

SCENERY RECONSTRUCTION IN TWO DIMENSIONS WITH MANY COLORS

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Kesten has observed that the known reconstruction methods of random sceneries seem to strongly depend on the one-dimensional setting of the problem and asked whether a construction still is possible in two dimensions. In this paper we answer this question in the affirmative under the condition that the number of colors in the scenery is large enough.

1. Introduction and the main result. The following problem has its roots in ergodic theory but may also be considered interesting in its own right. Consider a graph (V, E) and color its vertices in an arbitrary way (so we do not only concentrate on proper colorings in the strict sense that any two adjacent vertices need to have a different color). This coloring will be called a scenery on (V, E) . Then we run a random walk on (V, E) of which we only know the color record (i.e., the sequence of colors it reads at the vertices) but not where it actually reads them. The question then is: Can we still say anything about how V was colored?

This problem, which at first glance might seem a bit hopeless, was first investigated independently by Benjamini and by den Hollander and Keane [2]. From here the problem splits into basically three branches:

1. Can we distinguish two (known) sceneries by their random walk record? or, more ambitiously:
2. Can we even reconstruct (unknown) sceneries by the observations we obtain from a random walk? and:
3. Are there sceneries which cannot be reconstructed or distinguished by the color record of a random walk?

Basic answers to all three of these questions have been given already, while other aspects are still wide open. For example, Benjamini and Kesten [1] discovered the very strong result that almost surely any two given sceneries on the integer lattice \mathbb{Z} or \mathbb{Z}^2 can be distinguished by a simple random walk on these lattices given that the colors are selected by an i.i.d. process. Previous to that Howard [6] had already been able to show that in one dimension a periodic scenery can be distinguished from a periodic scenery with one defect.

Matzinger [10] showed that on \mathbb{Z} even more is true: Almost every i.i.d. two-color scenery can be reconstructed from the color record of a simple random

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walk (which even might have nonzero probability of standing still). This implies Benjamini's and Kesten's [7] result in one dimension as well as the earlier observation by Matzinger [9] that the same holds true for three and more colors. However, notice that Benjamini's and Kesten's techniques also work in a two-dimensional situation or when the random walk is allowed to jump. A remarkable answer to question 3 has been given by Lindenstrauss [8], who showed that there are still uncountably many sceneries on \mathbb{Z} which cannot be distinguished from the color record of a simple random walk.

To be more specific, in what follows (V, E) will always be the integer lattice \mathbb{Z}^2 and a function $\xi: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ will be called a two-dimensional scenery. For a subset $D \subset \mathbb{Z}^2$ we call $\xi: D \rightarrow \mathbb{Z}$ a piece of scenery. If the range of ξ contains exactly m elements we will say that ξ has m colors or that it is an m -color scenery. Two sceneries ξ and $\bar{\xi}$ will be called equivalent if there are $a \in \mathbb{Z}^2$ and

$$M \in \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$$

such that

$$\xi(x) = \bar{\xi}(Mx + a) \quad \forall x \in \mathbb{Z}^2.$$

Similarly, we call two pieces of scenery $\xi: D \rightarrow \mathbb{Z}$ and $\bar{\xi}: \bar{D} \rightarrow \mathbb{Z}$ equivalent if again

$$\xi(x) = \bar{\xi}(Mx + a) \quad \forall x \in D$$

holds true (a and M as above) and moreover $M(D) + a = \bar{D}$.

In other words ξ and $\bar{\xi}$ are equivalent (in symbols, $\xi \sim \bar{\xi}$) if they can be obtained from each other by translation and reflection on the coordinate axes. It is rather obvious that in general we cannot expect to distinguish equivalent sceneries by their color record and thus also reconstruction will work only up to equivalence. Throughout this paper we will consider ξ 's that result from an unbiased i.i.d. random process with m colors (thus we will also say that ξ has m colors); that is, the $\xi(v)$ are i.i.d. for all $v \in \mathbb{Z}^2$ and

$$P(\xi(0) = i) = \frac{1}{m}$$

for all colors $i \in \{0, \dots, m-1\}$. Moreover, let $(S_k)_{k \in \mathbb{N}}$ be a simple, symmetric random walk in two dimensions starting at the origin.

The main result of this paper states that if m is large enough, the color record of (S_k) , that is,

$$\chi := (\chi(k))_{k \in \mathbb{N}} := (\xi(S_k))_{k \in \mathbb{N}},$$

contains enough information to reconstruct ξ almost surely up to equivalence. Additionally, we will present a well-defined algorithm that given the scenery on a finite set reconstructs the whole scenery with probability larger than $1/2$. In the next section we will see why this actually suffices to prove the main theorem. This, in a more mathematical way, is expressed in the following theorem, which states that with sufficiently many colors reconstruction of ξ from χ (up to equivalence) is possible with probability 1.

THEOREM 1.1. *There exists $m_0 \in \mathbb{N}$ such that if $m \geq m_0$, there exists a measurable function (with respect to the canonical σ -fields)*

$$\mathcal{A}: \{0, \dots, m-1\}^{\mathbb{N}} \rightarrow \{0, \dots, m-1\}^{\mathbb{Z}^2}$$

such that

$$(1.1) \quad P(\mathcal{A}(\chi) \sim \xi) = 1.$$

Here the measure P lives on the product space of the outcomes of ξ and the space of all random walk paths.

REMARK 1.2. We have not calculated any lower bound for m_0 yet. We are also convinced that the methods presented here will lead to an m_0 which is terribly large and far off any reasonable number and, in particular, any of the “borderline” cases $m = 4, 5$ for which we have as many colors as (or one more color than, respectively) we have neighbors in \mathbb{Z}^2 or even $m = 2$ (for which we doubt that Theorem 1.1 is valid). This is basically so because we decided to keep the present proof as simple and transparent as possible and to use as many colors as necessary to this end. The specification of a good bound on m_0 will be subject to further research of the authors.

In Section 2 we present the basic ideas of the algorithm used to reconstruct a random scenery; Section 3 contains the rigorous proof of Theorem 1.1.

2. The main ideas and basic notation. The proof of Theorem 1.1 is crucially based on an induction argument. Given that we already know the scenery on a finite set A (for a special choice of A) we show how to extend this knowledge to the points sitting next to A . The following three lemmas are the building blocks of this induction. First we see that it suffices to exhibit an algorithm that reconstructs the scenery with probability larger than $1/2$ to be able to reconstruct the scenery almost surely.

LEMMA 2.1. *For all $m \geq 2$ (where m designates the number of colors in ξ), if there exists a measurable map*

$$\overline{\mathcal{A}}: \{0, \dots, m-1\}^{\mathbb{N}} \rightarrow \{0, \dots, m-1\}^{\mathbb{Z}^2}$$

such that

$$P(\overline{\mathcal{A}}(\chi) \sim \xi) > 1/2,$$

then there also exists a measurable

$$\mathcal{A} : \{0, \dots, m-1\}^{\mathbb{N}} \rightarrow \{0, \dots, m-1\}^{\mathbb{Z}^2}$$

with

$$P(\mathcal{A}(\chi) \sim \xi) = 1.$$

The proof of Lemma 2.1 is given in Section 3.

Lemma 2.1 will be useful, since we will soon see that with sufficiently many colors we are able to reconstruct with large probability the scenery on finite regions of \mathbb{Z}^2 such as the integer circle of radius n denoted by

$$B^n := \{x \in \mathbb{Z}^2 : \|x\| \leq n\}.$$

Here $\|\cdot\|$ stands for the standard Euclidean norm in \mathbb{Z}^2 . Moreover, in the following we will frequently use the following notation: we will write $f|B$ for the restriction of f to a subset B of the domain of definition of f ; for example, $\xi|B$ will be a piece of scenery (i.e., the scenery restricted to some subset B of \mathbb{Z}^2), while $\chi|B$ will be a part of the observations (here B will be a subset of \mathbb{N}).

The next two lemmas basically contain the induction. Lemma 2.2 below is the start of the induction, while Lemma 2.3 contains the induction step. So, first we show that we can reconstruct $\xi|B^n$ for each finite n with arbitrarily large probability, as long as the scenery contains sufficiently many colors.

LEMMA 2.2. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists $m_1 \in \mathbb{N}$ such that if $m \geq m_1$, there exists a measurable function*

$$\mathcal{A}^n : \{0, \dots, m-1\}^{\mathbb{N}} \rightarrow \{0, \dots, m-1\}^{B^n}$$

such that

$$P(\mathcal{A}^n(\chi) \sim \xi|B^n) \geq 1 - \varepsilon.$$

Lemma 2.2 is proven in the next section.

The next lemma is the induction step in the sense that it states that we can reconstruct $\xi|B^{n+1}$ with large probability provided we know $\xi|B^n$ up to equivalence and the number of colors is large enough.

LEMMA 2.3. *There exists $m_2 \in \mathbb{N}$ (random) such that for $m \geq m_2$ there is a sequence of measurable functions $(\tilde{\mathcal{A}}^n)_{n \in \mathbb{N}}$,*

$$\tilde{\mathcal{A}}^n : \bigcup_{a \in \mathbb{Z}^2} \{0, \dots, m-1\}^{B^n+a} \times \{0, \dots, m-1\}^{\mathbb{N}} \rightarrow \{0, \dots, m-1\}^{B^{n+1}}$$

such that, P -a.s.,

$$\tilde{\mathcal{A}}^n(\xi|B^n, \chi) \sim \xi|B^{n+1}$$

occurs for all but finitely many n .

REMARK 2.4. Note that given that m is large enough the critical n in Lemma 2.3 from which the algorithms work, that is, from which

$$\tilde{\mathcal{A}}^n(\xi|B^n, \chi) \sim \xi|B^{n+1},$$

is random.

Also note that Lemma 2.3 implies that for each $\varepsilon > 0$ we can find a number N (nonrandom) such that the probability that all $\tilde{\mathcal{A}}^n$ work for all $n \geq N$ is greater than $1 - \varepsilon$.

Roughly speaking, Lemma 2.3 means that the algorithm obtained by concatenating the different $\tilde{\mathcal{A}}^n$'s works well, in the sense that given $\xi|B^n$ up to equivalence and the observations χ it almost surely fails to reconstruct $\xi|B^{n+1}$ only for finitely many n .

To explain the proof of the induction step, which is crucial to the whole proof of Theorem 1.1, observe that the main difficulty in the reconstruction of sceneries is, of course, that we do not exactly know precisely where the random walk is. This is even more a problem in two dimensions than it is in one dimension as the random walk in one dimension by time N has returned to the origin about \sqrt{N} times and therefore produces a lot of information about the neighborhood of the origin. In two dimensions the local time of the origin at time N is only about $\log N$. Thus we have to find an accurate method for guessing when the random walk is close to the origin from the observations χ it produces. This will be achieved by using a set of signal words, that is, sequences of subsequent colors in B^n . Their frequent appearance in the observations will indicate that we really are in a neighborhood of B^n .

This “guessing that the random walk is inside B^n ” is the first step of the reconstruction algorithm. More accurately, these words which will indicate that we are inside B^n (the so called *signal words*) are horizontal, nonoverlapping words inside B^n of length proportional to $\log n$. The set of these words will be called \mathcal{S}^n . Whenever we read more than n^β words during a time interval of length n^2 whose endpoint is inside $[0, e^{n^\alpha}]$ (α and β are some numbers to be specified later), we will “guess” that the walk is inside B^{n^2+n} . The union of these time intervals will be called τ^n and the reconstruction will only take place during τ^n . Note that τ^n designates a random set.

More formally in the sequel let $c_1, c_2, c_3 > 0$ be positive constants (not depending on n) which we will specify later. For convenience we will assume

that $c_i \log n \in \mathbb{N}$ for each $i = 1, 2, 3$ (which of course means the c_i slightly depend on n but this dependence is irrelevant). Let

$$\begin{aligned} \mathcal{S}^n := \{w = (w_1, \dots, w_{c_1 \log n}) \mid \exists k \in \mathbb{Z} \text{ and } (x, y) \in \mathbb{Z}^2: x = kc_1 \log n, \\ (x + s, y) \in B^n \text{ and } w_s = \xi((x + s, y)), \\ \forall 0 \leq s \leq c_1 \log n - 1\}. \end{aligned}$$

In other words \mathcal{S}^n “partitions” $\xi|B^n$ into disjoint horizontal words of length $c_1 \log n$. Moreover let $1 < \alpha < \beta < 2$ be two real numbers close to 2 to be specified later, let

$$\begin{aligned} \mathcal{I}_{\alpha, \beta} := \{I = [t, t + n^2] \mid t \leq e^{n^\alpha} - n^2, \\ \chi|I \text{ contains more than } n^\beta \text{ different words from } \mathcal{S}^n\}, \end{aligned}$$

and

$$\tau^n := \tau_{\alpha, \beta}^n := \bigcup_{I \in \mathcal{I}_{\alpha, \beta}} I.$$

As sketched above, the point is that during the times $k \in \tau^n$ we can be pretty sure that the random walk is “close to B^n ”, more precisely that it is inside B^{n+n^2} . This will ensure that the reconstruction takes place at the boundary of B^n and not anywhere else.

The probability for the random walk to go right through a given signal word is equal to $(1/4)^{c_1 \log n}$. Thus for c_1 very small the random walk inside B^n typically reads $n^{2-\varepsilon_1}$ signal words during a time interval of length n^2 . Here $\varepsilon_1 > 0$ can be made arbitrarily small. This is basically so because the random walk typically visits about $n^2 / \log n$ distinct points in a time window of length n^2 , and thus during these time steps it would visit roughly about $n^2 / \log n \times (1/4)^{c_1 \log n} \geq n^{2-\varepsilon_1}$ (for c_1 small enough) signal words.

Now, if the number of colors m is large enough, we can choose c_1 small and the signal words still will be typical of B^n [i.e., the probability of reading them in a given ball $B_y^{n^2}$ (the ball of radius n^2 centered in y) is small, as long as the ball does not touch B^n]. Indeed, there are fewer than $\pi n^4 4^{c_1 \log n}$ different paths of length $c_1 \log n$ inside $B_y^{n^2}$. Thus by independence the probability for a given signal word to appear in $B_y^{n^2} \setminus B^n$ is less than $\pi n^4 (4/m)^{c_1 \log n}$, which is as small as we want, if only m is large enough. Exploiting the independence of the signals in a large-deviations argument we will be able to show that up to time e^{n^α} the random walk in a time interval of length n^2 will only be able to read more than n^β (α, β as above) signal words if it spends this time in B^{n^2+n} and that the probability of reading so many signals elsewhere is about e^{-n^α} . So, our test, to check when we are back in B^n , will not fail until time roughly e^{n^α} . However, by that time we will have returned to the origin about $n^{\alpha-\varepsilon_2}$ times ($\varepsilon_2 > 0$, small). Now if m were so large

that there were only different colors inside B^n , this would suffice to reconstruct ξ on the boundary of B^n . We simply would have to follow the walk until it exits B^n and read the first color outside as the color of a boundary point. If all colors were different, we would clearly know where this boundary point was. Moreover, there are on the order of n points in ∂B^n , so $n^{1+\varepsilon_3}$ ($\varepsilon_3 > 0$) returns to the origin would suffice to reconstruct the scenery on the boundary of B^n . As we already have seen that we have about n^α such returns, we would be done.

However, we are not allowed to choose m growing with n , so we cannot assume that all colors inside B^n are different. So we have to employ more subtle methods to reconstruct ξ on the boundary of B^n .

To describe this reconstruction part we have to introduce some more notation. Let

$$\underline{\partial} B^n := \{z \in B^n \mid \exists y \in \mathbb{Z}^2 \setminus B^n \text{ such that } z \text{ and } y \text{ are neighbors}\}$$

be the inner boundary of B^n and let

$$\overline{\partial} B^n := B^{n+1} \setminus B^n$$

be its outer boundary. Observe that $\overline{\partial} B^n$ may differ from the outer boundary of B^n in the lattice topology. Indeed, there might points at distance 1 from B^n without nearest neighbors in B^n . Moreover, using the lattice geometry of \mathbb{Z}^2 , it is easily checked that all points in B^{n+1} can be reached from a point in B^n by crossing at most two edges. Since by definition $B^n \cup \overline{\partial} B^n = B^{n+1}$ it clearly suffices to reconstruct $\overline{\partial} B^n$ with sufficiently large probability.

The strategy will be to guess the color of a point v in $\overline{\partial} B^n$ by extending a walk to a neighboring point in $\underline{\partial} B^n$ by two further steps. Of course, we have to be very careful both to walk to $v \in \underline{\partial} B^n$ and to extend the walk in the right direction. The principal idea behind this reconstruction can be described quite easily. Draw a straight (horizontal or vertical) line through v and suppose we know already the colors of a line segment of length approximately $\log n$ inside B^n and containing v as well as the colors of a line segment of about the same length outside B^n at distance 2 from v . Then we could figure out the two missing colors between these two segments by just waiting until the random walk first reads the colors of the segment inside B^n (in the right order) and then after a waiting time of 2 the colors of the segment outside B^n . Except, if the walk is far away from v (which we can exclude by the above arguments), the walk must have followed the straight line supporting the two segments at least partially and thus the missing two colors are the colors read between reading the colors of the two segments. Indeed, the “following partially” part above needs a little more technical work. We could deviate from the above line segment and just accidentally read the right colors. We will get rid of this nuisance by characterizing the missing two points as the shortest distance between two cones rather than between two line segments. This idea will be made more precise below.

Now a major difficulty is that we do not know the colors outside B^n . Thus we have to think of another characterization of the segment outside B^n (supported by the same line as the inner segment). It will turn out that it is useful to think of it as the segment whose colors can be read in shorter time by starting with the inner segment than by starting with any segment parallel to it.

To formalize this idea for $v \in \partial B^n$ we define a segment $\sigma(v)$ (the segment associated with v) in the following way: Let $\sigma(v)$ be the horizontal or vertical segment of length $(c_2 + c_3) \log n$ with endpoints v and $\sigma_0(v) \in B^n$, such that the angle between this segment and the tangent to the circle of radius $|v|$ centered on 0 at the point v is at least 45° (the latter is needed to ensure that the objects below are well defined).

The first $c_2 \log n$ lattice points [starting from $\sigma_0(v)$] are called the root segment of v and abbreviated as $\hat{\sigma}(v)$; the rest of $\sigma(v)$ is called the second root segment and denoted by the symbol $\bar{\sigma}(v)$; the left and right neighboring segments of $\hat{\sigma}(v)$ of the same length $c_2 \log n$ as $\hat{\sigma}(v)$ [or the lower and upper segment next to the root segment of v , if $\sigma(v)$ is a horizontal segment, respectively] are named the side segments of v . For these we reserve the symbols $\lambda(v)$ and $\rho(v)$, and their starting points [next to $\sigma_0(v)$] are denoted by $\lambda_0(v)$ and $\rho_0(v)$, respectively. Finally, the segment of length $c_2 \log n$ following $\sigma(v)$ after one step when we keep following the line supporting $\sigma(v)$ is called the invisible segment associated with v and denoted by $\varphi(v)$. Its endpoints are called v_2 and $\varphi_0(v)$. The words associated with these segments are called the root word, second root word, side words and invisible words, respectively. Finally, the lattice points we want to guess the color of, that is, the points on $\varphi(v)$ of distance 1 and 2 to v , are named v_1 and v_2 .

All this is illustrated in Figure 1.

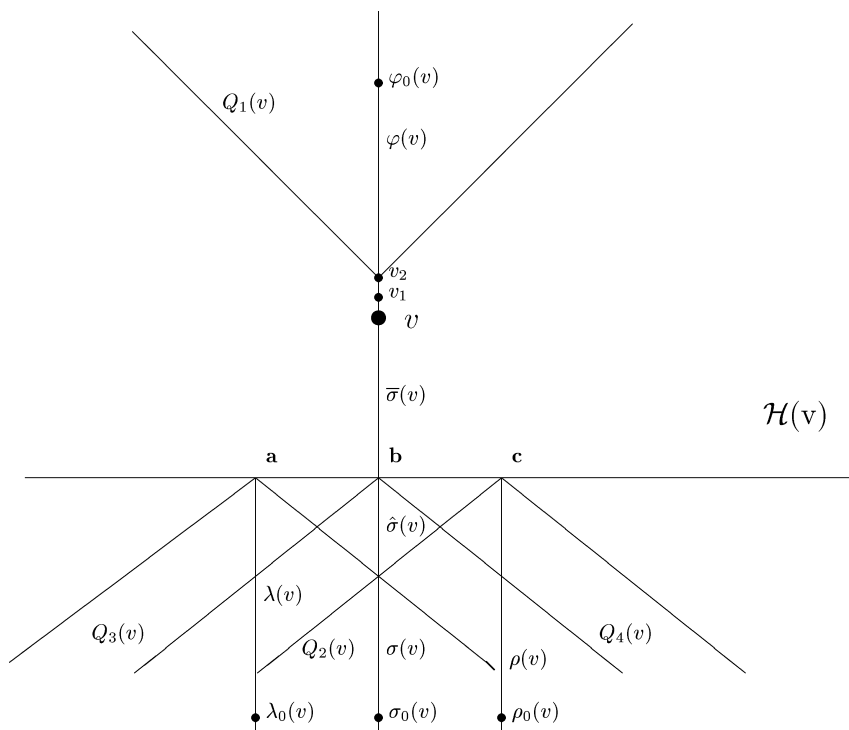
Let us now describe how this reconstruction works.

The idea behind the above setup is that, to read the color of v_1 and v_2 , we take a neighboring vertex $v \in \partial B^n$ and read the color of v_1 and v_2 as the next colors when we have read $\sigma(v)$ from $\sigma_0(v)$ to v . To guarantee that indeed we read the color of the right points we require that the algorithm picks a word w of length $c_2 \log n$ satisfying the following conditions:

1. w appears in $\chi|_{\tau^n}$ directly (one step) after the word supported by $\sigma(v)$.
2. In $\chi|_{\tau^n}$ the shortest time for w to appear after the root word of v is exactly equal to $c_3 \log n + 1$.
3. In $\chi|_{\tau^n}$ the shortest time for w to appear after the side word of v is exactly $c_3 \log n + 2$.

Condition 2 assures that we do not run backward after having read the word supported by $\sigma(v)$; condition 3 guarantees that we have not deviated from the segment from $\sigma_0(v)$ to v while reading the scenery.

Thus we estimate $\xi(v_2)$ to be the first color of w . The estimate for $\xi(v_1)$ will be the the color between $\sigma(v)$ and w , when they appear in $\chi|[0, e^{n^\alpha}]$ one step



apart from each other. If there is no word w satisfying the above conditions, we let the algorithm terminate (our conditions imply that this will happen only with extremely small probability).

To realize this idea, that is, to prove Theorem 1.1, we need some more definitions, which we give now. For $v \in \partial B^n$ the half-space associated with v [which we denote by $\mathcal{H}(v)$] is the half-space separating $\hat{\sigma}(v)$ from $\bar{\sigma}(v)$ orthogonal to $\sigma(v)$ and with $\bar{\sigma}(v)$ in $\mathcal{H}(v)$. The first quarter-space $Q_1(v)$ associated with v is the right-angular cone based on v_2 with bisecting line along $\varphi(v)$ such that the major part of $\varphi(v)$ is inside this cone. The second quarter-space $Q_2(v)$ associated with v is the right-angular cone based on the line separating $\mathcal{H}(v)$ from its complement such that $\hat{\sigma}(v)$ is on its bisecting line and $\hat{\sigma}(v)$ is in this cone. The third quarter-space $Q_3(v)$ associated with v is defined as the right-angular cone based on the line separating $\mathcal{H}(v)$ from its complement such that $\lambda(v)$ is on its bisecting line and $\lambda(v)$ is in this cone. Finally, the fourth quarter-space $Q_4(v)$ associated with v is the right-angular cone based on the line separating $\mathcal{H}(v)$ from its complement such that $\rho(v)$ is on its bisecting line and $\rho(v)$ is in this cone. The base points of Q_3 , Q_2 and Q_4 , respectively, are denoted by a , b and c , respectively.

All this is illustrated in Figure 1. In this figure the points $v, a, b, c, \lambda_0(v), \sigma_0(v)$ and $\rho_0(v)$ are inside B^n , while v_1, v_2 and $\varphi_0(v)$ are outside B^n .

As can be seen from Figure 1:

1. $Q_1(v)$ contains the segment $\varphi(v)$ which begins with v_2 and ends at $\varphi_0(v)$;
2. $Q_2(v)$ contains the segment $\hat{\sigma}(v)$ which begins with $\sigma_0(v)$ and ends at b ;
3. $Q_3(v)$ contains the segment $\lambda(v)$ which begins with $\lambda_0(v)$ and ends at a ;
4. $Q_4(v)$ contains the segment $\rho(v)$ which begins with $\rho_0(v)$ and ends at c ;
5. $\sigma(v)$ consists of $\hat{\sigma}(v)$ and $\bar{\sigma}(v)$;
6. All of the segments $\varphi(v), \hat{\sigma}(v), \rho(v)$ and $\lambda(v)$ contain $c_2 \log n$ lattice points, while $\bar{\sigma}(v)$ contains $c_3 \log n$ lattice points.

3. Proofs. In this section we give the proofs of Theorem 1.1 and Lemmas 2.1–2.3. Let us start with the proof of Lemma 2.1.

PROOF OF LEMMA 2.1. Let $X(l)$ be the indicator for the event that the reconstruction algorithm $\bar{\mathcal{A}}$ applied to the observations shifted by l give rise to a scenery which is equivalent to the actual scenery, that is, $X(l) = 1$ if $\bar{\mathcal{A}}(\Theta^l(\chi)) \sim \xi$ and $X(l) = 0$ otherwise. Obviously, $(X(l), l \in \mathbb{N})$ is stationary with

$$\mathbb{P}(X(l) = 1) = \mathbb{P}(\bar{\mathcal{A}}(\chi) \sim \xi) > \frac{1}{2}$$

for all l .

Furthermore let

$$\Omega = \{(+1, 0), (-1, 0), (0, +1), (0, -1)\}^{\mathbb{N}} \times \{0, \dots, m-1\}^{\mathbb{Z}^2}$$

and let \mathcal{F} be the standard σ -field on Ω . Let $\theta : \Omega \rightarrow \Omega$ be defined in the following way. For any

$$\omega = ((\bar{\Delta}_1, \bar{\Delta}_2, \dots), \psi),$$

where

$$\psi \in \{0, \dots, m-1\}^{\mathbb{Z}^2}$$

and

$$\bar{\Delta}_i \in \{(+1, 0), (-1, 0), (0, +1), (0, -1)\} \quad \text{for all } i \in \mathbb{N},$$

we define

$$\theta(\omega) := ((\bar{\Delta}_2, \bar{\Delta}_3, \dots), \psi + \bar{\Delta}_1).$$

Here $\psi + \bar{\Delta}_1$ stands for $2D$ scenery ψ shifted by $-\bar{\Delta}_1$, that is,

$$\psi + \bar{\Delta}_1(z) := \psi(\bar{\Delta}_1 + z).$$

Let Δ_i designate the i th increment of the random walk S , that is,

$$\Delta_i := S(i) - S(i-1).$$

Let μ be the measure describing the randomness of the object $((\Delta_1, \Delta_2, \dots), \xi)$. This means $(\Omega, \mathcal{F}, \mu)$ is a probability space. One easily verifies that θ is measure-preserving on $(\Omega, \mathcal{F}, \mu)$. Let $Z(l)$ designate the random vector

$$Z(l) = ((\Delta_{l+1}, \Delta_{l+2}, \dots), \xi + S(l)).$$

Note that $Z(l) = \theta^l(Z(1))$. Since θ is measure-preserving the sequence $Z(0), Z(1), Z(2), \dots$ is measure-preserving. Now $X(0), X(1), X(2), \dots$ is a stationary coding of the sequence $Z(0), Z(1), Z(2), \dots$. By this we mean that there exists a measurable function F such that for all $l \in \mathbb{N}$ we have

$$F(Z(l)) = X(l).$$

This implies stationarity of the sequence $X(0), X(1), X(2), \dots$. Now a stationary coding of an ergodic sequence is ergodic again. Thus to prove that $(X(l))_l$ is ergodic we will prove that $Z(0), Z(1), Z(2), \dots$ is ergodic. To do so we will show that $Z(0), Z(1), Z(2), \dots$ is actually mixing. For this it is enough to see that for any two $A, B \in \mathcal{F}$ that only depend on finitely many Δ_i we have

$$\lim_{k \rightarrow \infty} \mu(\theta^{-k}A \cap B) = \mu(A)\mu(B).$$

Let σ^n denote the σ -algebra

$$\sigma^n = \{\sigma(\Delta_1, \Delta_2, \dots, \Delta_n, \xi(z)) : z \in B_n\},$$

where

$$B_n := \{z \in \mathbb{Z}^2 : |z| \leq n\}.$$

Eventually let $C_{n,k}$ denote the event that

$$C_{n,k} := \{S(k) \notin B_{2n}\}.$$

Assume that $A, B \in \sigma^n$. Then, conditional on $C_{n,k}$, the events $\theta^{-k}(A)$ and B are independent. Also note that $\theta^{-k}(A)$ and $C_{n,k}$ are independent. Thus we obtain

$$\mu(\theta^{-k}(A) \cap B | C_{n,k}) = \mu(\theta^{-k}(A) | C_{n,k})\mu(B | C_{n,k}) = \mu(\theta^{-k}(A))\mu(B | C_{n,k}).$$

Hence

$$\mu(\theta^{-k}(A) \cap B \cap C_{n,k}) = \mu(\theta^{-k}(A))\mu(B \cap C_{n,k}).$$

This implies that

$$(3.1) \quad \mu(\theta^{-k}(A) \cap B) = \mu(\theta^{-k}(A) \cap B \cap C_{n,k}^c) + \mu(A)(\mu(B) - \mu(B \cap C_{n,k}^c)).$$

Keeping n fixed and taking k to infinity we obtain

$$\lim_{k \rightarrow \infty} \mu(C_{n,k}^c) = 0.$$

Hence also

$$\lim_{k \rightarrow \infty} \mu(B \cap C_{n,k}^c) = \lim_{k \rightarrow \infty} \mu(\theta^{-k}(A) \cap B \cap C_{n,k}^c) = 0.$$

Thus (3.1) implies

$$\mu(\theta^{-k}(A) \cap B) = \mu(A)\mu(B).$$

Hence the shift θ is mixing on $(\Omega, \mathcal{F}, \mu)$ and thus also ergodic. Therefore $Z(0), Z(1), \dots$ is an ergodic sequence of random variables. Since $X(0), X(1), \dots$ is a stationary coding of $Z(0), Z(1), \dots$ it inherits the property of ergodicity.

Hence by the ergodic theorem

$$\frac{X(1) + X(2) + \dots + X(l)}{l}$$

converges to a limit larger than $1/2$ almost surely. Thus under the assumption that

$$\mathbb{P}(\overline{\mathcal{A}}(\chi) \sim \xi) > \frac{1}{2}$$

we can identify the equivalence class of ξ as the only equivalence class which eventually is equivalent to the majority of the $\overline{\mathcal{A}}(\Theta^l(\chi))$'s. \square

Let us now prove Lemma 2.2.

PROOF OF LEMMA 2.2. The principal idea behind the proof of Lemma 2.2 is that with enough colors within a large area a certain color is typical of the point underlying it. This will help us to reconstruct the scenery on two basic shapes, which will help to reconstruct the scenery on the points of a three-by-three square and hence also on any other square. In a final step we will see this already suffices to reconstruct the scenery within a large ball.

To be more precise, let

$$E_{01}^n := \bigcap_{x \neq y \in B^n} \{\xi(x) \neq \xi(y)\},$$

and let

$$E_{02}^n := \bigcap_{\substack{x, y \in B^n, \\ \|x - y\| = 1}} \bigcap_{z \in B^n} \bigcap_{\substack{v \notin B^n: \\ \xi(v) = \xi(z)}} \{(S_k)_k \text{ passes from } x \text{ to } y \\ \text{in one step before visiting } v\}.$$

In words the event E_{01}^n says that all colors inside B^n are different, while E_{02}^n states that all edges inside B^n are crossed by $(S_k)_{k \in \mathbb{N}}$ before it visits a point outside B^n having the same color as one of the points inside B^n .

We now show that under the condition that E_{01}^n and E_{02}^n hold, we can reconstruct the scenery $\xi|_{B^n}$. The reconstruction will be based on the following two important cases.

Case I. Let $x, y, z, v \in B^n$ be the corners of a unit square with x and z (and, as well, y and v) across the diagonal. Then, if E_{01}^n and E_{02}^n hold, and we know the colors of x, y and z , we can figure out the color of v . The color of v is the first color appearing, neighboring both the color of x and the color of z , and different from the color of y . (Here and in the following we call two colors neighboring if they are read at consecutive times.)

Case II. Let $x_1, x_2, x_3, x_4, y \in B^n$ be a “cross” with center y ; that is, x_1, x_2, x_3, x_4, y are pairwise different and

$$|x_1 - y| = |x_2 - y| = |x_3 - y| = |x_4 - y| = 1.$$

Knowing that E_{01}^n and E_{02}^n hold as well as the colors of x_1, x_2, x_3 and y we can find the color of x_4 as the only color neighboring $\xi(y)$ different from $\xi(x_1), \xi(x_2)$ and $\xi(x_3)$.

We will now see that these two basic techniques suffice to reconstruct $\xi|B^n$, if E_{01}^n and E_{02}^n hold. Indeed, denoting by Q_j the $(2j+1)$ -by- $(2j+1)$ square with center zero, we can first reconstruct $\xi|Q_1$.

To this end we first recover the color of the origin (which is, of course, trivial) and the colors of $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$. Indeed, the colors themselves are known from the observations. Note that the only information we need is the relative positions of the colors of $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$ to each other because we only want to reconstruct up to equivalence. This means we only need to know which of the colors

$$\{\xi((1, 0)), \xi((0, 1)), \xi((-1, 0)), \xi((0, -1))\}$$

are from points across $(0, 0)$ and which of them are not. [Here we say that $(1, 0)$ and $(-1, 0)$ lie across $(0, 0)$ and that $(0, 1)$ and $(0, -1)$ lie across $(0, 0)$, while the other possible pairs do not.]

Now the following characterization holds:

Pairs from

$$\{\xi((1, 0)), \xi((0, 1)), \xi((-1, 0)), \xi((0, -1))\}$$

lie across $(0, 0)$ if and only if they have exactly one neighboring color (which is $\xi(0, 0)$), while the other pairs have exactly two neighboring colors.

Once we know $\xi|\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ up to equivalence we can reconstruct the scenery on Q_1 by applying Case I to the four corner points of Q_1 .

Now we can proceed inductively. Knowing $\xi|Q_j \cap B^n$, we want to reconstruct $\xi|Q_{j+1} \cap B^n$; that is, we want to find the color of the boundary points of Q_{j+1} (as far as they are inside B^n). For all points with at least one coordinate different from $j+1, j, -j-1$ or $-j$, this can be done by applying the technique of Case II. Then the color of the points with one coordinate equal to j or $-j$ can be reconstructed

by applying the technique of Case I. Finally the same technique yields the color of the corner points of Q_{j+1} .

This shows that under the condition that E_{01}^n and E_{02}^n hold we can reconstruct $\xi|B^n$ up to equivalence. It remains to understand that both E_{01}^n and E_{02}^n hold with arbitrarily large probability for fixed n and large enough m . Indeed, this is not very hard to see. For E_{01}^n , note that

$$\mathbb{P}((E_{01}^n)^c) \leq \text{const} n^2 \frac{1}{m},$$

which can be made arbitrarily small by choosing m large.

Similar techniques apply to E_{02}^n . Note that by taking T sufficiently large the random walk $(S_k)_{k \leq T}$ up to time T has visited each point in B^n , at least L times (L some number to be chosen soon; cf. [11] for similar results). Then the probability that there is an edge in B^n the random walk does not visit up to time T is bounded by

$$\text{const} n^2 \left(\frac{3}{4}\right)^L,$$

which is arbitrarily small for L sufficiently large. If we now first choose L , then take T as above and finally choose m so large that the probability that all colors in B^T are distinct (by the same techniques as above) is as large as we want, we see that

$$\mathbb{P}((E_{02}^n)^c) \leq \varepsilon$$

for each $\varepsilon > 0$ if only m is sufficiently large. This completes the proof of Lemma 2.2. \square

Next we will prove Lemma 2.3, which is indeed the key ingredient of the proof of Theorem 1.1.

PROOF OF LEMMA 2.3. Let E^n denote the event that given a piece of scenery ψ with $\psi \sim \xi|B^n$ the “reconstruction algorithm at step n ” $\overline{\mathcal{A}}^n$ produces a piece of scenery $\overline{\mathcal{A}}^n(\psi, \chi)$ with

$$\overline{\mathcal{A}}^n(\psi, \chi) \sim \xi|B^{n+1}.$$

We need to show that with probability 1 E^n holds for all but a finite number of n ’s (in the following we will also say that an event holds for almost all n if it holds for all but finitely many n).

To do so we decompose E^n for $n \in \mathbb{N}$ in such a way that

$$E^n \supset E_1^n \cap E_2^n \cap E_3^n.$$

We will then show that each E_i^n , $i = 1, 2, 3$, holds for all but finitely many n ’s.

In the sequel whenever we say about some observations κ that “ κ appears in A with starting point x ” or “ κ appears in A with endpoint y ”, respectively, where $\kappa \in \{0, \dots, m-1\}^l$ for some l , $A \subseteq \mathbb{Z}^2$ and $x, y \in \mathbb{Z}^2$, we mean that

$$\chi|T = \kappa$$

for some realization of the random walk S_n , some discrete time interval $T = [t_0, t_0 + l - 1]$ such that $S_{t_0} = x$ (or $S_{t_0+l-1} = y$, respectively) and $S|T \subset A$. In other words κ appears in A with starting point x (or endpoint y) if it can be read inside A by a nearest neighbor walk starting at x (ending at y). Moreover if, for one of the line segments $\sigma(v)$, $\hat{\sigma}(v)$, $\overline{\sigma}(v)$, $\varphi(v)$ or $\lambda(v)$, we refer to $\xi|_{\mathcal{L}}$ ($\mathcal{L} \in \{\sigma(v), \hat{\sigma}(v), \overline{\sigma}(v), \varphi(v), \lambda(v)\}$), we mean the observations obtained by reading ξ along \mathcal{L} from the center of B^n to the outside of B^n .

Now let

$$E_1^n := \bigcap_{x \in B^{\exp(n^\alpha)}} \{\text{there are fewer than } n^\beta \text{ different words from } \mathcal{S}^n \text{ appearing in } \xi|(B_x^{n^2} \setminus B^n)\},$$

where $B_x^{n^2}$ stands for the discrete ball of radius n^2 centered on x .

Observe that the definition of τ^n implies that on E_1^n we have that $S_k \in B^{n+n^2}$ for all $k \in \tau^n$.

Moreover let

$$E_2^n = E_{21}^n \cap E_{22}^n \cap E_{23}^n \cap E_{24}^n \cap E_{25}^n$$

with

$$E_{21}^n := \bigcap_{v \in \partial B^n} \{\xi|\overline{\sigma}(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with end point inside } \mathcal{H}(v)\},$$

$$E_{22}^n := \bigcap_{v \in \partial B^n} \{\xi|\hat{\sigma}(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with endpoint } x \in Q_2(v)\},$$

$$E_{23}^n := \bigcap_{v \in \partial B^n} \{\xi|\lambda(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with endpoint } x \in Q_3(v)\},$$

$$E_{24}^n := \bigcap_{v \in \partial B^n} \{\xi|\rho(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with endpoint } x \in Q_4(v)\},$$

$$E_{25}^n := \bigcap_{v \in \partial B^n} \{\xi|\varphi(v) \text{ appears in } \xi|B^{n^2+n} \text{ only with starting point } x \in Q_1(v)\}.$$

Finally let

$$E_3^n = E_{3,\sigma}^n \cap E_{3,\lambda}^n \cap E_{3,\rho}^n,$$

where

$$\begin{aligned}
 E_{3,\sigma}^n &:= \bigcap_{v \in \partial B^n} \{ \text{all nearest neighbor walks of length } (2c_2 + c_3) \log n + 1 \\
 &\quad \text{initially traversing } \hat{\sigma}(v) \text{ are realized at least once during } \tau^n \}, \\
 E_{3,\lambda}^n &:= \bigcap_{v \in \partial B^n} \{ \text{all nearest neighbor walks of length } (2c_2 + c_3) \log n + 1 \\
 &\quad \text{initially traversing } \lambda(v) \text{ are realized at least once during } \tau^n \}, \\
 E_{3,\rho}^n &:= \bigcap_{v \in \partial B^n} \{ \text{all nearest neighbor walks of length } (2c_2 + c_3) \log n + 1 \\
 &\quad \text{initially traversing } \rho(v) \text{ are realized at least once during } \tau^n \}.
 \end{aligned}$$

Before we show that $E_1^n \cap E_2^n \cap E_3^n$ indeed happens for all but a finite number of n 's, let us see that this will actually imply the desired result; that is, let us see that

$$E^n \supset E_1^n \cap E_2^n \cap E_3^n.$$

For each event in E_1^n we know that during τ^n we must be close to B^n ; more precisely, we know that during τ^n the walk is inside B^{n^2+n} . Then E_3^n ensures that in this time τ^n we read each sequence of length $(2c_2 + c_3) \log n + 1$ beginning with either $\xi|\hat{\sigma}(v)$, $\xi|\rho(v)$ or $\xi|\lambda(v)$ for each $v \in \partial B^n$ at least once. E_2^n now guarantees that during these times the walk is close to the points a , b and c (of the appropriate v). Finally E_2^n together with E_3^n ensures some of the walks actually pass the points a , b and c , correspondingly. Therefore, we are able to read the color of the vertices v_1 and v_2 next to v in the direction of $\sigma(v)$.

Let us explain this in detail, since this step is, indeed, the core of the reconstruction step. For fixed $v \in \partial B^n$ at the boundary of B^n we need to prove that the reconstruction method works correctly, that is, that the algorithm we give below reveals the colors of the corresponding v_1 and v_2 [i.e., $\xi(v_1)$ and $\xi(v_2)$] correctly, if E_1^n , E_2^n and E_3^n hold. Let us now define the reconstruction algorithm properly.

The algorithm is given as input $\xi|B^n$, the scenery restricted to B^n , which we assume to be known already.

ALGORITHM TO RECONSTRUCT $v_1 = v_1(v)$ AND $v_2 = v_2(v)$.

Step 1. Select all words w of length $c_3 \log n$ in $\xi \circ S|\tau^n$ with the following properties:

- (a) The shortest number of steps w appears after $\xi|\hat{\sigma}$ in $\xi \circ S|\tau^n$ is $c_3 \log n + 1$.
- (b) The shortest number of steps w appears after $\xi|\lambda$ in $\xi \circ S|\tau^n$ is $c_3 \log n + 2$.
- (c) The shortest number of steps w appears after $\xi|\rho$ in $\xi \circ S|\tau^n$ is $c_3 \log n + 2$.

(d) A word of the form $\xi|\sigma \diamond v \diamond w$ (where v is an arbitrary color and the symbol \diamond stands for the concatenation of two words) occurs in $\xi|\tau^n$, that is, the event that w is read precisely one step after $\xi|\sigma$ occurs in τ^n .

Step 2. Take the first letter of w as an estimator of $\xi(v_2)$.

Step 3. Take an occurrence of a word $\xi|\sigma \diamond v \diamond w$ in τ^n . Estimate $\xi(v_1)$ with v .

To prove that the above algorithm works and is well defined (Step 3) given that E_1^n , E_2^n and E_3^n hold, we will prove the following: for every word w selected by our algorithm its first letter is read at position v_2 . This automatically implies both that Step 2 of the above algorithm works and that Step 3 is well defined and works.

First assume that there is at least one word w selected by the first step of the above algorithm. Call the lattice point at which the first letter of w is read x . Assume that x is in \mathcal{H} but not on the line supporting σ . Then there is a path from either a or c to x which is strictly shorter than any path from any starting point in Q_2 to x (in particular it is shorter than a path from b to x). Now E_3^n holds, so in particular $E_{3,\lambda}^n$ and $E_{3,\rho}^n$ hold. Hence a path first reading λ (or ρ), crossing a (or c) and then walking to x in the shortest possible way in order to produce w from there will once be realized. Now E_1^n holds, ensuring that during τ^n the random walk is in B^{n^2+n} . Thus E_{22}^n holds. This guarantees that any time we read $\xi|\hat{\sigma}$ in χ we do this with an endpoint in Q_2 . Thus any time we read the word w (and still we assume that x is not on the line supporting σ) a time t' after having read $\xi|\hat{\sigma}$, this time t' will be strictly larger than the time to read w after having read one of $\xi|\lambda$ or $\xi|\rho$. This contradicts our selection criteria.

Thus we can only select words w with a first letter read at $x \in \mathcal{H}$, if x lies on the line supporting σ . Now from $E_{3,\sigma}^n$ we know that all paths of length $(2c_2 + c_3) \log n + 1$ are realized once during τ^n . From this together with the fact that we have selected w such that the *shortest* it appears in the observations after $\xi|\hat{\sigma}$ is $c_3 \log n + 1$, it follows that x is at distance $c_3 \log n + 1$ from b ; hence, given that $x \in \mathcal{H}$, we conclude that $x = v_2$. It only remains to show that x cannot be in \mathcal{H}^c , but this is guaranteed by E_{21}^n .

It remains to show that Step 1 of the above algorithm selects at least one word. However, as a consequence of E_1^n , E_{22}^n , E_{23}^n , E_{24}^n , E_{25}^n and $E_{3,\sigma}^n$ Step 1 of the above algorithm will select $\xi|\varphi$. Indeed, the shortest path from Q_3 or Q_4 to Q_1 is $c_3 \log n + 2$ steps long, while Q_1 can be reached from Q_2 in $c_3 \log n + 1$ steps. By E_1^n we know that we are in B^{n^2+n} during τ^n . Thus by E_{23}^n , E_{24}^n and E_{25}^n we know that the shortest possibility of reading $\xi|\varphi$ after $\xi|\rho$ or $\xi|\lambda$ is after $c_3 \log n + 2$ steps, while the shortest possibility of reading $\xi|\varphi$ after $\xi|\hat{\sigma}$ is after $c_3 \log n + 1$ step. Finally, $E_{3,\sigma}^n$ ensures that we will observe at least once the sequence $\xi|\sigma \diamond v \diamond \xi|\varphi$ for some color v . Thus $\xi|\varphi$ satisfies the selection criteria of Step 1 of the algorithm.

Hence we reconstruct the color of $v_1(v)$ and $v_2(v)$ if $E_1^n \cap E_2^n \cap E_3^n$ is satisfied.

As this works for all $v \in \partial B^n$ we are indeed able to reconstruct the scenery on B^{n+1} , proving that

$$E^n \supset E_1^n \cap E_2^n \cap E_3^n.$$

It remains to show that $E_1^n \cap E_2^n \cap E_3^n$ is true for all but finitely many n , if we choose α and β in the correct manner.

E_1^n holds for all but finitely many n . Let $\omega \in \mathcal{S}^n$ be any fixed signal word in B^n . By this we mean that ω is the signal word between two fixed starting points; so note that ω , although being fixed in this sense, will still be random. Let $y \notin B^n$ be any potential starting point for ω outside B^n . By independence of the colors

$$\mathbb{P}(\omega \text{ appears in } \xi|(\mathbb{Z}^2 \setminus B^n) \text{ with starting point } y) \leq \left(\frac{4}{m}\right)^{c_1 \log n}$$

as there are $4^{c_1 \log n}$ different walks of length $c_1 \log n$ starting in y . Thus for any y

$$\mathbb{P}(\omega \text{ appears in } \xi|(B_y^{n^2} \setminus B^n)) \leq \pi n^4 \left(\frac{4}{m}\right)^{c_1 \log n} = \pi n^{4+c_1(\log 4 - \log m)}$$

as there are πn^4 different points inside B^{n^2} .

Now the indicators I_w for the event that the word $w \in \mathcal{S}^n$ appears in $B_y^{n^2} \setminus B^n$ are conditionally independent (for different w) under \mathbb{P} given $\xi|(B_y^{n^2} \setminus B^n)$ as the different words have mutually disjoint support and therefore are independent. To understand this point correctly it is important to recall that \mathcal{S}^n is a random set (under \mathbb{P}). The independence claimed above would not be true for any fixed set of words or if we did not condition on knowing $\xi|(B_y^{n^2} \setminus B^n)$.

Hence the number of $w \in \mathcal{S}^n$ appearing in $B_y^{n^2} \setminus B^n$ is stochastically bounded by a binomial random variable with $N = n^2/c_1 \log n$ different trials and success probability $p = \pi n^{2+c_1(\log 4 - \log m)}$. However, for n, m sufficiently large,

$$\frac{n^2}{c_1 \log n} \leq n^2$$

as well as

$$p = \pi n^{2+c_1(\log 4 - \log m)} \leq \frac{1}{n^2}.$$

However, then the number of $w \in \mathcal{S}^n$ appearing in $B_y^{n^2} \setminus B^n$ is stochastically bounded by a Binomial random variable X with n^2 different trials and success probability $\frac{1}{n^2}$. However, by Chebyshev's exponential inequality

$$\mathbb{P}(X \geq n^\beta) \leq e^{-n^\beta} \mathbb{E} e^X = e^{-n^\beta} \left(1 + \frac{e-1}{n^2}\right)^{n^2} = \mathcal{O}(e^{-n^\beta}).$$

It follows that

$$(3.2) \quad \mathbb{P}[(E_1^n)^c] = \mathcal{O}(e^{2n^\alpha - n^\beta}),$$

which is summable for $\beta > \alpha$. By the Borel–Cantelli lemma this implies that E_1^n holds for all but finitely many n .

E_2^n holds for all but a finite number of n . Since the proofs of that E_{2i}^n holds for almost all n are very similar for each i , we just show the proof for E_{22}^n and leave the other proofs to the reader.

To this end consider any $v \in \partial B^n$ and any oriented connected segment s in \mathbb{Z}^2 of length $c_2 \log n$. Note that if the endpoint of s is not in Q_2 , the i th point of $\hat{\sigma}(v)$ is different from all the j th points of s , $j \leq i$, and thus $\xi(\hat{\sigma}_i(v))$ is a “fresh random variable”. Thus by conditional independence the probability of reading $\hat{\sigma}(v)$ along $\xi|s$ is bounded by

$$\mathbb{P}(\xi|s = \xi|\hat{\sigma}(v)) = \left(\frac{1}{m}\right)^{c_2 \log n},$$

and therefore, for every fixed $x \in B^{n^2+n} \setminus Q_2(v)$,

$$\mathbb{P}(\xi|\hat{\sigma}(v) \text{ appears with endpoint } x) \leq \left(\frac{4}{m}\right)^{c_2 \log n}.$$

As there are at most $\pi(n^2 + n)^2$ points in B^{n^2+n} and there are at most $\text{const} \times n$ points $v \in \partial B^n$, we obtain

$$\mathbb{P}((E_{22}^n)^c) \leq \left(\frac{4}{m}\right)^{c_2 \log n} \text{const} \times n \pi(n^2 + n)^2 \leq n^{c_2(\log 4 - \log m) + 6}.$$

The right-hand side of this inequality becomes summable if we choose m sufficiently large (depending on c_2 or c_3). More precisely, we choose m such that

$$c_2(\log 4 - \log m) + 6 < -2.$$

Note that this choice does not depend on n . This choice of m will basically be the proof of Theorem 1.1. Thus (again by a Borel–Cantelli argument) E_{22}^n holds for all but finitely many n .

Note that until now we have been free to choose c_1, c_2, c_3 .

E_3^n holds for all but finitely many n . Again we only give the proof in detail for one of the events, which will be $E_{3,\sigma}^n$. The proof for the other two events follows the same lines.

We split this proof into several parts.

First let us prove that in a certain (stricter than usual) sense the random walk by time e^{n^α} has returned to the origin more than n^γ times, where $\gamma < \alpha < \beta$. A result like this seems to be very much in the spirit of a result of Erdős and Taylor [5], who showed that almost surely a random walk at time e^n has returned to the origin between $n/(\log n)^{1+\varepsilon}$ and $(1+\varepsilon)n \log \log n$ times for all but finitely many n and every positive $\varepsilon > 0$. The reason we cannot simply refer to this result is that we also want these returns to the origin to be at least n^2 apart from each other. So, more precisely, let us introduce a sequence ϑ_i^n of stopping times such that $\vartheta_0^n = 0$ for all n and ϑ_{i+1}^n is the time of the first return of the random walk S_k to the origin

after time $\vartheta_i^n + n^2$. This will ensure that in the meantime the random walk is able to hit one of the boundary points of B^n . So we want to check that for $\gamma < \alpha < \beta$ (γ appropriately chosen afterward) the event

$$E_{31}^n := \{\vartheta_{n^\gamma}^n \leq e^{n^\alpha}\}$$

happens for all but finitely many n . Indeed, choosing $\delta = \frac{\alpha-\gamma}{2}$ the result of Erdős and Taylor [5] quoted above states that the event

$$E_{311}^n := \{\text{up to time } e^{n^\alpha} \text{ there are more than } n^{\gamma+\delta} \text{ returns to the origin}\}$$

holds almost surely for all but a finite number of n . Next we will show that the same is true for the event

$$E_{312}^n := \bigcap_{i=1}^{n^\gamma} \{\text{in the interval } [\vartheta_i^n, \vartheta_i^n + n^2] \text{ there are fewer than } n^\delta \text{ returns to the origin}\}.$$

The probability of a simple random walk starting at the origin not returning for t steps is bounded below by $\frac{2\pi}{\log t}$ for t large enough [3], [12], page 167. Applying this yields

$$\begin{aligned} & \mathbb{P}(\text{in the interval } [\vartheta_i^n, \vartheta_i^n + n^2] \text{ there are more than } n^\delta \text{ returns to the origin}) \\ & \leq \left(1 - \frac{\pi}{\log n}\right)^{n^\delta} \leq e^{-n^{\delta/2}} \end{aligned}$$

for each $i = 1, \dots, n^\gamma$ and n , large sufficiently. Hence, by bounding the probability of a union by the sum of the probabilities,

$$\mathbb{P}((E_{312}^n)^c) \leq n^\gamma e^{-n^{\delta/2}},$$

which is finitely summable. Therefore E_{312}^n holds for all but a finite number of n . As E_{311}^n and E_{312}^n together imply E_{31}^n also E_{31}^n holds for almost all n .

Next we will show that many of the intervals $[\vartheta_i^n, \vartheta_i^n + n^2]$ above are indeed signal times; that is, we will show that we read more than n^β different signals in all of these time intervals. To this end introduce random variables Y_i which are indicators for the event that the interval $[\vartheta_i^n, \vartheta_i^n + n^2]$ is a signal time, that is, for the event that there are more than n^β signal words read in $[\vartheta_i^n, \vartheta_i^n + n^2]$. To avoid the dependence among reading different signal words we only concentrate on such words as are “far apart” from each other. To this end we partition the inner part of B^n , that is, $B^n \setminus \partial_{(\log n)^3} B^n$, where

$$\partial_{(\log n)^3} B^n := \{x \in B^n, d(x, \partial B^n) \leq (\log n)^3\}$$

and $d(\cdot, \cdot)$ is the lattice distance in \mathbb{Z}^2 , into boxes of lengths $c_1 \log n$ and $(\log n)^3$. Let

$$W_{k,l}^n := \{(x, y) \in B^n : c_1 k \log n \leq x < c_1(k+1) \log n, \\ l(\log n)^3 \leq y < (l+1)(\log n)^3\} \quad (k, l \in \mathbb{Z}).$$

For $i = 1, \dots, n^\gamma$ consider the following indicators: let $\mathbb{I}^{1,n}(i)$ be the indicator for the event that $S_{\vartheta_i^n + n^{(4+\beta)/3}} \in B^{n/\log n}$; let $\mathbb{I}^{2,n}(i)$ denote the indicator for the event that the whole trajectory $(S_k)_{k=\vartheta_i^n, \dots, \vartheta_i^n + n^{(4+\beta)/3}}$ is contained in B^n ; let $\mathbb{I}^{3,n}(i)$ be one if the random walk visits more than $n^{2(1+\beta)/3} / (\frac{2(1+\beta)}{3} \log n)$ distinct points in $[\vartheta_i^n, \vartheta_i^n + n^{(4+\beta)/3}]$ and zero otherwise; let $\mathbb{I}_{k,l}^{4,n}(i)$ be the indicator for the event that in the time interval $[\vartheta_i^n, \vartheta_i^n + n^{(4+\beta)/3}]$ the walk enters $W_{k,l}^n$ and within $(\log n)^3$ steps after the first entrance time touches one of the lines $x = k \log n$ or $x = (k+1) \log n$, and finally follows the straight line supporting the the word associated with the starting point it touched.

First consider the event $\{\mathbb{I}^{1,n}(i) = 0\}$. By concentration of measure (cf. [13]) we have, for every fixed i ,

$$\mathbb{P}(\mathbb{I}^{1,n}(i) = 0) = \mathbb{P}\left(\|S_{\vartheta_i^n + n^{(4+\beta)/3}}\| \geq \frac{n}{\log n}\right) \leq \exp\left[-\text{const} \frac{n^{(2-\beta)/3}}{(\log n)^2}\right].$$

Therefore, as $\beta < 2$ and $\gamma < 2$,

$$\mathbb{P}\left(\left(\bigcap_i \{\mathbb{I}^{1,n}(i) = 1\}\right)^c\right) \leq n^2 \exp\left[-\text{const} \frac{n^{(2-\beta)/3}}{(\log n)^2}\right],$$

which is finitely summable and thus $\bigcap_i \{\mathbb{I}^{1,n}(i) = 1\}$ holds for almost all n . Here and in the following \bigcap_i refers to the intersection over $i = 1, \dots, n^\gamma$.

By the same argument

$$\mathbb{P}(\mathbb{I}^{2,n}(i) = 0) = \mathbb{P}(\exists t \in [\vartheta_i^n, \vartheta_i^n + n^{(4+\beta)/3}] : \|S_t\| \geq n) \\ \leq n^2 \mathbb{P}(\|S_{\vartheta_i^n + n^{(4+\beta)/3}}\| \geq n) \leq n^2 e^{-\text{const}(n^{2-\beta})/3}.$$

Thus, also $\bigcap_i \{\mathbb{I}^{2,n}(i) = 0\}$ holds for all but finitely many n .

To bound the probability that $\mathbb{I}^{3,n}(i)$ is equal to zero, first observe that the number of distinct points D_t visited by a simple symmetric random walk starting at the origin by time t satisfies (cf. [3, 4])

$$ED_t \geq \frac{2t}{\log t}$$

for all t sufficiently large. Moreover such a random walk clearly can only have visited at most t points (i.e., $D_t \leq t$) up to time t . Together these imply

$$(3.3) \quad \mathbb{P}\left(D_t \geq \frac{t}{\log t}\right) \geq \frac{1}{\log t}.$$

Partitioning the interval $[\vartheta_i^n, \vartheta_i^n + n^{(4+\beta)/3}]$ into $n^{(2-\beta)/3}$ intervals of length $n^{2(1+\beta)/3}$ and applying (3.3) with $t = n^{2(1+\beta)/3}$ (observe that $\log t = 2\frac{1+\beta}{3} \log n$) yields, for any fixed i ,

$$\mathbb{P}(\mathbb{I}^{3,n}(i) = 0) \leq \left(1 - \frac{\text{const}}{\log n}\right)^{n^{(2-\beta)/3}} \leq \exp\left[-\text{const} \frac{n^{(2-\beta)/3}}{\log n}\right].$$

Hence by the same summability argument as above $\bigcap_i \{\mathbb{I}^{3,n}(i) = 1\}$ holds for almost all n .

Next let us have a closer look at $\{\mathbb{I}_{k,l}^{4,n}(i) = 1\}$. Suppose that we already know that S_n enters the sector $W_{k,l}^n$ within $[\vartheta_i^n, \vartheta_i^n + n^{(4+\beta)/3}]$. Considering just the projection of the walk to the x -axes, we see a nearest neighbor random walk on \mathbb{Z} with holding probability $1/2$. The points $k \log n$ and $(k+1) \log n$ obtained by projecting the vertical limiting lines of $W_{k,l}^n$ may be considered absorbing barriers for this random walk. As the expected hitting time of one of these barriers is of order $(\log n)^2$, after time $(\log n)^3$ we will have hit one of the boundaries with a probability bounded away from zero (in n). In other words, S_n conditioned on its visiting $W_{k,l}^n$ at all will touch one of its left and right boundary lines within $(\log n)^3$ after the first entrance time into this sector with probability bounded away from zero. As the word associated with this boundary point has length $c_1 \log n$ the probability that the walk touches a boundary point and then follows the walk associated with it is bounded by $\text{const}(1/4)^{c_1 \log n}$.

Note that the events $\{\mathbb{I}_{k,l}^{4,n}(i) = 1\}$ are not independent for different choices of (k, l) and the same i and n . First, due to the fact that (S_k) is a Markov chain the event $\{\mathbb{I}_{k,l}^{4,n}(i) = 1\}$ increases the chances that we also hit a square close to $W_{k,l}^n$. However, also given that we visit both $W_{k,l}^n$ and $W_{(k+1),l}^n$, for example, the events $\{\mathbb{I}_{k,l}^{4,n}(i) = 1\}$ and $\{\mathbb{I}_{k+1,l}^{4,n}(i) = 1\}$ are dependent since reading a word associated with a boundary point of $W_{k,n}^n$ might easily coincide with touching a boundary point of $W_{k+1,n}^n$ fewer than $(\log n)^3$ steps after the first entrance time. To cope with this effect we disregard every other square; that is, we consider the indicators

$$\hat{\mathbb{I}}_{k,l}^{4,n}(i) := \mathbb{I}_{k,l}^{4,n}(i) \mathbb{I}(k, l),$$

where $\mathbb{I}(k, l)$ is $+1$ if k and l are even and 0 otherwise, instead.

Now observe that on $\{\mathbb{I}^{2,n}(i) = 1\} \cap \{\mathbb{I}^{3,n}(i) = 1\}$ the random walk visits more than $n^{2(1+\beta)/3} / \frac{2(1+\beta)}{3} \log n$ distinct points within $[\vartheta_i^n, \vartheta_i^n + n^{(4+\beta)/3}]$ —all of them lying in B^n —and therefore, as each of the $W_{k,l}^n$ has $c_1(\log n)^4$ points, also $n^{2(1+\beta)/3} / (2c_1 \frac{1+\beta}{3} (\log n)^5)$ distinct $W_{k,l}^n$'s. As one fourth of them will have both k and l even $\hat{\mathbb{I}}_{k,l}^{4,n}(i)$ has a chance to become $+1$ for $n^{2(1+\beta)/3} / (8c_1 \frac{1+\beta}{3} (\log n)^5)$ different choices of (k, l) . Given the indices (k, l) for which this is true the events $\{\hat{\mathbb{I}}_{k,l}^{4,n}(i) = 1\}$ indeed are independent and have probability at least

$\text{const}(1/4)^{c_1 \log n}$. Hence again by moderate deviations or concentration of measure on $\{\mathbb{I}^{2,n}(i) = 1\} \cap \{\mathbb{I}^{3,n}(i) = 1\}$

$$\mathbb{P}\left(\sum_{k,l} \hat{\mathbb{I}}_{k,l}^{4,n}(i) \leq n^\beta\right) \leq \exp\left(-\text{const} \frac{n^{(1/3)(2-\beta)-c_1 \log 4}}{(\log n)^{10}}\right) \leq e^{-n^\varepsilon}$$

for some small ε , if c_1 is sufficiently small (depending on how large we have chosen β before). As e^{-n^ε} is finitely summable even after multiplication with the number of different ϑ_i^n , we obtain that on the event $\bigcap_i \{\{\mathbb{I}^{2,n}(i) = 1\} \cap \{\mathbb{I}^{3,n}(i) = 1\}\}$ we have $\sum_{k,l} \hat{\mathbb{I}}_{k,l}^{4,n}(i) \geq n^\beta$ for all i and all but finitely many n . As also $\bigcap_i \{\{\mathbb{I}^{2,n}(i) = 1\} \cap \{\mathbb{I}^{3,n}(i) = 1\}\}$ holds for almost all n ,

$$\sum_{k,l} \hat{\mathbb{I}}_{k,l}^{4,n}(i) \geq n^\beta$$

also is true for almost all n . Finally, as also $\bigcap_i \{\mathbb{I}^{1,n}(i) = 1\}$ for all but a finite number of n , we arrive at

$$\bigcap_i \{Y_i = 1\} \cap \{\mathbb{I}^{1,n}(i) = 1\}$$

for all but finitely many n . (Recall that the random variables Y_i are the indicators for the event that the interval $[\vartheta_i^n, \vartheta_i^n + n^2]$ is a signal time, that is, for the event that there are more than n^β signal words read in $[\vartheta_i^n, \vartheta_i^n + n^2]$.)

Let us summarize what we know already. For almost all n the following holds: Until time e^{n^α} we have more than n^γ (γ smaller than α) different intervals of length n^2 of signal times. The signals are read in the first $n^{(4+\beta)/3}$ steps, after which the random walks stops in a distance at most $n/\log n$ from the origin.

Finally we have to show that in these time intervals $[\vartheta_i^n, \vartheta_i^n + n^2]$ we also read all words of length $(2c_2 + c_3) \log n$ beginning with either a root word or a side word associated with any of the boundary points. To avoid trouble with independence we will concentrate only on events where this happens in one of the time intervals $J_i^n := [\vartheta_i^n + n^{(4+\beta)/3}, \vartheta_i^n + n^2]$, $i = 1, 2, \dots$

To this end, first observe that on a time interval of length $|J_i^n|$ the random walk (S_k) deviates from its starting point by the variance of a sum $|J_i^n|$ many independent random variables with variance 1. This is immediately computed as

$$\sqrt{n^2 - n^{(4+\beta)/3}} \geq \frac{n}{2}$$

for n sufficiently large. Therefore and since “in the worst case” $S_{\vartheta_i^n + n^{(4+\beta)/3}} = 0$ with positive probability bounded away from zero (S_k) exits B^n during J_i . This bound will be used to estimate the probability of hitting the beginning $\sigma_0(v)$ of a root word for a boundary point $v \in \partial B^n$ or the beginning of one of its side words. This probability can be computed as the probability of hitting this point conditioned on our hitting the (discrete) sphere it is contained in, times the

probability that we hit this sphere at all. The latter probability is bounded from below by a constant away from zero, by the above considerations. On the other hand the probability of hitting a certain point in ∂B^n conditioned on our leaving B^n is bounded below by $\frac{\varkappa}{n}$ for some constant $\varkappa > 0$ no matter where in $B^{n/\log n}$ we started. Of course, it suffices to understand that this is true for large n . However, observing that, under the scaling $\mathbb{Z}^2 \rightarrow \frac{1}{n}\mathbb{Z}^2$, the boundary ∂B^n converges to the unit sphere, $B^{n/\log n}$ shrinks to the origin and (S_k) converges (after also rescaling the time axes, which is irrelevant for our argument) to Brownian motion $W^0(t)$ starting at the origin; moreover, taking into account that the harmonic measure on the unit sphere (any sphere centered on zero) with respect to $W^0(t)$ is the uniform distribution on it, shows that the above bound indeed holds. So, as all starting points of root words and side words lie in $B^{n^2} \setminus B^{n^2-(c_2+c_3)\log n}$ we see that the probability of hitting any fixed starting point is bounded from below by $\frac{\varkappa'}{n^2}$ for some $\varkappa' > 0$ (\varkappa' results from multiplying \varkappa by the probability of exiting B^{n^2} in a certain J_i).

Now the probability of reading $\hat{\sigma}(v)$ and after that any fixed continuation of length $(c_2 + c_3)\log n$ given that we first read $\sigma_0(v)$ has (for any fixed $v \in \partial B^n$) probability

$$\left(\frac{1}{4}\right)^{(2c_2+c_3)\log n} = n^{-(2c_2+c_3)\log 4}.$$

So the (unconditioned) probability of reading $\hat{\sigma}(v)$ and after that any fixed continuation of length $(c_2 + c_3)\log n$ is bounded from below by

$$\frac{\varkappa}{n} \left(\frac{1}{4}\right)^{(2c_2+c_3)\log n} = \varkappa n^{-1-(2c_2+c_3)\log 4}.$$

On the other hand there are n^γ different time intervals where we can read such a word. So the probability of not reading $\hat{\sigma}(v)$ and after that any fixed continuation of length $(c_2 + c_3)\log n$ in all of these intervals behaves like

$$(1 - \varkappa n^{-1-(2c_2+c_3)\log 4})^{n^\gamma} \leq \exp(-\varkappa n^{\gamma-1-(2c_2+c_3)\log 4}).$$

As we can choose c_2 and c_3 as small as we want to and $\gamma > 1$ (and still γ, α) this probability is smaller than e^{-n^ε} for some $\varepsilon > 0$. The same holds for the probability of reading a side word and then any fixed continuation of length $(c_2 + c_3)\log n$ given that we read its first letter. As for fixed n there are only polynomially many such nearest neighbor walk paths (more precisely, as there are fewer than

$$6\pi n 4^{(c_2+c_3)\log n} = 6\pi n^{1+(c_2+c_3)\log 4}$$

such nearest neighbour walk paths) the probability of not reading all of them is bounded by

$$6\pi n^{1+(c_2+c_3)\log 4} e^{-n^\varepsilon},$$

which is finitely summable in n . Therefore, by the Borel–Cantelli lemma, E_n^3 also holds for all but finitely many n . This completes the proof of Lemma 2.3. \square

The proof of the main theorem now consists only of choosing the constants in the correct order.

PROOF OF THEOREM 1.1. To complete the proof we specify the order in which we choose the constants. So first we choose α, β, γ with $\beta > \alpha$ [such that right-hand side of (3.2) is finitely summable] and $1 < \gamma < \alpha$. Then we choose c_1, c_2 and c_3 to make the last part of the above proof of Lemma 2.3 work (note that this part does not depend on the number of colors m). If we now choose m larger than a certain number m_2 (coming from the arguments which guarantee that E_1^n and E_2^n hold for all but a finite number of n), this procedure ensures that the reconstruction in Lemma 2.3 works with probability 1 for all but a finite number of n .

Thus for $1/2 > \varepsilon > 0$ we can choose N (nonrandom) such that the probability that we have

$$\tilde{\mathcal{A}}^n(\xi|B^n, \chi) \sim \xi|B^{n+1}$$

for all $n \geq N$ is greater than $1 - \frac{\varepsilon}{2}$ given $m \geq m_2$.

Now for N there exists m_1 such that for $m \geq m_1$ the reconstruction algorithm from Lemma 2.2, \mathcal{A}^N , ensures that we can reconstruct $\xi|B^N$ with probability greater than $1 - \frac{\varepsilon}{2}$. If we now choose $m \geq \max\{m_1, m_2\}$ and concatenate \mathcal{A}^N from Lemma 2.2 with the different \mathcal{A}^n for $n \geq N + 1$ from Lemma 2.3, we obtain an algorithm \mathcal{A} which reconstructs ξ with probability greater than $1 - \varepsilon$.

In view of Lemma 2.1 this suffices to prove Theorem 1.1. \square

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