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RISK-SENSITIVE DYNAMIC PORTFOLIO OPTIMIZATION WITH PARTIAL INFORMATION ON INFINITE TIME HORIZON

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We consider an optimal investment problem for a factor model treated by Bielecki and Pliska (*Appl. Math. Optim.* **39** 337–360) as a risk-sensitive stochastic control problem, where the mean returns of individual securities are explicitly affected by economic factors defined as Gaussian processes. We relax the measurability condition assumed as Bielecki and Pliska for the investment strategies to select. Our investment strategies are supposed to be chosen without using information of factor processes but by using only past information of security prices. Then our problem is formulated as a kind of stochastic control problem with partial information. The case on a finite time horizon is discussed by Nagai (*Stochastics in Finite and Infinite Dimension* 321–340. Birkhäuser, Boston). Here we discuss the problem on infinite time horizon.

1. Introduction There have been several works applying the idea of risk-sensitive control to problems of mathematical finance. Among them, Fleming [7], Fleming and Sheu [9], and Bielecki and Pliska [3] have studied risk-sensitive control problems arising from portfolio management. In particular, Bielecki and Pliska [3], which treated a factor model where the mean returns of individual securities are explicitly affected by economic factors defined as ergodic Gauss–Markov processes, motivates the present paper. For such model they considered an optimal investment problem maximizing the risk-sensitized expected growth rate per unit time of the value of the capital the investor possess under the condition that security prices and factors have independent randomness. Since their works there have been several works [4, 10, 11, 13] improving the independence condition assumed in [3]. In these works as well the investment strategies are assumed to be chosen by observing all past information of factor processes as well as security prices. On the other hand, in the previous work [14] we relaxed the measurability conditions for the investment strategies with no constraint as the ones to be selected without using information of factor processes but by using only past information of security prices in the case of a finite time horizon. Then the problem is formulated as a kind of risk-sensitive stochastic control with partial information. Indeed we can formulate our problem by regarding the factor processes as system processes and security prices observation processes in terms

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of stochastic control. Under such setting up we have constructed the optimal strategies for the optimal investment problem on a finite time horizon, which are explicitly represented by the solutions of the ordinary differential equations with the Riccati equations concerning filter and the value function. The results are summarized in Section 3.

In the present paper we shall discuss the optimal investment problem on infinite time horizon under such formulation with partial information. To consider the problem it is necessary to study asymptotic behavior of the solution $U(t;T)$ of inhomogeneous Riccati differential equation (3.6) with the terminal condition at T, related to the value function of the problem with the time horizon T. The inhomogeneity of the coefficients of the equations comes from filter, which is also the solution $\Pi(t)$ of a Riccati differential equation (3.3) with an initial condition, whose coefficients are all constant matrices. Under the very natural condition (i.e., stability of the factor process under the minimal martingale measure) we can see the solution $\Pi(t)$ has exponential stability as $t \to \infty$ (cf. Section 4). The difficulty lies in the study of asymptotics of the solution $U(t;T)$. Specific feature of asymptotics of the solution $U(t;T)$ we obtained here is stability as t and $T-t$ tend to ∞ (cf. Section 5), from which we can see the asymptotic behavior of the value function as $T \to \infty$ and we can construct the optimal strategy for the problem on infinite time horizon by using the solutions of the limit equations and filter (cf. Section 6). Known results on Riccati differential equations and necessary related notions for obtaining our results are completed in Appendix.

2. Setting up. We consider a market with $m+1 \geq 2$ securities and $n \geq 1$ factors. We assume that the set of securities includes one bond, whose price is defined by the ordinary differential equation

$$dS^0(t) = r(t)S^0(t)\,dt, \qquad S^0(0) = s^0, \tag{2.1}$$

where $r(t)$ is a deterministic function of t. The other security prices and factors are assumed to satisfy the following stochastic differential equations:

$$\begin{aligned} dS^i(t) &= S^i(t)\left\{(a+AX_t)^i\,dt + \sum_{k=1}^{n+m} \sigma_k^i\,dW_t^k\right\}, \\ S^i(0) &= s^i, \qquad i = 1,\dots,m \end{aligned} \tag{2.2}$$

and

$$dX_t = (b+BX_t)\,dt + \Lambda\,dW_t, \qquad X(0) = x \in \mathbb{R}^n, \tag{2.3}$$

where $W_t = (W_t^k)_{k=1,\dots,(n+m)}$ is an $m+n$ dimensional standard Brownian motion process defined on a filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$. Here A, B, Λ are respectively $m \times n$, $n \times n$, $n \times (m+n)$ constant matrices and $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$.

The constant matrix $(\sigma^i_k)_{i=1,2,\dots,m;k=1,2,\dots,(n+m)}$ will be often denoted by Σ in what follows. In the present paper we always assume that

$$\Sigma\Sigma^* > 0, \tag{2.4}$$

where Σ^* stands for the transposed matrix of Σ.

Let us denote investment strategy to ith security $S^i(t)$ by $h^i(t)$, $i = 0, 1, \dots, m$, and set

$$S(t) = \big(S^1(t), S^2(t), \dots, S^m(t)\big)^*,$$

$$h(t) = \big(h^1(t), h^2(t), \dots, h^m(t)\big)^*$$

and

$$\mathcal{G}_t = \sigma\big(S(u); u \le t\big).$$

Here S^* stands for transposed matrix of S.

DEFINITION 2.1. $(h^0(t), h(t)^*)_{0\le t\le T}$ is said an investment strategy if the following conditions are satisfied:

(i) $h(t)$ is a R^m valued $\mathcal{G}_t$ progressively measurable stochastic process such that

$$\sum_{i=1}^m h^i(t) + h^0(t) = 1; \tag{2.5}$$

(ii)

$$P\big(\exists\, c(\omega) \text{ such that } |h(s)| \le c(\omega),\ 0 \le s \le T\big) = 1.$$

The set of all investment strategies will be denoted by $\mathcal{H}(T)$. For simplicity when $(h^0(t), h(t)^*)_{0\le t\le T} \in \mathcal{H}(T)$ we will often write $h \in \mathcal{H}(T)$ since h^0 is determined by (2.5). For given $h \in \mathcal{H}(T)$ the process $V_t = V_t(h)$ representing the investor's capital at time t is determined by the stochastic differential equation

$$\begin{aligned}\frac{dV_t}{V_t} &= \sum_{i=0}^m h^i(t)\frac{dS^i(t)}{S^i(t)}\\ &= h^0(t)r(t)\,dt + \sum_{i=1}^m h^i(t)\left\{(a + AX_t)^i\,dt + \sum_{k=1}^{m+n}\sigma^i_k\,dW^k_t\right\},\\ &\qquad V_0 = v.\end{aligned}$$

Then, taking (2.5) into account it turns out to be a solution of

$$\begin{aligned}\frac{dV_t}{V_t} &= r(t)\,dt + h(t)^*\big(a + AX_t - r(t)\mathbf{1}\big)\,dt + h(t)^*\Sigma\,dW_t,\\ &\qquad V_0 = v,\end{aligned} \tag{2.6}$$

where $\mathbf{1} = (1, 1, \dots, 1)^*$.

We first consider the following problem. For a given constant $\theta > -2,\ \theta \neq 0$, maximize the following risk-sensitized expected growth rate up to time horizon T:

$$J(v, x; h; T) = -\frac{2}{\theta} \log E\left[\exp\left\{-\frac{\theta}{2} \log V_T(h)\right\}\right], \tag{2.7}$$

where h ranges over the set $\mathcal{A}(T)$ of all investment strategies defined later. Then we consider the problem maximizing the risk-sensitized expected growth rate per unit time

$$J(v, x; h) = \limsup_{T\to\infty}\left(-\frac{2}{\theta T}\right) \log E\left[\exp\left\{-\frac{\theta}{2} \log V_T(h)\right\}\right], \tag{2.8}$$

where h ranges over the set of all investment strategies such that $h \in \mathcal{A}(T)$ for each T.

Note that in our problem a strategy h is to be chosen as $\sigma(S(u); u \leq t)$ measurable process, different from the case of Bielecki–Pliska where it is $\sigma((S(u), X_u), u \leq t)$ measurable. Namely, in our case the strategy is to be selected without using past information of the factor process X_t.

Since V_t satisfies (2.6) we have

$$\begin{aligned} V_t^{-\theta/2} = v^{-\theta/2} \exp\bigg\{ &\frac{\theta}{2} \int_0^t \eta\big(X_s, h_s, r(s); \theta\big)\, ds \\ &- \frac{\theta}{2} \int_0^t h_s^* \Sigma\, dW_s - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 \int_0^t h_s^* \Sigma \Sigma^* h_s\, ds \bigg\}, \end{aligned}$$

where

$$\eta(x, h, r; \theta) = \frac{1}{2}\left(\frac{\theta}{2} + 1\right) h^* \Sigma \Sigma^* h - r - h^*(a + Ax - r\mathbf{1}).$$

Therefore, if $\theta > 0$ (resp. $-2 < \theta < 0$) our problem maximizing $J(v, x; h; T)$ is reduced to the one minimizing (resp. maximizing) the following criterion:

$$\begin{aligned} I(x, h; T) = v^{-\theta/2} E\bigg[\exp\bigg\{ &\frac{\theta}{2} \int_0^T \eta\big(X_s, h_s, r(s); \theta\big)\, ds \\ &- \frac{\theta}{2} \int_0^T h_s^* \Sigma\, dW_s - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 \int_0^T h_s^* \Sigma \Sigma^* h_s\, ds \bigg\}\bigg]. \end{aligned} \tag{2.9}$$

Now we shall reformulate the above problem as a partially observable risk-sensitive stochastic control problem. For that we set

$$Y_t^i = \log S^i(t),$$

then we can see that $Y_t = (Y_t^1, \dots, Y_t^m)^*$ satisfies the following stochastic differential:

$$dY_t^i = \{a^i - \tfrac{1}{2}(\Sigma\Sigma^*)^{ii} + (AX_t)^i\}\,dt + \sum_{k=1}^{m+n} \sigma_k^i \, dW_t^k, \tag{2.10}$$

$i = 1, \dots, m$, by using Itô formula. So, setting $d = (d^i) \equiv (a^i - \frac{1}{2}(\Sigma\Sigma^*)^{ii})$, we have

$$dY_t = (d + AX_t)\,dt + \Sigma\, dW_t, \tag{2.11}$$

which we shall regard as the SDE defining the observation process in terms of stochastic control with partial observation. On the other hand, X_t defined by (2.3) is regarded as a system process. In the present setting system noise $\Lambda\, dW_t$ and observation noise $\Sigma\, dW_t$ are correlated in general. Note that $\sigma(Y_u, ; u \le t) = \sigma(S(u); u \le t)$ holds since log is a strictly increasing function, so our problem is to minimize (or maximize) the criterion (2.9) while looking at the observation process Y_t and choosing a $\sigma(Y_u, ; u \le t)$ measurable strategy $h(t)$. Though there is no control in the SDE (2.3) defining system process X_t criterion $I(x, h; T)$ is defined as a functional of the strategy $h(t)$ measurable with respect to observation and the problem is the one of stochastic control with partial observation.

Now let us introduce a new probability measure $\hat{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{F}_T} = \rho_T,$$

where

$$\begin{aligned} \rho_t = \exp\bigg\{ &-\int_0^t (d + AX_s)^*(\Sigma\Sigma^*)^{-1}\Sigma\, dW_s \\ &- \tfrac{1}{2}\int_0^t (d + AX_s)^*(\Sigma\Sigma^*)^{-1}(d + AX_s)\, ds \bigg\}. \end{aligned} \tag{2.12}$$

We see that $\hat{P}$ is a probability measure since it can be seen by standard arguments (cf. [1]) that ρ_t is a martingale and $E[\rho_T] = 1$. Moreover, according to Girsanov theorem,

$$\hat{W}_t = W_t + \int_0^t \Sigma^*(\Sigma\Sigma^*)^{-1}(d + AX_s)\, ds \tag{2.13}$$

turns out to be a standard Brownian motion process under the probability measure $\hat{P}$ and we have

$$dY_t = \Sigma\, d\hat{W}_t, \tag{2.14}$$

$$dX_t = \{b + BX_t - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}(d + AX_t)\}\,dt + \Lambda\, d\hat{W}_t. \tag{2.15}$$

Set $\xi_t = 1/\rho_t$; then we have

$$\xi_t = \exp\left\{\int_0^t (d + AX_s)^*(\Sigma\Sigma^*)^{-1}\, dY_s - \tfrac{1}{2}\int_0^t (d + AX_s)^*(\Sigma\Sigma^*)^{-1}(d + AX_s)\, ds\right\}. \tag{2.16}$$

Let us rewrite our criterion $I(x, h; T)$ by using new probability measure $\hat{P}$. We have

$$\begin{aligned} I(x, h; T) &= v^{-\theta/2}\hat{E}\left[\xi_T \exp\left\{\frac{\theta}{2}\int_0^t \eta\big(X_s, h_s, r(s); \theta\big)\, ds - \frac{\theta}{2}\int_0^t h_s^*\Sigma\, dW_s - \frac{1}{2}\left(\frac{\theta}{2}\right)^2 \int_0^t h_s^*\Sigma\Sigma^* h_s\, ds\right\}\right] \\ &= v^{-\theta/2}\hat{E}\left[\exp\left\{\frac{\theta}{2}\int_0^T \eta\big(X_s, h_s; r(s); \theta\big)\, ds + \int_0^T Q(X_s, h_s)^*\, dY_s - \frac{1}{2}\int_0^T Q(X_s, h_s)^*(\Sigma\Sigma^*)Q(X_s, h_s)\, ds\right\}\right] \\ &= v^{-\theta/2}\hat{E}\left[\hat{E}\left[\exp\left\{\frac{\theta}{2}\int_0^T \eta\big(X_s, h_s; r(s); \theta\big)\, ds\right\}\Psi_T \big| \mathcal{G}_T\right]\right], \end{aligned} \tag{2.17}$$

where

$$\Psi_t = \exp\left\{\int_0^t Q(X_s, h_s)^*\, dY_s - \tfrac{1}{2}\int_0^t Q(X_s, h_s)^*(\Sigma\Sigma^*)Q(X_s, h_s)\, ds\right\}$$

and

$$Q(x, h) = (\Sigma\Sigma^*)^{-1}(Ax + d) - \frac{\theta}{2}h = (\Sigma\Sigma^*)^{-1}\left\{(Ax + d) - \frac{\theta}{2}(\Sigma\Sigma^*)h\right\}.$$

Set

$$q^h(t)\big(\varphi(t)\big) = \hat{E}\left[\exp\left\{\frac{\theta}{2}\int_0^t \eta\big(X_s, h_s; r(s); \theta\big)\, ds\right\}\Psi_t \varphi(t, X_t) \big| \mathcal{G}_t\right], \tag{2.18}$$

then (2.17) reads

$$I(x, h; T) = v^{-\theta/2}\hat{E}[q^h(T)(1)]. \tag{2.19}$$

Hence, if $\theta > 0$ (resp. $-2 < \theta < 0$) our problem is reduced to minimize (resp. maximize) I of (2.19) when taking h over $\mathcal{H}(T)$.

3. Finite time horizon case. In the present section we summarize the results obtained in the previous paper [14]. Let us set

$$L\varphi = \tfrac{1}{2}(\Lambda\Lambda^*)^{ij} D_{ij}\varphi + (b + Bx)^i D_i \varphi. \tag{3.1}$$

Then, the following proposition can be obtained by using Itô calculus in a standard way.

PROPOSITION 3.1 [14]. *$q(t)(\varphi(t)) \equiv q^h(t)(\varphi(t))$ satisfies the following stochastic partial differential equation (SPDE):*

$$\begin{aligned} q(t)\big(\varphi(t)\big) &= q(0)\big(\varphi(0)\big) \\ &\quad + \int_0^t q(s)\Big(\frac{\partial\varphi}{\partial t}(s,\cdot) + L\varphi(s,\cdot) - \frac{\theta}{2}h_s^*\Sigma\Lambda^* D\varphi(s,\cdot) + \frac{\theta}{2}\eta_s(\cdot)\varphi(s,\cdot)\Big)\,ds \\ &\quad + \int_0^t q(s)\big(\varphi(s,\cdot)Q(\cdot,h_s)\big)\,dY_s + \int_0^t q(s)\big((D\varphi)^*\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\big)\,dY_s, \end{aligned} \tag{3.2}$$

where $\eta_s(\cdot) = \eta(\cdot, h_s; r(s); \theta)$.

Now let us give the explicit representation to the solution of SPDE (3.2). For that let us introduce the matrix Riccati equation

$$\begin{aligned} &\dot\Pi + (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*) - \Lambda\Lambda^* - B\Pi - \Pi B^* = 0, \\ &\Pi(0) = 0 \end{aligned} \tag{3.3}$$

and the stochastic differential equation

$$\begin{aligned} d\gamma_t &= \big\{B\gamma_t + b - (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\gamma_t + d)\big\}\,dt \\ &\quad + (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}\,dY_t, \\ \gamma_0 &= x. \end{aligned} \tag{3.4}$$

The following theorem can be seen by using the methods developed in [1].

THEOREM 3.1 [14]. *The solution of the SPDE (3.2) with $q(0)(\varphi(0)) = \varphi(0,x)$ has the following representation:*

$$q(t)\big(\varphi(t)\big) = \alpha_t \int \varphi(t, \gamma_t + \Pi_t^{1/2} z)\frac{1}{(2\pi)^{n/2}}\, e^{-|z|^2/2}\,dz,$$

where

$$\begin{aligned} \alpha_t = \exp\bigg\{&\int_0^t Q(\gamma_s, h_s)^*\,dY_s \\ &- \frac{1}{2}\int_0^t Q(\gamma_s,h_s)^*(\Sigma\Sigma^*)Q(\gamma_s,h_s)\,ds + \frac{\theta}{2}\int_0^t \eta(\gamma_s, h_s; r(s);\theta)\,ds\bigg\}. \end{aligned}$$

REMARK. It is known that (3.3) has a unique solution (cf. [5, 8]).

Now we shall construct optimal strategy minimizing (resp. maximizing) the criterion (2.19) for $\theta > 0$ (resp. $-2 < \theta < 0$). Because of Theorem 3.1 (2.19) reads

$$I(x, h; T) = v^{-\theta/2}\hat{E}[\alpha_T]. \tag{3.5}$$

Let us introduce the following $n \times n$ matrix Riccati differential equation:

$$\begin{aligned}
&\dot{U} + U\left\{B - \frac{\theta}{\theta+2}(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\right\}\\
&\quad + \left\{B^* - \frac{\theta}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)\right\}U\\
&\quad - \frac{2\theta}{\theta+2}U(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)U\\
&\quad + \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A = 0, \qquad U(T) = 0.
\end{aligned} \tag{3.6}$$

When we have a solution U of (3.6) we get a solution g of the following linear differential equation on R^n:

$$\begin{aligned}
&\dot{g} + B^*g - \frac{2\theta}{\theta+2}U(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)g\\
&\quad - \frac{\theta}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)g\\
&\quad + Ub + \frac{1}{\theta+2}\{A - \theta(A\Pi + \Sigma\Lambda^*)U\}^*(\Sigma\Sigma^*)^{-1}\big(a - r(t)\mathbf{1}\big) = 0,\\
&\qquad\qquad g(T) = 0.
\end{aligned} \tag{3.7}$$

Furthermore, for given solutions U of (3.6) and g of (3.7) we have a solution k of the following differential equation:

$$\begin{aligned}
&\dot{k} + r(t) + \mathrm{tr}\big[U(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)\big]\\
&\quad - \theta g^*(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)g + 2g^*b\\
&\quad + \frac{1}{\theta+2}c_t^*(\Sigma\Sigma^*)^{-1}c_t = 0, \qquad k(T) = 0,
\end{aligned} \tag{3.8}$$

where

$$c_t = a - r(t)\mathbf{1} - \theta(A\Pi + \Sigma\Lambda^*)g.$$

DEFINITION. Let us denote by $\mathcal{A}(T)$ the set of all investment strategy satisfying

$$\hat{E}\left[\exp\left\{\int_0^T \Xi_s^*(h)\,dY_s - \tfrac{1}{2}\int_0^T \Xi_s(h)\Sigma\Sigma^*\Xi_s(h)\,ds\right\}\right] = 1$$

where

$$\Xi_t^* = \left[(\gamma_t^* A^* + d^*) - \theta(\gamma_t^* U + g^*)(\Pi A^* + \Lambda\Sigma^*) - \frac{\theta}{2}h_t^*(\Sigma\Sigma^*)\right](\Sigma\Sigma^*)^{-1}.$$

THEOREM 3.2 [14]. *If* (3.6) *has a solution U, then there exists an optimal strategy $\hat{h} \in \mathcal{A}(T)$ maximizing the criterion* (2.7) *and it is explicitly represented as*

$$\begin{aligned}\hat{h}_t = \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}\big[&a - r(t)\mathbf{1} - \theta(A\Pi + \Sigma\Lambda^*)g \\ &+ \{A - \theta(A\Pi + \Sigma\Lambda^*)U\}\gamma_t\big]\end{aligned} \tag{3.9}$$

where g is a solution to (3.7) *and* Π (*resp.* γ_t) *is the solution to* (3.3) (*resp.* 3.4). *Moreover*

$$\begin{aligned}J(v, x; \hat{h}; T) &= \sup_{h\in\mathcal{A}(T)} J(v, x; h; T) \\ &= \log v + x^* U(0)x + 2g^*(0)x + k(0)\end{aligned} \tag{3.10}$$

where k is a solution to (3.8).

REMARK. It is known that (3.6) has a unique solution if $\theta > 0$ (cf. [5, 8]).

4. Stability of filter. In the present section we study asymptotic behavior of the solution $\Pi(t)$ of (3.3) as $t \to \infty$.

LEMMA 4.1. *Assume that*

$$G := B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A \quad \textit{is stable}, \tag{4.1}$$

then $\Pi(t) \to \bar{\Pi} \geq 0$, $t \to \infty$, *where $\bar{\Pi}$ is a unique nonnegative definite solution of the algebraic Riccati equation*

$$\begin{aligned}&\big(B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A\big)\bar{\Pi} + \bar{\Pi}\big(B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A\big)^* \\ &\quad - \bar{\Pi}A^*(\Sigma\Sigma^*)^{-1}A\bar{\Pi} + \Lambda\big(I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\big)\Lambda^* = 0.\end{aligned} \tag{4.2}$$

Moreover, $B - (\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A$ *is stable*.

PROOF. We first note that (3.3) can be written as

$$\dot{\Pi} = \big(B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A\big)\Pi + \Pi\big(B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}A\big)^*$$
$$- \Pi A^*(\Sigma\Sigma^*)^{-1}A\Pi + \Lambda\big(I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\big)\Lambda^*, \qquad \Pi(0) = 0$$

and that

$$\big\{\big(I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\big)\Lambda^*\big\}^*\big(I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\big)\Lambda^*$$
$$= \Lambda\big(I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\big)\Lambda^*.$$

Because of (4.1) we see that $((I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma)\Lambda^*, B^* - A^*(\Sigma\Sigma^*)^{-1}\Sigma\Lambda^*)$ is detectable and the present lemma follows from the results by Wonham [15] and Kucera [12] (cf. Appendix). $\square$

REMARKS. (i) $\Pi(t)$ converges exponentially fast to $\bar{\Pi}$. In fact, by the expressions

$$\Pi(t) = -\int_0^t e^{(t-s)G}\{\Pi(s)A^*(\Sigma\Sigma^*)^{-1}A\Pi(s)\}e^{(t-s)G^*}\,ds$$
$$+ \int_0^t e^{(t-s)G}\Lambda\big(I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\big)\Lambda^* e^{(t-s)G^*}\,ds$$

and

$$\bar{\Pi} = -\int_0^\infty e^{sG}\bar{\Pi}A^*(\Sigma\Sigma^*)^{-1}A\bar{\Pi}e^{sG^*}\,ds$$
$$+ \int_0^\infty e^{sG}\Lambda\big(I_{m+n} - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\big)\Lambda^* e^{sG^*}\,ds$$

we see it since G is stable.

(ii) Condition (4.1) means stability of the factor process X_t under the minimal martingale measure $\tilde{P}$ (cf. [6], Proposition 1.8.2 as for minimal martingale measures), which is defined by

$$\frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_T} = \exp\bigg\{-\int_0^T \theta_s^*\,dW_s - \frac{1}{2}\int_0^T |\theta_s|^2\,ds\bigg\},$$

$\theta_s = \Sigma^*(\Sigma\Sigma^*)^{-1}(a + AX_s - r(s)\mathbf{1})$.

5. Asymptotics of inhomogeneous Riccati equations. In what follows we always assume that $\theta > 0$. Then we have always a solution of (3.6). To study asymptotics of the solution of (3.6) we first consider the equation

$$
\begin{aligned}
&\dot{\tilde{U}} + \tilde{U}\Big(B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\Big) \\
&\quad + \Big(B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\Big)^* \tilde{U} \\
&\quad - \frac{2\theta}{\theta+2}\tilde{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\tilde{U} \\
&\quad + \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A = 0, \qquad \tilde{U}(T) = 0.
\end{aligned}
\tag{5.1}
$$

LEMMA 5.1. *Under assumption* (4.1) $\tilde{U}(t;T)$ *converges to* $\overline{U} \geq 0$ *as* $T - t \to \infty$, *where* $\overline{U}$ *is a unique nonnegative definite solution of the algebraic Riccati equation*

$$
\begin{aligned}
&\bar{U}\Big(B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\Big) \\
&\quad + \Big(B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\Big)^* \bar{U} \\
&\quad - \frac{2\theta}{\theta+2}\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U} \\
&\quad + \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A = 0
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
&B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A \\
&\quad - \frac{2\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}
\end{aligned}
\tag{5.3}
$$

is stable.

PROOF. Because of Lemma 4.1 we see that $B - (\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A$ is stable and therefore we see that $(B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A, \bar{\Pi}A^* + \Lambda\Sigma^*)$ is stabilizable and $(\sqrt{\frac{1}{\theta+2}}\Sigma^*(\Sigma\Sigma^*)^{-1}A, B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A)$ is detectable. Indeed, by setting $K = \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}A$, we see that

$$
\begin{aligned}
&B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A - (\bar{\Pi}A^* + \Lambda\Sigma^*)K \\
&\quad = B - (\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A
\end{aligned}
$$

is stable. Furthermore we see that for $K' = \frac{2}{\sqrt{\theta+2}}\Sigma^*(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)$,

$$B^* - \frac{\theta}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*) - \sqrt{\frac{1}{\theta+2}}A^*(\Sigma\Sigma^*)^{-1}\Sigma K'$$
$$= \big(B - (\bar{\Pi}A^*\Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\big)^*$$

is stable. Thus the present lemma follows from the results by Wonham [15] and Kucera [12]. □

REMARK 1. We can actually see more on the convergence in the above lemma since $\dot{\tilde{U}} \leq 0$ (cf. [5]). In fact, we have $\sup_{\delta T \leq t \leq (1-\varepsilon)T} |\tilde{U}(t;T) - \bar{U}| \to 0$ as $T \to \infty$ for each $\delta > 0$ and $0 < \varepsilon < 1$.

To study asymptotics of the solution U of (3.6) we shall introduce a general equation rather than specific one. For given continuous matrix valued functions $C(t)$, $D(t)$ and $R(t) \geq 0$ and a constant matrix $N > 0$ we consider the inhomogeneous Riccati equation

$$\begin{aligned} 0 &= \dot{K}_T + C(t)^*K_T + K_T C(t) \\ &\quad - K_T D(t) N^{-1} D(t)^* K_T + R(t)^* R(t), \qquad K_T(T) = 0. \end{aligned} \tag{5.4}$$

On asymptotics of the solution of this equation we have the following lemma.

LEMMA 5.2. *Assume that $C(t)$, $D(t)$ and $R(t)$ converge exponentially fast to $\bar{C}, \bar{D}, \bar{R}$ respectively as $t \to \infty$ and that $(\bar{C}, \bar{D})$ is stabilizable and $(\bar{R}, \bar{C})$ is detectable. Then there exists $\kappa > 0$, $\beta > 0$ and $T_* > 0$ such that for each $T > T_0 > T_*$ the solution of* (5.4) *on* $[T_0, T]$ *satisfies*

$$\begin{aligned} &\tilde{K}_T(t) + \kappa e^{-\beta T_0} K_T^-(t) \leq K_T(t) \\ &\qquad \leq \tilde{K}_T(t) + \kappa e^{-\beta T_0} K_T^+(t), \qquad t \in [T_0, T] \end{aligned} \tag{5.5}$$

where $\tilde{K}_T(t)$, $t \in [T_0, T]$, is the solution of

$$0 = \dot{\tilde{K}}_T + \bar{C}^*\tilde{K}_T + \tilde{K}_T\bar{C} - \tilde{K}_T\bar{D}N^{-1}\bar{D}^*\tilde{K}_T + \bar{R}^*\bar{R}, \qquad \tilde{K}_T(T) = 0 \tag{5.6}$$

and $K_T^-(t)$ and $K_T^+(t)$ are the solutions of

$$\begin{aligned} 0 &= \dot{K}_T^- + \bar{C}^*K_T^- + K_T^-\bar{C} - K_T^-\bar{D}N^{-1}\bar{D}^*\tilde{K}_T \\ &\quad - \tilde{K}_T\bar{D}N^{-1}\bar{D}^*K_T^- - I_n, \qquad K_T^-(T) = 0 \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} 0 &= \dot{K}_T^+ + \bar{C}^*K_T^+ + K_T^+\bar{C} - K_T^+\bar{D}N^{-1}\bar{D}^*\tilde{K}_T \\ &\quad - \tilde{K}_T\bar{D}N^{-1}\bar{D}^*K_T^+ + I_n, \qquad K_T^+(T) = 0, \end{aligned} \tag{5.8}$$

respectively.

PROOF. We first note that $K_T(t) \geq 0$. Let us take a matrix H such that $\bar{C} - \bar{D}H$ is stable and rewrite (5.4) by

$$0 = \dot{K}_T + \big(C(t) - D(t)H\big)^* K_T + K_T\big(C(t) - D(t)H\big)$$

$$- \big(D(t)^* K_T - NH\big)^* N^{-1}\big(D(t)^* K_T - NH\big) + H^* NH + R(t)^* R(t), \tag{5.9}$$

$$K_T(T) = 0.$$

Let $S(t)$ be a solution of

$$\begin{aligned} &\dot{S} + \big(C(t) - D(t)H\big)^* S + S\big(C(t) - D(t)H\big) \\ &\quad + H^* NH + R(t)^* R(t) = 0, \qquad S(T) = 0, \end{aligned} \tag{5.10}$$

then we have

$$(S - \dot{K}_T) + \big(C(t) - D(t)H\big)^*(S - K_T) + (S - K_T)\big(C(t) - D(t)H\big) \leq 0$$

and we see that

$$0 \leq K_T(t) \leq S(t).$$

Let $\mu(t)$ be the maximum of the real part of the eigenvalues of $C(t) - D(t)H$. Then, since $C(t) - D(t)H \to \bar{C} - \bar{D}H$ there exists T_1 such that $\mu(t) \leq \bar{\mu} < 0$, $t \geq T_1$. Therefore we have

$$\begin{aligned} \|S(t)\| &\leq c \int_t^T e^{-\bar{\mu}(s-t)} \|H^* NH + R(s)^* R(s)\| \, ds \\ &\leq c' \int_t^T e^{-\bar{\mu}(s-t)} \, ds, \qquad t \geq T_1. \end{aligned}$$

Thus we see that $S(t),\ t \geq T_1$, is uniformly bounded with respect to t and T, accordingly so is $K_T(t),\ t \geq T_1$, and also is $K_T(t),\ t \geq 0$.

We rewrite (5.4) by

$$\begin{aligned} 0 &= \dot{K}_T + \bar{C}^* K_T + K_T \bar{C} \\ &\quad - K_T \bar{D} N^{-1} \bar{D}^* K_T + \bar{R}^* \bar{R} + \tilde{R}(t), \qquad K_T(T) = 0. \end{aligned} \tag{5.11}$$

Since $K_T(t)$ is uniformly bounded and $C(t)$, $D(t)$ and $R(t)$ converges exponentially fast to $\bar{C}$, $\bar{D}$ and $\bar{R}$ respectively we see that there exist $\kappa, \beta > 0$ such that

$$-\frac{\kappa}{2} e^{-\beta t} I_n \leq \tilde{R}(t) \leq \frac{\kappa}{2} e^{-\beta t} I_n. \tag{5.12}$$

We note that under our assumptions $\tilde{K}_T(t)$ converges to a constant matrix $\bar{K}$ as $T - t \to \infty$ and $\bar{C} - \bar{D}N^{-1}\bar{D}^*\bar{K}$ is stable (cf. Appendix). We set

$$K_T^1(t) = \tilde{K}_T(t) + \kappa e^{-\beta T_0} K_T^-(t), \qquad t \geq T_0 > 0,$$

then it satisfies

$$0 = \dot{K}_T^1 + \bar{C}^* K_T^1 + K_T^1 \bar{C} - K_T^1 \bar{D} N^{-1} \bar{D}^* K_T^1 + \bar{R}^* \bar{R} - \kappa e^{-\beta T_0} I_n \\ + \kappa^2 e^{-2\beta T_0} Q_1(t), \qquad K_T^1(T) = 0, \tag{5.13}$$

where $Q_1(t) = Q_1(t;T) := K_T^- \bar{D} N^{-1} \bar{D}^* K_T^-(t)$. Note that $K_T^-(t)$ satisfies

$$\dot{K}_T^- + (\bar{C} - \bar{D} N^{-1} \bar{D}^* \tilde{K}_T)^* K_T^- + K_T^- (\bar{C} - \bar{D} N^{-1} \bar{D}^* \tilde{K}_T) - I_n = 0$$

and $\bar{C} - \bar{D} N^{-1} \bar{D}^* \tilde{K}_T(t) \to \bar{C} - \bar{D} N^{-1} \bar{D}^* \bar{K}$ as $T - t \to \infty$ and $\bar{C} - \bar{D} N^{-1} \bar{D}^* \bar{K}$ is stable. We can see that $K_T^-(t)$ is uniformly bounded with respect to t and T in a similar way to the above and therefore so is $Q_1(t;T)$. Thus we see that by taking sufficiently large $T_* > 0$,

$$\kappa^2 e^{-2\beta T_0} Q_1(t;T) < \frac{\kappa}{2} e^{-\beta T_0} I_n, \qquad t \geq T_0 \geq T_*.$$

Hence we obtain

$$-\kappa e^{-\beta T_0} I_n + \kappa^2 e^{-2\beta T_0} Q_1(t) \leq \tilde{R}(t), \qquad t \geq T_0.$$

Therefore by comparison theorem of Riccati differential equation we have

$$K_T^1(t) \leq K_T(t), \qquad t \in [T_0, T].$$

On the other hand, if we set

$$K_T^2(t) = \tilde{K}_T(t) + \kappa e^{-\beta T_0} K_T^+(t), \qquad t \geq T_0,$$

then it satisfies

$$0 = \dot{K}_T^2 + \bar{C}^* K_T^2 + K_T^2 \bar{C} - K_T^2 \bar{D} N^{-1} \bar{D}^* K_T^2 + \bar{R}^* \bar{R} \\ + \kappa e^{-\beta T_0} I_n + \kappa^2 e^{-2\beta T_0} Q_2(t), \qquad K_T^2(T) = 0, \tag{5.14}$$

where $Q_2(t) = Q_2(t;T) = K_T^+(t) \bar{D} N^{-1} \bar{D}^* K_T^+(t)$. Because of (5.12) we can see that

$$K_T(t) \leq K_T^2(t), \qquad t \in [T_0, T],$$

by comparison theorem of Riccati differential equation and we conclude the proof of the lemma. □

THEOREM 5.1. *Assume* (4.1), $\theta > 0$ *and* $\lim_{t\to\infty} r(t) = \bar{r}$. *Then for the solutions* $U(t;T)$, $g(t;T)$ *and* $k(t;T)$ *of equations* (3.6), (3.7) *and* (3.8) *respectively it follows that*

$$\lim_{T-t\to\infty,\, t\to\infty} U(t;T) = \bar{U}, \tag{5.15}$$

$$\lim_{T-t\to\infty,\, t\to\infty} g(t;T) = \bar{g}, \tag{5.16}$$

$$-\lim_{T-t\to\infty,\, t\to\infty} \dot{k}(t;T) = \rho(\theta), \tag{5.17}$$

where $\bar{U} \geq 0$ *is the solution of* (5.2), $\bar{g}$ *the one of*

$$\begin{aligned} &\Big\{B - \frac{2\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U} \\ &\quad - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\Big\}^* \bar{g} \\ &\qquad + \bar{U}b + \frac{1}{\theta+2}\{A - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}\}^*(\Sigma\Sigma^*)^{-1}(a - \bar{r}\mathbf{1}) = 0 \end{aligned} \tag{5.18}$$

and $\rho(\theta)$ *is defined by*

$$\begin{aligned} \rho(\theta) = r &+ \mathrm{tr}[\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)] \\ &- \theta\bar{g}^*(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} + 2\bar{g}^*b \\ &+ \frac{1}{\theta+2}\big(a - \bar{r}\mathbf{1} - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g}\big)^*(\Sigma\Sigma^*)^{-1} \\ &\quad \times \big(a - \bar{r}\mathbf{1} - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g}\big). \end{aligned} \tag{5.19}$$

PROOF. Equation (5.14) is a consequence of Lemma 5.1 and 5.2. In fact, if we set in (5.4) $C(t) = B - \frac{\theta}{\theta+2}(\Pi A^* + \Lambda\Sigma)(\Sigma\Sigma^*)^{-1}A$, $D(t) = \Pi A^* + \Lambda\Sigma^*$, $N^{-1} = \frac{2\theta}{\theta+2}(\Sigma\Sigma^*)^{-1}$ and

$$R(t) = \sqrt{\frac{1}{\theta+2}}\Sigma^*(\Sigma\Sigma^*)^{-1}A,$$

then it can be seen that Lemma 5.2 applies, taking into account Lemma 4.1 and Lemma 5.1.

As for (5.15), owing to Lemma 5.1

$$\begin{aligned} B &- \frac{2\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U} \\ &- \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A \end{aligned}$$

is stable and (5.17) has a solution $\bar{g}$. Therefore we can see that $g(t;T)$ is uniformly bounded and

$$g(t;T) \to \bar{g}, \qquad T - t \to \infty, t \to \infty$$

since

$$\begin{aligned} B &- \frac{2\theta}{\theta+2}(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)U \\ &- \frac{\theta}{\theta+2}(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A \end{aligned}$$

converges to a stable matrix

$$B - \frac{2\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}$$
$$- \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A$$

and

$$Ub + \frac{1}{\theta+2}\{A - \theta(A\Pi + \Sigma\Lambda^*)U\}^*(\Sigma\Sigma^*)^{-1}(a - \bar{r}\mathbf{1})$$

to

$$\bar{U}b + \frac{1}{\theta+2}\{A - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}\}^*(\Sigma\Sigma^*)^{-1}(a - \bar{r}\mathbf{1})$$

as $T - t \to \infty,\ t \to \infty$.

Since $U(t;T)$ converges to $\bar{U}$, $\Pi(t)$ to $\bar{\Pi}$ and $g(t;T)$ to $\bar{g}$ we conclude (5.16). □

REMARK 2. Because of Remark 1 we can see more on the above convergence. We have that $\sup_{\delta T \le t \le (1-\varepsilon)T} |U(t;T) - \bar{U}| \to 0$ as $T \to \infty$ for each $\delta > 0$ and $0 < \varepsilon < 1$ and therefore $\sup_{\delta T \le t \le (1-\varepsilon)T} |g(t;T) - \bar{g}| \to 0$ as $T \to \infty$ for each $\delta > 0$ and $0 < \varepsilon < 1$, $\sup_{\delta T \le t \le (1-\varepsilon)T} |\dot{k}(t;T) + \rho(\theta)| \to 0$ as $T \to \infty$ for each $\delta > 0$ and $0 < \varepsilon < 1$.

6. Infinite time horizon case.

THEOREM 6.1. (i) *Under the assumptions of Theorem* 5.1,

$$\sup_h J(v, x; h) \le \rho(\theta), \tag{6.1}$$

where $\rho(\theta)$ is a constant defined by (5.18).

(ii) *In addition to the conditions above we assume that*

$$\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U} < \frac{1}{\theta^2}A^*(\Sigma\Sigma^*)^{-1}A, \tag{6.2}$$

then

$$\bar{h}_t = \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}\{a - r\mathbf{1} - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} + [A - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}]\gamma_t\}$$

is optimal:

$$J(v, x; \bar{h}) = \sup_h J(v, x; h) = \rho(\theta).$$

PROOF. (i) We already know that

$$-\frac{2}{\theta T}\log \hat{E}[q^h(T)(1)] = -\frac{2}{\theta T}\log \hat{E}[\alpha_T(h)] \\ \le \frac{1}{T}\big(x^*U(0;T)x + 2g^*(0;T)x + k(0;T)\big),$$

where U, g and k are solutions of (3.6), (3.7) and (3.8) respectively. Since $U(t;T)$ and $g(t;T)$ are uniformly bounded we see that

$$\lim_{T\to\infty}\frac{1}{T}\big(x^*U(0;T)x + 2g(0;T)^*x\big) = 0.$$

On the other hand, for each $\delta > 0$ and $\varepsilon > 0$,

$$k(0;T) = -\int_0^{\delta T}\dot{k}(s;T)\,ds - \int_{\delta T}^{(1-\varepsilon)T}\dot{k}(s;T)\,ds - \int_{(1-\varepsilon)T}^{T}\dot{k}(s;T)\,ds,$$

$\lim_{T\to\infty} -\dot{k}(s;T) = \rho(\theta)$, uniformly on $\delta T \le s \le (1-\varepsilon)T$ and $\dot{k}(s;T)$ is bounded and therefore we have

$$\limsup_{T\to\infty}\left|\frac{k(0;T)}{T} - \rho(\theta)\right| \le c(\delta+\varepsilon).$$

Since $\delta > 0$, $\varepsilon > 0$ are arbitrary, we see that

$$\lim_{T\to\infty}\frac{k(0;T)}{T} = \rho(\theta).$$

(ii) Let us set

$$\bar{h}_t = \bar{h}(\gamma_t) = \frac{2}{\theta+2}(\Sigma\Sigma^*)^{-1}\{a - r\mathbf{1} - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} \\ + [A - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}]\gamma_t\},$$

where γ_t is the solution of (3.4). We define a probability measure $P^{\bar{h}}$ by

$$\left.\frac{dP^{\bar{h}}}{d\hat{P}}\right|_{\mathcal{G}_T} = \exp\left\{\int_0^T Q(\gamma_s,\bar{h}_s)^*\,dY_s - \frac{1}{2}\int_0^T Q(\gamma_s,\bar{h}_s)^*(\Sigma\Sigma^*)Q(\gamma_s,\bar{h}_s)\,ds\right\},$$

then

$$Z_t = Y_t - \left\langle Y_\cdot, \int_0^\cdot Q(\gamma_s,\bar{h}_s)^*\,dY_s\right\rangle_t \\ = Y_t - \int_0^t \Sigma\Sigma^* Q\big(\gamma_s,\bar{h}(\gamma_s)\big)\,ds$$

is a Brownian motion with covariance $\Sigma\Sigma^*$ under the probability measure $P^{\bar{h}}$ and γ_t can be written as

$$
\begin{aligned}
d\gamma_t = \{&B\gamma_t + b - (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\gamma_t + d) \\
&+ (\Pi A^* + \Lambda\Sigma^*)Q(\gamma_t, \bar{h}(\gamma_t))\}\,dt \\
&+ (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}\,dZ_t.
\end{aligned}
\tag{6.3}
$$

Then

$$
\hat{E}_x[\alpha_{T-t}(\bar{h})] = E_x^{P^{\bar{h}}}\left[\exp\left\{\frac{\theta}{2}\int_0^{T-t} \eta\big(\gamma_s, \bar{h}(\gamma_s); r; \theta\big)\,ds\right\}\right] \equiv u(t,x)
\tag{6.4}
$$

and u satisfies

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{1}{2}\mathrm{tr}[(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)D^2u] \\
&\qquad + \big[Bx + b - (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(Ax + d) \\
&\qquad\qquad\qquad + (\Pi A^* + \Lambda\Sigma^*)Q\big(x, \bar{h}(x)\big)\big]^* Du \\
&\qquad + \frac{\theta}{2}\eta(x, \bar{h}(x); r; \theta) = 0, \\
&u(T,x) = 1,
\end{aligned}
\tag{6.5}
$$

and therefore $\mu(t,x) \equiv -\frac{2}{\theta}\log u(t,x)$ satisfies

$$
\begin{aligned}
&\frac{\partial \mu}{\partial t} + \frac{1}{2}\mathrm{tr}\big[(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)D^2\mu\big] \\
&\qquad + \big[Bx + b - (\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(Ax + d) \\
&\qquad\qquad\qquad + (\Pi A^* + \Lambda\Sigma^*)Q\big(x, \bar{h}(x)\big)\big]^* D\mu \\
&\qquad - \frac{\theta}{4}(D\mu)^*(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)D\mu \\
&\qquad - \eta\big(x, \bar{h}(x); r; \theta\big) = 0, \qquad \mu(T,x) = 0,
\end{aligned}
\tag{6.6}
$$

provided that (6.4) has a finite value. On the other hand, if we have a solution $\mu(t,x)$ of (6.6), then $u(t,x) = e^{-\frac{\theta}{2}\mu(t,x)}$ satisfies (6.5) and (6.4) has a finite value. We set

$$
\bar{h}(x) \equiv \frac{2}{\theta+2}(\Sigma\Sigma)^{-1}(Hx + \zeta),
$$

$H = A - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U},\ \zeta = a - r\mathbf{1} - \theta(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g}$. Then, (6.6) has the solution having an explicit representation such that

$$
\mu(t,x) = x^*P(t)x + 2q(t)^*x + \kappa(t).
$$

If the following equations (6.7), (6.8) and (6.9) have solutions

$$
\begin{aligned}
&\dot{P} + P\left\{B - \frac{\theta}{\theta+2}(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}H\right\} \\
&\quad + \left\{B^* - \frac{\theta}{\theta+2}H^*(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)\right\}P \\
&\quad - \frac{2\theta}{\theta+2}P(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)P \\
&\quad + \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A \\
&\quad - \frac{\theta^2}{\theta+2}\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U} = 0, \qquad P(T) = 0;
\end{aligned} \tag{6.7}
$$

$$
\begin{aligned}
&\dot{q} + \left(B - \frac{\theta}{\theta+2}(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}H\right)^* q \\
&\quad - \theta P(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)q + Pb \\
&\quad - \frac{\theta}{\theta+2}P(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}\zeta \\
&\quad - \frac{\theta^2}{\theta+2}\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} \\
&\quad + \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) = 0, \qquad q(T) = 0;
\end{aligned} \tag{6.8}
$$

and

$$
\begin{aligned}
&\dot{\kappa} + \mathrm{tr}[P(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)] \\
&\quad - \theta q^*(\Pi A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)q + 2b^*q \\
&\quad - \frac{2\theta}{\theta+2}\zeta^*(\Sigma\Sigma^*)^{-1}(A\Pi + \Sigma\Lambda^*)q \\
&\quad - \left\{\frac{1}{\theta+2}\zeta^*(\Sigma\Sigma^*)^{-1}\zeta - r - \frac{2}{\theta+2}\zeta^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1})\right\} = 0, \\
&\qquad\qquad \kappa(T) = 0.
\end{aligned} \tag{6.9}
$$

Because of our assumption (6.2) we have a unique solution of the Riccati differential equation (6.7) and therefore linear equations (6.8) and (6.9) have always unique solutions.

To study asymptotics of the solution $P(t)$ of (6.7) we consider the equation

$$
\begin{aligned}
&\dot{\tilde{P}} + \tilde{P}\left\{B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}H\right\} \\
&\quad + \left\{B^* - \frac{\theta}{\theta+2}H^*(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\right\}\tilde{P} \\
&\quad - \frac{2\theta}{\theta+2}\tilde{P}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\tilde{P} \\
&\quad + \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A \\
&\quad - \frac{\theta^2}{\theta+2}\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U} = 0, \qquad \tilde{P}(T) = 0.
\end{aligned}
\tag{6.10}
$$

Set

$$
K_1 = B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}H,
$$

$$
\Gamma = \bar{\Pi}A^* + \Lambda\Sigma^*, \qquad N^{-1} = \frac{2\theta}{\theta+2}(\Sigma\Sigma^*)^{-1}
$$

and

$$
R^*R = \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}A - \frac{\theta^2}{\theta+2}\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U},
$$

then (6.10) reads

$$
\dot{\tilde{P}} + K_1^*\tilde{P} + \tilde{P}K_1 - \tilde{P}\Gamma N^{-1}\Gamma^*\tilde{P} + R^*R = 0. \tag{6.11}
$$

When setting $K = \theta(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}$ we see that

$$
\begin{aligned}
K_1 - \Gamma K = B &- \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A \\
&- \frac{2\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}
\end{aligned}
$$

is stable because of Lemma 5.1 and we see that (K_1, Γ) is stabilizable, and also that (R, K_1) is observable under assumption (6.2). Therefore we see that $\tilde{P}(t;T)$ converges to $\bar{P}$ as $T \to \infty$, where $\bar{P}$ is the unique nonnegative definite solution of the algebraic Riccati equation

$$
\bar{P} + K_1^*\bar{P} + \bar{P}K_1 - \bar{P}\Gamma N^{-1}\Gamma^*\bar{P} + R^*R = 0. \tag{6.12}
$$

On the other hand, by direct calculation we see that $\bar{U}$ satisfies (6.12) and $\bar{P} = \bar{U}$. Thus, owing to Lemma 5.2 we see that $P(t;T)$ is uniformly bounded and converges to $\bar{U}$ as $T - t \to \infty,\ t \to \infty$.

To study (6.8) we first note that

$$\begin{aligned}
&B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}H \\
&\quad - \theta(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{P} \\
&\qquad = B - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A \\
&\qquad\quad - \frac{2\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U}
\end{aligned}$$

and it is stable. Therefore the solution $q(t;T)$ of (6.8) is uniformly bounded and converges to $\bar{q}$ as $T-t \to \infty,\ t \to \infty$, where $\bar{q}$ is the solution of

$$\begin{aligned}
&\Big\{B - \frac{2\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{U} \\
&\qquad\qquad - \frac{\theta}{\theta+2}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}A\Big\}^* \bar{q} \\
&\quad + \bar{U}b - \frac{\theta}{\theta+2}\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}\zeta + \frac{1}{\theta+2}A^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1}) \\
&\quad - \frac{\theta^2}{\theta+2}\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} = 0
\end{aligned}$$

and it is seen to be identical to equation (5.17). Hence we get that $\bar{q} = \bar{g}$.

As a consequence of the above consideration we have

$$\begin{aligned}
&\lim_{T-t\to\infty,\ t\to\infty} -\dot{\kappa}(t;T) \\
&\quad = \operatorname{tr}\big[\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\big] \\
&\qquad - \theta\bar{g}^*(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} + 2\bar{g}^*b \\
&\qquad - \frac{2\theta}{\theta+2}\zeta(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} \\
&\qquad - \Big\{\frac{1}{\theta+2}\zeta^*(\Sigma\Sigma^*)^{-1}\zeta - r - \frac{2}{\theta+2}\zeta^*(\Sigma\Sigma^*)^{-1}(a - r\mathbf{1})\Big\} \\
&\quad = r + \operatorname{tr}\big[\bar{U}(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\big] \\
&\qquad - \theta\bar{g}^*(\bar{\Pi}A^* + \Lambda\Sigma^*)(\Sigma\Sigma^*)^{-1}(A\bar{\Pi} + \Sigma\Lambda^*)\bar{g} + 2\bar{g}^*b \\
&\qquad + \zeta^*(\Sigma\Sigma^*)^{-1}\zeta \\
&\quad = \rho(\theta).
\end{aligned}$$

Therefore, in the same way as the proof of (i), we have

$$\lim_{T\to\infty} \frac{\kappa(0;T)}{T} = \rho(\theta),$$

which implies that

$$-\frac{2}{\theta T}\log u(0,x) = \frac{1}{T}\mu(0,x) \to \rho(\theta), \qquad T \to \infty. \qquad \square$$

APPENDIX

DEFINITION A.1. (i) The pair (L, M) of $n \times n$ matrix L and $n \times l$ matrix M is said stabilizable if there exists $l \times n$ matrix K such that $L - MK$ is stable.

(ii) The pair (L, F) of $l \times n$ matrix L and $n \times n$ matrix F is called detectable if (F^*, L^*) is stabilizable.

Let us consider the Riccati differential equation

$$\dot{P} + K_1^* P + PK_1 - P\Lambda N^{-1}\Lambda^* P + C^*C = 0, \qquad P(T) = 0. \tag{A.1}$$

Then, the following theorem would be well known in engineering.

THEOREM A.2 [15, 12]. *Assume that* $N > 0$ *and* (K_1, Λ) *is stabilizable, then for the solution of* (A.1) $\exists \lim_{T\to\infty} P(t;T) \equiv \lim_{T\to\infty} P(t) \equiv \tilde{P}$ *and* $\tilde{P}$ *satisfies the algebraic Riccati equation*

$$K_1^*\tilde{P} + \tilde{P}K_1 - \tilde{P}\Lambda N^{-1}\Lambda^*\tilde{P} + C^*C = 0. \tag{A.2}$$

Moreover, if (C, K_1) *is detectable* $K_1^* - \tilde{P}\Lambda N^{-1}\Lambda^*$ *is stable and the nonnegative definite solution* $\tilde{P}$ *of* (A.2) *is unique*.

DEFINITION A.2. (i) The pair (K, L) of $n \times n$ matrix K and $n \times l$ matrix L is said controllable if $n \times nl$ matrix $(L, KL, K^2L, \ldots, K^{n-1}L)$ has rank n.

(ii) The pair (L, K) of $l \times n$ matrix and $n \times n$ matrix is said observable if (K^*, L^*) is controllable.

It is known that if the pair (K, L) of matrices is controllable (resp. observable) then it is stabilizable (resp. detectable) (cf. [15]).

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