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LIMIT THEOREMS FOR QUADRATIC FORMS WITH APPLICATIONS TO WHITTLE'S ESTIMATE

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We establish strong and weak approximations for quadratic forms of weakly and strongly dependent random variables and obtain necessary and sufficient conditions for the weak convergence of weighted functions of quadratic forms. The results are applied to get the asymptotic distributions of some tests which can be used to detect possible changes in the long-memory parameter.

1. Introduction. Let $\{\xi_k,\ -\infty < k < \infty\}$ be a sequence of independent, identically distributed random variables with $E\xi_0 = 0$ and $E\xi_0^2 = \tau^2$. Throughout this paper $\mathbb{Z}$ stands for the set of integers. Define the moving average

$$X_k = \sum_{j\in\mathbb{Z}} a(k-j)\xi_j, \qquad k \in \mathbb{Z},$$

where $a(k)$, $k \in \mathbb{Z}$, are real weights satisfying $\sum_k a^2(k) < \infty$ ($\sum_k = \sum_{-\infty<k<\infty}$). The sequence X_k, $k \in \mathbb{Z}$, is stationary with $EX_0 = 0$ and $EX_0^2 = \sum_k a^2(k)$. The covariance function of X_k is denoted by $r(k) = EX_0X_k$. Let

$$Q(x) = \sum_{1\le i,\, j\le [x]} b(i-j)\{X_iX_j - r(i-j)\} \quad \text{if } 0 \le x < \infty$$

and

$$Q^*(x, y) = \sum_{[x]<i,\, j\le [y]} b(i-j)\big\{X_iX_j - r(i-j)\big\} \quad \text{if } 0 \le x \le y < \infty$$

($\sum_\varnothing = 0$), where $b(-k) = b(k)$ for all $k \in \mathbb{Z}$ and $\sum_k b^2(k) < \infty$.

Anderson and Walker (1964) and Hannan and Heyde (1972) established central limit theorems for $Q(n)$ when $\{X_k,\ k \in \mathbb{Z}\}$ is weakly dependent. Assuming that $\{X_k,\ k \in \mathbb{Z}\}$ is a strongly dependent stationary Gaussian sequence, Avram (1988) and Fox and Taqqu (1987) proved again the asymptotic normality of $Q(n)$. Their results were extended by Giraitis and Surgailis (1990) to include the stationary, long dependent non-Gaussian case. Kouritzin (1995) obtained strong approximation for cross-covariance of linear variables which is a special case of $Q(n)$.

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Motivated by detection of possible change in the long-memory parameter of strongly dependent observations, Beran and Terrin (1996) studied the asymptotic behavior of

$$Z_n^*(t) = n^{1/2}(t(1-t))^{1/2}\left\{\frac{Q(nt)}{nt} - \frac{Q^*(nt,n)}{n(1-t)}\right\}, \qquad 0 < t < 1.$$

Beran and Terrin (1996) claimed that

$$Z_n^*(t) \longrightarrow_{\mathscr{D}[0,1]} c_0(t(1-t))^{1/2}\left\{\frac{W_1(t)}{t} - \frac{W_2(1-t)}{1-t}\right\}, \tag{1.1}$$

where $0 < c_0 < \infty$ is a constant and W_1 and W_2 are independent Wiener processes (Brownian motions). However, Lévy's law of the iterated logarithm [cf. Theorem 1.3.3 in Csörgő and Révész (1981)] yields that

$$\sup_{0<t<1}(t(1-t))^{1/2}\left\{\frac{W_1(t)}{t} - \frac{W_2(1-t)}{1-t}\right\} = \infty$$

with probability 1 and therefore (1.1) must be false. In this paper we obtain approximations for $Q(n)$ as well as for

$$Z_n(t) = n^{1/2}t(1-t)\left\{\frac{Q(nt)}{nt} - \frac{Q^*(nt,n)}{n(1-t)}\right\}, \qquad 0 < t < 1.$$

The weighted approximations for $Z_n(t)$ will result in limit theorems for functionals of $Z_n^*(t)$.

We can and shall assume without loss of generality that all random variables and processes introduced in this paper can be defined on the same probability space. Let

$$c(k) = b(0)a(k) + 2\sum_{1\le j<\infty} b(j)a(k-j) \tag{1.2}$$

and

$$\begin{aligned}\sigma^2 = {} & E(\xi_0^4 - \tau^4)\Big(\sum_{j\in\mathbb{Z}} a(j)c(j)\Big)^2 \\ & + \tau^4 \sum_{1\le l<\infty}\Big(\sum_{j\in\mathbb{Z}}\big\{a(j)c(j+l) + c(j)a(j+l)\big\}\Big)^2.\end{aligned} \tag{1.3}$$

THEOREM 1.1. *Assume that there exist* $\alpha > 0$, $\beta > 0$ *and* $\theta > 0$ *satisfying* $\alpha + \beta > 1/2$ *and* $\theta + 2\alpha > 1$ *such that*

$$a(k) = O\big(|k|^{-1/2-\alpha}\big), \qquad c(k) = O\big(|k|^{-1/2-\beta}\big), \qquad b(k) = O\big(|k|^{-1/2-\theta}\big) \tag{1.4}$$

as $|k| \to \infty$ *and* $E|\xi_0|^{4+r} < \infty$ *for some* $r > 0$. *Then we can define a Wiener process* $\{W(t),\ 0 \le t < \infty\}$ *such that*

$$Q(n) - \sigma W(n) = O(n^{1/2-\varepsilon}) \quad a.s., \tag{1.5}$$

with some $\varepsilon > 0$.

Next we use Theorem 1.1 to derive the following weighted approximation of $Z_n(t)$.

THEOREM 1.2. *If the conditions of Theorem* 1.1 *are satisfied, then we can define a sequence of Brownian bridges* $\{B_n(t), 0 \le t \le 1\}$ *such that*

$$\sup_{\lambda/n \le t \le 1-\lambda/n} |Z_n(t) - \sigma B_n(t)|/(t(1-t))^{1/2-\varepsilon} = O_P(n^{-\varepsilon}) \tag{1.6}$$

for all $0 < \lambda < \infty$ *and* $0 \le \varepsilon \le \varepsilon_0$, *where* $\varepsilon_0 > 0$.

To get asymptotics for $Z_n^*(t)$, we must look at the behavior of the weighted $Z_n(t)$. The first corollary gives the necessary and sufficient condition for the convergence in distribution of the weighted supremum of $Z_n(t)$. Let

$$FC_{0,1} = \Big\{q\colon \inf_{\delta \le t \le 1-\delta} q(t) > 0 \text{ for all } 0 < \delta < 1/2, q \text{ is nondecreasing in a neighborhood of 0 and nonincreasing in a neighborhood of 1}\Big\}$$

and

$$I(q,c) = \int_0^1 \frac{1}{t(1-t)} \exp\left(-\frac{cq^2(t)}{t(1-t)}\right) dt.$$

COROLLARY 1.1. *We assume that the conditions of Theorem* 1.1 *are satisfied and* $q \in FC_{0,1}$. *Then*

$$\max_{1 \le k \le n} n^{-3/2} k(n-k) \left| \frac{Q(k)}{k} - \frac{Q^*(k,n)}{n-k} \right| \Big/ q(k/n) \longrightarrow_{\mathscr{D}} \sigma \sup_{0<t<1} |B(t)|/q(t) \tag{1.7}$$

if and only if $I(q,c) < \infty$ *with some* $c > 0$, *where* $\{B(t),\ 0 \le t \le 1\}$ *is a Brownian bridge.*

Corollary 1.1 yields that $Z_n^*(t)$ cannot converge in $\mathscr{D}[0,1]$ and

$$\sup_{0<t<1} |Z_n^*(t)| \longrightarrow_P \infty.$$

Let

$$A(x) = (2\log x)^{1/2}$$

and

$$D(x) = 2\log x + \tfrac{1}{2}\log\log x - \tfrac{1}{2}\log \pi$$

if $x > 1$ and $A(x) = D(x) = 1$ when $x \le 1$.

COROLLARY 1.2. *We assume that the conditions of Theorem* 1.1 *are satisfied. Then*

$$\lim_{n\to\infty} P\left\{\frac{A(\log n)}{\sigma n^{1/2}} \max_{1\le k<n} \big(k(n-k)\big)^{1/2}\left|\frac{Q(k)}{k} - \frac{Q^*(k,n)}{n-k}\right| \le x + D(\log n)\right\}$$
$$= \exp(-2e^{-x})$$

for all x.

The approximation in (1.5) and the law of the iterated logarithm for W yield immediately the law of the iterated logarithm for $Q(n)$ under the conditions of Theorem 1.1. However, the applications in Section 2 will need only a strong law of large numbers with rate under somewhat weaker conditions.

THEOREM 1.3. *Assume that there exist* $\alpha > 0$ *and* $\theta > 0$ *with* $\alpha + \theta > 1/2$ *such that*

$$a(k) = O\big(|k|^{-\alpha-1/2}\big) \quad \textit{and} \quad b(k) = O\big(|k|^{-\theta-1/2}\big)$$

as $|k| \to \infty$. *If* $E\xi_0^4 < \infty$, *then*

$$Q(n) = o(n^{1-\varepsilon}) \quad \textit{a.s.}$$

for all $0 < \varepsilon < \min(1/2, 4\alpha + 2\theta - 1)$.

The proofs of Theorems 1.1–1.3 and Corollaries 1.1 and 1.2 are given in Section 3. These results are used to detect possible changes in the long-memory parameter in the next section.

2. Change in the long-memory parameter. In this section we assume that the distribution of X_k can be characterized by a finite-dimensional parameter $(\kappa_k, \boldsymbol{\lambda}_k)$, $\boldsymbol{\lambda}_k = (\lambda_{k,1}, \dots, \lambda_{k,p})$. Namely,

$$X_k = \kappa_k \sum_{j\in\mathbb{Z}} R(k-j; \boldsymbol{\lambda}_k)\xi_j.$$

The long-memory behavior of the sequence is characterized by the first coordinate of $\boldsymbol{\lambda}_k$. We wish to test the null hypothesis

H_0: $(\kappa_1, \boldsymbol{\lambda}_1) = (\kappa_2, \boldsymbol{\lambda}_2) = \cdots = (\kappa_n, \boldsymbol{\lambda}_n)$

against the alternative

H_A: there is an integer k^*, $1 \le k^* < n$, such that

$$\lambda_{1,1} = \lambda_{2,1} = \cdots = \lambda_{k^*,1} \ne \lambda_{k^*+1,1} = \cdots = \lambda_{n,1}.$$

Let

$$D(t; \boldsymbol{\lambda}) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} e^{itx}\frac{1}{f(t;\boldsymbol{\lambda})}\,dt,$$

where $f(t;\boldsymbol{\lambda}) = 2\pi|\hat{R}(t;\boldsymbol{\lambda})|^2$,

$$R(t;\boldsymbol{\lambda}) = \int_{-\pi}^{\pi} e^{itx}\hat{R}(x;\boldsymbol{\lambda})\,dx.$$

Next we define

$$\Lambda_k(\boldsymbol{\lambda}) = \frac{1}{k}\sum_{1\le i,\, j\le k} D(i-j;\boldsymbol{\lambda})X_iX_j$$

and

$$\Lambda_k^*(\boldsymbol{\lambda}) = \frac{1}{n-k}\sum_{k<i,\, j\le n} D(i-j;\boldsymbol{\lambda})X_iX_j.$$

Using $X_1,\dots,X_k$, Whittle's estimates $(\hat{\kappa}_k, \hat{\boldsymbol{\lambda}}_k)$, $\hat{\boldsymbol{\lambda}}_k = (\hat{\lambda}_{k,1},\dots,\hat{\lambda}_{k,p})$ are the solutions of the equations

$$\sum_{1\le i,\, j\le k}\frac{\partial}{\partial\boldsymbol{\lambda}}D(i-j;\hat{\boldsymbol{\lambda}}_k)X_iX_j = 0$$

and

$$\hat{\kappa}_k^2 = \Lambda_k(\hat{\boldsymbol{\lambda}}_k).$$

Similarly, the estimators $(\tilde{\kappa}_k, \tilde{\boldsymbol{\lambda}}_k)$, $\tilde{\boldsymbol{\lambda}}_k = (\tilde{\lambda}_{k,1},\dots,\tilde{\lambda}_{k,p})$ based on $X_{k+1},\dots,X_n$ are the solutions of

$$\sum_{k<i,\, j\le n}\frac{\partial}{\partial\boldsymbol{\lambda}}D(i-j;\tilde{\boldsymbol{\lambda}}_k)X_iX_j = 0$$

and

$$\tilde{\kappa}_k^2 = \Lambda_k^*(\tilde{\boldsymbol{\lambda}}_k).$$

If $|\hat{\lambda}_{k,1} - \tilde{\lambda}_{k,1}|$ is large for some k, then we reject H_0 in favor of H_A. We consider the behavior of $\hat{\lambda}_{k,1} - \tilde{\lambda}_{k,1}$ under the null hypothesis.

We assume that the parameter set $\mathscr{K}\times\mathscr{L}\subset(0,\infty)\times R^p$ is an open, relatively compact set. The true value $(\kappa_0, \boldsymbol{\lambda}_0)$ lies in the interior of $\mathscr{K}\times\mathscr{L}$. We also assume the normalization $R(0;\boldsymbol{\lambda}) = 1$, $\boldsymbol{\lambda}\in\mathscr{L}$ or, equivalently,

$$\int_{-\pi}^{\pi}\log f(t;\boldsymbol{\lambda})\,dt = 0, \qquad \boldsymbol{\lambda}\in\mathscr{L}.$$

The following set of regularity conditions are taken from Giraitis and Surgailis (1990): There exist $0<\gamma=\gamma(\boldsymbol{\lambda})<1$ and $0<C=C(\boldsymbol{\lambda})<\infty$ such that

(2.1) $\int_{-\pi}^{\pi}\log f(t;\boldsymbol{\lambda})\,dt(\equiv 0)$ is twice differentiable in $\boldsymbol{\lambda}$ under the sign of the integral,

(2.2) $f(t;\boldsymbol{\lambda})$ is continuous at all $(t;\boldsymbol{\lambda})$, $t\ne 0$ and $|f(t;\boldsymbol{\lambda})|\le C|t|^{-\gamma}$,

(2.3) $1/f(t;\boldsymbol{\lambda})$ is continuous at all $(t;\boldsymbol{\lambda})$, (2.4) $$\frac{\partial}{\partial\lambda_i}\frac{1}{f(t;\boldsymbol{\lambda})} \text{ is continuous at all } (t;\boldsymbol{\lambda}),\ 1 \le i \le p \text{ and}$$

$$\left|\frac{\partial}{\partial\lambda_i}\frac{1}{f(t;\boldsymbol{\lambda})}\right| \le C|t|^{\gamma}, \qquad 1 \le i \le p,$$

(2.5) $$\frac{\partial^2}{\partial\lambda_i\,\partial\lambda_j}\frac{1}{f(t;\boldsymbol{\lambda})} \text{ is continuous at all } (t;\boldsymbol{\lambda}),\ 1 \le i,\ j \le p$$

and

(2.6) $$\left|\frac{\partial^2}{\partial t\,\partial\lambda_i}\frac{1}{f(t;\boldsymbol{\lambda})}\right| \le C|t|^{\gamma-1}, \qquad 1 \le i \le p.$$

Let $\mathscr{W}(\boldsymbol{\lambda})$ be a $p \times p$ matrix with entries

$$w_{ij}(\boldsymbol{\lambda}) = \int_{-\pi}^{\pi} f(t;\boldsymbol{\lambda})\frac{\partial^2}{\partial\lambda_i\,\partial\lambda_j}\frac{1}{f(t;\boldsymbol{\lambda})}\,dt$$

and let w^* be the first element in the first row of $\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)$. In the next theorem, we assume that the assumptions of Theorem 1.1 are satisfied. Let

$$c_i^*(k) = R(0;\boldsymbol{\lambda}_0)\frac{\partial}{\partial\lambda_i}D(0;\boldsymbol{\lambda}_0) + 2\sum_{1\le j<\infty} R(k-j;\boldsymbol{\lambda}_0)\frac{\partial}{\partial\lambda_i}D(j;\boldsymbol{\lambda}_0).$$

We assume that

(2.7) there exist $\alpha > 0$, $\beta > 0$ and $\theta > 0$ satisfying $\alpha + \beta > 1/2$ and $\theta + 2\alpha > 1$ such that

$$R(k;\boldsymbol{\lambda}_0) = O(|k|^{-\alpha-1/2}),$$

$$\max_{1\le i\le p}\left|\frac{\partial}{\partial\lambda_i}D(k;\boldsymbol{\lambda}_0)\right| = O\big(|k|^{-\theta-1/2}\big)$$

and

$$\max_{1\le i\le p}\big|c_i^*(k)\big| = O\big(|k|^{-\beta-1/2}\big).$$

THEOREM 2.1. *We assume that* $E|\xi_0|^{4+r} < \infty$ *with some* $r > 0$ *and* H_0 *and* (2.1)–(2.7) *are satisfied. Then*

$$n^{1/2}t(1-t)\left(\hat{\lambda}_{[nt],1} - \tilde{\lambda}_{[nt],1}\right) \longrightarrow_{\mathscr{D}[0,1]} (4\pi w^*)^{1/2}B(t),$$

where $\{B(t), 0 \le t \le 1\}$ *is a Brownian bridge.*

To increase the power of tests, we consider weighted versions of Theorem 2.1.

THEOREM 2.2. *We assume that* $E|\xi_0|^{4+r} < \infty$ *with some* $r > 0$, $q \in FC_{0,1}$ *and* H_0 *and* (2.1)–(2.7) *are satisfied. Then*

$$\max_{1\le k<n} n^{-3/2}k(n-k)|\hat\lambda_{k,1}-\tilde\lambda_{k,1}|/q(k/n) \longrightarrow_{\mathscr{D}} (4\pi w^*)^{1/2}\sup_{0<t<1}|B(t)|/q(t) \tag{2.8}$$

if and only if $I(q,c) < \infty$ *with some* $c > 0$, *where* $\{B(t),\ 0\le t\le 1\}$ *is a Brownian bridge.*

The standard deviation of $\hat\lambda_{k,1}-\tilde\lambda_{k,1}$ would be the "optimal" weight function in (2.8). The weighted statistic has higher power if the change occurs early or late. For a discussion on the application of weight functions in change-point analysis, we refer to Csörgő and Horváth (1997). Since the variance of $\hat\lambda_{k,1}-\tilde\lambda_{k,1}$ is proportional to $n/(k(n-k))$, this case is not covered in Theorem 2.2. The next result considers this case. In addition to (2.1)–(2.7), we assume

(2.9) there exists $\theta' > 0$ satisfying $\alpha+\theta' > 1/2$ such that

$$\max_{1\le i,\,j\le p}\left|\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}D(k;\boldsymbol\lambda_0)\right| = O\big(|k|^{-\theta'-1/2}\big),$$

(2.10) $\dfrac{\partial^2}{\partial\lambda_i\,\partial\lambda_j}\dfrac{1}{f(t;\boldsymbol\lambda)}$ is continuous at all $(t;\boldsymbol\lambda)$, $1\le i,\ j\le p$ and

$$\left|\frac{\partial^2}{\partial\lambda_i\,\partial\lambda_j}\frac{1}{f(t;\boldsymbol\lambda_0)}\right| \le C|t|^{\gamma}, \qquad 1\le i,\ j\le p$$

and

(2.11) $\dfrac{\partial^3}{\partial\lambda_i\,\partial\lambda_j\,\partial\lambda_k}\dfrac{1}{f(t;\boldsymbol\lambda)}$ is continuous at all $(t;\boldsymbol\lambda)$, $1\le i,\ j,\ k\le p$.

THEOREM 2.3. *We assume that* $E|\xi_0|^{4+r} < \infty$ *with some* $r > 0$ *and* H_0, (2.1)–(2.7) *and* (2.9)–(2.11) *are satisfied. Then*

$$\begin{aligned}\lim_{n\to\infty} P\left\{\frac{A(\log n)}{(4\pi w^*)^{1/2}}\max_{1\le k<n} n^{-1/2}(k(n-k))^{1/2}|\hat\lambda_{k,1}-\tilde\lambda_{k,1}| \le x + D(\log n)\right\}\\ = \exp(-2e^{-x})\end{aligned} \tag{2.12}$$

for all x.

REMARK 2.1. If

$$\max_{1\le i,\,j\le p}\left|\frac{\partial^3}{\partial\lambda_i\,\partial\lambda_j\,\partial t}\frac{1}{f(t;\boldsymbol\lambda_0)}\right| \le C, \tag{2.13}$$

with some $C < \infty$, then

$$\max_{1\le i,\,j\le p}\left|\frac{\partial^2}{\partial\lambda_i\partial\lambda_j}D(k;\boldsymbol\lambda_0)\right| = O(k^{-1}).$$

We note that the statistic $\max_{1\le k<n} n^{-1/2}(k(n-k)^{1/2}|\hat{\lambda}_{k,1} - \tilde{\lambda}_{k,1}|$ was also suggested by Beran and Terrin (1996). However, they could not get this correct limit distribution.

3. Proofs of Theorems 1.1–1.3 and Corollaries 1.1 and 1.2. The proofs of Theorems 1.1 and 1.2 are based on decompositions of Q and Q^* into martingales and smaller remainder terms. First we note that

$$(3.1)\quad \begin{aligned} Q(n) &= \sum_{1\le j\le n}\left\{b(0)(X^2(j)-r(0)) + 2\sum_{1\le i<j} b(i)(X(j)X(j-i)-r(i))\right\}\\ &= Q_1(n) - 2Q_2(n), \end{aligned}$$

where

$$Q_1(n) = \sum_{1\le j\le n}\left\{b(0)(X^2(j)-r(0)) + 2\sum_{1\le i<\infty} b(i)\big(X(j)X(j-i)-r(i)\big)\right\}$$

and

$$Q_2(n) = \sum_{1\le j\le n}\sum_{j\le i<\infty} b(i)\{X(j)X(j-i)-r(i)\}.$$

It follows from the definition of $X(k)$ and $c(k)$ that

$$\begin{aligned} Q_1(n) &= \sum_{1\le j\le n}\left\{b(0)\sum_{k,l} a(j-k)a(j-l)(\xi_k\xi_l - E\xi_k\xi_l).\right.\\ &\qquad\left. + 2\sum_{i\ge 1}\sum_{k,l} b(i)a(j-k)a(j-i-l)(\xi_k\xi_l - E\xi_k\xi_l)\right\}\\ &= \sum_{k,l}\sum_{1\le j\le n} a(j-k)c(j-l)(\xi_k\xi_l - E\xi_k\xi_l)\\ &= \sum_{k}\sum_{1\le j\le n} a(j-k)c(j-k)(\xi_k^2-\tau^2)\\ &\quad + \sum_{-\infty<l<k}\sum_{1\le j\le n}\big\{a(j-k)c(j-l)+a(j-l)c(j-k)\big\}\xi_l\xi_k\\ &= \sum_{1\le k\le n}\sum_{1\le j\le n} a(j-k)c(j-k)(\xi_k^2-\tau^2) \qquad (3.2)\\ &\quad + \sum_{1\le l<k\le n}\sum_{1\le j\le n}\big\{a(j-k)c(j-l)+a(j-l)c(j-k)\big\}\xi_l\xi_k\\ &\quad + \sum_{k\le 0 \text{ or } k>n}\sum_{1\le j\le n} a(j-k)c(j-k)(\xi_k^2-\tau^2)\\ &\quad + \sum_{l<k, l\le 0 \text{ or } k>n}\sum_{1\le j\le n}\big\{a(j-k)c(j-l)+a(j-l)c(j-k)\big\}\xi_l\xi_k\\ &= Q_3(n)+\cdots+Q_7(n), \end{aligned}$$

where

$$Q_3(n) = \sum_{1\le k\le n} \sum_{1\le j<\infty} a(j-k)c(j-k)(\xi_k^2 - \tau^2)$$
$$+ \sum_{1\le l<k\le n} \sum_{1\le j<\infty} \{a(j-k)c(j-l) + a(j-l)c(j-k)\}\xi_l \xi_k,$$

$$Q_4(n) = -\sum_{1\le k\le n} \sum_{n<j<\infty} a(j-k)c(j-k)(\xi_k^2 - \tau^2),$$

$$Q_5(n) = -\sum_{1\le l<k\le n} \sum_{n<j<\infty} \{a(j-k)c(j-l) + a(j-l)c(j-k)\}\xi_l \xi_k,$$

$$Q_6(n) = \sum_{k\le 0 \text{ or } k>n} \sum_{1\le j\le n} a(j-k)c(j-k)(\xi_k^2 - \tau^2)$$

and

$$Q_7(n) = \sum_{l<k,\, l\le 0 \text{ or } k>n} \sum_{1\le j\le n} \{a(j-k)c(j-l) + a(j-l)c(j-k)\}\xi_l \xi_k.$$

The following two lemmas show that $Q_3(n)$ is the leading term in (3.1) and (3.2).

LEMMA 3.1. *If the conditions of Theorem* 1.1 *are satisfied*, *then we can define a Wiener process* $\{W(t),\ 0 \le t < \infty\}$ *such that*

$$Q_3(n) - W(\sigma^2 n) = O(n^{1/2-\varepsilon}) \quad a.s., \tag{3.3}$$

with some $\varepsilon > 0$.

LEMMA 3.2. *If the conditions of Theorem* 1.1 *are satisfied*, *then*

$$|Q_4(n)| + |Q_6(n)| = O(n^{1/2-\delta}) \quad a.s., \tag{3.4}$$

$$|Q_5(n)| + |Q_7(n)| = O(n^{1/2-\delta}) \quad a.s. \tag{3.5}$$

and

$$Q_2(n) = O(n^{1/2-\delta}) \quad a.s., \tag{3.6}$$

with some $\delta > 0$.

The proofs of Lemmas 3.1 and 3.2 are given in the next section.

PROOF OF THEOREM 1.1. Putting together the decompositions in (3.1) and (3.2) and Lemmas 3.1 and 3.2, we get that

$$Q(n) - W(\sigma^2 n) = O(n^{1/2-\varepsilon}) \quad \text{a.s.},$$

with some $\varepsilon > 0$. Since $\{W(\sigma^2 x),\ 0 \le x < \infty\} =_{\mathscr{D}} \{\sigma W(x),\ 0 \le x < \infty\}$, the proof of Theorem 1.1 is complete. □

The proof of Theorem 1.2 is based on decompositions similar to (3.1) and (3.2). One can write

$$
\begin{aligned}
Q^*(m,n) &= \sum_{1\le j\le n-m} \Big\{ b(0)(X^2(j+m)-r(0)) \\
&\qquad + 2\textstyle\sum_{1\le i<j} b(i)(X(j+m)X(j+m-i)-r(i)) \Big\} \qquad (3.7)\\
&= Q_1^*(m,n) - 2Q_2^*(m,n),
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1^*(m,n) = \sum_{1\le j\le n-m} \Big\{ & b(0)(X^2(j+m)-r(0)) \\
& + 2\sum_{1\le i<\infty} b(i)(X(j+m)X(j+m-i)-r(i)) \Big\}
\end{aligned}
$$

and

$$
Q_2^*(m,n) = \sum_{1\le j\le n-m} \sum_{j\ge i} b(i)\big\{X(j+m)X(j+m-i)-r(i)\big\}.
$$

Similarly to (3.2), we have

$$
Q_1^*(m,n) = Q_3^*(m,n) + \cdots + Q_8^*(m,n), \tag{3.8}
$$

with

$$
\begin{aligned}
Q_3^*(m,n) &= \sum_{m<k\le n} \sum_{1\le j<\infty} a(j-k)c(j-k)(\xi_k^2-\tau^2) \\
&\quad + \sum_{m<k\le n} \sum_{1\le l<k} \sum_{1\le j<\infty} \big\{a(j-k)c(j-l)+a(j-l)c(j-k)\big\}\xi_l\xi_k, \\
Q_4^*(m,n) &= -\sum_{m<k\le n} \sum_{1\le j\le m \text{ or } n<j<\infty} a(j-k)c(j-k)(\xi_k^2-\tau^2), \\
Q_5^*(m,n) &= -\sum_{m<k\le n} \sum_{1\le l<k} \sum_{1\le j\le m \text{ or } n<j<\infty} \big\{a(j-k)c(j-l) \\
&\qquad + a(j-l)c(j-k)\big\}\xi_l\xi_k, \\
Q_6^*(m,n) &= \sum_{k\le m \text{ or } k>n} \sum_{m<j\le n} a(j-k)c(j-k)(\xi_k^2-\tau^2), \\
Q_7^*(m,n) &= \sum_{k\le m \text{ or } k>n} \sum_{l<k} \sum_{m<j\le n} \big\{a(j-k)c(j-l)+a(j-l)c(j-k)\big\}\xi_l\xi_k
\end{aligned}
$$

and

$$
Q_8^*(m,n) = -\sum_{m<k\le n} \sum_{-\infty<l\le 0} \sum_{m<j\le n} \big\{a(j-k)c(j-l)+a(j-l)c(j-k)\big\}\xi_l\xi_k.
$$

The following two lemmas are needed in the proof of Theorem 1.2.

LEMMA 3.3. *If the conditions of Theorem* 1.1 *are satisfied*, *then*, *for each* n, *we can define two independent Wiener processes* $\{W_n^{(1)}(t),\ 0 \le t < \infty\}$ *and* $\{W_n^{(2)}(t),\ 0 \le t < \infty\}$ *such that*

$$\sup_{1 \le m \le n/2} \big|Q_3(m) - \sigma W_n^{(1)}(m)\big| / m^{1/2-\varepsilon} = O_P(1) \tag{3.9}$$

and

$$\sup_{n/2 \le m < n} \big|Q_3^*(m,n) - \sigma W_n^{(2)}(n-m)\big| / (n-m)^{1/2-\varepsilon} = O_P(1), \tag{3.10}$$

with some $\varepsilon > 0$.

LEMMA 3.4. *If the conditions of Theorem* 1.1 *are satisfied*, *then*

$$\max_{1 \le k \le n} \big\{|Q_2(k)| + |Q_4(k)| + \cdots + |Q_7(k)|\big\} / k^{1/2-\delta} = O_P(1), \tag{3.11}$$

$$\sup_{1 \le m < n} \big\{|Q_4^*(m,n)| + |Q_6^*(m,n)|\big\} / (n-m)^{1/2-\delta} = O_P(1), \tag{3.12}$$

$$\sup_{1 \le m < n} \big\{|Q_5^*(m,n)| + |Q_7^*(m,n)| + |Q_8^*(m,n)|\big\} / (n-m)^{1/2-\delta} = O_P(1) \tag{3.13}$$

and

$$\sup_{1 \le m < n} \big|Q_2^*(m,n)\big| / (n-m)^{1/2-\delta} = O_P(1), \tag{3.14}$$

with some $\delta > 0$.

The proofs of Lemmas 3.3 and 3.4 are postponed until Section 4.

PROOF OF THEOREM 1.2. Let

$$\begin{aligned} Y_k = &\sum_{1 \le j < \infty} a(j-k)c(j-k)(\xi_k^2 - \tau^2) \\ &+ \sum_{1 \le l < k} \sum_{1 \le j < \infty} \big\{a(j-k)c(j-l) + a(j-l)c(j-k)\big\}\xi_l \xi_k. \end{aligned}$$

It follows from the definitions of $Q_3(m)$ and $Q_3^*(m,n)$ that

$$Q_3(m) = \sum_{1 \le k \le m} Y_k \quad \text{and} \quad Q_3^*(m,n) = \sum_{m < k \le n} Y_k$$

and therefore

$$Q_3(m) + Q_3^*(m,n) = Q_3(n/2) + Q_3^*(n/2,n) \quad \text{for all } 1 \le m < n.$$

Hence by Lemma 3.3 we have

$$\begin{aligned} &\max_{1 \le m < n} \big|Q_3(m) + Q_3^*(m,n) - \sigma(W_n^{(1)}(n/2) + W_n^{(2)}(n/2))\big| \\ &\qquad = O_P(n^{1/2-\varepsilon}). \end{aligned} \tag{3.15}$$

Let $\mu = \min(\varepsilon, \delta)$. Lemmas 3.3 and 3.4 and the modulus of continuity of Wiener processes [cf. Csörgő and Révész (1981)] yield

$$\sup_{\lambda/n \le t \le 1/2} \left| Q(nt) - \sigma W_n^{(1)}(nt) \right| / (nt)^{1/2-\mu} = O_P(1), \tag{3.16}$$

$$\sup_{1/2 \le t \le 1-\lambda/n} \left| Q^*(nt, n) - \sigma W_n^{(2)}(n(1-t)) \right| / (n(1-t))^{1/2-\mu} = O_P(1) \tag{3.17}$$

for all $\lambda > 0$ and by (3.15) we have

$$\sup_{0 \le t \le 1} \left| Q(nt) + Q^*(nt, n) - \sigma(W_n^{(1)}(n/2) + W_n^{(2)}(n/2)) \right| = O_P(n^{1/2-\mu}). \tag{3.18}$$

Let

$$B_n(t) = \begin{cases} n^{-1/2}\big(W_n^{(1)}(nt) - t(W_n^{(1)}(n/2) + W_n^{(2)}(n/2))\big), & 0 \le t \le 1/2, \\ n^{-1/2}\big(-W_n^{(2)}(nt) + (1-t)(W_n^{(1)}(n/2) + W_n^{(2)}(n/2))\big), & 1/2 < t \le 1. \end{cases}$$

It is easy to see that

$$Z_n(t) = \begin{cases} n^{-1/2}\big(Q(nt) - t(Q(nt) + Q^*(nt, n))\big), & 0 \le t \le 1/2, \\ n^{-1/2}\big(-Q^*(nt, n) + (1-t)(Q(nt) + Q^*(nt, n))\big), & 1/2 \le t \le 1. \end{cases}$$

Let $0 \le \varepsilon < \mu$ and $\delta > 0$ so small that $\varepsilon + \delta < \mu$. By (3.16) and (3.18) we have

$$\begin{aligned} n^{\varepsilon} & \sup_{\lambda/n \le t \le 1/2} \left| Z_n(t) - \sigma B_n(t) \right| / t^{1/2-\varepsilon} \\ & \le \lambda^{-\delta} n^{\varepsilon+\delta} \sup_{\lambda/n \le t \le 1/2} \left| Z_n(t) - \sigma B_n(t) \right| / t^{1/2-\varepsilon-\delta} \\ & \le \lambda^{-\delta} \sup_{\lambda/n \le t \le 1/2} \left| Q(nt) - \sigma W_n^{(1)}(nt) \right| / (nt)^{1/2-\varepsilon-\delta} \\ & \quad + \lambda^{-\delta} n^{\varepsilon+\delta-1/2} \sup_{\lambda/n \le t \le 1/2} t^{1/2+\varepsilon+\delta} \big| Q(nt) + Q^*(nt, n) \\ & \qquad\qquad - \sigma(W_n^{(1)}(n/2) + W_n^{(2)}(n/2)) \big| \\ & = O_P(1). \end{aligned}$$

Replacing (3.16) with (3.17), we obtain that

$$n^{\varepsilon} \sup_{1/2 \le t \le 1-\lambda/n} \left| Z_n(t) - \sigma B_n(t) \right| / (1-t)^{1/2-\varepsilon} = O_P(1)$$

for all $0 \le \varepsilon < \mu$. Computing the covariance function, one can easily verify that $B_n(t)$ is a Brownian bridge. This completes the proof of Theorem 1.2. □

PROOF OF COROLLARY 1.1. If $I(q, c) < \infty$ with some $c > 0$, then

$$\lim_{t \to 0} t^{1/2}/q(t) = 0 \quad \text{and} \quad \lim_{t \to 1} (1-t)^{1/2}/q(t) = 0$$

[cf. page 180 in Csörgő and Horváth (1993)]. Theorem 1.2 yields for all $0 < \delta < 1/2$ that

$$\sup_{\delta \le t \le 1-\delta} \big|Z_n(t) - \sigma B_n(t)\big| = o_P(1). \tag{3.19}$$

Also,

$$\begin{aligned} &\sup_{1/(2n) \le t \le \delta} \big|Z_n(t) - \sigma B_n(t)\big| / q(t) \\ &\quad \le \sup_{1/(2n) \le t \le 1-/(2n)} \big|Z_n(t) - \sigma B_n(t)\big| / t^{1/2} \sup_{0<t\le\delta} t^{1/2}/q(t) \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\sup_{1-\delta \le t \le 1-1/(2n)} \big|Z_n(t) - \sigma B_n(t)\big| / q(t) \\ &\qquad \le \sup_{1/(2n) \le t \le 1-1/(2n)} \big|Z_n(t) - \sigma B_n(t)\big| / (1-t)^{1/2} \\ &\qquad \times \sup_{1-\delta < t < 1} (1-t)^{1/2}/q(t). \end{aligned} \tag{3.21}$$

Applying again Theorem 1.2 with $\varepsilon = 0$, we obtain that

$$\sup_{1/(2n) \le t \le 1-1/(2n)} \big|Z_n(t) - \sigma B_n(t)\big| / q(t) = o_P(1),$$

which gives (1.7).

If (1.7) holds, then the limiting random variable is finite. So using Theorem 4.1.1 in Csörgő and Horváth [(1993), page 181] we get that $I(q, c) < \infty$ for some $c > 0$. □

PROOF OF COROLLARY 1.2. By Lemma 5.1.3 in Csörgő and Horváth [(1993), page 257], we have that

$$(2 \log\log n)^{-1/2} \sup_{1/(2n) \le t \le 1-1/(2n)} |B_n(t)|/(t(1-t))^{1/2} \longrightarrow_P 1, \tag{3.22}$$

$$\sup_{1/(2n) \le t \le (\log n)/n} |B_n(t)|/(t(1-t))^{1/2} = O_P\big((\log\log\log n)^{1/2}\big) \tag{3.23}$$

and

$$\sup_{1-(\log n)/n \le t \le 1-1/(2n)} |B_n(t)|/(t(1-t))^{1/2} = O_P\big((\log\log\log n)^{1/2}\big). \tag{3.24}$$

Using Theorem 1.2 with $\varepsilon = 0$, we get from (3.22)–(3.24) that

$$(2 \log\log n)^{-1/2} \sup_{1/(2n) \le t \le 1-1/(2n)} |Z_n(t)|/(t(1-t))^{1/2} \longrightarrow_P \sigma, \tag{3.25}$$

$$\sup_{1/(2n) \le t \le (\log n)/n} |Z_n(t)|/(t(1-t))^{1/2} = O_P\big((\log\log\log n)^{1/2}\big) \tag{3.26}$$

and

$$\sup_{1-(\log n)/n \le t \le 1-1/(2n)} |Z_n(t)|/(t(1-t))^{1/2} = O_P\big((\log\log\log n)^{1/2}\big). \tag{3.27}$$

Let l_n and b_n be defined by

$$\sup_{1/(2n) \le t \le 1-1/(2n)} |Z_n(t)|/(t(1-t))^{1/2} = |Z(l_n)|/(l_n(1-l_n))^{1/2}$$

and

$$\sup_{1/(2n) \le t \le 1-1/(2n)} |B_n(t)|/(t(1-t))^{1/2} = |B(b_n)|/(b_n(1-b_n))^{1/2}.$$

Now (3.22)–(3.27) imply that

$$\lim_{n\to\infty} P\big\{(\log n)/n \le l_n,\ b_n \le 1-(\log n)/n\big\} = 1. \tag{3.28}$$

Theorem 1.2 with $0 < \varepsilon \le \varepsilon_0$ gives that

$$\begin{aligned} &\sup_{(\log n)/n \le t \le 1-(\log n)/n} \big|Z_n(t) - \sigma B_n(t)\big|/(t(1-t))^{1/2} \\ &\quad\le \sup_{(\log n)/n \le t \le 1-(\log n)/n} (t(1-t))^{-\varepsilon} \\ &\quad\quad \times \sup_{1/(2n) \le t \le 1-1/(2n)} \big|Z_n(t) - \sigma B_n(t)\big|/(t(1-t))^{1/2-\varepsilon} \\ &\quad= O_P((\log n)^{-\varepsilon}). \end{aligned} \tag{3.29}$$

Putting together (3.28) and (3.29), we get that

$$\begin{aligned} &\Bigg|\frac{1}{\sigma n^{1/2}} \max_{1\le k<n} (k(n-k))^{1/2} \left|\frac{Q(k)}{k} - \frac{Q^*(k,n)}{n-k}\right| \\ &\qquad - \sup_{1/(2n)\le t\le 1-1/(2n)} |B_n(t)|/(t(1-t))^{1/2}\Bigg| = O_P((\log n)^{-\varepsilon}). \end{aligned}$$

Since [cf. Lemma 5.1.3 in Csörgő and Horváth (1993), page 257]

$$\begin{aligned} &\lim_{n\to\infty} P\left\{A(\log n) \sup_{1/(2n)\le t\le 1-1/(2n)} |B_n(t)|/(t(1-t))^{1/2} \le x + D(\log n)\right\} \\ &\quad = \exp(-2e^{-x}) \end{aligned}$$

for all x, the proof of Corollary 1.2 is complete. □

PROOF OF THEOREM 1.3. In light of Lemma 4.3, it suffices to show that

$$E\big(Q(n+m) - Q(m)\big)^2 \le C_1(n + n^{3-4\alpha+2\theta}) \tag{3.30}$$

for all $0 \le m < \infty$ and $1 \le n < \infty$ with some C_1. It is easy to see that

$$\begin{aligned} Q(n+m) - Q(m) = \sum_{k,l} (\xi_k\xi_l - E\xi_k\xi_l) \Bigg\{ &\sum_{1\le i,\, j\le n+m} b(i-j)a(i-k)a(j-l) \\ &- \sum_{1\le i,\, j\le m} b(i-j)a(i-k)a(j-l)\Bigg\}. \end{aligned}$$

Thus we get [cf. Giraitis and Surgailis (1990), page 91]

$$
\begin{aligned}
&E(Q(n+m)-Q(m))^2\\
&\quad \le C_2 \sum_{k,l} \Bigg\{ \sum_{1\le i,\, j\le n+m} b(i-j)a(i-k)a(j-l) - \sum_{1\le i,\, j\le m} b(i-j)a(i-k)a(j-l) \Bigg\}^2\\
&\quad \le 4C_2\big(I_1(m,n)+I_2(m,n)+I_3(m,n)\big),
\end{aligned}
$$

with

$$
I_1(m,n) = \sum_{k,l} \Bigg\{ \sum_{1\le i\le m} \sum_{m<j\le n+m} b(i-j)a(i-k)a(j-l) \Bigg\}^2,
$$

$$
I_2(m,n) = \sum_{k,l} \Bigg\{ \sum_{m<i\le n+m} \sum_{m<j\le n+m} b(i-j)a(i-k)a(j-l) \Bigg\}^2
$$

and

$$
I_3(m,n) = \sum_{k,l} \Bigg\{ \sum_{m<i\le n+m} \sum_{1\le j\le m} b(i-j)a(i-k)a(j-l) \Bigg\}^2.
$$

Using Lemma 4.5, we obtain that

$$
\begin{aligned}
&I_1(m,n)\\
&\quad \le C_3 \sum_{k,l} \Bigg\{ \sum_{1\le i\le m} \sum_{m<j\le n+m} (j-i)^{-1/2-\theta}(|i-k|+1)^{-1/2-\alpha}\\
&\qquad\qquad \times (|j-l|+1)^{-1/2-\alpha} \Bigg\}^2\\
&\quad = C_3 \sum_{k,l} \Bigg\{ \sum_{1\le i\le m} \sum_{0<j\le n} (j+m-i)^{-1/2-\theta}(|i-k|+1)^{-1/2-\alpha}\\
&\qquad\qquad \times (|j+m-l|+1)^{-1/2-\alpha} \Bigg\}^2\\
&\quad = C_3 \sum_{k,l} \Bigg\{ \sum_{1\le i\le m} \sum_{0<j\le n} (j+i)^{-1/2-\theta}(|m-i-k|+1)^{-1/2-\alpha}\\
&\qquad\qquad \times (|j+m-l|+1)^{-1/2-\alpha} \Bigg\}^2\\
&\quad = C_3 \sum_{k,l} \Bigg\{ \sum_{1\le i\le m} \sum_{0<j\le n} (j+i)^{-1/2-\theta}(|i+k|+1)^{-1/2-\alpha}\\
&\qquad\qquad \times (|j+l|+1)^{-1/2-\alpha} \Bigg\}^2
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
&\le C_4 \sum_{k,l} \biggl\{ \sum_{1\le i<\infty} \sum_{0<j\le n} (j+i)^{-1/2-\theta}(|i+k|+1)^{-1/2-\alpha} \\
&\qquad\qquad\qquad\qquad \times (|j+l|+1)^{-1/2-\alpha} \biggr\}^2 \\
&\le C_5 \sum_{k,l} \biggl\{ \sum_{0<j\le n} (j+|k|+1)^{-\alpha-\theta}(|j+l|+1)^{-1/2-\alpha} \biggr\}^2 \\
&= C_5 \sum_{0<i,\, j\le n} \sum_{k,l} (j+|k|+1)^{-\alpha-\theta}(|j+l|+1)^{-1/2-\alpha} \\
&\qquad\qquad \times (i+|k|+1)^{-\alpha-\theta}(|i+l|+1)^{-1/2-\alpha} \\
&\le C_6 \sum_{0<i,\, j\le n} (i+j)^{1-2(\alpha+\theta)}(|j-i|+1)^{-2\alpha} \\
&\le C_7 n^{3-2(2\alpha+\theta)}.
\end{aligned}
$$

Similarly,

$$
I_3(m,n) \le C_8 n^{3-2(2\alpha+\theta)}.
$$

Following the arguments in (3.31), we get that

$$
\begin{aligned}
I_2(m,n) &\le C_9 \sum_{k,l} \biggl\{ \sum_{m<i\le n+m} \sum_{m<j\le n+m} (|j-i|+1)^{-1/2-\theta} \\
&\qquad\qquad \times (|i-k|+1)^{-1/2-\alpha}(|j-l|+1)^{-1/2-\alpha} \biggr\}^2 \\
&= C_9 \sum_{k,l} \biggl\{ \sum_{0<i,\, j\le n} (|j-i|+1)^{-1/2-\theta}(|i+k|+1)^{-1/2-\alpha} \\
&\qquad\qquad\qquad\qquad \times (|j+l|+1)^{-1/2-\alpha} \biggr\}^2 \\
&= C_9 \sum_{0<i,\, j\le n} \sum_{0<u,\, v\le n} (|j-i|+1)^{-1/2-\theta}(|u-v|+1)^{-1/2-\theta} \\
&\qquad\qquad \times \sum_{k,l} (|i+k|+1)^{-1/2-\alpha}(|j+l|+1)^{-1/2-\alpha} \\
&\qquad\qquad \times (|u+k|+1)^{-1/2-\alpha}(|v+l|+1)^{-1/2-\alpha} \\
&= C_9 \sum_{0<i,\, j\le n} \sum_{0<u,\, v\le n} (|j-i|+1)^{-1/2-\theta}(|u-v|+1)^{-1/2-\theta} \\
&\qquad\qquad \times \sum_{k} (|k|+1)^{-1/2-\alpha}(|k+u-i|+1)^{-1/2-\alpha} \\
&\qquad\qquad \times \sum_{l} (|l|+1)^{-1/2-\alpha}(|l+v-j|+1)^{-1/2-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_{10} \sum_{0<i,\, j\le n} \sum_{0<u,\, v\le n} (|j-i|+1)^{-1/2-\theta}(|i-u|+1)^{-2\alpha} \\
&\qquad\qquad \times (|u-v|+1)^{-1/2-\theta}(|v-j|+1)^{-2\alpha} \\
&\le C_{11} \sum_{-n<l_1,\, l_2,\, l_3,\, l_4<n} (|l_1|+1)^{-1/2-\theta}(|l_2|+1)^{-2\alpha}(|l_3|+1)^{-1/2-\theta} \\
&\qquad\qquad \times (|l_1+l_2+l_3|+1)^{-2\alpha} \\
&\le 2nC_{11} \sum_{-n<l_1,\, l_2,\, l_3<n} (|l_1|+1)^{-1/2-\theta}(|l_2|+1)^{-2\alpha}(|l_3|+1)^{-1/2-\theta} \\
&\qquad\qquad \times (|l_1+l_2+l_3|+1)^{-2\alpha} \\
&\le nC_{12}\Bigg\{ \sum_{0\le l_1,\, l_2,\, l_3<n} (l_1+1)^{-1/2-\theta}(l_2+1)^{-2\alpha}(l_3+1)^{-1/2-\theta} \\
&\qquad\qquad \times (|l_1-l_2-l_3|+1)^{-2\alpha} \\
&\qquad + \sum_{0\le l_1,\, l_2,\, l_3<n} (l_1+1)^{-1/2-\theta}(l_2+1)^{-2\alpha}(l_3+1)^{-1/2-\theta} \\
&\qquad\qquad\qquad \times (|l_2-l_1-l_3|+1)^{-2\alpha}\Bigg\} \\
&= nC_{12}\big(I_{2,1}(n)+I_{2,2}(n)+I_{2,3}(n)+I_{2,4}(n)\big),
\end{aligned}
$$

with

$$
\begin{aligned}
I_{2,1}(n) &= \sum_{0\le l_1,\, l_2,\, l_3<n,\, l_1>l_2+l_3} (l_1+1)^{-1/2-\theta}(l_2+1)^{-2\alpha}(l_3+1)^{-1/2-\theta} \\
&\qquad \times (|l_1-l_2-l_3|+1)^{-2\alpha}, \\
I_{2,2}(n) &= \sum_{0\le l_1,\, l_2,\, l_3<n,\, l_1\le l_2+l_3} (l_1+1)^{-1/2-\theta}(l_2+1)^{-2\alpha}(l_3+1)^{-1/2-\theta} \\
&\qquad \times (|l_1-l_2-l_3|+1)^{-2\alpha}, \\
I_{2,3}(n) &= \sum_{0\le l_1,\, l_2,\, l_3<n,\, l_2>l_1+l_3} (l_1+1)^{-1/2-\theta}(l_2+1)^{-2\alpha}(l_3+1)^{-1/2-\theta} \\
&\qquad \times (|l_2-l_1-l_3|+1)^{-2\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
I_{2,4}(n) &= \sum_{0\le l_1,\, l_2,\, l_3<n,\, l_2\le l_1+l_3} (l_1+1)^{-1/2-\theta}(l_2+1)^{-2\alpha}(l_3+1)^{-1/2-\theta} \\
&\qquad \times (|l_2-l_1-l_3|+1)^{-2\alpha}.
\end{aligned}
$$

Next we note that

$$
\begin{aligned}
I_{2,1}(n) &= O(1) \sum_{0<l_1<n} l_1^{1-4\alpha-2\theta}, \\
I_{2,2}(n) &= O(1) \sum_{0\le l_2,\, l_3<n} (l_2+1)^{-2\alpha}(l_3+1)^{-1/2-\theta}(l_2+l_3+1)^{1/2-\theta-2\alpha}, \\
I_{2,3}(n) &= O(1) \sum_{0<l_2<n} l_2^{1-4\alpha-2\theta}
\end{aligned}
$$

and

$$I_{2,4}(n) = O(1) \sum_{0 \le l_1, l_3 < n} (l_1+1)^{-1/-\theta} l_3^{-1/2-\theta} (l_1+l_3+1)^{1-4\alpha}.$$

Thus we conclude

$$I_2(m,n) \le C_{13} n(1+n^{2-4\alpha-2\theta}),$$

which completes the proof of (3.30). □

4. Proofs of Lemmas 3.1–3.4. We start with five technical lemmas which will be used in the proofs. The first two results are moment inequalities for sums and quadratic forms of independent random variables. Lemmas 4.3 and 4.4 are moment inequalities for increments of partial sums.

LEMMA 4.1. *Let $\eta_1, \eta_2, \ldots$ be independent, identically distributed random variables (i.i.d. r.v.'s) with $E\eta_1 = 0$ and $E|\eta_1|^p < \infty$ for some $2 \le p < \infty$, and let $\{a_i,\ 1 \le i < \infty\}$ be a sequence of real numbers such that $\sum a_i^2 < \infty$. Then there exists a constant K_p such that*

$$E\left|\sum_{1\le i<\infty} a_i\eta_i\right|^p \le K_p E|\eta_1|^p \left(\sum_{1\le i<\infty} a_i^2\right)^{p/2}. \tag{4.1}$$

PROOF. It follows from Rosenthal's inequality [cf. Theorem 2.12 in Hall and Heyde (1980)].

LEMMA 4.2. *Let $\eta_1, \eta_2, \ldots, \eta_n$ be i.i.d. r.v.'s with $E\eta_1 = 0$ and $E\eta_1^4 < \infty$, and let $\{a_{ij},\ 1 \le i, j \le n\}$ be an array of real numbers with $a_{ii} = 0$. Then there exists a constant K^* such that*

$$E\left|\sum_{1\le i,\, j\le n} a_{ij}\eta_i\eta_j\right|^4 \le K^*(E\eta_1^4)^2 \left(\sum_{1\le i, j\le n} a_{i,j}^2\right)^2. \tag{4.2}$$

PROOF. Let

$$\hat{a}_{ij} = \begin{cases} a_{ij}, & \text{if } i \le j, \\ a_{ji}, & \text{if } j \le i. \end{cases}$$

Clearly, $\hat{a}_{ij} = \hat{a}_{ji}$ for all $1 \le i,\ j \le n$ and

$$\sum_{1\le i<j\le n} a_{i,j}\eta_i\eta_j = (1/2) \sum_{1\le i,\, j\le n} \hat{a}_{ij}\eta_i\eta_j.$$

Let $\{\tilde{\eta}_i,\ 1 \le i \le n\}$ be an independent copy of $\{\eta_i,\ 1 \le i \le n\}$. According to Theorem 2.3 of de la Peña and Klass (1994), there exists an absolute constant

A such that

$$
\begin{aligned}
E\Big|\sum_{1\le i<j\le n} a_{ij}\eta_i\eta_j\Big|^4 &\le E\Big|\sum_{1\le i,\,j\le n} \hat a_{ij}\eta_i\eta_j\Big|^4\\
&\le AE\biggl(\sum_{1\le i,\,j\le n}\hat a_{ij}^2\eta_i^2\tilde\eta_j^2\biggr)^2\\
&= A\sum_{1\le i,\,j,\,k,\,l\le n}\hat a_{ij}^2\hat a_{kl}^2E(\eta_i^2\eta_k^2)E(\tilde\eta_j^2\tilde\eta_l^2)\\
&\le A\sum_{1\le i,\,j\le n}\hat a_{ij}^4E\eta_i^4E\eta_j^4 + A\biggl(\sum_{1\le i,\,j\le n}\hat a_{ij}^2E\eta_i^2E\eta_j^2\biggr)^2\\
&\quad + 2A\sum_{1\le i,\,j,\,k\le n}\hat a_{ij}^2\hat a_{ik}^2E\eta_i^4E\eta_j^2E\eta_k^2\\
&\le 4A\biggl(\sum_{1\le i,\,j\le n}\hat a_{ij}^2\biggr)^2(E\eta_1^4)^2\\
&= 16A\biggl(\sum_{1\le i<j\le n} a_{ij}^2\biggr)^2(E\eta_1^4)^2.
\end{aligned}
$$

Similar arguments yield that

$$
E\Big|\sum_{1\le j<i\le n} a_{ij}\eta_i\eta_j\Big|^4 \le 16A\biggl(\sum_{1\le j<i\le n} a_{ij}^2\biggr)^2(E\eta_1^4)^2,
$$

which completes the proof of Lemma 4.2. □

LEMMA 4.3. *Let $\{T_n,\ 1\le n<\infty\}$ be a sequence of r.v.'s. Assume that there exist $v\ge 1$, $\gamma>1$ and C such that*

$$
E\big|T_{m+n}-T_m\big|^v \le Cn^\gamma \quad \textit{for all } m+n>m\ge 0. \tag{4.3}
$$

Then

$$
T_n = O\bigl(n^{\gamma/v}(\log n)^{2\gamma/v}\bigr) \quad \textit{a.s.} \tag{4.4}
$$

PROOF. It follows from Theorem 12.2 of Billingsley (1968) and the subsequence method [cf. also Stout (1974)]. □

LEMMA 4.4. *Let $\{T_n,\ 1\le n<\infty\}$ be a sequence of r.v.'s. Assume that there exist $v\ge 1$, $\gamma>1$ and C such that*

$$
E\big|T_{m+k}-T_m\big|^v \le Ck^\gamma \quad \textit{for all } 0\le m<k+m\le n. \tag{4.5}
$$

Then

$$
\max_{1\le k\le n}|T_k|/k^{\gamma/v}(\log k)^{2\gamma/v} = O_P(1) \tag{4.6}
$$

and

$$\max_{1\le k<n} |T_n - T_k| \big/ \big((n-k)^{\gamma/v}(\log(n-k))^{2\gamma/v}\big) = O_P(1). \tag{4.7}$$

PROOF. Similarly to Lemma 4.3, Theorem 12.2 of Billingsley (1968) and the subsequence method yield (4.6) and (4.7). □

LEMMA 4.5. *Let* $1 \le k,\ l < \infty$ *and* $0 < a,\ b < 1$ *with* $a+b>1$. *Then*

$$\sum_{0\le j<\infty} (j+k)^{-a}(j+l)^{-b} \le C_1(k+l)^{1-a-b} \tag{4.8}$$

and

$$\sum_{0\le j<\infty} (j+k)^{-a}(|j-l|+1)^{-b} \le C_2(k+l)^{1-a-b} \tag{4.9}$$

with some C_1 *and* C_2.

PROOF. We can assume without loss of generality that $k \ge l$. One can write

$$\begin{aligned}&\sum_{0\le j<\infty} (j+k)^{-a}(j+l)^{-b}\\ &\quad= \sum_{0\le j\le k} (j+k)^{-a}(j+l)^{-b} + \sum_{k<j<\infty} (j+k)^{-a}(j+l)^{-b}\\ &\quad\le k^{-a}\sum_{0\le j\le k} (j+l)^{-b} + \sum_{k<j<\infty} j^{-a-b},\end{aligned}$$

which gives immediately (4.8).

To prove (4.9), first we assume that $k \ge l$. Then we have

$$\begin{aligned}&\sum_{0\le j<\infty} (j+k)^{-a}(|j-l|+1)^{-b}\\ &\quad= \sum_{0\le j\le 4k} (j+k)^{-a}(|j-l|+1)^{-b} + \sum_{4k<j<\infty} (j+k)^{-a}(|j-l|+1)^{-b}\\ &\quad\le k^{-a}\sum_{0\le j\le 4k} (|j-l|+1)^{-b} + \sum_{4k<j<\infty} j^{-a-b}\\ &\quad\le C_{2,1}k^{1-a-b} \le 2^{a+b-1}C_{2,1}(k+l)^{1-a-b}.\end{aligned}$$

If $1 \le k < l$, then we write

$$\begin{aligned}&\sum_{0\le j<\infty} (j+k)^{-a}(|j-l|+1)^{-b}\\ &\quad\le \sum_{0\le j<\infty} (j+1)^{-a}(|j-l|+1)^{-b}\\ &\quad= \sum_{0\le j\le l/2} (j+1)^{-a}(|j-l|+1)^{-b} + \sum_{l/2<j\le l} (j+1)^{-a}(|j-l|+1)^{-b}\end{aligned}$$

$$+ \sum_{l<j<\infty} (j+1)^{-a}(|j-l|+1)^{-b}$$
$$\le (l/2)^{-b} \sum_{0\le j\le l/2} (j+1)^{-a} + (l/2)^{-a} \sum_{l/2<j\le l} (|j-l|+1)^{-b}$$
$$+ \sum_{1\le j<\infty} (j+l)^{-a} j^{-b},$$

which completes the proof of (4.9). □

After these preliminaries we are ready to work on the proofs of Lemmas 3.1–3.4. Throughout the rest of this section, we shall assume without loss of generality that

$$0<\alpha<1/2,\quad 0<\beta<1/2,\quad 1/2<\alpha+\beta<3/4,\quad 0<r<1, \text{ and } 1<\theta+2\alpha<5/4. \tag{4.10}$$

Let $\mathscr{F}_m$ be the σ-algebra generated by $\xi_1, \xi_2, \ldots, \xi_m$ and put

$$d_k = \sum_{1\le j<\infty} a(j-k)c(j-k),$$
$$d_{k,l} = \sum_{1\le j<\infty} \{a(j-k)c(j-l) + a(j-l)c(j-k)\}$$

and

$$Y_k = d_k(\xi_k^2-\tau^2) + \xi_k \sum_{1\le l\le k-1} d_{k,l}\xi_l.$$

Using Lemma 4.5, we get under conditions (4.10) that

$$\sum_{|j|\ge n} |a(j)c(j)| = O(n^{-\alpha-\beta}), \tag{4.11}$$

$$\sum_{j\in\mathbb{Z}} \{|a(j)c(j+n)| + |c(j)a(j+n)|\} = O(n^{-\alpha-\beta}), \tag{4.12}$$

$$\sum_{|j|\ge n} \{|a(j)c(j+m)| + |a(j+m)c(j)|\} \le C_1(n+m)^{-\alpha-\beta} \tag{4.13}$$

and

$$|d_{k,l}| \le C_2(1+|k-l|)^{-\alpha-\beta}, \tag{4.14}$$

with some constants C_1 and C_2.

Proof of Lemma 3.1. It is easy to see that

$$Q_3(n) = \sum_{1\le k\le n} Y_k$$

and that $\{Q_3(n), \mathscr{F}_n,\ 1 \le n < \infty\}$ is a martingale sequence. By the Skorokhod embedding theorem [cf. Appendix I in Hall and Heyde (1980)], there are a Wiener process $\{W(t),\ 0 \le t < \infty\}$ and a sequence of stopping times $\{\nu_n,\ 1 \le n < \infty\}$ such that

$$Q_3(n) = W\biggl(\sum_{1 \le i \le n} \nu_i\biggr), \qquad 1 \le n < \infty, \tag{4.15}$$

$$E(\nu_k | \mathscr{F}_{k-1}) = E(Y_k^2 | \mathscr{F}_{k-1}) \quad \text{a.s.}, \qquad 1 \le k < \infty \tag{4.16}$$

and

$$E(\nu_k^p) \le c_p E|Y_k|^{2p} \quad \text{for all } 1 < p < \infty \text{ and } 1 \le k < \infty. \tag{4.17}$$

Next we show that

$$\sum_{1 \le i \le n} \nu_i - \sigma^2 n = O(n^{1-\varepsilon}) \quad \text{a.s.}, \tag{4.18}$$

with some $\varepsilon > 0$. First we note that

$$\begin{aligned}
&\sum_{1 \le i \le n} \nu_i - n\sigma^2 \\
&\qquad = \sum_{1 \le i \le n} \bigl\{\nu_i - E(\nu_i | \mathscr{F}_{i-1})\bigr\} + \sum_{1 \le i \le n} \bigl(E(Y_i^2 | \mathscr{F}_{i-1}) - EY_i^2\bigr) \\
&\qquad\quad + \sum_{1 \le i \le n} EY_i^2 - n\sigma^2.
\end{aligned}$$

By (4.17), Lemma 4.1 and (4.14) we have

$$\begin{aligned}
E|\nu_i|^{1+r/4} &\le c_{2+r/2} E|Y_i|^{2+r/2} \\
&= O(1)E|\xi_1|^{4+r} + O(1)E|\xi_1|^{2+r/2} E\biggl|\sum_{1 \le l \le i-1} d_{i,l}\xi_l\biggr|^{2+r/2} \\
&= O(1)\biggl(1 + \biggl(\sum_{1 \le l \le i-1} d_{i,l}^2\biggr)^{1+r/4}\biggr) \\
&= O(1)\biggl(1 + \biggl(\sum_{1 \le l \le i-1} (i-l)^{-2\alpha-2\beta}\biggr)^{1+r/4}\biggr) \\
&= O(1).
\end{aligned}$$

Hence Theorem 2.18 in Hall and Heyde (1980) yields

$$\sum_{1 \le i \le n} \bigl\{\nu_i - E(\nu_i | \mathscr{F}_{i-1})\bigr\} = O(n^{(8+r)/(8+2r)}) \quad \text{a.s.} \tag{4.19}$$

Next we show

$$E\Big(\sum_{k<i\le k+m}\big\{E(Y_i^2|\mathscr{F}_{i-1})-EY_i^2\big\}\Big)^2\le Cm^{3-2(\alpha+\beta)} \tag{4.20}$$

for all $k+m>k\ge 0$, with some C. It is easy to see that

$$\begin{aligned}
&E\big(Y_i^2|\mathscr{F}_{i-1}\big)-EY_i^2\\
&\quad=E\xi_i^2\bigg(\Big(\sum_{1\le l\le i-1}d_{i,l}\xi_l\Big)^2-E\Big(\sum_{1\le l\le i-1}d_{i,l}\xi_l\Big)^2\bigg)\\
&\qquad+2d_iE\xi_i^3\sum_{1\le l\le i-1}d_{i,l}\xi_l\\
&\quad=E\xi_1^2\sum_{1\le l\le i-1}d_{i,l}^2(\xi_l^2-\tau^2)+2E\xi_1^2\sum_{1\le l<j\le i-1}d_{i,l}d_{i,j}\xi_l\xi_j\\
&\qquad+2d_iE\xi_1^3\sum_{1\le l\le i-1}d_{i,l}\xi_l.
\end{aligned}$$

Hence, using the notation $a\vee b=\max(a,b)$, we have

$$\begin{aligned}
&E\Big(\sum_{k<i\le k+m}\big\{E(Y_i^2|\mathscr{F}_{i-1})-EY_i^2\big\}\Big)^2\\
&\quad\le 8E\xi_1^4E\Big(\sum_{k<i\le k+m}\sum_{1\le l\le i-1}d_{i,l}^2(\xi_l^2-\tau^2)\Big)^2\\
&\qquad+32E\xi_1^4E\Big(\sum_{k<i\le k+m}\sum_{1\le l<j\le i-1}d_{i,l}d_{i,j}\xi_l\xi_j\Big)^2\\
&\qquad+32(E\xi_1^3)^2E\Big(\sum_{k<i\le k+m}\sum_{1\le l\le i-1}d_id_{i,l}\xi_l\Big)^2\\
&\quad\le C_1\bigg\{\sum_{1\le l<k+m}\Big(\sum_{l\vee k<i\le k+m}d_{i,l}^2\Big)^2+\sum_{1\le l<j<k+m}\Big(\sum_{j\vee k<i\le k+m}d_{i,l}d_{i,j}\Big)^2\\
&\qquad\qquad+\sum_{1\le l<k+m}\Big(\sum_{l\vee k<i\le k+m}d_id_{i,l}\Big)^2\bigg\}\\
&\quad\le C_2\bigg\{\sum_{1\le l<k+m}\Big(\sum_{l\vee k<i\le k+m}(i-l)^{-2(\alpha+\beta)}\Big)^2\\
&\qquad+\sum_{1\le l<j<k+m}\Big(\sum_{j\vee k<i\le k+m}(i-l)^{-\alpha-\beta}(i-j)^{-\alpha-\beta}\Big)^2\\
&\qquad\qquad+\sum_{1\le l<k+m}\Big(\sum_{l\vee k<i\le k+m}i^{-\alpha-\beta}(i-l)^{-\alpha-\beta}\Big)^2\bigg\},
\end{aligned}$$

where we also used (4.11) and (4.14). Similarly,

$$\begin{aligned}
&\sum_{1\le l<k+m}\biggl(\sum_{l\vee k<i\le k+m}(i-l)^{-2(\alpha+\beta)}\biggr)^2\\
&\quad=\sum_{1\le l<k+m}\ \sum_{l\vee k<i\le k+m}\ \sum_{l\vee k<j\le k+m}(i-l)^{-2(\alpha+\beta)}(j-l)^{-2(\alpha+\beta)}\\
&\quad\le C_3\sum_{k<i\le k+m}\ \sum_{k<j\le k+m}(|j-i|+1)^{1-2(\alpha+\beta)}\\
&\quad\le C_4 m^{3-2(\alpha+\beta)}.
\end{aligned}$$

With the notation $a\wedge b=\min(a,b)$, we can write

$$\begin{aligned}
&\sum_{1\le l<j<k+m}\biggl(\sum_{j\vee k<i\le k+m}(i-l)^{-\alpha-\beta}(i-j)^{-\alpha-\beta}\biggr)^2\\
&\quad=\sum_{1\le l<j<k+m}\ \sum_{j\vee k<i\le k+m}\ \sum_{j\vee k<v\le k+m}\bigl\{(i-l)(i-j)(v-l)(v-j)\bigr\}^{-\alpha-\beta}\\
&\quad\le\sum_{k<i\le k+m}\ \sum_{k<v\le k+m}\ \sum_{1\le l,\ j\le i\wedge v}\bigl\{(i-l)(i-j)(v-l)(v-j)\bigr\}^{-\alpha-\beta}\\
&\quad\le C_5\sum_{k<i\le k+m}\ \sum_{k<v\le k+m}(|i-v|+1)^{2-4(\alpha+\beta)}\\
&\quad=O\bigl(m^{4-4(\alpha+\beta)}\bigr)\\
&\quad=O\bigl(m^{3-2(\alpha+\beta)}\bigr),
\end{aligned}$$

which completes the proof of (4.20).

Putting together Lemma 4.3 and (4.20), we get

$$\sum_{1\le i\le n}\bigl\{E(Y_i^2|\mathscr{F}_{i-1})-EY_i^2\bigr\}=O\bigl(n^{3/2-(\alpha+\beta)}(\log n)^{3-2(\alpha+\beta)}\bigr)\quad\text{a.s.}\tag{4.21}$$

Let

$$g(k)=\sum_{j\in\mathbb{Z}}\bigl\{a(j)c(j+k)+a(j+k)c(j)\bigr\}.$$

Applying (4.11) and (4.12), we obtain that

$$\begin{aligned}
&\sum_{1\le k\le n}EY_k^2\\
&\quad=\sum_{1\le k\le n}\biggl\{d_k^2E(\xi_1^4-\tau^4)+\tau^4\sum_{1\le l\le k-1}d_{k,l}^2\biggr\}\\
&\quad=E(\xi_1^4-\tau^4)\sum_{1\le k\le n}\biggl\{\sum_{j\in\mathbb{Z}}a(j)c(j)-\sum_{j\le -k}a(j)c(j)\biggr\}^2\\
&\qquad+\tau^4\sum_{1\le k\le n}\ \sum_{1\le l\le k-1}g^2(k-l)
\end{aligned}\tag{4.22}$$

$$
\begin{aligned}
&+\tau^4 \sum_{1\le k\le n}\ \sum_{1\le l\le k-1} (d_{k,l}-g(k-l))(d_{k,l}+g(k-l))\\
&= nE(\xi_1^4-\tau^4)\Big(\sum_{j\in\mathbb{Z}} a(j)c(j)\Big)^2 + O(n^{1-\alpha-\beta})\\
&\quad+\tau^4 n\sum_{l\ge 1} g^2(l) - \tau^4\sum_{1\le k\le n}\ \sum_{k\le l<\infty} g^2(l)\\
&\quad+\tau^4\sum_{1\le k\le n}\ \sum_{1\le l\le k-1}(d_{k,l}+g(k-l))\\
&\qquad\qquad\times\sum_{-\infty<v\le 0}\{a(v-k)c(v-l)+a(v-l)c(v-k)\}\\
&= n\sigma^2 + O(n^{1-\alpha-\beta}) + O(n^{2-2(\alpha+\beta)})\\
&\quad+O(1)\sum_{1\le k\le n}\ \sum_{1\le l\le k-1}(k-l)^{-\alpha-\beta}\big(k^{-\alpha}l^{-\beta}+k^{-\beta}l^{-\alpha}\big)\\
&= n\sigma^2 + O(n^{2-2(\alpha+\beta)}).
\end{aligned}
$$

Putting together (4.16), (4.19), (4.21) and (4.22), we conclude (4.18). Now Theorem 1.2.1 of Csörgő and Révész (1981) and (4.18) yield that

$$
\left|W\Big(\sum_{1\le i\le n}\nu_i\Big)-W(n\sigma^2)\right| = O(n^{1/2-\varepsilon'})\quad \text{a.s.}
$$

with any $\varepsilon' < \varepsilon/2$, which completes the proof of Lemma 3.1. □

PROOF OF LEMMA 3.2. We start with the proof of (3.4). By Lemma 4.1 and (4.11) we have

$$
\begin{aligned}
E|Q_4(n)|^{2+r/2} &= O(1)\bigg(\sum_{1\le k\le n}\Big\{\sum_{n<j<\infty} a(j-k)c(j-k)\Big\}^2\bigg)^{1+r/4}\\
&= O(1)\bigg(\sum_{1\le k\le n}(n+1-k)^{-2(\alpha+\beta)}\bigg)^{1+r/4}\\
&= O(1).
\end{aligned}
\tag{4.23}
$$

Similarly,

$$
\begin{aligned}
E|Q_6(n)|^{2+r/2} &= O(1)\bigg(\sum_{k\le 0 \text{ or } k>n}\Big\{\sum_{1\le j\le n} a(j-k)c(j-k)\Big\}^2\bigg)^{1+r/4}\\
&= O(1)\bigg(\sum_{-\infty<k\le 0}(1-k)^{-2(\alpha+\beta)}+\sum_{n<k<\infty}(k-n)^{-2(\alpha+\beta)}\bigg)^{1+r/4}\\
&= O(1).
\end{aligned}
\tag{4.24}
$$

Using (4.23) and (4.24), we can find a constant C_1 such that

$$E\left|Q_4(m+n)-Q_4(m)\right|^{2+r/2}+E\left|Q_6(m+n)-Q_6(m)\right|^{2+r/2}\le C_1$$

for all $m+n>m\ge 0$, and therefore Lemma 4.3 gives (3.4).

Next we prove (3.5). Let

$$h_{j,k,l}=a(j-k)c(j-l)+a(j-l)c(j-k).$$

For any $n+m>m\ge 0$, we can write

$$Q_5(n+m)-Q_5(m)=-\sum_{1\le l<k\le m}\sum_{m<j\le n+m}h_{j,k,l}\xi_l\xi_k + \sum_{1\le l<k\le n+m,\,k>m}\sum_{n+m<j<\infty}h_{j,k,l}\xi_l\xi_k.$$

By Lemma 4.2 we conclude

$$\begin{aligned}E&\left|Q_5(n+m)-Q_5(m)\right|^4\\ &\le C_2(E\xi_1^4)^2\left\{\left(\sum_{1\le l<k\le m}\left(\sum_{m<j\le n+m}h_{j,k,l}\right)^2\right)^2\right.\\ &\qquad\left.+\left(\sum_{1\le l<k\le n+m,\,k>m}\left(\sum_{n+m<j<\infty}h_{j,k,l}\right)^2\right)^2\right\}.\end{aligned}\tag{4.25}$$

By (1.4) and (4.8) we have

$$\begin{aligned}&\sum_{1\le l<k\le m}\left(\sum_{m<j\le n+m}h_{j,k,l}\right)^2\\ &\quad=\sum_{m<j\le n+m}\sum_{m<i\le n+m}\sum_{1\le l<k\le m}h_{j,k,l}h_{i,k,l}\\ &\quad\le C_3\sum_{m<j\le n+m}\sum_{m<i\le n+m}\sum_{1\le l<k\le m}\{(j-k)^{-\alpha-1/2}(j-l)^{-\beta-1/2}\\ &\qquad\qquad+(j-k)^{-\beta-1/2}(j-l)^{-\alpha-1/2}\}\\ &\qquad\times\{(i-k)^{-\alpha-1/2}(i-l)^{-\beta-1/2}+(i-k)^{-\beta-1/2}(i-l)^{-\alpha-1/2}\}\\ &\quad\le C_4\sum_{m<j\le n+m}\sum_{m<i\le n+m}(j+i-2m)^{-2(\alpha+\beta)}\\ &\quad\le C_5\sum_{0<j\le n}\sum_{0<i\le n}(j+i)^{-2(\alpha+\beta)}\\ &\quad\le C_6n^{2-2(\alpha+\beta)}.\end{aligned}$$

Similarly, by (4.13) we conclude

$$\begin{aligned}&\sum_{1\le l<k\le n+m,\,k>m}\left(\sum_{n+m<j<\infty}h_{j,k,l}\right)^2\\ &\qquad\le C_7\sum_{m<k\le n+m}\sum_{1\le l<k}(n+m-l)^{-2(\alpha+\beta)}\end{aligned}$$

$$\le C_8 \sum_{m<k\le n+m} (n+m+1-k)^{1-2(\alpha+\beta)}$$
$$\le C_9 n^{2-2(\alpha+\beta)}.$$

Going back to (4.25), we obtain that

$$E\big(Q_5(n+m)-Q_5(m)\big)^4 \le C_{10} n^{4-4(\alpha+\beta)}$$

and therefore Lemma 4.3 implies

$$|Q_5(n)| = O(n^{1/2-\delta}) \quad \text{a.s.,} \tag{4.26}$$

with some $\delta > 0$.

Following the proof of (4.26), one can show that

$$|Q_7(n)| = O(n^{1/2-\delta}) \quad \text{a.s.,}$$

with some $\delta > 0$, which also completes the proof of (3.5).

To prove (3.6), we write

$$\begin{aligned}
Q_2(n) &= \sum_{1\le k\le n} \sum_{k\le j<\infty} \sum_{l,\,v} b(j)a(k-l)a(k-j-v)\big(\xi_l\xi_v - E(\xi_l\xi_v)\big)\\
&= \sum_l \sum_{1\le k\le n} \sum_{k\le j<\infty} b(j)a(k-l)a(k-j-l)(\xi_l^2-\tau^2)\\
&\quad + \sum_{l\ne v} \sum_{1\le k\le n} \sum_{k\le j<\infty} b(j)a(k-l)a(k-j-v)\xi_l\xi_v\\
&= Q_{2,1}(n) + Q_{2,2}(n).
\end{aligned}$$

Applying Lemma 4.1, we get for all $m+n>m\ge 0$ that

$$E|Q_{2,1}(n)|^{2+r/2} \le C_{11}\Bigg(\sum_l \bigg\{\sum_{0<k\le n}\sum_{k\le j<\infty} b(j)a(k-l)a(k-j-l)\bigg\}^2\Bigg)^{1+r/4}.$$

By (1.4), (4.8) and (4.10) we have

$$\begin{aligned}
&\sum_l \bigg\{\sum_{0<k\le n} \bigg\{\sum_{k\le j<\infty} b(j)a(k-l)a(k-j-l)\bigg\}^2\\
&\quad\le C_{12}\sum_l \bigg\{\sum_{0<k\le n} (|k-l|+1)^{-1/2-\alpha} \sum_{k\le j<\infty} j^{-1/2-\theta}(|k-j-l|+1)^{-1/2-\alpha}\bigg\}^2\\
&\quad= C_{13}\bigg\{\sum_{-\infty<l\le 0}\bigg(\sum_{0<k\le n}(k-l+1)^{-1/2-\alpha}\\
&\qquad\qquad \times \sum_{k\le j<\infty} j^{-1/2-\theta}(|k-j-l|+1)^{-1/2-\alpha}\bigg)^2
\end{aligned} \tag{4.27}$$

$$+\sum_{0<l<\infty}\bigg(\sum_{0<k\le n}(|k-l|+1)^{-1/2-\alpha}$$

$$\times\sum_{k\le j<\infty} j^{-1/2-\theta}(j+l-k+1)^{-1/2-\alpha}\bigg)^2\bigg\}$$

$$\le C_{14}\bigg\{\sum_{-\infty<l\le 0}\bigg(\sum_{0<k\le n}(k-l+1)^{-1/2-\alpha}(k-l+1)^{-\theta-\alpha}\bigg)^2$$

$$+\sum_{0<l<\infty}\bigg(\sum_{0<k\le n}(|k-l|+1)^{-1/2-\alpha}(k+l)^{-\theta-\alpha}\bigg)^2\bigg\}$$

$$\le C_{15}\sum_{0\le l<\infty}\bigg(\sum_{0<k\le n}(|k-l|+1)^{-1/2-\alpha}(k+l+1)^{-\theta-\alpha}\bigg)^2$$

$$= C_{15}\sum_{0<k\le n}\sum_{0<i\le n}\sum_{0\le l<\infty}(|k-l|+1)^{-1/2-\alpha}(k+l+1)^{-\theta-\alpha}$$

$$\times(|i-l|+1)^{-1/2-\alpha}(i+l+1)^{-\theta-\alpha}$$

$$\le C_{15}\sum_{0<k\le n}\sum_{0<i\le n} k^{-\theta-\alpha}i^{-\theta-\alpha}$$

$$\times\sum_{0\le l<\infty}(|k-l|+1)^{-1/2-\alpha}(|i-l|+1)^{-1/2-\alpha}$$

$$\le C_{16}\sum_{0<k\le n}\sum_{0<i\le n} k^{-\theta-\alpha}i^{-\theta-\alpha}(|k-i|+1)^{-2\alpha}$$

$$\le C_{17}.$$

Hence

$$E\big|Q_{2,1}(m+n)-Q_{2,1}(m)\big|^{2+r/2}\le C_{11}C_{17}^{1+r/4}$$

and therefore Lemma 4.3 yields

$$Q_{2,1}(n)=O(n^{1/2-\delta})\quad\text{a.s.,}\tag{4.28}$$

with some $\delta>0$.

Lemma 4.2 gives for all $m+n>m\ge 0$ that

$$E\big|Q_{2,2}(n+m)-Q_{2,2}(m)\big|^4$$
$$\le C_{18}(E\xi_1^4)^2\bigg(\sum_{l\ne v}\bigg\{\sum_{m<k\le n+m}\sum_{k\le j<\infty} b(j)a(k-l)a(k-j-v)\bigg\}^2\bigg)^2.$$

Following the proof of (4.27), we get

$$
\begin{aligned}
&\sum_{l\neq v}\left\{\sum_{m<k\leq n+m}\sum_{k\leq j<\infty} b(j)a(k-l)a(k-j-v)\right\}^2\\
&\quad\leq C_{19}\sum_{l,\,v}\left\{\sum_{m<k\leq n+m}(|k-l|+1)^{-1/2-\alpha}\right.\\
&\qquad\qquad\left.\times\sum_{k<j<\infty} j^{-1/2-\theta}(|k-j-v|+1)^{-1/2-\alpha}\right\}^2\\
&\quad\leq C_{20}\sum_{l,\,v}\left\{\sum_{m<k\leq n+m}(|k-l|+1)^{-1/2-\alpha}(k+|v|)^{-\theta-\alpha}\right\}^2\\
&\quad= 4C_{20}\sum_{0\leq v,\,l<\infty}\left\{\sum_{m<k\leq n+m}(|k-l|+1)^{-1/2-\alpha}(k+v)^{-\theta-\alpha}\right\}^2\\
&\quad= 8C_{20}\sum_{m<k\leq j\leq n+m}\sum_{0\leq v,\,l<\infty}(|k-l|+1)^{-1/2-\alpha}(|j-l|+1)^{-1/2-\alpha}\\
&\qquad\qquad\times(k+v)^{-\theta-\alpha}(j+v)^{-\theta-\alpha}\\
&\quad\leq C_{21}\sum_{m<k\leq j\leq n+m} j^{1-2(\theta+\alpha)}(j-k+1)^{-2\alpha}\\
&\quad= C_{21}\sum_{0<k\leq j\leq n}(j+m)^{1-2(\theta+\alpha)}(j-k+1)^{-2\alpha}\\
&\quad\leq C_{22}\sum_{1\leq j\leq n} j^{1-2(\theta+\alpha)+1-2\alpha}\\
&\quad\leq C_{23}n^{3-2(\theta+2\alpha)}.
\end{aligned}
$$

Hence

$$E|Q_{2,2}(n+m)-Q_{2,2}(m)|^4\leq C_{18}(E\xi_1)^4)^2C_{23}^2n^{6-4(\theta+2\alpha)}$$

and therefore Lemma 4.3 implies

$$Q_{2,2}(n)=O(n^{1/2-\delta})\quad\text{a.s.}\tag{4.29}$$

Now (3.6) follows from (4.28) and (4.29). □

PROOF OF LEMMA 3.3. We rewrite $Q_3^*(m,n)$ as

$$Q_3^*(m,n)=\sum_{m<k\leq n} Y_k = Q_{3,1}^*(m,n)+Q_{3,2}^*(m,n),$$

with

$$Q_{3,1}^*(m,n)=\sum_{m<k\leq n} d_k(\xi_k^2-\tau^2)+\sum_{m<k<n}\sum_{k<l\leq n} d_{k,l}\xi_l\xi_k$$

and

$$Q^*_{3,2}(m,n) = \sum_{1\le l\le m}\sum_{m<k\le n} d_{k,l}\xi_k\xi_l.$$

Let

$$z_k = d_k(\xi_k^2 - \tau^2) + \sum_{k<l\le n} d_{k,l}\xi_l\xi_k, \qquad z_{k,n} = z_{n+1-k}, \quad \text{for } 1\le k\le n$$

and let $\mathscr{F}_k(n)$ be the σ-field generated by $\xi_n, \xi_{n-1}, \dots, \xi_{n+1-k}$. It is easy to see that $\{z_{k,n}, \mathscr{F}_k(n), 1\le k< n/2\}$ is a martingale difference sequence and that $\{Q_3(k), 1\le k\le n/2\}$ and $\{z_{k,n}, 1\le k<n/2\}$ are independent for each n. Using again the Skorokhod embedding theorem for each n, we can define two independent Wiener processes $\{W_{n,1}(t), t\ge 0\}$ and $\{W_{n,2}(t), t\ge 0\}$ and two independent sequences of stopping times $\{\nu_k(n), 1\le k\le n/2\}$ and $\{\nu_k^*(n), 1\le k<n/2\}$ such that

$$Q_3(k) = W_{n,1}\bigg(\sum_{1\le i\le k}\nu_i(n)\bigg), \qquad 1\le k\le n/2$$

and

$$\sum_{1\le i\le k} z_{i,n} = W_{n,2}\big(\sum_{1\le i\le k}\nu_i^*(n)\big), \qquad 1\le k<n/2.$$

Following the proof of Lemma 3.1, one can easily verify that

$$\max_{1\le k\le n/2}\big|Q_3(k) - W_{n,1}(\sigma^2 k)\big|/k^{1/2-\varepsilon} = O_P(1) \tag{4.30}$$

and

$$\max_{1\le k<n/2}\bigg|\sum_{1\le i\le k} z_{i,n} - W_{n,2}(\sigma^2 k)\bigg|\bigg/k^{1/2-\varepsilon} = O_P(1), \tag{4.31}$$

with some $\varepsilon>0$. Since

$$Q^*_{3,1}(m,n) = \sum_{1\le i\le n-m} z_{i,n},$$

it follows from (4.31) that

$$\max_{n/2<m<n}\big|Q^*_{3,1}(m,n) - W_{n,2}(\sigma^2(n-m))\big|/(n-m)^{1/2-\varepsilon} = O_P(1). \tag{4.32}$$

Using Lemma 4.2 and (4.14), we obtain that

$$\begin{aligned}
E\big|Q^*_{3,2}&(m+i,n) - Q^*_{3,2}(i,n)\big|^4\\
&= E\bigg(\sum_{1\le l\le i}\sum_{i<k\le i+m} d_{k,l}\xi_k\xi_l + \sum_{i<l\le i+m}\sum_{i+m<k\le n} d_{k,l}\xi_k\xi_l\bigg)^4\\
&\le C_1\bigg(\sum_{1\le l\le i}\sum_{i<k\le i+m} d^2_{k,l} + \sum_{i<l\le i+m}\sum_{i+m<k\le n} d^2_{k,l}\bigg)^2
\end{aligned}$$

$$
\begin{aligned}
&\le C_2\biggl(\sum_{i<k\le i+m}\ \sum_{1\le l\le i}(k-l)^{-2(\alpha+\beta)}\\
&\qquad\qquad+\sum_{i<l\le i+m}\ \sum_{i+m<k\le n}(k-l)^{-2(\alpha+\beta)}\biggr)^2\\
&\le C_3 m^{4-4(\alpha+\beta)}
\end{aligned}
$$

for all $0<m\le m+i<n$. Thus by Lemma 4.4 we have

$$
\max_{1\le m<n}\bigl|Q^*_{3,2}(m,n)\bigr|/(n-m)^{1/2-\varepsilon}=O_P(1), \tag{4.33}
$$

with some $\varepsilon>0$. Defining $W_n^{(1)}(t)=(1/\sigma)W_{n,1}(\sigma^2 t)$ and $W_n^{(2)}(t)=(1/\sigma)W_{n,2}(\sigma^2 t)$, Lemma 4.3 follows from (4.30), (4.32) and (4.33). □

PROOF OF LEMMA 3.4. Lemma 3.3 implies immediately (3.11). According to Lemma 4.4, if we show that

$$
E\bigl|Q^*_4(i+m,n)-Q^*_4(i,n)\bigr|^{2+r/2}\le C_1, \tag{4.34}
$$

$$
E\bigl|Q^*_6(i+m,n)-Q^*_6(i,n)\bigr|^{2+r/2}\le C_2, \tag{4.35}
$$

$$
E\bigl|Q^*_5(i+m,n)-Q^*_5(i,n)\bigr|^{4}\le C_3 m^{4-4(\alpha+\beta)}, \tag{4.36}
$$

$$
E\bigl|Q^*_7(i+m,n)-Q^*_7(i,n)\bigr|^{4}\le C_4 m^{4-4(\alpha+\beta)}, \tag{4.37}
$$

$$
E\bigl|Q^*_8(i+m,n)-Q^*_8(i,n)\bigr|^{4}\le C_5 m^{4-4(\alpha+\beta)} \tag{4.38}
$$

and

$$
E\bigl|Q^*_2(i+m,n)-Q^*_2(i,n)\bigr|^{4}\le C_6 m^{6-4(\theta+2\alpha)} \tag{4.39}
$$

for all $0\le i<i+m\le n$, then (3.12)–(3.14) are established.

Lemma 4.1 and (1.4) yield

$$
\begin{aligned}
E\bigl|Q^*_4(m,n)\bigr|^{2+r/2}&\le C_7\biggl(\sum_{m<k\le n}\biggl\{\sum_{1\le j\le m\ \text{or}\ j>n}a(j-k)c(j-k)\biggr\}^2\biggr)^{1+r/4}\\
&\le C_8\biggl(\sum_{m<k\le n}(n+1-k)^{-2(\alpha+\beta)}+(k-m)^{-2(\alpha+\beta)}\biggr)^{1+r/4}\\
&<\infty,
\end{aligned}
$$

which gives (4.34). Similarly,

$$
\begin{aligned}
E\bigl|Q^*_6(m,n)\bigr|^{2+r/2}&\le C_9\biggl(\sum_{k\le m\ \text{or}\ k>n}\biggl\{\sum_{m<j\le n}a(j-k)c(j-k)\biggr\}^2\biggr)^{1+r/4}\\
&\le C_{10}\biggl(\sum_{k\le m}(m+1-k)^{-2(\alpha+\beta)}+\sum_{k>n}(k-n)^{-2(\alpha+\beta)}\biggr)^{1+r/4}\\
&\le C_{11}
\end{aligned}
$$

and therefore (4.35) is proven.

To prove (4.36), first we recall the definitions of $h_{j,k,l}$ from the proof of Lemma 3.2. By Lemma 4.2 we have for all $0 \le i < i+m \le n$ that

$$
\begin{aligned}
&E\big|Q_5^*(i+m) - Q_5^*(i,n)\big|^4 \\
&\quad \le C_{12}\bigg\{E\bigg|\sum_{i<k\le i+m}\ \sum_{1\le l<k}\ \sum_{j>n} h_{j,k,l}\xi_l\xi_k\bigg|^4 \\
&\qquad + E\bigg|\sum_{i<k\le i+m}\ \sum_{1\le l<k}\ \sum_{1\le j\le i} h_{j,k,l}\xi_l\xi_k\bigg|^4 \\
&\qquad + E\bigg|\sum_{i+m<k\le n}\ \sum_{1\le l<k}\ \sum_{i<j\le i+m} h_{j,k,l}\xi_l\xi_k\bigg|^4\bigg\} \qquad (4.40)\\
&\quad \le C_{13}\bigg\{\bigg(\sum_{i<k\le i+m}\ \sum_{1\le l<k}\bigg\{\sum_{j>n} h_{j,k,l}\bigg\}^2\bigg)^2 \\
&\qquad + \bigg(\sum_{i<k\le i+m}\ \sum_{1\le l<k}\bigg\{\sum_{1\le j\le i} h_{j,k,l}\bigg\}^2\bigg)^2 \\
&\qquad + \bigg(\sum_{i+m<k\le n}\ \sum_{1\le l<k}\bigg\{\sum_{i<j\le i+m} h_{j,k,l}\bigg\}^2\bigg)^2\bigg\}.
\end{aligned}
$$

Next we apply (4.13) and get

$$
\begin{aligned}
&\textstyle\sum_{i<k\le i+m}\sum_{1\le l<k}\Big\{\sum_{j>n} h_{j,k,l}\Big\}^2 \\
&\quad \le C_{14}\sum_{i<k\le i+m}\ \sum_{1\le l<k}\bigg(\sum_{j>n}\big\{(j-k)^{-1/2-\alpha}(j-l)^{-1/2-\beta} \\
&\qquad\qquad + (j-k)^{-1/2-\beta}(j-l)^{-1/2-\alpha}\big\}\bigg)^2 \qquad (4.41)\\
&\quad \le C_{15}\sum_{i<k\le i+m}\ \sum_{1\le l<k}(2n+2-k-l)^{-2(\alpha+\beta)} \\
&\quad \le C_{16}\sum_{i<k\le i+m}(2n+2-2k)^{1-2(\alpha+\beta)} \\
&\quad \le C_{17}m^{2-2(\alpha+\beta)}.
\end{aligned}
$$

Similarly to (4.41), we have

$$
\begin{aligned}
&\sum_{i<k\le i+m}\ \sum_{1\le l<k}\bigg\{\sum_{1\le j\le i} h_{j,k,l}\bigg\}^2 \\
&\quad \le C_{18}\sum_{i<k\le i+m}\ \sum_{1\le l<k}\bigg\{\sum_{1\le j\le i}\big\{(k-j)^{-1/2-\alpha}(|j-l|+1)^{-1/2-\beta} \\
&\qquad\qquad + (k-j)^{-1/2-\beta}(|j-l|+1)^{-1/2-\alpha}\big\}\bigg\}^2 \qquad (4.42)
\end{aligned}
$$

$$\begin{aligned}
&\le C_{19} \sum_{0<k\le m} \sum_{1\le l<k+i} \left\{ \sum_{0\le j<i} \{(k+j)^{-1/2-\alpha}(|j+l-i|+1)^{-1/2-\beta} \right. \\
&\qquad\qquad \left. + (k+j)^{-1/2-\beta}(|j+l-i|+1)^{-1/2-\alpha}\} \right\}^2 \\
&\le C_{20} \sum_{0<k\le m} \sum_{1\le l<k+i} (k+|l-i|)^{-2(\alpha+\beta)} \\
&\le C_{21} \sum_{0<k\le m} k^{1-2(\alpha+\beta)} \\
&= C_{22} m^{2-2(\alpha+\beta)}.
\end{aligned}$$

Following the calculations in (4.41), we arrive at

$$\sum_{i+m<k\le n} \sum_{1\le l<k} \left\{ \sum_{i<j\le i+m} h_{j,k,l} \right\}^2 \le C_{23}(I_1+I_2), \tag{4.43}$$

where

$$\begin{aligned}
I_1 &= \sum_{i+m<k\le n} \sum_{1\le l<k} \left\{ \sum_{i<j\le i+m} a(j-k)c(j-l) \right\}^2 \\
&\le C_{24} \sum_{i<j,\, v\le i+m} \sum_{i+m<k\le n} \sum_{l\ge 1} (k-j)^{-1/2-\alpha}(|j-l|+1)^{-1/2-\beta} \\
&\qquad \times (k-v)^{-1/2-\alpha}(|v-l|+1)^{-1/2-\beta} \\
&\le C_{25} \sum_{i<j,\, v\le i+m} (i+m+1-j+i+m+1-v)^{-2\alpha}(|j-v|+1)^{-2\beta} \\
&\le C_{25} \sum_{0\le j,\, v<m} (j+l+1)^{-2\alpha}(|j-v|+1)^{-2\beta} \\
&\le C_{26} m^{2-2(\alpha+\beta)}
\end{aligned} \tag{4.44}$$

and

$$\begin{aligned}
I_2 &= \sum_{i+m<k\le n} \sum_{1\le l\le k} \left(\sum_{i<j\le i+m} a(j-l)c(j-k) \right)^2 \\
&\le C_{27} m^{2-2(\alpha+\beta)}.
\end{aligned} \tag{4.45}$$

Putting together (4.40)–(4.45), we obtain (4.36).

As to (4.37), we have

$$\begin{aligned}
&E\left|Q_7^*(i+m,n) - Q_7^*(i,n)\right|^4 \\
&\quad \le C_{28} \left\{ E\left| \sum_{k>n} \sum_{l<k} \sum_{i<j\le i+m} h_{j,k,l}\xi_l\xi_k \right|^4 \right. \\
&\qquad\qquad + E\left| \sum_{k\le i} \sum_{l<k} \sum_{i<j\le i+m} h_{j,k,l}\xi_l\xi_k \right|^4
\end{aligned}$$

$$+ E\left|\sum_{i<k\le i+m}\sum_{l<k}\sum_{m+i<j\le n} h_{j,k,l}\xi_l\xi_k\right|^4\Bigg\}$$

$$\le C_{29}\Bigg\{\Bigg(\sum_{k>n}\sum_{l<k}\Big\{\sum_{i<j\le i+m} h_{j,k,l}\Big\}^2\Bigg)^2$$

$$+\Bigg(\sum_{k\le i}\sum_{l<k}\Big\{\sum_{i<j\le i+m} h_{j,k,l}\Big\}^2\Bigg)^2$$

$$+\Bigg(\sum_{i<k\le i+m}\sum_{l<k}\Big\{\sum_{m+i<j\le n} h_{j,k,l}\Big\}\Bigg)^2\Bigg\}$$

$$\le C_{30}m^{4-4(\alpha+\beta)},$$

which gives (4.37).

Using again Lemma 4.2, one can easily verify that

$$E\left|Q_8^*(i+m,n)-Q_8^*(i,n)\right|^4$$

$$\le 16\Bigg\{E\left|\sum_{i<k\le i+m}\sum_{l\le 0}\sum_{i<j\le n} h_{j,k,l}\xi_l\xi_k\right|^4 + E\left|\sum_{i+m<k\le n}\sum_{l\le 0}\sum_{i<j\le i+m} h_{j,k,l}\xi_l\xi_k\right|^4\Bigg\}$$

$$\le C_{31}\Bigg\{\Bigg(\sum_{i<k\le i+m}\sum_{l\le 0}\Big(\sum_{i<j\le n} h_{j,k,l}\Big)^2\Bigg)^2+\Bigg(\sum_{i+m<k\le n}\sum_{l\le 0}\Big(\sum_{i<j\le i+m} h_{j,k,l}\Big)^2\Bigg)^2\Bigg\}$$

$$\le C_{32}m^{4-4(\alpha+\beta)},$$

which is just the statement in (4.38).

Next we note that

$$Q_2^*(m,n)=\sum_{1\le j\le n-m}\sum_{v\ge j} b(v)\sum_{k,l} a(j+m-k)a(j+m-v-l)(\xi_k\xi_l-E\xi_k\xi_l)$$

$$=\sum_k\sum_{1\le j\le n-m}\sum_{v\ge j} b(v)a(j+m-k)a(j+m-v-k)(\xi_k^2-\tau^2)$$

$$+\sum_{k\ne l}\sum_{1\le j\le n-m}\sum_{v\ge j} b(v)a(j+m-k)a(j+m-v-l)(\xi_k\xi_l-E\xi_k\xi_l)$$

$$=\sum_k\sum_{m<j\le n}\sum_{v\ge j-m} b(v)a(j-k)a(j-v-k)(\xi_k^2-\tau^2)$$

$$+\sum_{k\ne l}\sum_{m<j\le n}\sum_{v\ge j-m} b(v)a(j-k)a(j-v-l)\xi_k\xi_l$$

$$=Q_{2,1}^*(m,n)+Q_{2,2}^*(m,n).$$

By Lemma 4.1 we have

$$
\begin{aligned}
E\left|Q_{2,1}^{*}(m,n)\right|^{2+r/2}
&\le C_{33}\left(\sum_k\left(\sum_{m<j\le n}\sum_{v\ge j-m} b(v)a(j-k)a(j-v-k)\right)^2\right)^{1+r/2}\\
&\le C_{34}\sum_k\left(\sum_{m<j\le n}\sum_{v\ge j-m} v^{-1/2-\theta}(|j-k|+1)^{-1/2-\alpha}(|j-v-k|+1)^{-1/2-\alpha}\right)^2\\
&\le C_{35}\sum_k\left(\sum_{m<j\le n}(j-m+|k-m|)^{-\alpha-\theta}(|j-k|+1)^{-1/2-\alpha}\right)^2\\
&\le C_{35}\sum_k\left(\sum_{j>0}(j+|k-m|)^{-\alpha-\theta}(|j+m-k|+1)^{-1/2-\alpha}\right)^2\\
&= C_{35}\sum_k\left(\sum_{j>0}(j+|k|)^{-\alpha-\theta}(|j-k|+1)^{-1/2-\alpha}\right)^2\\
&\le C_{36}\sum_k(1+|k|)^{1-2(2\alpha+\theta)}<\infty.
\end{aligned}
$$

Lemma 4.2 yields

$$
\begin{aligned}
&E\left|Q_{2,2}^{*}(m+i,n)-Q_{2,2}^{*}(i,n)\right|^4\\
&\quad\le C_{37}\left\{\left(\sum_{k\ne l}\left(\sum_{i<j\le i+m}\sum_{v\ge j-i} b(v)a(j-k)a(j-v-l)\right)^2\right)^2\right.\\
&\qquad\left.+\left(\sum_{k\ne l}\left(\sum_{m+i<j\le n}\;\sum_{j-(m+i)\le v<j-i} b(v)a(j-k)a(j-v-l)\right)^2\right)^2\right\}.
\end{aligned}
$$

By Lemma 4.5 we have

$$
\begin{aligned}
&\sum_{k\ne l}\left(\sum_{i<j\le i+m}\sum_{v\ge j-i} b(v)a(j-k)a(j-v-l)\right)^2\\
&\quad\le C_{38}\sum_{k\ne l}\Big(\sum_{i<j\le i+m}\sum_{v\ge j-i} v^{-1/2-\theta}(1+|j-k|)^{-1/2-\alpha}(1+|j-v-l|)^{-1/2-\alpha}\Big)^2\\
&\quad\le C_{39}\sum_{k,l}\left(\sum_{i<j\le i+m}(j-i+|i-l|)^{-\alpha-\theta}(1+|j-k|)^{-1/2-\alpha}\right)^2\\
&\quad\le C_{39}\sum_{i<j,\,v\le i+m}\;\sum_{k,l}(j-i+|i-l|)^{-\alpha-\theta}(1+|j-k|)^{-1/2-\alpha}\\
&\qquad\qquad\times(v-i+|i-l|)^{-\alpha-\theta}(1+|v-k|)^{-1/2-\alpha}
\end{aligned}
$$

$$\leq C_{40} \sum_{i<j,\, v\leq i+m} (j-i+v-i)^{1-2(\alpha+\theta)}(1+|j-v|)^{-2\alpha}$$
$$\leq C_{41} m^{3-2(\theta+2\alpha)}.$$

Similarly, arguments give

$$\sum_{k\neq l}\biggl(\sum_{m+i<j\leq n}\ \sum_{j-(m+i)\leq v<j-i} b(v)a(j-k)a(j-v-l)\biggr)^2$$
$$\leq C_{42}\sum_{k\neq l}\biggl(\sum_{m+i<j\leq n}\ \sum_{j-(m+i)\leq v<j-i} v^{-1/2-\theta}(1+|j-k|)^{-1/2-\alpha} \times (1+|j-v-l|)^{-1/2-\alpha}\biggr)^2$$
$$= C_{42}\sum_{k\neq l}\biggl(\sum_{0\leq v<m}\ \sum_{m+i<j\leq n} (v+j-i-m)^{-1/2-\theta}(1+|j-k|)^{-1/2-\alpha} \times (1+|i+m-v-l|)^{-1/2-\alpha}\biggr)^2$$
$$\leq C_{43}\sum_{k\neq l}\biggl(\sum_{0\leq v<m} (v+1+|m+i-k|)^{-\alpha-\theta}(1+|i+m-v-l|)^{-1/2-\alpha}\biggr)^2$$
$$\leq C_{44}\sum_{0\leq v,\, j<m}\ \sum_{k,\, l}(v+1+|m+i-k|)^{-\alpha-\theta}(1+|i+m-v-l|)^{-1/2-\alpha} \times (j+1+|m+i-k|)^{-\alpha-\theta}(1+|i+m-j-l|)^{-1/2-\alpha}$$
$$\leq C_{45}\sum_{0\leq v,\, j<m}\ \sum_{k,\, l}(v+j+1)^{1-2(\alpha+\theta)}(1+|j-v|)^{-2\alpha}$$
$$\leq C_{46} m^{3-2(\theta+2\alpha)}.$$

Thus we have shown

$$E\bigl|Q^*_{2,2}(m+i,n)-Q^*_{2,2}(i,n)\bigr|^4 \leq C_{47} m^{6-4(\theta+2\alpha)},$$

which gives (4.39). □

5. Proofs of Theorems 2.1–2.3.

PROOF OF THEOREM 2.1. By the mean value theorem, we have

$$\mathbf{0} = \frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_k(\hat{\boldsymbol{\lambda}}_k) = \frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_k(\boldsymbol{\lambda}_0) + \frac{\partial^2}{\partial\boldsymbol{\lambda}^2}\Lambda_k(\hat{\boldsymbol{\lambda}}^*_k)(\hat{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0) \tag{5.1}$$

and

$$\mathbf{0} = \frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda^*_k(\tilde{\boldsymbol{\lambda}}_k) = \frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda^*_k(\boldsymbol{\lambda}_0) + \frac{\partial^2}{\partial\boldsymbol{\lambda}^2}\Lambda^*_k(\tilde{\boldsymbol{\lambda}}^*_k)(\tilde{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0), \tag{5.2}$$

where $\|\hat{\boldsymbol\lambda}_k^* - \boldsymbol\lambda_0\| \le \|\hat{\boldsymbol\lambda}_k - \boldsymbol\lambda_0\|$ and $\|\tilde{\boldsymbol\lambda}_k^* - \boldsymbol\lambda_0\| \le \|\tilde{\boldsymbol\lambda}_k - \boldsymbol\lambda_0\|$ ($\|\cdot\|$ denotes the maximum norm of vectors and matrices). Let $\varepsilon_1, \varepsilon_2 > 0$. By Hannan (1973) [cf. also Giraitis and Surgailis (1990), page 98], there is an integer $n_0 = n_0(\varepsilon_1, \varepsilon_2)$ such that for all $n > n_0$ we have

$$P\biggl\{\max_{n_0\le k\le n-n_0}\biggl\|\biggl(\frac{\partial^2}{\partial\boldsymbol\lambda^2}\Lambda_k(\hat{\boldsymbol\lambda}_k^*)\biggr)^{-1} - \frac{2\pi}{\kappa_0^2}\mathscr{W}^{-1}(\boldsymbol\lambda_0)\biggr\| \ge \varepsilon_1\biggr\} \le \varepsilon_2 \tag{5.3}$$

and

$$P\biggl\{\max_{n_0\le k\le n-n_0}\biggl\|\biggl(\frac{\partial^2}{\partial\boldsymbol\lambda^2}\Lambda_k^*(\tilde{\boldsymbol\lambda}_k^*)\biggr)^{-1} - \frac{2\pi}{\kappa_0^2}\mathscr{W}^{-1}(\boldsymbol\lambda_0)\biggr\| \ge \varepsilon_1\biggr\} \le \varepsilon_2. \tag{5.4}$$

By (5.1) and (5.2) we have

$$\begin{aligned}
&n^{1/2}t(1-t)\bigl(\hat{\boldsymbol\lambda}_{[nt]} - \tilde{\boldsymbol\lambda}_{[nt]}\bigr)\\
&\quad= n^{1/2}t(1-t)\biggl\{\biggl(\frac{\partial^2}{\partial\boldsymbol\lambda^2}\Lambda_{[nt]}^*(\tilde{\boldsymbol\lambda}_{[nt]}^*)\biggr)^{-1}\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}^*(\boldsymbol\lambda_0)\\
&\qquad\qquad - \biggl(\frac{\partial^2}{\partial\boldsymbol\lambda^2}\Lambda_{[nt]}(\hat{\boldsymbol\lambda}_{[nt]})\biggr)^{-1}\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}(\boldsymbol\lambda_0)\biggr\}\\
&\quad= n^{1/2}t(1-t)\biggl\{-\biggl(\frac{\partial^2}{\partial\boldsymbol\lambda^2}\Lambda_{[nt]}^*(\tilde{\boldsymbol\lambda}_{[nt]}^*)\biggr)^{-1}\biggl(\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}(\boldsymbol\lambda_0) - \frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}^*(\boldsymbol\lambda_0)\biggr)\biggr\}\\
&\qquad + n^{1/2}t(1-t)\biggl\{\biggl(\frac{\partial^2}{\partial\boldsymbol\lambda^2}\Lambda_{[nt]}^*(\tilde{\boldsymbol\lambda}_{[nt]}^*)\biggr)^{-1} - \biggl(\frac{\partial^2}{\partial\boldsymbol\lambda^2}\Lambda_{[nt]}(\tilde{\boldsymbol\lambda}_{[nt]}^*)\biggr)^{-1}\biggr\}\\
&\qquad\quad\times\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}(\boldsymbol\lambda_0),
\end{aligned}\tag{5.5}$$

and therefore (5.3) and (5.4) imply

$$\begin{aligned}
P\biggl\{&\sup_{n_0/n\le t\le 1-n_0/n}\biggl\|n^{1/2}t(1-t)\bigl(\hat{\boldsymbol\lambda}_{[nt]} - \tilde{\boldsymbol\lambda}_{[nt]}\bigr) - n^{1/2}t(1-t)\frac{2\pi}{\kappa_0^2}\mathscr{W}^{-1}(\boldsymbol\lambda_0)\\
&\qquad\qquad\times\biggl(\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}^*(\boldsymbol\lambda_0) - \frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}(\boldsymbol\lambda_0)\biggr)\biggr\|\\
&\le \varepsilon_1\sup_{n_0/n\le t\le 1-n_0/n}\biggl\|n^{1/2}t(1-t)\biggl(\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}^*(\boldsymbol\lambda_0) - \frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}(\boldsymbol\lambda_0)\biggr)\biggr\|\\
&\qquad + 2\varepsilon_1\sup_{n_0/n\le t\le 1-n_0/n}\biggl\|n^{1/2}t(1-t)\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}(\boldsymbol\lambda_0)\biggr\|\biggr\}\\
&\ge 1-2\varepsilon_2.
\end{aligned}$$

Using Lemma 4 of Giraitis and Surgailis (1990), we can assume that n_0 is so large that

$$n^{1/2}\sup_{n_0/n\le t\le 1-n_0/n}\biggl\|E\frac{\partial}{\partial\boldsymbol\lambda}\Lambda_{[nt]}(\boldsymbol\lambda_0)\biggr\| \le \varepsilon, \tag{5.6}$$

and

$$n^{1/2} \sup_{n_0/n \le t \le 1-n_0/n} \left\| E \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda^*_{[nt]}(\boldsymbol{\lambda}_0) \right\| \le \varepsilon \tag{5.7}$$

if $n \ge n_0$. Theorems 1.1 and 1.2 imply that

$$\sup_{0 \le t \le 1} \left\| n^{1/2} t(1-t) \left\{ \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda_{[nt]}(\boldsymbol{\lambda}_0) - E \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda_{[nt]}(\boldsymbol{\lambda}_0) - \left(\frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda^*_{[nt]}(\boldsymbol{\lambda}_0) - E \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda^*_{[nt]}(\boldsymbol{\lambda}_0) \right) \right\} \right\| = O_P(1)$$

and

$$\sup_{0 \le t \le 1} \left\| n^{1/2} t(1-t) \left\{ \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda_{[nt]}(\boldsymbol{\lambda}_0) - E \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda_{[nt]}(\boldsymbol{\lambda}_0) \right\} \right\| = O_P(1).$$

Hence it is enough to obtain the weak convergence of the first coordinate of

$$n^{1/2} t(1-t) \frac{2\pi}{\kappa_0^2} \mathscr{W}^{-1}(\boldsymbol{\lambda}_0) \left\{ \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda^*_{[nt]}(\boldsymbol{\lambda}_0) - E \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda^*_{[nt]}(\boldsymbol{\lambda}_0) - \left(\frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda_{[nt]}(\boldsymbol{\lambda}_0) - E \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda_{[nt]}(\boldsymbol{\lambda}_0) \right) \right\}.$$

Thus, by Theorem 1.2, there is a sequence of Brownian bridges $\{B_n(t),\ 0 \le t \le 1\}$ and a constant c_0 such that

$$\sup_{n_0/n \le t \le 1-n_0/n} \left| n^{1/2} t(1-t) \left(\hat{\lambda}_{[nt],1} - \tilde{\lambda}_{[nt],1} \right) - c_0 B_n(t) \right| = o_P(1).$$

Giraitis and Surgailis (1990) showed that $c_0 = (4\pi w^*)^{1/2}$. Observing that

$$\sup_{0 \le t \le n_0/n} n^{1/2} t(1-t) \left| \hat{\lambda}_{[nt],1} - \tilde{\lambda}_{[nt],1} \right| + \sup_{1-n_0/n \le t \le 1} n^{1/2} t(1-t) \left| \hat{\lambda}_{[nt],1} - \tilde{\lambda}_{[nt],1} \right| = o_P(1)$$

and

$$\sup_{0 \le t \le n_0/n} |B_n(t)| + \sup_{1-n_0/n \le t \le 1} |B_n(t)| = o_P(1)$$

for any n_0, the proof of Theorem 2.1 is complete. □

PROOF OF THEOREM 2.2. We follow the proof of Theorem 2.1. It follows from Giraitis and Surgailis [(1990), page 100] that there is a $\delta_0 > 0$ such that

$$\left\| E k^{1/2} \frac{\partial}{\partial \boldsymbol{\lambda}} \Lambda_k(\boldsymbol{\lambda}_0) \right\| \le C_1 k^{-\delta_0},$$

with some $C_1 > 0$. Thus we get

$$\begin{aligned}
&\sup_{n_0/n\le t\le 1-n_0/n} n^{1/2}t(1-t)\left\|E\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_k(\boldsymbol{\lambda}_0)-E\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_k^*(\boldsymbol{\lambda}_0)\right\|\Big/q(t)\\
&\quad\le C_1 n^{1/2}\sup_{n_0/n\le t\le 1-n_0/n}\frac{t(1-t)}{q(t)}\left\{(nt)^{-1/2-\delta_0}+(n(1-t))^{-1/2-\delta_0}\right\}\\
&\quad=o(1)\quad\text{as } n\to\infty
\end{aligned}$$

for all n_0, on account of (3.18). Similarly,

$$\sup_{n_0/n\le t\le 1-n_0/n} n^{1/2}t(1-t)\left\|E\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_k(\boldsymbol{\lambda}_0)\right\|\Big/q(t)=o(1)\quad\text{as } n\to\infty.$$

Hence

$$\begin{aligned}
&P\Bigg\{\sup_{n_0/n\le t\le 1-n_0/n} n^{1/2}\frac{t(1-t)}{q(t)}\\
&\qquad\times\Bigg\|\hat{\boldsymbol{\lambda}}_{[nt]}-\tilde{\boldsymbol{\lambda}}_{[nt]}-\frac{2\pi}{\kappa_0^2}\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)\\
&\qquad\qquad\times\left\{\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda^*_{[nt]}(\boldsymbol{\lambda}_0)-\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_{[nt]}(\boldsymbol{\lambda}_0)-E\left(\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda^*_{[nt]}(\boldsymbol{\lambda}_0)-\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_{[nt]}(\boldsymbol{\lambda}_0)\right)\right\}\Bigg\|\\
&\quad\le 2\varepsilon_1+\varepsilon_1\sup_{n_0/n\le t\le 1-n_0/n} n^{1/2}\frac{t(1-t)}{q(t)}\\
&\qquad\times\left\|\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda^*_{[nt]}(\boldsymbol{\lambda}_0)-\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_{[nt]}(\boldsymbol{\lambda}_0)-E\left(\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda^*_{[nt]}(\boldsymbol{\lambda}_0)-\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_{[nt]}(\boldsymbol{\lambda}_0)\right)\right\|\\
&\qquad+2\varepsilon_1\sup_{n_0/n\le t\le 1-n_0/n} n^{1/2}\frac{t(1-t)}{q(t)}\left\|\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_{[nt]}(\boldsymbol{\lambda}_0)-E\frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_{[nt]}(\boldsymbol{\lambda}_0)\right)\Bigg\|\Bigg\}\\
&\quad\ge 1-2\varepsilon_2.
\end{aligned}$$

So, by Corollary 1.1, we have

$$\max_{n_0\le k\le n-n_0} n^{-3/2}k(n-k)|\hat{\lambda}_{k,1}-\tilde{\lambda}_{k,1}|/q(k/n)\xrightarrow{\mathscr{D}}(4\pi w^*)^{1/2}\sup_{0<t<1}|B(t)|/q(t),$$

where $\{B(t),\ 0\le t\le 1\}$ is a Brownian bridge. It is easy to see that [cf. (3.18)]

$$\max_{1\le k\le n_0} n^{-3/2}k(n-k)|\hat{\lambda}_{k,1}-\tilde{\lambda}_{k,1}|/q(k/n)=o_P(1)$$

and

$$\max_{n-n_0\le k\le n} n^{-3/2}k(n-k)|\hat{\lambda}_{k,1}-\tilde{\lambda}_{k,1}|/q(k/n)=o_P(1)$$

for all n_0, which gives (2.8).

If (2.8) holds, then the limit is finite with probability 1 and therefore $I(q,c)<\infty$ with some $c>0$ [cf. Csörgő and Horváth (1993), page 181]. □

PROOF OF THEOREM 2.3. Combining (5.1)–(5.4) with Theorem 1.1 and the law of the iterated logarithm, we conclude

$$\max_{1<k\le n} k^{1/2}||\hat{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0||/(\log\log k)^{1/2} = O_P(1) \tag{5.8}$$

and

$$\max_{1\le k<n} (n-k)^{1/2}||\tilde{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0||/(\log\log(n-k))^{1/2} = O_P(1). \tag{5.9}$$

Similarly to (5.3) and (5.4), we have

$$\max_{1\le i,\, j,\, l\le p}\ \max_{1\le k<n}\ \sup_{\boldsymbol{\lambda}\in\mathscr{L}}\left|\frac{\partial^3}{\partial\lambda_i\partial\lambda_j\partial\lambda_l}\Lambda_k(\boldsymbol{\lambda})\right| = O_P(1) \tag{5.10}$$

and

$$\max_{1\le i,\, j,\, l\le p}\ \max_{1\le k<n}\ \sup_{\boldsymbol{\lambda}\in\mathscr{L}}\left|\frac{\partial^3}{\partial\lambda_i\partial\lambda_j\partial\lambda_l}\Lambda_k^*(\boldsymbol{\lambda})\right| = O_P(1). \tag{5.11}$$

Replacing (5.1) and (5.2) with two-term Taylor expansions, we conclude by (5.10) and (5.11) that

$$\mathbf{0} = \frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_k(\boldsymbol{\lambda}_0) + \frac{\partial^2}{\partial\boldsymbol{\lambda}^2}\Lambda_k(\boldsymbol{\lambda}_0)(\hat{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0) + \boldsymbol{u}_k, \tag{5.12}$$

$$\mathbf{0} = \frac{\partial}{\partial\boldsymbol{\lambda}}\Lambda_k^*(\boldsymbol{\lambda}_0) + \frac{\partial^2}{\partial\boldsymbol{\lambda}^2}\Lambda_k^*(\boldsymbol{\lambda}_0)(\tilde{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0) + \boldsymbol{u}_k^* \tag{5.13}$$

and

$$\max_{1<k\le n} k\|\boldsymbol{u}_k\|/\log\log k = O_P(1), \tag{5.14}$$

$$\max_{1\le k<n} (n-k)\|\boldsymbol{u}_k^*\|/\log\log(n-k) = O_P(1). \tag{5.15}$$

Theorem 1.3 yields

$$\max_{1\le k<n}\left\|\left(\frac{\partial^2}{\partial\boldsymbol{\lambda}^2}\Lambda(\boldsymbol{\lambda}_0)\right)^{-1} - \frac{2\pi}{\kappa_0^2}\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)\right\|\Big/ k^{\varepsilon} = O_P(1) \tag{5.16}$$

and

$$\max_{1\le k<n}\left\|\left(\frac{\partial^2}{\partial\boldsymbol{\lambda}^2}\Lambda^*(\boldsymbol{\lambda}_0)\right)^{-1} - \frac{2\pi}{\kappa_0^2}\mathscr{W}^{-1}(\boldsymbol{\lambda}_0)\right\|\Big/ (n-k)^{\varepsilon} = O_P(1), \tag{5.17}$$

with some $\varepsilon > 0$. Applying Theorem 1.1 and the law of the iterated logarithm for each coordinate of $(\partial/\partial\boldsymbol{\lambda})\Lambda_k(\boldsymbol{\lambda}_0)$, by (5.16) we have

$$\max_{1\le k\le \log n} k^{1/2}\|\hat{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0\| = O_P\big((\log\log\log n)^{1/2}\big), \tag{5.18}$$

$$\max_{1\le k<n} k^{1/2}\|\hat{\boldsymbol{\lambda}}_k - \boldsymbol{\lambda}_0\| = O_P\big((\log\log n)^{1/2}\big) \tag{5.19}$$

and, similarly,

$$\max_{n-\log n\le k<n}(n-k)^{1/2}\|\tilde{\boldsymbol{\lambda}}_k-\boldsymbol{\lambda}_0\|=O_P\big((\log\log\log n)^{1/2}\big), \tag{5.20}$$

$$\max_{1\le k<n}(n-k)^{1/2}\|\tilde{\boldsymbol{\lambda}}_k-\boldsymbol{\lambda}_0\|=O_P\big((\log\log n)^{1/2}\big). \tag{5.21}$$

Next, using Theorem 1.2, (5.12)–(5.17), (5.19) and (5.21), we can construct a Brownian bridge $\{B_n(t),\ 0\le t\le 1\}$ such that

$$\left|\frac{1}{(4\pi w^*)^{1/2}}\max_{\log n\le k\le n-\log n} n^{-1/2}(k(n-k))^{1/2}\big|\hat{\lambda}_{k,1}-\tilde{\lambda}_{k,1}\big|\right. \tag{5.22}$$

$$\left.-\sup_{(\log n)/n\le t\le 1-(\log n)/n}\frac{|B_n(t)|}{(t(1-t))^{1/2}}\right|=O_P\big((\log n)^{-\varepsilon'}\big), \tag{5.23}$$

with some $\varepsilon'>0$. Lemma 1.3 in Csörgő and Horváth [(1993), page 257] gives

$$\lim_{n\to\infty}P\left\{A(\log n)\sup_{(\log n)/n\le t\le 1-(\log n)/n}\frac{|B_n(t)|}{(t(1-t))^{1/2}}\le x+D(\log n)\right\}$$
$$=\exp\big(-2e^{-x}\big)$$

for all x and therefore by (5.22) we have

$$\begin{aligned}\lim_{n\to\infty}P\bigg\{&\frac{A(\log n)}{(4\pi w^*)^{1/2}}\max_{\log n\le k\le n-\log n} n^{-1/2}(k(n-k))^{1/2}\big|\hat{\lambda}_{k,1}-\tilde{\lambda}_{k,1}\big|\\ &\qquad\le x+D(\log n)\bigg\}\\ &=\exp\big(-2e^{-x}\big).\end{aligned} \tag{5.24}$$

It follows from (5.18) and (5.20) that

$$\frac{A(\log n)}{(4\pi w^*)^{1/2}}\max_{1\le k\le \log n} n^{-1/2}(k(n-k))^{1/2}\big|\hat{\lambda}_{k,1}-\tilde{\lambda}_{k,1}\big|-(x+D(\log n))\longrightarrow_P-\infty$$

and

$$\frac{A(\log n)}{(4\pi w^*)^{1/2}}\max_{n-\log n\le k\le n} n^{-1/2}(k(n-k))^{1/2}\big|\hat{\lambda}_{k,1}-\tilde{\lambda}_{k,1}\big|-(x+D(\log n))\longrightarrow_P-\infty$$

for all x and therefore (5.24) implies Theorem 2.3. □

PROOF OF REMARK 2.1. It suffices to show that if $\sup_{0\le t\le 1}|g'(t)|<\infty$, then

$$l(n)=O(1/n), \tag{5.25}$$

where $l(n) = \int_0^1 \exp(i2\pi nx) g(x)\, dx$. We have

$$
\begin{aligned}
l(n) &= \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \exp(i2n\pi x) g(x)\, dx \\
&= \sum_{k=0}^{n-1} \int_0^{1/n} \exp(i2n\pi(x+k/n)) g(x+k/n)\, dx \\
&= \sum_{k=0}^{n-1} \int_0^{1/n} \exp(i2n\pi x) g(x+k/n)\, dx \\
&= \sum_{k=0}^{n-1} \left\{ \int_0^{1/(4n)} + \int_{1/(4n)}^{1/(2n)} + \int_{1/(2n)}^{3/(4n)} + \int_{3/(4n)}^{1/n} \right\} \exp(i2n\pi x) g(x+k/n)\, dx \\
&= \sum_{k=0}^{n-1} \left\{ \int_0^{1/(4n)} \exp(i2n\pi x) \big\{ g(x+k/n) - g(1/(2n) - x + k/n) \big\}\, dx \right. \\
&\qquad \left. + \int_{1/(4n)}^{1/(2n)} \exp(i2n\pi x) \big\{ g(x+k/n) - g(1/n - x + k/n) \big\}\, dx \right\},
\end{aligned}
$$

which yields (5.25) by the assumption $\sup_{0\le t\le 1} |g'(t)| < \infty$. $\square$

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