ has been completely defined.

We will now repeat this description a bit more formally in order to introduce notation used in the rest of the paper. Thus we consider the algorithm to proceed in stages. At the start of stage i there is a set $N_i \subseteq [n]$ such that if $j \in \overline{N}_i = [n] \setminus N_i$, then $\tau(j)$ has been determined; initially, $N_1 = [n]$. For $j \in N_i$ we let

$$f_i(j) = \min\{u \colon \pi_j(u) \in N_i\},$$

that is, $f_i(j)$ points to the good in N_i that is most desired by trader j and $f_1(j) = \pi_j(1)$ for $j \in [n]$.

Now let $D_i = (N_i, A_i)$ be the functional digraph of f_i. That is, $A_i = \{(x, f_i(x)) \colon x \in N_i\}$. D_i can be described as a collection of vertex disjoint cycles $\mathscr{C}_i$ with $C_i = V(\mathscr{C}_i)$, plus a rooted forest F_{i+1} on $N_{i+1} = N_i \setminus C_i$. The roots K_{i+1} of F_{i+1} are precisely those vertices in $x \in N_{i+1}$ which have $f_i(x) \in C_i$. An iteration consists of assigning $\tau(x) = f_i(x)$ for $x \in C_i$ and then replacing N_i by N_{i+1}. The definition of f_{i+1} implies that $f_{i+1}(x) = f_i(x)$ for $x \in N_{i+1} \setminus K_{i+1}$ and it is only $f_{i+1}(x)$ for $x \in K_{i+1}$ that needs to be recomputed. The process continues for r iterations until the first time we find $N_{r+1} = \varnothing$.

It is not hard to show that τ belongs to the core. In fact, τ is the only core allocation. (The uniqueness of a core allocation was first observed by Roth and Postlewaite [15]; see also Roth [14].) The outcome of the algorithm is the partially ordered set of cycles (trading groups) that allows formation of competitive prices as follows. The goods eliminated at the same iteration are priced equally and higher than the goods eliminated at any later iteration.

In this paper we elucidate some of the characteristics of a typical run of this simple but fundamental algorithm. To do this we define a probability space on the set $\Omega = S_n^n = \{(\pi_1, \pi_2, \dots, \pi_n)\}$ of possible sequences of preference permutations. We will assume each sample point of Ω is equally likely, that is, $\pi_1, \pi_2, \dots, \pi_n$ are independent random permutations on $[n]$. At first glance this distribution seems highly unrealistic because most traders would not be expected to exchange expensive champagne for cheap table wine. Clearly though, the goods are being traded in the same market precisely because some of the traders may prefer other traders' goods to their own. This makes the uniformity assumption more palatable. Besides, no obvious alternative distribution is in sight. Also our results, especially Theorem 1, are rather precise and so we hope that what we may lose in reality, we make up for in depth of analysis.

Our first theorem concerns the number of iterations of the process. So let $X_n = X_n(\pi_1, \pi_2, \ldots, \pi_n)$ denote the number of iterations in a particular instance. We prove the following central limit theorem:

THEOREM 1. (a) $\mathbf{E}(X_n) = \sqrt{8/\pi}\, n^{1/2} - (3/\pi)\log n + O(1)$.

(b) $\operatorname{Var}(X_n) = (56\sqrt{2}/(3\pi^{3/2}) - 2\sqrt{2/\pi})n^{1/2} + o(n^{1/2})$.

(c) $X_n^* = (X_n - \sqrt{8/\pi}\, n^{1/2}) / \sqrt{(56\sqrt{2}/(3\pi^{3/2}) - 2\sqrt{2/\pi})n^{1/2}}$ *converges in distribution to the standard normal variable* $\mathcal{N}(0,1)$ *with mean* 0 *and variance* 1. *Moreover*, $\mathbf{E}((X_n^*)^l) \to \mathbf{E}((\mathcal{N}(0,1))^l)$, *for* $l = 0, 1, \ldots$.

Thus, with high probability, there are about $\sqrt{8n/\pi}$ classes of equipriced goods.

Our second result concerns the total number Y_n of cycles formed during the algorithm. In the light of the discussion above, this is the number of trading groups formed by the allocation process. Our study is not as comprehensive as that for X_n: as yet we can only obtain the limiting behavior of the expectation.

THEOREM 2.

$$\mathbf{E}(Y_n) = \sqrt{2\pi n} + O(\log n).$$

The proof shows that on average about $\pi/2$ cycles are deleted at each iteration, except for the terminal iterations. Thus the average number of cycles deleted per iteration is bounded and this surprising fact deserves a heuristic explanation. At each stage the average number of cyclic vertices is $O(\sqrt{\nu})$ (ν denotes the number of vertices left). The average degree is bounded and so the average number of trees in the forest left after the deletion of cycles is $O(\sqrt{\nu})$ too. Now Pavlov [9] has shown that a uniform random forest on ν vertices with $O(\sqrt{\nu})$ trees has giant tree(s). Assuming that most of the forests produced by the algorithm are close to being uniform, we are led to the conclusion that in a typical iteration the dangling roots are likely to be mapped into these large trees. Hence, there will likely be few cycles after the reselection made by the roots.

Our final result concerns the ranks of the goods chosen by the traders. Suppose trader i goes away with the good which is the $R(i)$th on his list. Let $R_n = \sum_{i=1}^{n} R(i)$.

THEOREM 3.

$$\left(\tfrac{1}{2} + o(1)\right) n \log n \le \mathbf{E}(R_n) \le (1 + o(1))\, n \log n.$$

In fact $R_n \geq (\frac{1}{2} - \varepsilon) n \log n$ *with subexponentially high probability, for every fixed* $\varepsilon > 0$. *That is,*

$$\mathbf{P}\big(R_n \geq \big(\tfrac{1}{2} - \varepsilon\big) n \log n\big) \geq 1 - \exp(-n^c), \qquad c = c(\varepsilon) > 0.$$

NOTE. Here (and elsewhere) we use the word "subexponentially" to underscore the fact that the probability of the complementary event $\{R_n < (0.5 - \varepsilon) n \log n\}$ converges to zero at a rate somewhat slower than, but not too far from, an exponential rate.

2. Markov chains. The algorithm produces a random sequence $F_1, D_1, F_2, \dots, F_{r+1}$ of forests of rooted trees and their functional digraphs, so $\{F_i\}$ is a "forest-valued" Markov process. Here F_1 consists of the unique (trivial) forest consisting of n isolated vertices. Denote the set of roots of F_i by K_i, and the vertex set of F_i by N_i. The process can be summarized:

$F_i \to D_i$: each vertex $v \in K_i$ chooses a random neighbor $\phi_i(v) \in N_i$.

$D_i \to F_{i+1}$: delete the cyclic vertices C_i.

As mentioned previously, F_1 is trivial and $f_1 = \phi_1$ is randomly chosen from the n^n functions in $[n] \to [n]$. It would be simpler if we could say "f_i is always a random function, given N_i." This is not true. However, as we shall see (Corollary 2), the Markov chain $\{F_i\}$ induces a simpler Markov chain $\{(n_i, k_i)\}$, where n_i is the number of vertices of F_i and k_i is the number of its components. This is not at all obvious and without this Markovian property it would be difficult, not to say impossible, to do any nontrivial analysis. Even more surprising is the fact (see Lemma 2) that the transition probability from (n_i, k_i) to (n_{i+1}, k_{i+1}) is *independent* of k_i. This allows us to prove that the sequence $\{n_i\}$ is also a Markov chain. The study of the transition probabilities of this final chain throws up a curious interpretation (Remark 1) of this part of the process as a simple urn model, with an unusual sampling procedure.

Let $\mathscr{F}_{N,k}$ denote the set of rooted forests with vertex set N and k trees.

LEMMA 1. *Given* (N_j, k_j), $j = 1, 2, \dots, i$, F_i *is a random member of* $\mathscr{F}_{N_i, k_i}$ *for all* $i \geq 1$. *That is, the conditional distribution of* F_i *is uniform.*

PROOF. We prove this by induction on i. It is trivially true for $i = 1$. Fix i, $N_{i+1} \subseteq N_i$, $\kappa = k_i$, $\lambda = k_{i+1}$, and denote $\nu = n_i = |N_i|$, $\mu = n_{i+1} = |N_{i+1}|$. We start by showing that each forest $F \in \mathscr{F}_{N_{i+1}, k_{i+1}}$ arises from the same number of pairs (F', ϕ), where $F' \in \mathscr{F}_{N_i, k_i}$ and ϕ maps the roots of F' to its vertices. Given F we can construct all such pairs by the following choices:

(a) choosing t *old* roots from N_{i+1} for some $t \in [\mu]$;

(b) choosing $\kappa - t$ *old* roots from $N_i \setminus N_{i+1}$;

(c) choosing a permutation of $N_i \setminus N_{i+1}$, each cycle of which contains at least one old root from (b);

(d) choosing a mapping of the λ new roots to $N_i \setminus N_{i+1}$.

This gives a total number of choices as

$$a(\nu,\kappa;\mu,\lambda) = \sum_{t=0}^{\kappa}\binom{\mu}{t}\binom{\nu-\mu}{\kappa-t}\left(\frac{\kappa-t}{\nu-\mu}(\nu-\mu)!\right)(\nu-\mu)^{\lambda}$$
$$= \binom{\nu}{\kappa}\frac{\kappa}{\nu}(\nu-\mu)!(\nu-\mu)^{\lambda},$$
$$= \binom{\nu-1}{\kappa-1}(\nu-\mu)!(\nu-\mu)^{\lambda},$$

which is independent of F.

NOTE. It is known (Lovász [6], Exercise 3.6) that the total number of permutations of $[\alpha]$ such that each cycle contains at least one element of $[\beta] \subseteq [\alpha]$ is $(\beta/\alpha)\alpha!$. This explains the third factor in the sum.

If $F \in \mathscr{F}_{N_{i+1},k_{i+1}}$, then the inductive assumption and the Markov property of the process $\{F_j\}$ implies (via conditioning on F_i) that

$$\mathbf{P}(F_{i+1} = F|(N_1,k_1),\dots,(N_i,k_i)) = \frac{1}{|\mathscr{F}_{N_i,k_i}|}\sum_{F'\in\mathscr{F}_{N_i,k_i}}\mathbf{P}(F_{i+1} = F|F_i = F').$$

Now, let ϕ_i be the random mapping of the roots of F_i into the set N_i and let ϕ be a generic mapping of that sort. Since, conditioned on $F_i = F'$, the mapping ϕ_i is uniform, we get

$$\mathbf{P}(F_{i+1} = F|F_i = F') = \frac{1}{|N_i|^{k_i}}\sum_{\phi}\mathbf{P}(F_{i+1} = F|F_i = F', \phi_i = \phi).$$

The conditional probability in the sum is 1 or 0, dependent upon whether the forest F arises from the pair (F', ϕ) or not. As we know, the number of such pairs depends only on k_i, k_{i+1}, $|N_i|$ and $|N_{i+1}|$. Therefore we obtain

$$\mathbf{P}(F_{i+1} = F|(N_1,k_1),\dots,(N_i,k_i)) = \frac{a(|N_i|,k_i;|N_{i+1}|,k_{i+1})}{|\mathscr{F}_{N_i,k_i}|\cdot|N_i|^{k_i}}.$$

Thus this probability is independent of $F \in \mathscr{F}_{N_{i+1},k_{i+1}}$. However, then so is $\mathbf{P}(F_{i+1} = F|(N_1,k_1),\dots,(N_{i+1},k_{i+1}))$, since it equals the ratio of the above probability and $\mathbf{P}(F_{i+1} \in \mathscr{F}_{N_{i+1},k_{i+1}}|(N_1,k_1),\dots,(N_i,k_i))$. □

COROLLARY 1. *The random sequence (N_i,k_i) is a Markov chain.*

PROOF. Slightly abusing notation,

$$\mathbf{P}((N_{i+1},k_{i+1})|(N_1,k_1),\dots,(N_i,k_i))$$
$$= \sum_{F\in\mathscr{F}_{N_{i+1},k_{i+1}}}\mathbf{P}(F|(N_1,k_1),\dots,(N_i,k_i))$$

$$\begin{aligned}
&= \sum_{F \in \mathscr{F}_{N_{i+1}, k_{i+1}}} \sum_{F' \in \mathscr{F}_{N_i, k_i}} \mathbf{P}(F, F' | (N_1, k_1), \dots, (N_i, k_i)) \\
&= \sum_{F \in \mathscr{F}_{N_{i+1}, k_{i+1}}} \sum_{F' \in \mathscr{F}_{N_i, k_i}} \mathbf{P}(F | (N_1, k_1), \dots, (N_{i-1}, k_{i-1}), F') \\
&\qquad\qquad \times \mathbf{P}(F' | (N_1, k_1), \dots, (N_i, k_i)) \\
&= \sum_{F \in \mathscr{F}_{N_{i+1}, k_{i+1}}} \sum_{F' \in \mathscr{F}_{N_i, k_i}} |\mathscr{F}_{N_i, k_i}|^{-1} \mathbf{P}(F | F'),
\end{aligned}$$

which depends only on $N_i, k_i, N_{i+1}, k_{i+1}$. □

Now, by symmetry, given $(n_1, k_1), (n_2, k_2), \dots, (n_i, k_i)$, the set N_i is chosen uniformly at random from among all of the $\binom{n}{n_i}$ possible sets. So, using both Corollary 1 and the argument which proves it, we establish the following result.

COROLLARY 2. *The random sequence (n_i, k_i) is a Markov chain.*

We need to determine the one-step transition probabilities

$$p(\nu, \kappa; \mu, \lambda) = \mathbf{P}(n_{i+1} = \mu, k_{i+1} = \lambda | n_i = \nu, k_i = \kappa).$$

The following lemma does this.

LEMMA 2.

$$p(\nu, \kappa; \mu, \lambda) = \frac{\mu^{\mu - \lambda - 1}}{(\lambda - 1)!(\mu - \lambda)!} \frac{\nu!}{\nu^{\nu}} (\nu - \mu)^{\lambda} \tag{1}$$

for $1 \le \mu < \nu$, $1 \le \kappa \le \nu$, $1 \le \lambda \le \mu$ and $p(\nu, \kappa; 0, 0) = \nu!/\nu^{\nu}$.

PROOF. It follows from Lemma 1 that

$$p(\nu, \kappa; \mu, \lambda) = \frac{\Theta}{\Upsilon},$$

where

$$\Upsilon = \binom{\nu}{\kappa} (\kappa \nu^{\nu - \kappa - 1}) \nu^{\kappa}$$

is the number of ways of choosing a forest in $\mathscr{F}_{[\nu], \kappa}$ and then choosing a mapping from its roots to its vertices. [The middle factor in Υ is the number of forests on $[\nu]$ with κ rooted trees, each of which contains one of a prescribed set of κ vertices as its root (see, e.g., Moon [8]).] Furthermore,

$$\begin{aligned}
\Theta &= \left[\binom{\nu}{\mu} \binom{\mu}{\lambda} \lambda \mu^{\mu - \lambda - 1} \right] \times \left[\sum_{t=0}^{\kappa} \binom{\mu}{t} \binom{\nu - \mu}{\kappa - t} \left(\frac{\kappa - t}{\nu - \mu} (\nu - \mu)! \right) (\nu - \mu)^{\lambda} \right] \\
&= \left[\binom{\nu}{\mu} \binom{\mu}{\lambda} \lambda \mu^{\mu - \lambda - 1} \right] \times \left[\binom{\nu - 1}{\kappa - 1} (\nu - \mu)! (\nu - \mu)^{\lambda} \right]
\end{aligned}$$

is the number of choices which lead to a forest F with μ vertices and λ trees.

EXPLANATION. The quantity Θ is the number of ways to choose a forest of λ rooted trees on a subset of μ vertices and then go back to a forest from $\mathscr{F}_{[\nu],\kappa}$ (see the proof of Lemma 1). □

Notice the remarkable fact that the r.h.s. of (1) does not depend on κ. This means that $\{n_i\}$ is also a Markov chain. So now let

$$p_{\nu,\mu} = \mathbf{P}(n_{i+1} = \mu | n_i = \nu).$$

Taking an arbitrary choice for κ in (1) and summing over λ we obtain

$$\begin{aligned} p_{\nu,\mu} &= \sum_{\lambda=1}^{\mu} p(\nu,\kappa;\mu,\lambda) \\ &= \frac{\nu!\mu^{\mu}}{\nu^{\nu}\mu!}\left(\frac{\nu-\mu}{\mu}\right)\sum_{\lambda=1}^{\mu}\binom{\mu-1}{\lambda-1}\left(\frac{\nu-\mu}{\mu}\right)^{\lambda-1} \\ &= \frac{\nu!}{\nu^{\nu-\mu}\mu!}\left(\frac{\nu-\mu}{\nu}\right). \end{aligned} \tag{2}$$

As a partial check

$$\begin{aligned} \sum_{\mu=1}^{\nu} p_{\nu,\mu} &= \nu!\left(\sum_{\mu=1}^{\nu}\frac{1}{\nu^{\nu-\mu}\mu!} - \sum_{\mu=0}^{\nu-1}\frac{1}{\nu^{\nu-\mu}\mu!}\right) \\ &= 1 - \frac{\nu!}{\nu^{\nu}} = 1 - p_{\nu,0}. \end{aligned}$$

REMARK 1. As already mentioned, our process is intimately connected to a curious urn model: Suppose we have ν balls numbered 1 to ν in an urn. We repeatedly and randomly select a ball from the urn, note its number and then replace it in the urn. The process continues until a ball is selected which has been selected before. Then *all* balls which have been selected are thrown away. Let μ be the number of balls left. A simple computation reveals that $p_{\nu,\mu}$ is equal to the probability that there are μ balls left in the urn. Call the removal of the $\nu-\mu$ balls one iteration of the urn model. We can now apply the same procedure to the μ remaining balls. Let X'_ν denote the number of iterations before the urn becomes empty. It is clear from our observation about $p_{\nu,\mu}$ that X'_ν and X_ν have the same distribution.

Some of the mystery may be explained by the fact that since $p(\nu,\kappa;\mu,\lambda)$ is independent of κ, we may as well assume $\kappa=\nu$ and then we see that $\nu-\mu$ is distributed as the number of cyclic vertices in a random functional digraph, regardless of the number of trees in the forest. So in particular,

$$\sum_{\mu=1}^{\nu-1} \mu p_{\nu,\mu} = \nu - c\nu^{1/2} + O(1), \tag{3}$$

where $c = \sqrt{\pi/2}$; see, for example, Bollobás [1, Theorem XIV.33(vi)].

So the Markov chain $\{n_i\}$ has the same distribution as a Markov chain $\{\nu_i\}$: Here ν_{i+1} is the number of elements of the set $[\nu_i]$ left when we delete all the cyclic vertices of a random mapping ϕ_i from $[\nu_i]$ to $[\nu_i]$, but the cyclic decomposition of $N_i \setminus N_{i+1}$ in the algorithm and the cycles of the random mapping ϕ_i are typically quite different. Indeed, for large ν_i, the number of cycles in ϕ_i is close, with high conditional probability to $\frac{1}{2}\log \nu_i$ (Stepanov [18]). As for the algorithm, the number of cycles deleted in one iteration is close, on average to $\pi/2$, and thus is bounded.

3. Proof of Theorem 1. We now introduce the generating function $g_n(z) = \mathbf{E}(z^{X_n})$, $z \geq 0$. Then, by the Markov property, for $\nu \geq 1$,

$$g_\nu(z) = z \sum_{\mu=0}^{\nu-1} p_{\nu,\mu} g_\mu(z) \tag{4}$$

and $g_0(z) = 1$ by definition. Even though the recurrence (4) does not seem explicitly solvable, we will be able to find some $\hat{g}_\nu(z)$ which almost (i.e., asymptotically as $\nu \to \infty$) satisfies it.

Since $\mathbf{E}(X_\nu) = g'_\nu(1)$, differentiating both sides of (4) at $z = 1$, we obtain

$$\mathbf{E}(X_\nu) = 1 + \sum_{\mu=0}^{\nu-1} p_{\nu,\mu}\mathbf{E}(X_\mu). \tag{5}$$

A recurrence for $\mathbf{E}(X_\nu(X_\nu - 1))$ can be obtained by twice differentiating (4) at $z = 1$:

$$\mathbf{E}(X_\nu(X_\nu - 1)) = 2(\mathbf{E}(X_\nu) - 1) + \sum_{\mu=0}^{\nu-1} p_{\nu,\mu}\mathbf{E}\big(X_\mu(X_\mu - 1)\big). \tag{6}$$

Setting

$$f_\nu(z) = \frac{\nu^\nu}{\nu!} g_\nu(z),$$

we obtain

$$f_\nu(z) = z \sum_{\mu=0}^{\nu-1} f_\mu(z) \left(\frac{\nu}{\mu}\right)^\mu \left(\frac{\nu-\mu}{\nu}\right),$$

for $\nu \geq 1$, $f_0(z) = 1$ [$(\nu/\mu)^\mu = 1$ for $\mu = 0$, by definition].

Observe that

$$\mathbf{E}(X_\nu) = g'_\nu(1) = \frac{\nu!}{\nu^\nu} f'_\nu(1),$$

and that the sequence $[a_\nu = f'_\nu(1)]$ satisfies

$$\begin{aligned} a_\nu &= f_\nu(1) + \sum_{\mu=1}^{\nu-1} a_\mu \left(\frac{\nu}{\mu}\right)^\mu \left(\frac{\nu-\mu}{\nu}\right) \\ &= \frac{\nu^\nu}{\nu!} + \sum_{\mu=1}^{\nu-1} a_\mu \left(\frac{\nu}{\mu}\right)^\mu \left(\frac{\nu-\mu}{\nu}\right), \qquad \nu \geq 1,\ a_0 = 0. \end{aligned} \tag{7}$$

Here by Stirling's formula,

$$\frac{\nu^{\nu}}{\nu!} = e^{\nu}\nu^{-1/2}\left(\frac{1}{\sqrt{2\pi}} - \frac{1}{12\sqrt{2\pi}}\nu^{-1} + O(\nu^{-2})\right). \tag{8}$$

The shape of this term makes the following lemma appropriate for (almost) solving (7), whence (5).

LEMMA 3. *For a given $\delta \in \mathbf{R}$, there are values $\{\sigma_{i,u,j}\}$ which depend only on δ such that the following is true: Suppose the sequence (γ_ν) satisfies*

$$\gamma_\nu = e^{\nu}\nu^{\delta}\sum_{i=0}^{N}(\alpha_i + \beta_i \log \nu)\nu^{-i/2} + O(e^{\nu}\nu^{\delta-(N+1)/2}\log \nu),$$

for some (α_i, β_i). Then provided the following equations in $(\hat{\alpha}_i, \hat{\beta}_i)$,

$$\sum_{j=0}^{i-1}\sigma_{j,0,i-j}\hat{\beta}_j = -\beta_{i-1}, \qquad 1 \le i \le N+1, \tag{9}$$

$$\sum_{j=0}^{i-1}\sigma_{j,0,i-j}\hat{\alpha}_j - \sum_{\substack{u+j+k=i\\ u\ge 1;\, j,k\ge 0}} u^{-1}\sigma_{j,u,k}\hat{\beta}_j = -\alpha_{i-1} \qquad 1 \le i \le N+1, \tag{10}$$

have a solution, the sequence

$$\hat{\eta}_\nu = \frac{\nu!}{\nu^{\nu}}e^{\nu}\nu^{\delta+1/2}\sum_{i=0}^{N}\left(\hat{\alpha}_i + \hat{\beta}_i \log \nu\right)\nu^{-i/2} \tag{11}$$

satisfies

$$\hat{\eta}_\nu = \frac{\nu!}{\nu^{\nu}}\gamma_\nu + \sum_{\mu=1}^{\nu-1}\hat{\eta}_\mu p_{\nu,\mu} + O(\nu^{\delta-N/2}\log \nu). \tag{12}$$

The proof of this lemma is given in the Appendix.

There is no error term in the recurrence (5) satisfied by $\mathbf{E}(X_\nu)$. Lemma 3 only guarantees that $(\hat{\eta}_\nu)$ will solve (5) approximately. The next lemma will relate the approximate solution to the exact solution.

LEMMA 4. *Keeping the notation of Lemma 3, suppose η_ν satisfies $\eta_0 = 0$ and*

$$\eta_\nu = \frac{\nu!}{\nu^{\nu}}\gamma_\nu + \sum_{\mu=1}^{\nu-1}\eta_\mu p_{\nu,\mu}, \qquad \nu \ge 1. \tag{13}$$

If $\delta - (N+1)/2 < -3/2$, then

$$|\eta_\nu - \hat{\eta}_\nu| = O(1).$$

PROOF. Let $\theta_\nu = \eta_\nu - \hat{\eta}_\nu$. It follows from (12) and (13) that

$$\theta_\nu = \sum_{\mu=1}^{\nu-1}\theta_\mu p_{\nu,\mu} + O(\nu^{\rho}\log \nu), \tag{14}$$

where $\rho = \delta - N/2 < -1$. Let A be the hidden constant in the error term of (14) and let $B = |\theta_1|$. Let

$$\zeta_\nu = B + A \sum_{\mu=1}^{\nu} \mu^\rho \log \mu. \tag{15}$$

We show by an easy induction that $|\theta_\nu| \le \zeta_\nu$. The lemma follows as the r.h.s. of (15) is bounded as $\nu \to \infty$. Now $\zeta_1 = B = |\theta_1|$ and then, by induction,

$$\begin{aligned}
|\theta_\nu| &\le \sum_{\mu=1}^{\nu-1} p_{\nu,\mu}\zeta_\mu + A\nu^\rho \log \nu \\
&= \sum_{\mu=1}^{\nu-1} p_{\nu,\mu}\left(B + A \sum_{\mu'=1}^{\mu} (\mu')^\rho \log \mu'\right) + A\nu^\rho \log \nu \\
&\le B + A \sum_{\mu'=1}^{\nu-1} (\mu')^\rho \log \mu' \left(\sum_{\mu=\mu'}^{\nu-1} p_{\nu,\mu}\right) + A\nu^\rho \log \nu \\
&\le \zeta_\nu.
\end{aligned}$$

□

The constants $\{\sigma_{i,u,j}\}$ will be given explicitly in the proof of Lemma 3, but to apply Lemmas 3 and 4 we need the following values:

$$\begin{aligned}
&\sigma_{t,0,0} = 1, \qquad \sigma_{t,1,0} = \frac{\sqrt{2\pi}}{2}, \qquad \sigma_{t,0,1} = \frac{\sqrt{2\pi}}{4}(t - 2 - 2\delta), \\
&\sigma_{0,0,2} = -\frac{1}{3} + \frac{4}{3}\delta + \delta^2 = \begin{cases} -3/4 & (\delta = -1/2), \\ -1/3 & (\delta = 0). \end{cases}
\end{aligned} \tag{16}$$

We first apply Lemmas 3 and 4 to get an expression for a_ν of (7). Examining (8) we see that the relevant *input* values are $\delta = -1/2$, $N = 2$, $\alpha_0 = 1/\sqrt{2\pi}$, $\alpha_1 = 0$, $\alpha_2 = -1/12\sqrt{2\pi}$ and $\beta_0 = \beta_1 = \beta_2 = 0$.

Equations (9) become

$$-\frac{\sqrt{2\pi}}{4}\hat{\beta}_0 = 0, \tag{17}$$

$$-\frac{3}{4}\hat{\beta}_0 = 0, \tag{18}$$

$$\sigma_{0,0,3}\hat{\beta}_0 + \sigma_{1,0,2}\hat{\beta}_1 + \frac{\sqrt{2\pi}}{4}\hat{\beta}_2 = 0. \tag{19}$$

Equations (17) and (18) are satisfied by $\hat{\beta}_0 = 0$. Equations (10) become, after removing $\hat{\beta}_0$,

$$-\frac{\sqrt{2\pi}}{4}\hat{\alpha}_0 = -\frac{1}{\sqrt{2\pi}}, \tag{20}$$

$$-\frac{3}{4}\hat{\alpha}_0 - \frac{\sqrt{2\pi}}{2}\hat{\beta}_1 = 0, \tag{21}$$

$$\sigma_{0,0,3}\hat{\alpha}_0 + \sigma_{1,0,2}\hat{\alpha}_1 + \frac{\sqrt{2\pi}}{4}\hat{\alpha}_2 - \left(\sigma_{1,1,1} + \frac{1}{2}\sigma_{1,2,0}\right)\hat{\beta}_1 - \frac{\sqrt{2\pi}}{2}\hat{\beta}_2 = \frac{1}{12\sqrt{2\pi}}; \tag{22}$$

(20) determines $\hat{\alpha}_0 = 2/\pi$, and (21) determines $\hat{\beta}_1 = -3/\pi\sqrt{2\pi}$. The remaining equations (19) and (22) are solvable for $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_2$, but not uniquely solvable. Thus there exist $\hat{A}_1, \hat{A}_2, \hat{B}_2$ (whose exact values are not important to us) such that (Lemma 3)

$$\begin{aligned}\hat{m}_\nu &= e^\nu\left(\frac{\nu!}{\nu^\nu}\right)\left(\frac{2}{\pi} + \hat{A}_1\nu^{-1/2} + \hat{A}_2\nu^{-1} - \left(\frac{3}{\pi\sqrt{2\pi}}\nu^{-1/2} + \hat{B}_2\nu^{-1}\right)\log\nu\right)\\ &= \sqrt{\frac{8}{\pi}}\,\nu^{1/2} - \frac{3}{\pi}\log\nu + \hat{A}_1\sqrt{2\pi} - \hat{B}_2\sqrt{2\pi}\,\nu^{-1/2}\log\nu + \frac{1}{\sqrt{18\pi}}\nu^{-1/2}\\ &\quad + O(\nu^{-1}\log\nu)\end{aligned} \tag{23}$$

satisfies

$$\hat{m}_\nu = 1 + \sum_{\mu=1}^{\nu-1} p_{\nu,\mu}\hat{m}_\mu + O(\nu^{-3/2}\log\nu). \tag{24}$$

[Later $\hat{m}_\nu$ will be used as a *surrogate* for $\mathbf{E}(X_\nu)$.] Furthermore, Lemma 4 and (7) imply

$$\begin{aligned}\mathbf{E}(X_\nu) &= a_\nu\frac{\nu!}{\nu^\nu}\\ &= \sqrt{\frac{8}{\pi}}\,\nu^{1/2} - \frac{3}{\pi}\log\nu + O(1).\end{aligned} \tag{25}$$

This completes the proof of part (a) of Theorem 1.

To estimate the surrogate variance we let

$$b_\nu = f_\nu''(1) = \frac{\nu^\nu}{\nu!}g_\nu''(1) = \frac{\nu^\nu}{\nu!}\mathbf{E}(X_\nu(X_\nu - 1)).$$

Then

$$f_\nu''(z) = 2\sum_{\mu=1}^{\nu-1} f_\mu'(z)\left(\frac{\nu}{\mu}\right)^\mu\left(\frac{\nu-\mu}{\nu}\right) + z\sum_{\mu=1}^{\nu-1} f_\mu''(z)\left(\frac{\nu}{\mu}\right)^\mu\left(\frac{\nu-\mu}{\nu}\right),$$

and so

$$\begin{aligned}b_\nu &= 2\sum_{\mu=1}^{\nu-1} a_\mu\left(\frac{\nu}{\mu}\right)^\mu\left(\frac{\nu-\mu}{\nu}\right) + \sum_{\mu=1}^{\nu-1} b_\mu\left(\frac{\nu}{\mu}\right)^\mu\left(\frac{\nu-\mu}{\nu}\right)\\ &= 2\left(a_\nu - \frac{\nu^\nu}{\nu!}\right) + \sum_{\mu=1}^{\nu-1} b_\mu\left(\frac{\nu}{\mu}\right)^\mu\left(\frac{\nu-\mu}{\nu}\right).\end{aligned}$$

We now apply Lemma 3 with

$$
(26)\quad \begin{aligned}\gamma_\nu &= 2\left(\frac{\nu^\nu}{\nu!}\right)(\hat{m}_\nu - 1)\\ &= e^\nu\left(\frac{4}{\pi} + \left(2\hat{A}_1 - \sqrt{\frac{2}{\pi}}\right)\nu^{-1/2} - \frac{6}{\pi\sqrt{2\pi}}\nu^{-1/2}\log\nu + O(\nu^{-1}\log\nu)\right)\end{aligned}
$$

and $\delta = 0$, $N = 1$, $\alpha_0 = 4/\pi$, $\alpha_1 = 2\hat{A}_1 - \sqrt{2/\pi}$, $\beta_0 = 0$ and $\beta_1 = -6/\pi\sqrt{2\pi}$. Equations (9) become

$$
\begin{aligned}
-\frac{\sqrt{2\pi}}{2}\hat{\beta}_0 &= 0,\\
-\frac{1}{3}\hat{\beta}_0 - \frac{\sqrt{2\pi}}{4}\hat{\beta}_1 &= \frac{6}{\pi\sqrt{2\pi}},
\end{aligned}
$$

whose solution is $\hat{\beta}_0 = 0$, $\hat{\beta}_1 = -12/\pi^2$.

Equations (10) become, after removing $\hat{\beta}_0$,

$$
\begin{aligned}
-\frac{\sqrt{2\pi}}{2}\hat{\alpha}_0 &= -\frac{4}{\pi},\\
-\frac{1}{3}\hat{\alpha}_0 - \frac{\sqrt{2\pi}}{4}\hat{\alpha}_1 - \frac{\sqrt{2\pi}}{2}\hat{\beta}_1 &= -2\hat{A}_1 + \sqrt{\frac{2}{\pi}},
\end{aligned}
$$

which has solution $\hat{\alpha}_0 = 8/\pi\sqrt{2\pi}$, $\hat{\alpha}_1 = 8\hat{A}_1/\sqrt{2\pi} - 4/\pi + 56/3\pi^2$, $\hat{\beta}_0 = 0$. Thus if

$$
(27)\quad \begin{aligned}\hat{s}_\nu &= e^\nu \nu^{1/2}\left(\frac{\nu!}{\nu^\nu}\right)\\ &\times\left(\frac{8}{\pi\sqrt{2\pi}} + \left(\frac{8\hat{A}_1}{\sqrt{2\pi}} - \frac{4}{\pi} + \frac{56}{3\pi^2}\right)\nu^{-1/2} - \frac{12}{\pi^2}\nu^{-1/2}\log\nu\right),\end{aligned}
$$

then (Lemma 3)

$$
(28)\quad \hat{s}_\nu = 2(\hat{m}_\nu - 1) + \sum_{\mu=1}^{\nu-1} p_{\nu,\mu}\hat{s}_\mu + O(\nu^{-1/2}\log\nu).
$$

Comparing (28) with (6) makes it natural to define a surrogate variance $\hat{\sigma}_\nu^2$ by

$$
(29)\quad \begin{aligned}\hat{\sigma}_\nu^2 &= \hat{s}_\nu + \hat{m}_\nu - \hat{m}_\nu^2\\ &= \left(\frac{56\sqrt{2}}{3\pi^{3/2}} - 2\sqrt{\frac{2}{\pi}}\right)\nu^{1/2} + O\left((\log\nu)^2\right).\end{aligned}
$$

[Note the fortunate cancellation of terms involving $\hat{A}_1$.]

Note that (29) suggests, but does not prove, part (b) of Theorem 1. The proof of this part will be completed in conjunction with the proof of part (c) and it will be based on (29) and other estimates.

Our next task is to prove a concentration result for the random variable X_n, which will in turn be useful in the proof of part (c) of Theorem 1 and also Theorem 3. To do this we consider the urn model of Remark 1. We will prove that the number of iterations in this model is highly concentrated around its expected value.

LEMMA 5. *There exists a constant $\kappa > 0$ such that, for any $t \geq 0$,*

$$\mathbf{P}(|X_n - \mathbf{E}(X_n)| \geq t) \leq 2\exp\left\{-\frac{\kappa t^2}{n}\right\}. \tag{30}$$

PROOF. Let $b_1, b_2, \dots, b_r$ be the sequence of ball drawings in the urn model. Here $n + 1 \leq r \leq 2n$. If $r < 2n$ we can pad out this sequence to length $2n$ by joining the $(2n - r)$-long tail $(b_r, b_r, \dots, b_r)$ to its end.

Let $X(b_1, b_2, \dots, b_{2n})$ be the corresponding number of iterations and $E_n = E(X_n)$. Following a general idea of Shamir and Spencer [16] or Rhee and Talagrand [13], we will apply a martingale tail inequality to a Doob martingale in a way which has recently proved most useful in probabilistic combinatorics; see also Bollobás [2] or McDiarmid [7].

Let $Z_i(b_1, b_2, \dots, b_i) = E(X_n | b_1, b_2, \dots, b_i)$, $0 \leq i < 2n$. This sequence is a martingale. We show that there exists an absolute constant $K > 0$ such that

$$|Z_i - Z_{i+1}| \leq K \quad \text{for } 0 \leq i < 2n. \tag{31}$$

It then follows from [2] and [7] that

$$\mathbf{P}(|X - E(X)| \geq t) \leq 2\exp\left\{-\frac{t^2}{K^2 n}\right\}.$$

We prove (31) by showing that

$$|Z_{i+1}(b_1, b_2, \dots, b_i, b) - Z_{i+1}(b_1, b_2, \dots, b_i, b')| \leq K, \tag{32}$$

for all $b_1, b_2, \dots, b_i, b, b'$. We can assume without loss of generality that b finishes an iteration and b' does not; otherwise $Z_{i+1}(b_1, b_2, \dots, b_i, b) = Z_{i+1}(b_1, b_2, \dots, b_i, b')$.

Assume that there have been k iterations including the one finished by using ball b and that there are n' balls that have not been selected at all during the whole process $b_1, b_2, \dots, b_i, b$. Thus.

$$Z_{i+1}(b_1, b_2, \dots, b_i, b) = k + E_{n'}. \tag{33}$$

We observe next that part (a) implies that for some finite absolute constant $L > 0$,

$$E_{n_1+1} - L \leq E_{n_1} \leq E_{n_2} + L \tag{34}$$

whenever $1 \leq n_1 < n_2$.

REMARK 2. One can in fact show that

$$E_{n-1} \leq E_n \leq E_{n-1} + 1,$$

but (34) is available without effort and will suffice.

We deduce immediately from (34) that

$$Z_{i+1}(b_1, b_2, \dots, b_i, b') \leq k + E_{n'} + L, \tag{35}$$

because after the iteration that takes b', there will have been k iterations and there will now be at most $n'' \leq n'$ balls left to draw.

To finish the argument we prove

$$Z_{i+1}(b_1, b_2, \dots, b_i, b') \geq k + E_{n'} - (2L + 1) \tag{36}$$

and then take $K = 2L + 1$ in (31).

To prove (36), let m denote the number of balls in the urn which have previously been selected, immediately after b' is drawn. Thus $m \geq 1$ because b' is such a ball. The total number of balls in the urn at this point is $n' - 1 + m$, with $n' - 1$ being the number of balls not yet selected.

Let Y' denote the number of balls out of these $n' - 1$ balls that are deleted in the kth iteration and let Y denote the number of balls deleted in the first iteration of a process starting with $n' - 1$ balls. Then, for any $j \geq 1$,

$$\begin{aligned} \mathbf{P}(Y' \geq j | b_1, b_2, \dots, b_i, b') &= \prod_{i=0}^{j-1} \frac{n' - 1 - i}{n' - 1 + m} \\ &\leq \prod_{i=0}^{j-1} \frac{n' - 1 - i}{n' - 1} \\ &= \mathbf{P}(Y \geq j) \end{aligned}$$

and so (conditioned on $b_1, b_2, \dots, b_i, b'$) Y' is stochastically dominated by Y. However,

$$\begin{aligned} Z_{i+1}(b_1, b_2, \dots, b_i, b') &= k - 1 + 1 + \mathbf{E}(E_{n'-1-Y'} | b_1, b_2, \dots, b_i, b') \\ &\geq k - 1 + 1 + \mathbf{E}(E_{n'-1-Y} - L) \\ &= k - 1 + E_{n'-1} - L \\ &\geq k + E_{n'} - (2L + 1). \end{aligned}$$

The inequality $\mathbf{E}(E_{n'-1-Y'}|\cdot) \geq \mathbf{E}(E_{n'-1-Y} - L)$ follows from (34) (the right case), the fact that $n' - 1 - Y'$ dominates $n' - 1 - Y$ and the observation that if a random variable U dominates a random variable V, then there exists a probability space with U, V defined on it in such a way that $U \geq V$ sample pointwise. The last inequality follows from (34) (the left case). $\square$

We continue now with the proof of part (c) of Theorem 1. Define

$$h_\nu(z) = g_\nu(\exp z) = \mathbf{E}(\exp(zX_\nu)) \quad \text{and} \quad \psi_\nu(z) = \exp\left\{z\hat{m}_\nu + \frac{z^2}{2}\hat{\sigma}_\nu^2\right\},$$

where $\hat{m}_\nu$ and $\hat{\sigma}_\nu^2$ are defined in (23) and (29) [with $\hat{s}_\nu$ therein defined at (27)], respectively.

Note that $h_\nu(z)$ and $\psi_\nu(z)$ are the moment generating functions of X_ν and the normal random variable with mean $\hat{m}_\nu$ and standard deviation $\hat{\sigma}_\nu$, respectively. The proof of Theorem 1 will be completed by showing that if $z = vn^{-1/4}$, for fixed $v \in \mathbf{R}$, then

$$h_n(z) = (1 + o(1))\psi_n(z) \tag{37}$$

as $n \to \infty$. Indeed, since $\hat{m}_n - \mathbf{E}(X_n) = o(\hat{\sigma}_n)$,

$$\lim_{n\to\infty} \mathbf{E}\left\{\exp\left[v\frac{X_n - \mathbf{E}(X_n)}{\hat{\sigma}_n}\right]\right\} = \lim_{n\to\infty} \mathbf{E}\left\{\exp\left[v\frac{X_n - \hat{m}_n}{\hat{\sigma}_n}\right]\right\}$$
$$= \exp\left(\frac{v^2}{2}\right), \qquad \forall v \in \mathbf{R}.$$

Therefore (Curtiss [3]), $X_n' = (X_n - \mathbf{E}(X_n))/\hat{\sigma}_n \to \mathscr{N}(0,1)$. In addition, since

$$|x|^k \le k!(e^x + e^{-x}),$$

convergence of $(\mathbf{E}(e^{vX_n^*}))_{n\ge 1}$ implies the existence of $(c_k)_{k\ge 1}$ such that, for all $n \ge 1$,

$$\mathbf{E}\big(|X_n^*|^k\big) \le c_k.$$

Thus,

$$\lim_{n\to\infty} \mathbf{E}\big((X_n^*)^k\big) = \mathbf{E}\big(\mathscr{N}(0,1)^k\big), \qquad k \ge 1.$$

Consequently,

$$\lim_{n\to\infty} \frac{\operatorname{Var}(X_n)}{\hat{\sigma}_n^2} = 1,$$

proving part (b), and furthermore the convergence of the moment generating functions holds if $\hat{m}_n, \hat{\sigma}_n$ are replaced by the leading terms in their expansions, proving part (c).

We will first show that $\psi_\nu(z)$ $[z = O(n^{-1/4})]$ "almost" satisfies—in terms of relative error—the equation (4) for $h_\nu(z) = g_\nu(e^z)$ uniformly for $\nu \le n$.

We use two constants $5/8 < \delta < 3/4$ and $0 < \delta_1 < 1/2$, and we let $n_1 = \lfloor n^{\delta_1} \rfloor$. We start by noticing that, for $\nu \ge 1$ and $\nu \ge l \ge 0$,

$$\sum_{j\le l} p_{\nu,j} \le A\exp\left\{-\frac{1}{2\nu}(\nu - l)^2\right\}, \qquad A = e^{1/2}. \tag{38}$$

Indeed, by considering our urn model,

$$\sum_{j\le l} p_{\nu,j} = \prod_{j=1}^{\nu-l-1}\left(1 - \frac{j}{\nu}\right)$$
$$\le \exp\left\{-\frac{1}{2\nu}(\nu - l)(\nu - l - 1)\right\},$$

which implies (38).

Now put $l = \nu_0 = \lfloor \nu - \nu^{\delta} \rfloor$ and apply (38) to obtain

$$\sum_{j \le \nu_0} p_{\nu, j} \le A \exp\{-\nu^{2\delta - 1}/2\}. \tag{39}$$

Let us, in our pursuit of (37), deal first with $\nu \le n_1$.

To this end, observe that since $\hat{m}_\nu = O(\nu^{1/2})$ and $\hat{\sigma}_\nu^2 = O(\nu^{1/2})$,

$$\begin{aligned} \psi_\nu(z) &= \exp\{O(n^{\delta_1/2 - 1/4})\} \\ &= 1 + O(n^{(\delta_1/2)-(1/4)}) = 1 + o(1) \quad \text{as } n \to \infty, \end{aligned} \tag{40}$$

uniformly for $\nu \le n_1 = \lfloor n^{\delta_1} \rfloor$. We want to show that $h_\nu(z)$ behaves similarly for those ν. Uniformly for $\nu \le n^{\delta_1/2}$, using $0 \le X_\nu \le \nu$,

$$\begin{aligned} h_\nu(z) &= \mathbf{E}\{\exp(zX_\nu)\} = \exp\{O(n^{-1/4}\nu)\} \\ &= \exp\{O(n^{\delta_1/2 - 1/4})\} \\ &= 1 + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

Suppose $n^{\delta_1/2} \le \nu \le n^{\delta_1}$. Pick $\delta_2 \in (1/2, 1/(4\delta_1))$ and write

$$\begin{aligned} h_\nu(z) &= \mathbf{E}(\exp(zX_\nu)); \{X_\nu \le \nu^{\delta_2}\} + \mathbf{E}\{\exp(zX_\nu); X_\nu > \nu^{\delta_2}\} \\ &= E_1 + E_2, \quad \text{say}. \end{aligned}$$

Here

$$E_1 = \exp\{O(n^{-1/4}\nu^{\delta_2})\} = \exp\{O(n^{-1/4}n^{\delta_1\delta_2})\} = 1 + o(1) \quad \text{as } n \to \infty,$$

since $\delta_1\delta_2 < 1/4$. Further, using (25), Lemma 5, (30) and $\delta_2 > 1/2$,

$$\begin{aligned} E_2 &= \int_{(\nu^{\delta_2}, \infty)} \exp(zx)\, dF_{X_\nu}(x) \\ &\le \exp(z\nu^{\delta_2})\mathbf{P}(X_\nu \ge \nu^{\delta_2}) + |z| \int_{\nu^{\delta_2}}^{\infty} \exp(|z|x)\mathbf{P}(X_\nu \ge x)\, dx \\ &\le 2\exp(f(\nu^{\delta_2})) + 2|z| \int_{\nu^{\delta_2}}^{\infty} \exp(f(x))\, dx, \end{aligned}$$

where $f(x) = |z|x - \kappa x^2/(2\nu)$.

The function $f(x)$ is concave, and

$$\begin{aligned} f(\nu^{\delta_2}) &= -\frac{\kappa}{2}\nu^{2\delta_2 - 1}(1 + O(|z|\nu^{1-\delta_2})) \\ &\le -\frac{\kappa}{3}\nu^{2\delta_2 - 1}, \end{aligned}$$

for all large n and ν in the range under discussion, since

$$|z|\nu^{1-\delta_2} = O(n^{-1/4}n^{\delta_1(1-\delta_2)}) = o(1).$$

Likewise,

$$f'(\nu^{\delta_2}) = |z| - \kappa\nu^{\delta_2 - 1} \le -\frac{\kappa}{2}\nu^{\delta_2 - 1}.$$

So, using

$$f(x) \le f(\nu^{\delta_2}) + f'(\nu^{\delta_2})(x - \nu^{\delta_2}), \tag{41}$$

we arrive at

$$\begin{aligned} E_2 &\le o(1) + 2|z|\exp\left\{-\frac{\kappa}{3}\nu^{2\delta_2 - 1}\right\}\int_0^\infty \exp\left\{-\frac{\kappa}{2}\nu^{\delta_2 - 1}y\right\} dy \\ &= o(1) + \frac{4}{\kappa}(|z|\nu^{1-\delta_2})\exp\left\{-\frac{\kappa}{3}\nu^{2\delta_2 - 1}\right\} \\ &= o(1), \end{aligned} \tag{42}$$

uniformly over $\nu \in [n^{\delta_1/2}, n^{\delta_1}]$.

Summarizing, uniformly for $\nu \le n^{\delta_1}$, and $\delta_2 \in (1/2, 1/(4\delta_1))$,

$$h_\nu(z) = 1 + O(n^{\delta_1\delta_2 - 1/4}) = 1 + o(1) \quad \text{as } n \to \infty. \tag{43}$$

The relations (40) and (43) imply that, uniformly for $\nu \le n_1$,

$$\psi_\nu(z)/h_\nu(z) = 1 + O(n^{\delta_1\delta_2 - 1/4}) = 1 + o(1) \quad \text{as } n \to \infty. \tag{44}$$

We now consider $n_1 \le \nu \le n$. We know that

$$h_\nu(z) = e^z \sum_{j=0}^{\nu-1} p_{\nu,j} h_j(z) \tag{45}$$

and so we now estimate

$$\begin{aligned} e^z \sum_{j=0}^{\nu-1} p_{\nu,j}\psi_j(z) &= e^z \sum_{j=0}^{\nu_0} p_{\nu,j}\psi_j(z) + e^z \sum_{j=\nu_0+1}^{\nu-1} p_{\nu,j}\psi_j(z) \\ &= \Sigma_1 + \Sigma_2, \quad \text{say}. \end{aligned}$$

Now $\psi_j(z) = \exp\{O(\sqrt{\nu}/n^{1/4})\}$, uniformly for $0 \le j \le \nu - 1 < n$, so using (39) and $\delta > 5/8$ we have

$$\Sigma_1 = O\left(\exp\{-\nu^{2\delta-1}/4\}\right). \tag{46}$$

Turn to Σ_2. It follows easily from (23) and (29) that

$$\hat{m}_\nu - \hat{m}_j = O\left(\sqrt{\nu} - \sqrt{j}\right), \qquad \hat{\sigma}_\nu^2 - \hat{\sigma}_j^2 = O\left(\sqrt{\nu} - \sqrt{j} + (\log \nu)^2\right),$$

uniformly over $\nu_0 \le j \le \nu$. Thus, uniformly over $\nu_0 \le j \le \nu$,

$$\begin{aligned} \frac{\exp(z)\psi_j(z)}{\psi_\nu(z)} &= \exp\left\{z\left(\hat{m}_j - \hat{m}_\nu + 1\right) + \frac{z^2}{2}\left(\hat{\sigma}_j^2 - \hat{\sigma}_\nu^2\right)\right\} \\ &= 1 + z\left(\hat{m}_j - \hat{m}_\nu + 1\right) + \frac{z^2}{2}\left(\left(\hat{m}_j - \hat{m}_\nu + 1\right)^2 + \hat{\sigma}_j^2 - \hat{\sigma}_\nu^2\right) \\ &\quad + O\left(|z|^3\left[\left(\sqrt{\nu} - \sqrt{j}\right)^3 + (\log \nu)^2 + \left(\sqrt{\nu} - \sqrt{j}\right)(\log \nu)^2\right]\right), \end{aligned} \tag{47}$$

since (uniformly)

$$\begin{aligned}\left|z\left(\sqrt{\nu}-\sqrt{j}\right)\right| &= O\left(|z|\frac{\nu^{\delta}}{\nu^{1/2}}\right)\\ &= O\left(\frac{\nu^{\delta-1/2}}{n^{1/4}}\right)\\ &= O(n^{\delta-3/4})\\ &= o(1).\end{aligned}$$

Now by (39),

$$\begin{aligned}&\sum_{j=\nu_0+1}^{\nu-1} p_{\nu,j}\left(\hat{m}_j-\hat{m}_\nu+1\right)\\ &\qquad=\sum_{j=0}^{\nu-1} p_{\nu,j}\left(\hat{m}_j-\hat{m}_\nu+1\right)+O\left(\sqrt{\nu}\exp\{-\nu^{2\delta-1}/2\}\right)\\ &\qquad=S_1+O\left(\sqrt{\nu}\exp\{-\nu^{2\delta-1}/2\}\right)\end{aligned} \tag{48}$$

and

$$\begin{aligned}&\sum_{j=\nu_0}^{\nu-1} p_{\nu,j}\left(\left(\hat{m}_j-\hat{m}_\nu+1\right)^2+\left(\hat{\sigma}_j^2-\hat{\sigma}_\nu^2\right)\right)\\ &\quad=\sum_{j=0}^{\nu-1} p_{\nu,j}\left(\left(\hat{m}_j-\hat{m}_\nu+1\right)^2+\left(\hat{\sigma}_j^2-\hat{\sigma}_\nu^2\right)\right)+O\left(\nu\exp\{-\nu^{2\delta-1}/2\}\right)\\ &\quad=S_2+O\left(\nu\exp\{-\nu^{2\delta-1}/2\}\right),\end{aligned} \tag{49}$$

where from (24) and (28) we find

$$\begin{aligned}S_1&=\sum_{j=1}^{\nu-1} p_{\nu,j}\left(\hat{m}_j-\hat{m}_\nu+1\right)\\ &=O\left(\nu^{-3/2}\log\nu\right),\end{aligned} \tag{50}$$

$$\begin{aligned}S_2&=\sum_{j=1}^{\nu-1} p_{\nu,j}\left(\left(\hat{m}_j-\hat{m}_\nu+1\right)^2+\hat{\sigma}_j^2-\hat{\sigma}_\nu^2\right)\\ &=O\left(\nu^{-1/2}\log\nu\right).\end{aligned} \tag{51}$$

[That S_2 is expected to be small follows from

$$\operatorname{Var}(X_\nu)=\sum_{j=0}^{\nu-1} p_{\nu,j}\left[\operatorname{Var}(X_j)+\left(\mathbf{E}(X_j)-\mathbf{E}(X_\nu)+1\right)^2\right];$$

cf. (5) and (6).]

Furthermore, since

$$\sqrt{\nu} - \sqrt{j} = \frac{\nu - j}{\sqrt{\nu} + \sqrt{j}} \le \nu^{-1/2}(\nu - j),$$

we find that, for a fixed $r \ge 1$,

$$\begin{aligned}\sum_{j=\nu_0+1}^{\nu-1} p_{\nu,j}\big(\sqrt{\nu} - \sqrt{j}\big)^r &\le \nu^{-r/2}\sum_{j=0}^{\nu-1} p_{\nu,j}(\nu - j)^r \\ &= O\left(\nu^{-r/2}\sum_{t=1}^{\nu-1} t^{r-1}\left(\left(\sum_{j\le\nu-t} p_{\nu,j}\right)\right)\right) \\ &= O\left(\nu^{-r/2}\sum_{t=1}^{\infty} t^{r-1}\exp(-t^2/2\nu)\right) \\ &= O\left(\int_0^\infty x^{r-1}\exp(-x^2/2)\,dx\right) \\ &= O(1).\end{aligned}$$

Then (47)–(51) and $z = O(n^{-1/4})$ imply

$$\Sigma_2 = \psi_\nu(z)\Big(1 + O\big((\log \nu)^2(n^{-1/4}\nu^{-3/2} + n^{-1/2}\nu^{-1/2} + n^{-3/4})\big)\Big),$$

uniformly in $n_1 \le \nu \le n$, and together with (46) we obtain that, uniformly in $n_1 \le \nu \le n$,

$$\psi_\nu(z) = \Big(1 + O\big((\log n)^2 n^{-(1+\delta_1)/2}\big)\Big)e^z\sum_{j=0}^{\nu-1} p_{\nu,j}\psi_j(z). \tag{52}$$

[Compare with (45).]

The equations (44) and (52) and the optional sampling theorem for (sub, super) martingales provide the tools for the last step of the proof.

The deletion process produces a random sequence $S_0, S_1, \dots, S_k \dots$, where $S_0 = n$ and S_k is the size of the remaining set after k deletion steps, if the total number of deletion steps is at least k; otherwise $S_k = 0$. Introduce a stopping time $T = \min\{k\colon S_k < n_1\}$. Now define

$$\begin{aligned} &Y_k = e^{kz}h_{S_k}(z), \qquad Y'_k = e^{kz}\psi_{S_k}(z), \qquad k \le T \\ &Y_k = Y_T, \qquad Y'_k = Y'_T, \qquad k > T.\end{aligned}$$

Then if $S_0, S_1, \dots, S_k$ are such that $k < T$,

$$\begin{aligned}\mathbf{E}(Y_{k+1}|S_0, S_1, \dots, S_k) &= e^{(k+1)z}\mathbf{E}\big(h_{S_{k+1}}(z)|S_k\big) \\ &= e^{(k+1)z}\sum_{j=0}^{S_k-1} p_{S_{k,j}}h_j(z) \\ &= Y_k,\end{aligned} \tag{53}$$

on using (45). Furthermore, (53) holds trivially for $k \ge T$.

Similarly, (52) implies

$$\mathbf{E}(Y'_{k+1}|S_0, S_1, \dots, S_k) = \left(1 + O\left((\log n)^2 n^{-(1+\delta_1)/2}\right)\right)Y'_k. \tag{54}$$

Equation (53) states that the sequence (Y_k) is a martingale with respect to the sequence (S_k). Now by the optional stopping time theorem (see, e.g., Theorem 4.1 of Durrett [4]),

$$\mathbf{E}(Y_T) = \mathbf{E}(Y_0) = h_n(z).$$

Applying the same theorem to upper and lower estimates for $\mathbf{E}(Y'_k|\cdot)$ we see from (53) and (54) that

$$\mathbf{E}(Y'_T) = \psi_n(z)\mathbf{E}\left(\left(1 + O\left((\log n)^2 n^{-(1+\delta_1)/2}\right)\right)^T\right).$$

Furthermore, (44) implies

$$\mathbf{E}(Y'_T)/\mathbf{E}(Y_T) = 1 + o(1), \qquad n \to \infty,$$

and so

$$\psi_n(z)/h_n(z) = (1 + o(1))\mathbf{E}\left(\left(1 + O\left((\log n)^2 n^{-(1+\delta_1)/2}\right)\right)^T\right).$$

We then write

$$\begin{aligned}
&\mathbf{E}\left\{\left(1 + O\left((\log n)^2 n^{-(1+\delta_1)/2}\right)\right)^T\right\} \\
&\quad = \mathbf{E}\left\{\left(1 + O\left((\log n)^2 n^{-(1+\delta_1)/2}\right)\right)^T; T \le n^\delta\right\} \\
&\qquad + \mathbf{E}\left\{\left(1 + O\left((\log n)^2 n^{-(1+\delta_1)/2}\right)\right)^T; T > n^\delta\right\} \\
&\quad = E_3 + E_4.
\end{aligned}$$

Here, using Lemma 5,

$$\begin{aligned}
E_3 &= \left(1 + O\left((\log n)^2 n^{-(1+\delta_1)/2}\right)\right)^{n^\delta}\mathbf{P}(T \le n^\delta) \\
&= 1 + O\left(\exp\{-\kappa n^{2\delta-1}/2\} + (\log n)^2 n^{\delta-(1+\delta_1)/2}\right) \\
&= 1 + o(1),
\end{aligned}$$

for $\delta \in (5/8, 3/4)$ and δ_1 chosen sufficiently close (from below) to $1/2$. Also using Lemma 5,

$$\begin{aligned}
E_4 &\le \mathbf{E}\left\{\left(1 + c(\log n)^2 n^{-(1+\delta_1)/2}\right)^T; T > n^\delta\right\} \\
&\le 2\sum_{j>n^\delta} \exp\left\{c(\log n)^2 n^{-(1+\delta_1)/2} j - \kappa j^2/n\right\},
\end{aligned}$$

for some c and sufficiently large n, and arguing as in (41) and (42), we obtain

$$\begin{aligned}
E_4 &= O\left(n^{1-\delta}\exp\left\{c(\log n)^2 n^{\delta-(1+\delta_1)/2} - \kappa n^{2\delta-1}\right\}\right) \\
&= o(1),
\end{aligned}$$

since

$$\delta - (1 + \delta_1)/2 < 2\delta - 1,$$

for $\delta > 5/8$ and $\delta_1 > 0$. Thus

$$\frac{\psi_n(z)}{h_n(z)} = 1 + o(1),$$

and the proof of Theorem 1 is complete. □

REMARK 3. It is worth noticing that we could have proved the asymptotic normality of X_n via characteristic functions, rather than the moment generating functions. The advantage of using the latter is that we get a stronger result (convergence to the normal distribution together with the moments) which allows us, in particular, to establish the asymptotic formula for $\mathrm{Var}(X_n)$, thus reversing the usual course of events in proofs of central limit theorems.

4. Proof of Theorem 2. We first obtain an expression for $c(\nu, \kappa)$, the expected number of cycles produced in the next iteration if the current forest has ν vertices and κ trees. Let $[x^a y^b]f(x, y)$ denote the coefficient of $x^a y^b$ in the double power series expansion of $f(x, y)$.

LEMMA 6.

$$c(\nu, \kappa) = \frac{\nu!}{\binom{\nu}{\kappa}\kappa\nu^{\nu-1}}[x^\nu y^\kappa]\left\{(1-x)\exp(\nu(x+xy))\log\frac{1-x}{1-x-xy}\right\}.$$

PROOF. Let $p(\nu, \kappa; \mu, \lambda, m)$ denote the probability that starting with a random member of $\mathscr{F}_{[\nu], \kappa}$, m cycles are created and their deletion leads to a forest with μ vertices and λ trees. Then

$$p(\nu, \kappa; \mu, \lambda, m) = \frac{\Theta_1\Theta_2}{\Upsilon},$$

where:

(i) $\Upsilon = \binom{\nu}{\kappa}\kappa\nu^{\nu-1}$ (see Lemma 2) is the number of ways of choosing a forest in $\mathscr{F}_{[\nu], \kappa}$ and then choosing a mapping from its roots to its vertices;

(ii) $\Theta_1 = \binom{\nu}{\mu}\binom{\mu}{\lambda}\lambda\mu^{\mu-\lambda-1}(\nu-\mu)^\lambda$ is the number of ways of choosing a μ-subset of $[\nu]$ and then a forest F with these μ vertices and λ trees and a mapping from its roots to the $\nu - \mu$ excluded vertices;

(iii) $\Theta_2 = \sum_{\rho=0}^{\kappa}\binom{\mu}{\kappa-\rho}\binom{\nu-\mu}{\rho}\pi(\nu-\mu, \rho, m)$.

EXPLANATION. The symbol ρ denotes the number of roots of the (ν, κ)-forest which are on one of the m cycles, $\binom{\mu}{\kappa-\rho}\binom{\nu-\mu}{\rho}$ is the number of ways of choosing the κ roots in this way and $\pi(\xi, \rho, m)$ is the number of permutations of $[\xi]$ which have exactly m cycles, each containing at least one member of $[\rho]$. [Thus $\pi(\xi, \rho, m) = 0$, if $m > \rho$.]

We must therefore evaluate

$$\pi(\xi, \rho, m) = \frac{1}{m!} \sum_{\substack{i_1 + \cdots + i_m = \xi - \rho \\ j_1 + \cdots + j_m = \rho \\ i_1 \geq 0, \ldots, i_m \geq 0 \\ j_1 \geq 1, \ldots, j_m \geq 1}} \frac{(\xi - \rho)!}{i_1! \cdots i_m!} \frac{\rho!}{j_1! \cdots j_m!} \prod_{s=1}^{m} (i_s + j_s - 1)!.$$

Here the sth cycle has j_s vertices from $[\rho]$ and i_s vertices from $[\xi] \setminus [\rho]$. Thus we can write

$$\begin{aligned}
\pi(\xi, \rho, m) &= (\xi - \rho)!\rho! \frac{1}{m!} [x^{\xi-\rho} y^{\rho}] \left\{ \sum_{\substack{i_1 \geq 0, \ldots, i_m \geq 0 \\ j_1 \geq 1, \ldots, j_m \geq 1}} \prod_{s=1}^{m} \frac{x^{i_s} y^{j_s}}{i_s! j_s!} (i_s + j_s - 1)! \right\} \\
&= (\xi - \rho)!\rho! \frac{1}{m!} [x^{\xi-\rho} y^{\rho}] \left\{ \left(\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{x^i y^j}{i! j!} (i + j - 1)! \right)^m \right\} \\
&= (\xi - \rho)!\rho! [x^{\xi-\rho} y^{\rho}] \left\{ \frac{1}{m!} \left(\sum_{t=1}^{\infty} \frac{1}{t} \sum_{i=0}^{t-1} \binom{t}{i} x^i y^{t-i} \right)^m \right\} \\
&= (\xi - \rho)!\rho! [x^{\xi-\rho} y^{\rho}] \left\{ \frac{1}{m!} \left(\sum_{t=1}^{\infty} \frac{(x+y)^t}{t} - \sum_{t=1}^{\infty} \frac{x^t}{t} \right)^m \right\} \\
&= (\xi - \rho)!\rho! [x^{\xi-\rho} y^{\rho}] \left\{ \frac{1}{m!} \left(\log \frac{1-x}{1-x-y} \right)^m \right\}.
\end{aligned}$$

Hence

$$\sum_{m=1}^{\infty} \pi(\xi, \rho, m) z^m = (\xi - \rho)!\rho! [x^{\xi-\rho} y^{\rho}] \left\{ \left(\frac{1-x}{1-x-y} \right)^z \right\}$$

Now define

$$\begin{aligned}
\Theta_2(z) &= \sum_{m=1}^{\infty} \Theta_2 z^m \\
&= \sum_{\rho=0}^{\kappa} \binom{\mu}{\kappa - \rho} \binom{\nu - \mu}{\rho} \sum_{m=1}^{\infty} \pi(\nu - \mu, \rho, m) z^m \\
&= \sum_{\rho=0}^{\kappa} \binom{\mu}{\kappa - \rho} \binom{\nu - \mu}{\rho} (\nu - \mu - \rho)!\rho! [x^{\nu-\mu-\rho} y^{\rho}] \left\{ \left(\frac{1-x}{1-x-y} \right)^z \right\}
\end{aligned}$$

$$= (\nu - \mu)! \sum_{\rho=0}^{\kappa} \binom{\mu}{\kappa - \rho} [x^{\nu-\kappa} y^{\kappa}] \left\{ \left(\frac{1-x}{1-x-y} \right)^z x^{\mu-\kappa+\rho} y^{\kappa-\rho} \right\}$$

$$= (\nu - \mu)! [x^{\nu-\kappa} y^{\kappa}] \left\{ \left(\frac{1-x}{1-x-y} \right)^z (x+y)^{\mu} \right\}.$$

Hence (for $\mu \geq 1$)

$$\sum_{m=1}^{\infty} p(\nu, \kappa; \mu, \lambda, m) z^m = \frac{\Theta_1 \Theta_2(z)}{\Upsilon}$$

$$= \frac{\Theta_2(z)}{\Upsilon} \frac{\nu!(\nu-\mu)}{(\nu-\mu)!\mu!} \binom{\mu-1}{\lambda-1} (\nu-\mu)^{\lambda-1} \mu^{\mu-\lambda}.$$

Analogously,

$$\sum_{m=1}^{\infty} p(\nu, \kappa; 0, 0, m) z^m = \frac{\binom{\nu}{\kappa}}{\Upsilon} \sum_{m=1}^{\infty} \pi(\nu, \kappa, m) z^m$$

$$= \frac{\nu!}{\Upsilon} [x^{\nu-\kappa} y^{\kappa}] \left(\frac{1-x}{1-x-y} \right)^z.$$

Summing over λ, we obtain (for $\mu \geq 1$)

$$\sum_{\lambda=1}^{\mu} \sum_{m=1}^{\infty} p(\nu, \kappa; \mu, \lambda, m) z^m = \frac{\Theta_2(z)}{\Upsilon} \frac{\nu!(\nu-\mu)}{(\nu-\mu)!\mu!} \nu^{\mu-1},$$

and summing over μ,

$$\sum_{\mu=1}^{\nu-1} \sum_{\lambda=1}^{\mu} \sum_{m=1}^{\infty} p(\nu, \kappa; \mu, \lambda, m) z^m + \sum_{m=1}^{\infty} p(\nu, \kappa; 0, 0, m) z^m$$

$$= \frac{\nu!}{\Upsilon} [x^{\nu-\kappa} y^{\kappa}] \left\{ \left(\frac{1-x}{1-x-y} \right)^z \sum_{\mu=0}^{\infty} \frac{\nu-\mu}{\mu!} \nu^{\mu-1} (x+y)^{\mu} \right\}$$

$$= \frac{\nu!}{\Upsilon} [x^{\nu-\kappa} y^{\kappa}] \left\{ \left(\frac{1-x}{1-x-y} \right)^z \exp(\nu(x+y))(1-x-y) \right\}.$$

Differentiating with respect to z and then setting $z = 1$ gives

$$c(\nu, \kappa) = \frac{\nu!}{\binom{\nu}{\kappa} \kappa \nu^{\nu-1}} [x^{\nu-\kappa} y^{\kappa}] \left\{ (1-x) \exp(\nu(x+y)) \log \frac{1-x}{1-x-y} \right\}.$$

Then replace y by xy to obtain the statement of the lemma. $\square$

Now let $C(\nu, \kappa)$ denote the expected total number of cycles produced from the current iteration onwards if the current forest has ν vertices and κ trees. Thus

$$\mathbf{E}(Y_n) = C(n, n).$$

Now

$$C(\nu,\kappa) = c(\nu,\kappa) + \sum_{\mu=0}^{\nu-1}\sum_{\lambda=1}^{\mu} p(\nu,\kappa;\mu,\lambda)C(\mu,\lambda) \tag{55}$$
$$= c(\nu,\kappa) + S(\nu),$$

since the double sum in (55) does not depend on κ. However, then (55) implies

$$c(\nu,\kappa) + S(\nu) = c(\nu,\kappa) + \sum_{\mu=0}^{\nu-1}\sum_{\lambda=1}^{\mu} p(\nu,\kappa;\mu,\lambda)(c(\mu,\lambda) + S(\mu))$$

or

$$S(\nu) = s(\nu) + \sum_{\mu=0}^{\nu-1} p_{\nu,\mu}S(\mu),$$

where

$$s(\nu) = \sum_{\mu=0}^{\nu-1}\sum_{\lambda=1}^{\mu} p(\nu,\kappa;\mu,\lambda)c(\mu,\lambda). \tag{56}$$

LEMMA 7.

$$s(\nu) = \frac{\pi}{2} + O(\nu^{-1/2}).$$

PROOF.

$$\begin{aligned}
s(\nu) &= \frac{\nu!}{\nu^\nu}\sum_{\mu=1}^{\nu-1}\sum_{\lambda=1}^{\mu}\left(\frac{\nu-\mu}{\mu}\right)^{\lambda}[x^\mu y^\lambda] \\
&\quad\times\left\{(1-x)\exp(\mu(x+xy))\log\left(\frac{1-x}{1-x-xy}\right)\right\} \\
&= \frac{\nu!}{\nu^\nu}\sum_{\mu=1}^{\nu-1}[x^\mu]\left\{\sum_{\lambda=1}^{\mu}[y^\lambda]\left\{(1-x)\exp\left\{\mu\left(x+x\left(\frac{\nu-\mu}{\mu}\right)y\right)\right\}\right.\right. \\
&\qquad\left.\left.\times\log\left(\frac{1-x}{1-x-x((\nu-\mu)/\mu)y}\right)\right\}\right\} \qquad (57)\\
&= \frac{\nu!}{\nu^\nu}\sum_{\mu=1}^{\nu-1}[x^\mu]\left\{(1-x)\exp\left\{\mu\left(x+x\left(\frac{\nu-\mu}{\mu}\right)1\right)\right\}\right. \\
&\qquad\left.\times\log\left(\frac{1-x}{1-x-x((\nu-\mu)/\mu)1}\right)\right\} \\
&= \frac{\nu!}{\nu^\nu}\sum_{\mu=1}^{\nu-1}[x^\mu]\left\{(1-x)\exp(\nu x)\log\left(\frac{1-x}{1-x\nu/\mu}\right)\right\}.
\end{aligned}$$

We now estimate the summand in (57) via Cauchy's formula:

$$\begin{aligned}&[x^{\mu}]\left\{(1-x)e^{\nu x}\log\left(\frac{1-x}{1-x\nu/\mu}\right)\right\}\\ &\quad=\oint_C\frac{1}{2\pi i}\frac{(1-x)e^{\nu x}\log((1-x)/(1-x\nu/\mu))}{x^{\mu}}\frac{dx}{x},\end{aligned}\tag{58}$$

where C is the circle of radius $r=\mu/\nu$ with center at the origin in the complex x-plane ($x=r$ is the saddle point of $e^{\nu x}/x^{\mu}$). Here

$$\Im(\log z)=\arg z\in(-\pi,\pi).$$

Notice that $\log((1-x)/(1-x/r))$ is analytic in the disk $|x|\le r$, except at $x=r$, which is on C. So, to be precise, we apply Cauchy's formula to a contour that is C with a small circular dent which leaves the point $x=r$ outside, and then let the radius of the dent go to 0.

Since the μth summand (times $\nu!/\nu^{\nu}$) is

$$\sum_{\lambda=1}^{\mu}p(\nu,\kappa;\mu,\lambda)c(\mu,\lambda)\le\nu p_{\nu,\mu},$$

the inequality (38) shows that the overall contribution to $s(\nu)$ of $\mu\le\nu-(\log\nu)\sqrt{\nu}$ is

$$O\left(\nu\exp\left\{-\tfrac{1}{2}(\log\nu)^2\right\}\right)=O(\nu^{-K})\quad\text{for all } K>0.$$

So, substituting $\mu=\nu-\alpha\sqrt{\nu}$, we concentrate on $\alpha\le\log\nu$. We substitute

$$x=re^{i\theta},\qquad -\pi\le\theta<\pi,$$

into the integral in (58).

We will first examine the case of large θ. Now

$$\frac{1-re^{i\theta}}{1-e^{i\theta}}=\frac{r+1}{2}+\frac{i(1-r)}{2}\cot\left(\frac{\theta}{2}\right)$$

and we deduce that if $\varepsilon=(\log\nu)/\nu^{1/2}$, then

$$\left|\log\left(\frac{1-re^{i\theta}}{1-e^{i\theta}}\right)\right|=O(\log\nu),$$

uniformly for $|\theta|\ge\varepsilon$ and all μ. Further,

$$\left|\frac{\exp(\nu x)}{x^{\mu}}\right|=\frac{\exp(\nu r)}{r^{\mu}}\exp\{-\mu(1-\cos\theta)\},\tag{59}$$

where [cf. (63)] uniformly for $\alpha\le\log\nu$,

$$\frac{\exp(\nu r)}{r^{\mu}}=\exp\left\{\nu-\frac{1}{2}\alpha^2+O(\alpha^3\nu^{-1/2})\right\}\le\exp\nu,$$

if ν is sufficiently large. The second factor in (59) is at most

$$\exp\{-c\mu\theta^2\}\le\exp\{-c\mu\varepsilon^2\}\le\exp\left\{-c'(\log\nu)^2\right\}$$

($c > c' > 0$). So if C_ε represents the portion of C with $|\theta| \geq \varepsilon$, then

$$\left| \oint_{C_\varepsilon} \frac{(1-x)e^{\nu x} \log((1-x)/(1-x\nu/\mu))}{x^\mu} \frac{dx}{x} \right| = O(e^\nu \nu^{-K}), \tag{60}$$

for any constant $K > 0$.

Turn to the dominant contribution that comes from small θ, that is, $|\theta| \leq \varepsilon$, or (substituting $\theta = u/\sqrt{\nu}$) from $|u| \leq \log \nu$. We have

$$\begin{aligned} 1 - re^{i\theta} &= 1 - r - ri\theta + O(\theta^2) \\ &= \frac{\alpha}{\sqrt{\nu}} - \frac{\mu}{\nu^{3/2}} ui + O(\theta^2) \\ &= \frac{1}{\sqrt{\nu}}(\alpha - iu) + O\left(\frac{(\alpha + |u|)|u|}{\nu} \right) \end{aligned}$$

and

$$1 - e^{i\theta} = -\frac{iu}{\sqrt{\nu}} + O\left(\frac{u^2}{\nu} \right),$$

both estimates being uniform over $|u| \leq \log \nu$. Thus

$$(1 - re^{i\theta}) \log\left(\frac{1 - re^{i\theta}}{1 - e^{i\theta}} \right) = \frac{1}{\sqrt{\nu}}(\alpha - iu)\log\left(\frac{\alpha - iu}{-iu} \right) + O\left(\frac{R(\alpha, u)}{\nu} \right),$$

where

$$\begin{aligned} R(\alpha, u) &= |u|(\alpha + |u|)\left(1 + \log\left(\frac{\alpha + |u|}{|u|} \right)\right) \\ &\leq |u|(\alpha + |u|)\left(1 + \frac{\alpha}{|u|}\right) \\ &\leq (\alpha + |u|)^2. \end{aligned} \tag{61}$$

Also, since $\log x = O(\sqrt{x})$, for $x \geq 1$, and $\log z = \log|z| + O(1)$ ($z \in \mathbb{C}$), we have

$$\left| (\alpha - iu) \log\left(\frac{\alpha - iu}{-iu} \right) \right| = O(S(\alpha, u)),$$

where

$$S(\alpha, u) = \frac{(\alpha + |u|)^{3/2}}{|u|^{1/2}}. \tag{62}$$

The other factor in the integral is

$$
\begin{aligned}
\frac{\exp(\nu x)}{x^{\mu}} &= \exp\{\nu x - \mu \log x\} \\
&= \exp\{\mu \exp(i\theta) - \mu(\log r + i\theta)\} \\
&= \exp\left\{\mu - \mu \log r - \frac{\mu}{2}\theta^2 + O(\mu|\theta|^3)\right\} \qquad (63)\\
&= \exp\left\{\nu - \frac{\alpha^2}{2} - \frac{u^2}{2} + O\left(\frac{(\alpha + |u|)^3}{\sqrt{\nu}}\right)\right\}.
\end{aligned}
$$

Also

$$\frac{dx}{x} = i\,d\theta = \frac{i\,du}{\sqrt{\nu}}.$$

So the *real* part of the integrand in (58) (when the variable of integration is u, not x or θ) can be expressed as

$$
\begin{aligned}
&\frac{\exp(\nu - \alpha^2/2)}{2\pi\nu} \exp\left(-\frac{u^2}{2}\right)\left(\alpha \log\sqrt{\frac{\alpha^2 + u^2}{u^2}} + u \arctan\left(\frac{\alpha}{u}\right)\right) \qquad (64)\\
&\quad + O\left(\frac{Q(\alpha, u)}{\nu^{3/2}}\right).
\end{aligned}
$$

Here, using (61)–(63),

$$
\begin{aligned}
Q(\alpha, u) &= \exp\left\{\nu - \frac{\alpha^2}{2} - \frac{u^2}{2}\right\}\left(R(\alpha, u) + S(\alpha, u)(\alpha + |u|)^3\right) \\
&= O\left(\exp\left\{\nu - \frac{\alpha^2}{2} - \frac{u^2}{2}\right\}\left((\alpha + |u|)^2 + \frac{(\alpha + |u|)^{9/2}}{u^{1/2}}\right)\right) \\
&= O\left(\exp\left\{\nu - \frac{\alpha^2}{2} - \frac{u^2}{2}\right\}\left(\alpha^2 + u^2 + \alpha^{9/2}|u|^{-1/2} + u^4\right)\right).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\int_{-\infty}^{\infty} Q(\alpha, u)\, du &= O\left(\exp(\nu - \alpha^2/2)(1 + \alpha^2 + \alpha^{9/2})\right) \\
&= O\left(\exp(\nu - \alpha^2/2)(1 + \alpha^5)\right).
\end{aligned}
$$

Also, since $\alpha \le \log \nu$,

$$\int_{|u| \ge \log \nu} \exp\left(-\frac{u^2}{2}\right)\left(\alpha \log \sqrt{\frac{\alpha^2+u^2}{u^2}} + u \arctan\left(\frac{\alpha}{u}\right)\right) du$$

$$= O\left(\int_{|u| \ge \log \nu} |u| \exp\left(-\frac{u^2}{2}\right) du\right)$$

$$= O\left(\exp\left\{-\frac{(\log \nu)^2}{2}\right\}\right).$$

Therefore the real part of the integral in (58) becomes

$$\frac{\exp(\nu - \alpha^2/2)}{2\pi\nu} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right)\left(\alpha \log \sqrt{\frac{\alpha^2+u^2}{u^2}} + u \arctan\left(\frac{\alpha}{u}\right)\right) du$$

$$+ O\left(\frac{(1+\alpha^5)\exp(\nu - \alpha^2/2)}{\nu^{3/2}}\right).$$

Summing this expression over $\nu - (\log \nu)\sqrt{\nu} \le \mu \le \nu - 1$, we obtain, via the Euler summation formula,

$$s(\nu) = \frac{\nu!}{\nu^\nu} \frac{e^\nu}{2\pi\sqrt{\nu}} \int_{\alpha=0}^{\infty} \exp\left(-\frac{\alpha^2}{2}\right)$$

$$\times \int_{u=-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right)\left(\alpha \log \sqrt{\frac{\alpha^2+u^2}{u^2}} + u \arctan\left(\frac{\alpha}{u}\right)\right) du\, d\alpha$$

$$+ O\left(\frac{1}{\sqrt{\nu}}\right).$$

[By Stirling's formula, $\nu! = \sqrt{2\pi\nu}(\nu/e)^\nu(1 + O(\nu^{-1}))$.]

Setting $\alpha = \rho \cos \phi$, $u = \rho \sin \phi$ and $du\, d\alpha = \rho\, d\rho\, d\phi$ we see that the leading term in $s(\nu)$ is

$$\frac{1}{\sqrt{2\pi}}\left(\int_0^\infty \rho^2 \exp\left(-\frac{\rho^2}{2}\right) d\rho\right)$$

$$\times \left(\int_{-\pi/2}^{\pi/2} \left(\cos \phi \log(|\csc \phi|) + |\sin \phi|\left(\frac{\pi}{2} - |\phi|\right)\right) d\phi\right) = \frac{\pi}{2},$$

and the lemma follows. □

Consequently

$$S(\nu) = \frac{\pi}{2} + O(\nu^{-1/2}) + \sum_{\mu=0}^{\nu-1} p_{\nu,\mu} S(\mu), \tag{65}$$

but

$$\mathbf{E}(X_\nu) = 1 + \sum_{\mu=0}^{\nu-1} p_{\nu,\mu} \mathbf{E}(X_\mu). \tag{66}$$

We prove by induction that

$$S(\nu) = \frac{\pi}{2} \cdot \mathbf{E}(X_\nu) + O(\log \nu). \tag{67}$$

Let A be the hidden constant in (65) and $\zeta_\nu = |S(\nu) - \pi \mathbf{E}(X_\nu)/2|$. Then (65) and (66) imply

$$\zeta_\nu \le \sum_{\mu=0}^{\nu-1} p_{\nu,\mu} \zeta_\mu + A\nu^{-1/2}.$$

Assume inductively that $\zeta_\mu \le B \log(\mu + 1)$, for $\mu < \nu$, where B can be adjusted to handle values of $\nu \le 2$. Then

$$\zeta_\nu \le B \sum_{\mu=0}^{\nu-1} p_{\nu,\mu} \log(\mu + 1) + A\nu^{-1/2}.$$

Substituting

$$\begin{aligned} \log(\mu + 1) &= \log(\nu + 1) + \log\left(1 - \frac{\nu - \mu}{\nu + 1}\right) \\ &\le \log(\nu + 1) - \frac{\nu - \mu}{\nu + 1} \end{aligned}$$

yields

$$\begin{aligned} \zeta_\nu &\le B \log(\nu + 1) - \frac{B}{\nu + 1} \sum_{\mu=0}^{\nu-1} (\nu - \mu) p_{\nu,\mu} + A\nu^{-1/2} \\ &\le B \log(\nu + 1), \end{aligned}$$

if, in addition to the above-mentioned restriction on B, we choose

$$B \ge \sup_{\nu \ge 2} \frac{\nu^{1/2} + 1}{\sum_{\mu=0}^{\nu-1} (\nu - \mu) p_{\nu,\mu}}.$$

That the supremum is finite follows from (3). This completes the inductive proof of (67).

It remains to notice that

$$\mathbf{E}(Y_n) = C(n, n) = S(n) + c(n, n) \quad \text{and} \quad c(n, n) = O(\log n),$$

since $c(n, n)$ is the expected number of cycles in a random mapping from $[n]$ to $[n]$. Indeed,

$$\begin{aligned} c(n,n) &= \sum_{k=1}^{n} \binom{n}{k}(k-1)!\frac{n^{n-k}}{n^n} \\ &= \sum_{k=1}^{n} \frac{1}{k}\frac{(n)_k}{n^k} \\ &\le \sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

□

5. Proof of Theorem 3. First of all let $D(i)$ denote the number of iterations that expose the trader i as a root before i is finally deleted. Let $M(i)$ denote the number of traders remaining (including i) at the end of the iteration which makes i a root for the $D(i)$th time [with $M(i) = n$, if $D(i) = 0$]. By symmetry, $(D(i), M(i))$ are equidistributed for all i.

LEMMA 8.

$$\mathbf{E}(R(i)) = \mathbf{E}(D(i)) + \mathbf{E}\left(\frac{n - D(i) + 1}{M(i) + 1}\right). \tag{68}$$

PROOF. The trader i will go away with the best among the $M(i)$ goods. This good is preceded (on i's preference list) by all $D(i)$ goods lost for i and some of the remaining $n - D(i) - M(i)$ goods. Conditioned on $D(i)$ and $M(i)$, the latter has the same distribution as an occupancy number of a cell in the uniform allocation model with $n - D(i) - M(i)$ indistinguishable balls and $M(i) + 1$ cells. Thus

$$\begin{aligned} \mathbf{E}(R(i)|D(i), M(i)) &= 1 + D(i) + \frac{n - D(i) - M(i)}{M(i) + 1} \\ &= D(i) + \frac{n - D(i) + 1}{M(i) + 1}. \end{aligned}$$

Now remove the conditioning to obtain the lemma. □

We proceed to estimate the expected values of the quantities in the r.h.s. of (68).

LEMMA 9.

$$\limsup_{n \to \infty} \mathbf{E}(D(1)) \le 1.$$

[*In fact*, $\lim_{n\to\infty} \mathbf{E}(D(1)) = 1$, *but we do not need this*.]

PROOF. Let $T(n) = \sum_{i=1}^{n} D(i)$. We prove the lemma by showing that

$$\mathbf{E}(T(n)) \leq n + O(\sqrt{n}). \tag{69}$$

Now $T(n)$ is the total number of trees produced during the course of the algorithm, not counting the n trivial trees at the very start of the process. Let $t(\nu, \kappa)$ denote the expected number of trees produced starting with a random forest from $\mathscr{F}_{[\nu], \kappa}$, not counting the κ trees we begin with. Then if $\nu \geq 1$,

$$\begin{aligned} t(\nu, \kappa) &= \sum_{\mu=1}^{\nu-1} \sum_{\lambda=1}^{\mu} p(\nu, \kappa; \mu, \lambda)(\lambda + t(\mu, \lambda)) \\ &= \sum_{\mu=1}^{\nu-1} \sum_{\lambda=1}^{\mu} p(\nu, \kappa; \mu, \lambda)\lambda + \sum_{\mu=1}^{\nu-1} \sum_{\lambda=1}^{\mu} p(\nu, \kappa; \mu, \lambda) t(\mu, \lambda). \end{aligned} \tag{70}$$

Now the first sum in (70) is independent of κ and one can easily check by direct computation that conditional on $\nu, \mu \geq 1$, the distribution of λ is $1 + B(\mu - 1, 1 - \mu/\nu)$, where $B(\cdot, \cdot)$ stands for a binomial random variable. Hence

$$\begin{aligned} \sum_{\mu=1}^{\nu-1} \sum_{\lambda=1}^{\mu} p(\nu, \kappa; \mu, \lambda)\lambda &= \sum_{\mu=1}^{\nu-1} p_{\nu, \mu}\left(1 + (\mu - 1)\left(1 - \frac{\mu}{\nu}\right)\right) \\ &\leq 1 + \sum_{\mu=1}^{\nu-1} p_{\nu, \mu}(\nu - \mu). \end{aligned} \tag{71}$$

We see from (70) that $t(\nu, \kappa)$ is independent of κ and so we use $t(\nu)$ from now on. So (71) implies

$$t(\nu) \leq 1 + \sum_{\mu=0}^{\nu-1} p_{\nu, \mu}(\nu - \mu) + \sum_{\mu=1}^{\nu-1} p_{\nu, \mu} t(\mu).$$

Consequently,

$$\begin{aligned} t(\nu) &\leq \mathbf{E}(X_\nu) + \nu \\ &= \nu + O(\sqrt{\nu}). \end{aligned}$$

It remains only to notice that $\mathbf{E}(T(n)) = t(n)$. □

It follows from Lemmas 8 and 9 that

$$\mathbf{E}(R_n) = O(n) + n \sum_{i=1}^{n} \mathbf{E}\left(\frac{1}{M(i) + 1}\right). \tag{72}$$

Now let $M'(i) \leq M(i)$ denote the number of members present at the beginning of the iteration which results in the elimination of member i. The upper bound in the theorem follows directly from the following lemma.

LEMMA 10. *Uniformly in* $\omega \in \Omega$,

$$\sum_{i=1}^{n}\left(\frac{1}{M'(i)+1}\right) \le (1+o(1))\log n.$$

PROOF. Let Δ_k denote the number of members deleted at iteration k. Then

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{1}{M'(i)+1}\right) &= \frac{\Delta_1}{n+1} + \frac{\Delta_2}{n-\Delta_1+1} + \frac{\Delta_3}{n-\Delta_1-\Delta_2+1} + \cdots \\ &\le \sum_{t=1}^{n+1}\frac{1}{t} \\ &= (1+o(1))\log n. \qquad \square\end{aligned}$$

For the lower bound we shall assume (following a method of [10], [11] and [12]) that the random preferences are induced by an $n \times n$ matrix $[x_{i,j}]$, where the $x_{i,j}$ are i.i.d. uniform $[0,1]$ random variables. Thus member i orders the goods of other members (including his own) in the increasing order of the entries of his own row. The core allocation is an ordered sequence of groups of members of sizes $l_1, l_2, \dots, l_r$, where $l_1 + l_2 + \cdots + l_r = n$ and a sequence of permutations $\pi_1, \pi_2, \dots, \pi_r$ on each of the groups which must satisfy the following *necessary* condition. If a member i belongs to the sth group, then i prefers $\pi_s(i)$ to anything in the groups $t \ge s$.

Thus, for every $m \ge n$ and $\tau \ge 1$,

$$\begin{aligned}&\mathbf{P}(R \le m) \\ &\quad \le n!\sum \int_{x_1=0}^{1} \cdots \int_{x_n=0}^{1} \prod_{i=1}^{n}(1-x_i)^{L_i-1} \\ &\qquad \times \mathbf{P}\left(\sum_{i=1}^{n} B(n-L_i, x_i) \le m-n\right) dx_1\, dx_2 \cdots dx_n \\ &\qquad + \mathbf{P}(X_n \ge \tau),\end{aligned} \tag{73}$$

where the $B(n - L_i, x_i)$, $i = 1, 2, \dots, n$, are independent, and the sum is over all $1 \le r \le \tau, l_1, l_2, \dots, l_r$ such that $l_1 + l_2 + \cdots + l_r = n$ and

$$L_i = \begin{cases} l_1 + \cdots + l_r, & \text{if } 1 \le i \le l_1, \\ l_2 + \cdots + l_r, & \text{if } l_1 + 1 \le i \le l_1 + l_2, \\ \vdots & \\ l_r, & \text{if } l_1 + \cdots + l_{r-1} + 1 \le i \le n. \end{cases}$$

EXPLANATION. Having fixed $r, l_1, l_2, \dots, l_r$, the number of ways to partition $[n]$ into the ordered sequence of subsets of cardinality $l_1, \dots, l_r$ and then to choose a sequence of permutations $\pi_1, \dots, \pi_r$, one for each set, is

$$\binom{n}{l_1, l_2, \dots, l_r} l_1! \cdots l_r! = n!.$$

Given the values $x_1, \dots, x_n$ of the member's assignments, $(1 - x_i)^{L_i - 1}$ is the conditional probability that member i prefers his choice to those in the permutations $\pi_{i+1}, \dots, \pi_r$ and to other possible choices within his group. Finally, given $x_1, \dots, x_n, \{R(i) - 1: 1 \le i \le n\}$ is distributed as $\{B(n - L_i, x_i): 1 \le i \le n\}$.

Letting I denote the n-fold integral in (73), we estimate it from above by applying the Chernoff method to bound $\mathbf{P}(\sum_{i=1}^n B(n - L_i, x_i) \le m - n)$. For any $0 < z < 1$,

$$\begin{aligned}
I &\le \int_{x_1=0}^{1} \cdots \int_{x_n=0}^{1} \prod_{i=1}^{n} (1 - x_i)^{L_i - 1} \frac{\prod_{i=1}^n \mathbf{E}(z^{B(n-L_i, x_i)})}{z^{m-n}} \, dx_1 \, dx_2 \cdots dx_n \\
&= \int_{x_1=0}^{1} \cdots \int_{x_n=0}^{1} z^{n-m} \prod_{i=1}^{n} (1 - x_i)^{L_i - 1} (z x_i + 1 - x_i)^{n - L_i} \, dx_1 \, dx_2 \cdots dx_n \\
&\le z^{n-m} \prod_{i=1}^{n} \int_{x_i=0}^{\infty} \exp\{-x_i(L_i - 1) - x_i(1 - z)(n - L_i)\} \, dx_i \\
&= z^{n-m} \prod_{i=1}^{n} (L_i - 1 + (1 - z)(n - L_i))^{-1}.
\end{aligned}$$

The bound depends on z. Not surprisingly, we select it so as to get the best estimate. Denoting $\lambda_s = \sum_{t=s}^r l_t$, we proceed with

$$\begin{aligned}
I &\le z^{n-m} \prod_{s=1}^{r} (\lambda_s - 1 + (1 - z)(n - \lambda_s))^{-l_s} \\
&= z^{n-m} \exp\left\{ - \sum_{s=1}^{r} l_s \log(n - 1 - z(n - \lambda_s)) \right\} \\
&\le z^{n-m} \exp\left\{ - \int_0^n \log(n - 1 - z(n - x)) \, dx \right\} \quad (\text{if } z < 1 - n^{-1}) \qquad (74) \\
&\le z^{n-m} \exp\{-z^{-1}(n - 1)\log(n - 1) + n + z^{-1}(n - 1 - zn) \\
&\qquad\qquad \times \log(n - 1 - zn)\} \\
&\le \exp\{-n \log n + n - (\sigma - \gamma + O(n^{-\sigma}))n^{1-\sigma} \log n + O(\log n)\},
\end{aligned}$$

if $z = 1 - n^{-\sigma}$, $\sigma \in (0, 1)$ and $m = \gamma n \log n$. Let us choose $\tau = \lfloor n^\delta \rfloor$, $\delta \in (1/2, 1)$. Then the term $\mathbf{P}(X_n \ge \tau)$ in (73) is subexponentially small by Lemma 5. Next observe that the number of terms in the summation in (73) is

$$\sum_{r=1}^{\tau} \binom{n - 1}{r - 1} \le n^\tau.$$

So $n!$ times the sum is bounded by

$$n! n^{\tau} \exp\{-n \log n + n - (\sigma - \gamma + O(n^{-\sigma}))n^{1-\sigma} \log n + O(\log n)\}$$
$$\leq \exp\{n^{\delta} \log n - (\sigma - \gamma + O(n^{-\sigma}))n^{1-\sigma} \log n + O(\log n)\},$$

which is subexponentially small too, if $\delta < 1 - \sigma$ and $\sigma > \gamma$. If $\gamma < 1/2$, the conditions are met by choosing $\delta \in (1/2, 1)$ and $\sigma \in (\gamma, 1 - \delta)$ both sufficiently close to $1/2$.

This completes the proof of Theorem 3. □

NOTE. It seems reasonable to guess that $\mathbf{E}(R_n) \approx cn \log n$, but we are at a loss as to what the actual value $c \in [1/2, 1]$ is. It would also be very interesting to prove that R_n is concentrated around $\mathbf{E}(R_n)$.

Incidentally and importantly, the idea of generating random preferences via the matrix $X = \{x_{ij}\}$ makes it clear that the deletion algorithm can be used as a greedy heuristic for the $n \times n$ linear assignment problem with cost matrix X. The expected value of the assignment delivered by the algorithm is $n^{-1}\mathbf{E}(R_n)$; thus it is asymptotically between $\frac{1}{2} \log n$ and $\log n$. Is the algorithm better than two classic greedy algorithms which deliver expected values asymptotic to $\log n$?

APPENDIX

PROOF OF LEMMA 3. We let $\hat{\xi}_\nu = (\nu^\nu/\nu!)\hat{\eta}_\nu$ and show

$$\hat{\xi}_\nu - \gamma_\nu - \sum_{\mu=1}^{\nu-1} \hat{\xi}_\mu \left(\frac{\nu}{\mu}\right)^{\mu} \left(\frac{\nu-\mu}{\nu}\right) = O(e^\nu \nu^{\delta-(N+1)/2} \log \nu), \tag{75}$$

which is equivalent to (12). Let

$$S(\nu, a, u) = \sum_{\mu=1}^{\nu-1} \left(\frac{e\nu}{\mu}\right)^{\mu} \left(\frac{\nu-\mu}{\nu}\right)^{u+1} \mu^{-a}.$$

Express

$$\begin{aligned} \log \mu &= \log \nu + \log\left(1 - \frac{\nu-\mu}{\nu}\right) \qquad (1 \leq \mu < \nu) \\ &= \log \nu - \sum_{u=1}^{\infty} \frac{1}{u} \left(\frac{\nu-\mu}{\nu}\right)^{u} \end{aligned} \tag{76}$$

and

$$\begin{aligned} \mu^{-a} &= \nu^{-a}\left(1 + \frac{\mu-\nu}{\nu}\right)^{-a} \\ &= \nu^{-a} \sum_{t=0}^{\infty} \binom{a+t-1}{t} \left(\frac{\nu-\mu}{\nu}\right)^{t}. \end{aligned} \tag{77}$$

Then

$$S(\nu, a, u) = \sum_{t=0}^{\infty} \frac{1}{\nu^{u+a+t+1}} \binom{a+t-1}{t} R(\nu, t+u), \tag{78}$$

where

$$R(\nu, \tau) = \sum_{\mu=1}^{\nu} \left(\frac{e\nu}{\mu}\right)^{\mu} (\nu - \mu)^{\tau+1}.$$

With this notation, the sum in (75) becomes

$$\sum_{i=0}^{N} \left(\hat{\alpha}_i S(\nu, \delta_i, 0) + \hat{\beta}_i \left(S(\nu, \delta_i, 0) \log \nu - \sum_{u=1}^{\infty} u^{-1} S(\nu, \delta_i, u) \right) \right), \tag{79}$$

where $\delta_i = (i-1)/2 - \delta$.

We now proceed to obtain an asymptotic expansion for $R(\nu, t)$. First of all let $F(x) = (e\nu/x)^x$ and $G(x) = \log F(x)$. The Taylor expansion of G at $x = \nu$ is given by

$$G(x) = \nu - \frac{1}{2\nu}(x-\nu)^2 - \sum_{r=3}^{\infty} \frac{(\nu - x)^r}{r(r-1)\nu^{r-1}}. \tag{80}$$

Thus

$$F(x) = \exp(\nu) \exp\left\{ -\frac{(x-\nu)^2}{2\nu} \right\} \left(1 + \sum_{r=3}^{\infty} \chi_r \frac{(\nu - x)^r}{\nu^{r-1}} \right), \tag{81}$$

where, in particular,

$$\chi_3 = -\frac{1}{6}, \qquad \chi_4 = -\frac{1}{12}, \qquad \chi_5 = -\frac{1}{20}.$$

Using (81) in the definition of $R(\nu, t)$, we obtain

$$\begin{aligned} R(\nu, t) &= \exp(\nu) \sum_{\mu=1}^{\nu-1} \exp\left\{ -\frac{(\mu - \nu)^2}{2\nu} \right\} (\nu - \mu)^{t+1} \\ &\quad + \exp(\nu) \sum_{r=3}^{\infty} \frac{\chi_r}{\nu^{r-1}} \sum_{\mu=1}^{\nu} \exp\left\{ -\frac{(\mu-\nu)^2}{2\nu} \right\} (\nu - \mu)^{r+t+1}. \end{aligned} \tag{82}$$

We now need a result from Knuth and Pittel [5, Lemma 1]. For every fixed $y > -1$ and $a \geq 0$,

$$\begin{aligned} \sum_{\mu=1}^{\infty} \mu^y \exp\left\{ -\frac{\mu^2}{2\nu} \right\} &= 2^{(y-1)/2} \Gamma\left(\frac{y+1}{2} \right) \nu^{(y+1)/2} \\ &\quad + \sum_{i=0}^{a} \frac{(-1)^i}{2^i i!} \frac{\zeta(-y-2i)}{\nu^i} + O(\nu^{-a-1}). \end{aligned} \tag{83}$$

Here Γ is the gamma function and ζ is the Riemann ζ-function.

Now for fixed t, r,

$$\sum_{\mu=1}^{\nu-1} \exp\left\{-\frac{(\nu-\mu)^2}{2\nu}\right\}(\nu-\mu)^{r+t+1}$$
$$= \sum_{\mu=1}^{\nu-1} \exp\left\{-\frac{\mu^2}{2\nu}\right\}\mu^{r+t+1}$$
$$= \sum_{\mu=1}^{\infty} \exp\left\{-\frac{\mu^2}{2\nu}\right\}\mu^{r+t+1} + O\left(\exp\left(-\frac{\nu}{3}\right)\right).$$

Thus (82) and (83) give that, for every $a \geq 0$,

$$R(\nu t) = e^{\nu}\Bigg(2^{t/2}\Gamma\left(\frac{t+2}{2}\right)\nu^{(t+2)/2} + \sum_{i=0}^{a} \frac{(-1)^i}{2^i i!}\frac{\zeta(-t-1-2i)}{\nu^i}$$
$$\text{(84)} \qquad + \sum_{r=3}^{2a+t+5} \frac{\chi_r}{\nu^{r-1}}\Bigg(2^{(r+t)/2}\Gamma\left(\frac{r+t+2}{2}\right)\nu^{(r+t+2)/2}$$
$$+ \sum_{i=0}^{a-r+1} \frac{(-1)^i}{2^i i!}\frac{\zeta(-r-t-1-2i)}{\nu^i}\Bigg) + O(\nu^{-(a+1)})\Bigg).$$

Thus for any $A \geq 0$,

$$\text{(85)} \qquad R(\nu, t) = e^{\nu}\nu^{(t+2)/2}\left(\sum_{j=0}^{A} \rho_{t,j}\nu^{-j/2} + O(\nu^{-(A+1)/2})\right),$$

for some constants $\rho_{t,j}$. In particular,

$$R(\nu, 0) = e^{\nu}\nu\left(1 - \frac{\sqrt{2\pi}}{4}\nu^{-1/2} - \frac{3}{4}\nu^{-1} + O(\nu^{-3/2})\right),$$
$$\text{(86)} \qquad R(\nu, 1) = e^{\nu}\nu^{3/2}\left(\frac{\sqrt{2\pi}}{2} - \frac{4}{3}\nu^{-1/2} + O(\nu^{-1})\right),$$
$$R(\nu, 2) = e^{\nu}\nu^2(2 + O(\nu^{-1/2})).$$

To compute these quantities we needed to know certain values of the gamma and zeta functions. We remind the reader that, for nonnegative integer n, (i) $\Gamma(n+1) = n!$, $\Gamma(n+\frac{1}{2}) = ((2n)!/n!2^{2n})\sqrt{\pi}$ and (ii) $\zeta(-2(n+1)) = 0$, $\zeta(-1) = -1/12$.

It turns out that $R(\nu, t)$ is essentially of order $e^{\nu}\nu^{(t+2)/2}$ for moderately large t's as well. Indeed, by definition of $R(\nu, t)$ and (80),

$$R(\nu, t) \le \exp(\nu) \sum_{\mu=1}^{\nu} \exp\left\{-\frac{1}{2\nu}(\nu-\mu)^2\right\}(\nu-\mu)^{t+1}$$

$$\begin{aligned}(87)\qquad &= O\left(\exp(\nu)\left(\int_0^{\infty} f(x)\,dx + \max_{u\ge 0} f(u)\right)\right) \qquad \left[f(x) = \exp\left(-\frac{x^2}{2\nu}\right)x^{t+1}\right]\\ &= O\left(\exp(\nu)\left(2^{t/2}\Gamma\left(\frac{t}{2}+1\right)\nu^{(t+2)/2} + \left(\frac{t+1}{e}\right)^{(t+1)/2}\nu^{(t+1)/2}\right)\right)\\ &= O(R_1(\nu, t)),\end{aligned}$$

where $R_1(\nu, t) = e^{\nu}\nu^{(t+2)/2}(t/e)^{(t+1)/2}$.

We will need yet another bound for $R(\nu, t)$ that holds for $t \ge 3\nu/4$. The function $(e\nu/x)^x(\nu - x)^{t+1}$, $x \in (0, \nu]$, attains its maximum at the root $\bar{x} \in (0, \nu)$ of the equation

$$\log(\nu/x) = (t+1)/(\nu - x).$$

Clearly

$$\log(\nu/\bar{x}) \ge (t+1)/\nu,$$

so that

$$(88)\qquad \bar{x} \le \nu \exp(-(t+1)/\nu).$$

Therefore

$$\begin{aligned}(89)\qquad \left(\frac{e\nu}{\bar{x}}\right)^{\bar{x}}(\nu-\bar{x})^{t+1} &= \exp\left\{\bar{x} + (t+1)\left(\log(\nu-\bar{x}) + \frac{\bar{x}}{\nu-\bar{x}}\right)\right\}\\ &\le \nu^{t+1}\exp\left\{\bar{x} + \frac{(t+1)(\bar{x}/\nu)^2}{1-(\bar{x}/\nu)}\right\}.\end{aligned}$$

Here

$$\begin{aligned}\bar{x} + \frac{(t+1)(\bar{x}/\nu)^2}{1-(\bar{x}/\nu)} &\le \nu\left(\exp\left(-\frac{t+1}{\nu}\right) + \frac{t+1}{\nu}\,\frac{\exp(-2(t+1)/\nu)}{1-\exp(-(t+1)/\nu)}\right)\\ &\le 2\exp\left(-\frac{t+1}{\nu}\right)\nu\\ &\le \left(2\exp\left(-\frac{3}{4}\right)\right)\nu\\ &\le 0.95\nu.\end{aligned}$$

Using (89) and the last estimate,

$$R(\nu,t) \le R_2(\nu,t) := \nu^{t+2}\exp\{0.95\nu\}, \qquad t \ge 3\nu/4. \tag{90}$$

Finally, $\bar{x} \le 1$ for $t \ge \nu \log \nu$ [see (88)]. Therefore, for $t \ge 2\nu \log \nu$ we have

$$\begin{aligned} R(\nu,t) &\le \nu(e\nu)(\nu-1)^{t+1} \\ &\le R_3(\nu,t) := 3\nu^{t+3}\exp(-(t+1)/\nu). \end{aligned} \tag{91}$$

We use (85), (87), (90) and (91) to find an asymptotic expansion for $S(\nu, a, u)$ given in (78) for fixed a and u. [The reviewer observed correctly that truncating expansions (76) and (77) via the Taylor formula would allow us to get the desired expansion by using (84) only. However, the estimates (86), (89) and (90) are still needed to treat $\Sigma_{u\ge 1} u^{-1}S(\nu, a, u)$. An advantage of our approach is that it works for this sum in exactly the same way. So we can afford just to sketch the corresponding derivation that would have been quite protracted in this more complex case.] Fix $T > 1$ and write

$$\sum_{t\ge T}\frac{1}{\nu^{u+a+t+1}}\binom{a+t-1}{t}R(\nu,t+u) = O(\Sigma_1+\Sigma_2+\Sigma_3),$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{t=T}^{3\nu/4}\frac{1}{\nu^{u+a+t+1}}\binom{a+t-1}{t}R_1(\nu,t+u), \\ \Sigma_2 &= \sum_{t=3\nu/4}^{2\nu\log\nu}\frac{1}{\nu^{u+a+t+1}}\binom{a+t-1}{t}R_2(\nu,t+u), \\ \Sigma_3 &= \sum_{t\ge 2\nu\log\nu}\frac{1}{\nu^{u+a+t+1}}\binom{a+t-1}{t}R_3(\nu,t+u). \end{aligned}$$

The ratio of the $(t+1)$st term to the tth term in Σ_1 is at most

$$\begin{aligned} \frac{1}{\nu}\frac{a+t}{t+1}\nu^{1/2}\frac{t+u+1}{(t+u)^{1/2}} &= \left(\frac{t}{\nu}\right)^{1/2}\left(1+O\left(\frac{1}{T}\right)\right) \\ &\le \left(\frac{3}{4}\right)^{1/2}\left(1+O\left(\frac{1}{T}\right)\right), \\ &\le 0.87, \end{aligned}$$

if T is sufficiently large. Then [see (85)]

$$\begin{aligned} \Sigma_1 &= O\left(\frac{1}{\nu^{u+a+T+1}}\binom{a+T-1}{T}R_1(\nu,T+u)\right) \\ &= O(e^{\nu}\nu^{-(u+2a+T)/2}). \end{aligned} \tag{92}$$

By (90), the generic summand in Σ_2 is of order

$$O\left(\frac{1}{\nu^{u+a+t+1}} t^{a-1} \nu^{t+u+2} e^{0.95\nu}\right) = O\left((\log \nu)^{a-1} e^{0.95\nu}\right),$$

so

$$\Sigma_2 = O(e^{0.96\nu}) \tag{93}$$

Finally,

$$\begin{aligned} \Sigma_3 &= O\left(\sum_{t \geq 2\nu \log \nu} \frac{t^{a-1}}{\nu^{u+a+t+1}} \nu^{t+u+3} \exp\left(-\frac{t+u+1}{\nu}\right)\right) \\ &= O\left(\nu^2 \int_{2\log \nu}^{\infty} x^{a-1} \exp - x\, dx\right) \\ &= O(\nu). \end{aligned} \tag{94}$$

Consequently,

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = O(e^{\nu} \nu^{-(u+2a+T)/2}). \tag{95}$$

Furthermore, using (85), for every $A \geq 0$,

$$\begin{aligned} &\sum_{t<T} \frac{1}{\nu^{u+a+t+1}} \binom{a+t-1}{t} R(\nu, t+u) \\ &= e^{\nu} \nu^{-(a+u/2)} \left(\sum_{\substack{0 \leq t < T,\, j \leq A \\ j+t \leq A}} \nu^{-(j+t)/2} \binom{a+t-1}{t} \rho_{t+u,j} + O(\nu^{-(A+1)/2}) \right). \end{aligned} \tag{96}$$

So choosing $T > A$ and using (95), we establish that (96) is an asymptotic expansion for $S(\nu, a, u)$ (a, u being fixed) for every $A \geq 0$. Hence, for any $A \geq 0$, for fixed u,

$$S(\nu, \delta_i, u) = e^{\nu} \nu^{\delta - (i+u-1)/2} \left(\sum_{j=0}^{A} \sigma_{i,u,j} \nu^{-j/2} + O(\nu^{-(A+1)/2}) \right),$$

where

$$\sigma_{i,u,j} = \sum_{t=0}^{j} \binom{t - \delta + (i-3)/2}{t} \rho_{t+u,j-t}.$$

Thus, in particular, using (86),

$$\sigma_{i,0,0} = 1, \qquad \sigma_{i,1,0} = \frac{\sqrt{2\pi}}{2}, \qquad \sigma_{i,0,1} = \frac{\sqrt{2\pi}}{4}(i - 2 - 2\delta),$$

$$\sigma_{0,0,2} = -\frac{1}{3} + \frac{4}{3}\delta + \delta^2 = \begin{cases} -3/4, & \delta = -1/2, \\ -1/3, & \delta = 0. \end{cases}$$

We also need an analogous expansion for $\sum_{u=1}^{\infty} u^{-1} S(\nu, a, u)$, a being arbitrary and fixed. Choose $B > 0$ and write

$$\sum_{u+t \geq B} u^{-1} \frac{1}{\nu^{u+a+t+1}} \binom{a+t-1}{t} R(\nu, t+u) = \Sigma' + \Sigma'' + \Sigma''',$$

where Σ', Σ'' and Σ''' are the sums over u, t such that $B \leq t + u \leq 3\nu/4$, $3\nu/4 < u + t \leq 2\nu \log \nu$ and $u + t > 2\nu \log \nu$, respectively. Analogously to (95), we obtain

$$\Sigma' + \Sigma'' + \Sigma''' = O\left(e^{\nu} \nu^{b} / \nu^{B/2}\right), \tag{97}$$

where b is an absolute constant. The estimate (97) shows that we can get a required expansion for $\sum_{u=1}^{\infty} u^{-1} S(\nu, a, u)$ by choosing B large enough and writing—term by term—an expansion for $\sum_{u=1}^{B} u^{-1} S(\nu, a, u)$ based on (96). Let us now consider the l.h.s. of (75).

Coefficient of $e^{\nu} \nu^{\delta-(i-1)/2} \log \nu$: 0 for $i = 0$, while for $1 \leq i \leq N + 1$,

$$\hat{\beta}_i - \beta_{i-1} - \sum_{j=0}^{i} \hat{\beta}_j \sigma_{j,0,i-j} = -\beta_{i-1} - \sum_{j=0}^{i-1} \hat{\beta}_j \sigma_{j,0,i-j}.$$

Coefficient of $e^{\nu} \nu^{\delta-(i-1)/2}$: 0 for $i = 0$, while for $1 \leq i \leq N + 1$,

$$\hat{\alpha}_i - \alpha_{i-1} - \sum_{j=0}^{i} \hat{\alpha}_j \sigma_{j,0,i-j} + \sum_{\substack{u+j+k=i \\ u \geq 1,\, j,k \geq 0}} \hat{\beta}_j u^{-1} \sigma_{j,u,k}.$$

Equation (75) follows immediately if we can choose $(\hat{\alpha}_i, \hat{\beta}_i)$, $0 \leq i \leq N$, to satisfy (9) and (10) (the cases where $i = 0$ follow from $\sigma_{i,0,0} = 1$). The lemma follows by multiplying (75) by $\nu!/\nu^{\nu}$ and using (2) and (8). □

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Note added in proof. Don Knuth has recently proved that R_n of Theorem 3 is asymptotically equal to $n \log n$ in mean and in probability.

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