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SAMPLE PATH LARGE DEVIATIONS AND CONVERGENCE PARAMETERS

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In this paper we prove the local sample path large deviation estimates for a general class of Markov chains with discontinuous statistics. The local rate function is represented in terms of the convergence parameter of associated local transform matrices. Our method is illustrated by the case of perturbated random walks in $\mathbb{Z}^d$.

1. Introduction. This paper is devoted to the representation of the local sample path large deviation rate function for Markov processes with a discontinuous statistical behavior.

For the moment, there are few general results in this domain. Dupuis and Ellis [12] established the sample path large deviation principle for lattice-based jump Markov processes describing queueing systems. They give a quite general approach, that can be used to prove the sample path large deviation principle for a large class of Markov processes on $\mathbb{R}^d$ having constant or smooth statistical behavior on the regions separated by an arbitrary number of intersecting hyperplanes across which the statistical behavior can change discontinuously. In order to obtain the local large deviation estimates, Dupuis and Ellis described the local large deviation probabilities in terms of the minimal cost functions of associated stochastic optimal control problems, and studied the limits of these probabilities by using a sub-additivity-type argument. Such a method leads to a rather implicit description of the rate function.

In [8], Dupuis, Ellis and Weiss proved an explicit general upper large deviation bound. In this paper, the authors conjectured that their upper bound is tight, i.e the lower large deviation bound should be satisfied with the same rate function under some general conditions. The corresponding general lower bound has not been proved. It is known that this conjecture is wrong in general (see Alanyali and Hajek [1] or Blinovskii and Dobrushin [3] for example).

The sample path large deviation principle was proved and an explicit representation of the rate function were obtained in [1, 3, 10, 15, 21] for processes whose statistical behavior can be discontinuous across one $(d-1)$-dimensional hyperplane, or more generally across a smooth $(d-1)$-dimensional interface in $\mathbb{R}^d$.

The rate function is much more difficult to evaluate when the discontinuity in the transition mechanism of the process occurs across an arbitrary number of intersecting hyperplanes. This is the case for the Markov processes describ-

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ing queueing networks for example. Several techniques have been developed to resolve this problem in particular cases. For Markov processes describing tandem queues, the contraction principle can be applied (see [7, 23]). For the Markov processes describing Jackson networks, different approaches have been proposed. Dupuis, Ishii and Soner [9] used the method of viscosity solutions of Hamilton-Jacobi equations. In Ignatiouk [14], a closed form expression for the rate function is obtained by using the classical method of exponential change of measure and the explicit representation of the related fluid limits. Atar and Dupuis [2] evaluated the rate function for a more general class of networks for which the associated Skorohod problem has some regularity properties. In [5], Delcoigne and de La Fortelle obtained an explicit representation of the rate function for Polling Systems. All these methods use some special properties of the processes under study, and therefore do not seem to be generalizable.

Before describing our results, we recall the definition of the sample path large deviation principle and the main points of Dupuis and Ellis' approach [12].

For $\tau > 0$, the sequence of Markov processes $Z_n(t)$, $t \in [0, \tau]$ on $\mathbb{R}^d$ is said to satisfy the sample path large deviation principle with the good rate function $I_\tau(\cdot) : \mathscr{D}([0, \tau], \mathbb{R}^d) \to \mathbb{R}_+$ iff the following assertions hold:

(i) for any $c > 0$ the level set $\{\varphi \ : I_\tau(\varphi) \le c\}$ is a compact subset of $\mathscr{D}([0, \tau], \mathbb{R}^d)$;

(ii) for every open subset $\mathscr{O}$ of $\mathscr{D}([0, \tau], \mathbb{R}^d)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n(\cdot) \in \mathscr{O}) \ge -\inf_{\varphi\in\mathscr{O}} I_\tau(\varphi);$$

(iii) for every closed subset F of $\mathscr{D}([0, \tau], \mathbb{R}^d)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n(\cdot) \in F) \le -\inf_{\varphi\in F} I_\tau(\varphi).$$

$\mathscr{D}([0, \tau], \mathbb{R}^d)$ denotes the space of all functions from $[0, \tau]$ to $\mathbb{R}^d$ that are continuous from the right and have the limits from the left. This space is endowed with Skorohod topology.

For a Markov process $Z(t)$ describing a queueing system, $Z_n(t)$ is usually a renormalized process defined by

$$Z_n(t) = Z(nt)/n$$

given that $Z_n(0) = z_n$ where $z_n \to z$ as $n \to \infty$ for some $z \in \mathbb{R}^d_+$. The statistical behavior of such a model is discontinuous at the boundary set $\{z : z_i = 0, \text{ for some } 1 \le i \le d\}$.

The first step of Dupuis and Ellis' approach [12] consists in proving the local large deviation estimates for tubes centered at linear paths $\varphi(t) = z + vt$, $t \in [0, \tau]$, $z, v \in \mathbb{R}^d$ with $\tau > 0$ small enough. To perform this step, Dupuis and Ellis described the local large deviation behavior of the process in a neighborhood of $z \in \mathbb{R}^d$ in terms of the associated local process $\big(X(t), Y(t)\big)$. This is a

Markov process on $\mathbb{Z}^N \times E$ with $N \le d$ and $E \subseteq \mathbb{Z}^{d-N}$ depending on z. The transition probabilities of this Markov process are invariant with respect to the translations on the first coordinate and hence, following usual terminology (see Ney and Nummelin [17] for example) this is a Markov-additive process with additive part $X(t)$ on $\mathbb{Z}^N$ and with Markovian part $Y(t)$ on E (the Markovian part $Y(t)$ is a Markov process on E). For these Markov-additive processes, Dupuis and Ellis proved the lower large deviation estimate

$$\begin{aligned}(1.1)\qquad &\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{(0,0)}\left(\sup_{t\in[0,n\tau]}(|X(t)-vt|+|Y(t)|)\le\delta n\right)\\ &\quad\ge -\tau L(z,v)\end{aligned}$$

and the upper large deviation estimate

$$\begin{aligned}(1.2)\qquad &\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{(0,0)}\left(\sup_{t\in[0,n\tau]}(|X(t)-vt|+|Y(t)|)\le\delta n\right)\\ &\quad\le -\tau L(z,v)\end{aligned}$$

with the same local rate function $\tau L(z, v)$, where $|\cdot|$ denotes the Euclidean norm, and $\mathbb{P}_{(x,y)}$ is the distribution of the Markov process $(X(t), Y(t))$ starting at the point (x, y).

Under the assumptions of [12], the above estimates imply the local large deviation estimates for tubes centered at linear paths $z+vt$, $t \in [0, \tau]$ with the local rate function $\tau L(z, v)$ for the original Markov processes (see the proof of Proposition 3.7 and Proposition 5.1 in [12]). Using the last estimates together with the Markov property of the process, Dupuis and Ellis deduced the local large deviation estimates for tubes centered at piecewise linear, continuous paths $\varphi : [0, \tau] \to \mathbb{R}^d$ with the local rate function

$$\tilde{I}_\tau(\varphi) = \int_0^\tau L(\varphi(t), \dot{\varphi}(t))dt.$$

This is the second step of the proof. The third step completes the proof of the full large deviation principle by using an approximation argument. The rate function $I_\tau(\cdot)$ of the full large deviation principle is the lower semi-continuous regularization of $\tilde{I}_\tau(\cdot)$.

To identify the rate function of the full sample path large deviation principle, it is sufficient therefore, to identify the function $L(z, \cdot)$ satisfying inequalities (1.1) and (1.2) for every local Markov-additive process.

In the present paper, we show that the function $L(z, v)$ can be described in terms of the convergence parameters of associated transform matrices. This gives a good representation of the rate function because the properties of the convergence parameter are well known. The description of the convergence parameter in terms of ρ-superharmonic functions and its approximation by Perron-Frobenius eigenvalues allow to evaluate the rate function in many particular cases. In a different context, such a representation of the rate function has been obtained for the large deviations of additive functionals of Markov chains(see Ney and de Acosta [4] and Ney and Nummelin [17, 18]).

In order to represent the function $L(z, v)$ in terms of the convergence parameters, we propose an alternative proof of the local large deviation estimates (1.1) and (1.2). In this paper we consider discrete time Markov-additive processes $(X(t), Y(t))$ under quite general assumptions. For Markov processes with continuous time, our results can be extended in a straightforward way.

To prove of the lower estimate (1.1), we use a truncation argument and an approximation of the convergence parameter by Perron-Frobenius eigenvalues. The upper estimate (1.2) is proved via the method of the change of measure associated with ρ-superharmonic functions.

Our results show that to evaluate the function $L(z, v)$, one has to identify the infimum over all λ for which the corresponding local transform matrix has a positive e^{λ}-superharmonic function. In some particular cases, it is sufficient to consider exponential e^{λ}-superharmonic functions. When this is the case, the general upper large deviation bound of Dupuis Ellis and Weiss [8] is tight.

Two examples illustrate our results. The first one is the case when there is one discontinuity along an hyperplane of codimension one. We consider here a reflected random walk on $\mathbb{Z}^N \times \mathbb{Z}_+$. The explicit expression of the rate function for such a random walk was obtained in [15, 21]. We show how this result can be proved with our approach (for this example, it is sufficient to consider the exponential e^{λ}-superharmonic functions).

The second example concerns the local perturbation of a homogeneous random walk on $\mathbb{Z}^d$. To identify the rate function in this case, the existing methods can not be applied. With our approach the rate function is easily evaluated. For this example, the general large deviation upper bound of Dupuis, Ellis and Weiss [8] is not tight.

2. The main results. Let E be an arbitrary countable set equipped with an integer-valued metric $\text{dist}(\cdot, \cdot)$, and let $(X(t), Y(t))$ be a discrete time Markov chain on $\mathbb{Z}^N \times E$ with transition probabilities being invariant with respect to the translations on $x \in \mathbb{Z}^N$:

$$p((x, y), (x', y')) = p((0, y), (x' - x, y')) \qquad \text{for all } x, x' \in \mathbb{Z}^N \text{ and } y, y' \in E.$$

A Markov process $(X(t), Y(t))$ satisfying the above condition is usually called a Markov-additive process, $X(t)$ is its additive part, $Y(t)$ is a Markovian part (see Ney and Nummelin [17] for example). The Markovian part $Y(t)$ is a Markov chain on E with transition probabilities

$$p(y, y') = \sum_{x \in \mathbb{Z}^N} p\left((0, y), (x, y')\right), \qquad y, y' \in E.$$

The matrix $\mathscr{P}(\alpha) = (\mathscr{P}(\alpha; y, y');\ y, y' \in E)$ with

$$\mathscr{P}(\alpha; y, y') = \mathbb{E}_{(0,y)}\left(\exp\{\langle \alpha, X(1)\rangle\}\, \mathbb{1}_{\{Y(1)=y'\}}\right), \quad y, y' \in E, \alpha \in \mathbb{R}^N,$$

is usually called a transform matrix of the Markov-additive process $(X(t), Y(t))$. ($\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^N$.)

ASSUMPTION 1. We will assume that:

(i) the Markov chain $(Y(t))$ is irreducible;
(ii) the coefficients of the transform matrix $\mathscr{P}(\alpha)$ are finite for all $\alpha \in \mathbb{R}^N$.

Under the above assumption, the transform matrix $\mathscr{P}(\alpha)$ is irreducible and its convergence parameter $\rho(\alpha)$ is defined as follows.

DEFINITION 1. The convergence parameter $\rho(\alpha)$ of $\mathscr{P}(\alpha)$ is the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \mathscr{P}^{(n)}(\alpha; y, y')\rho^n,$$

where $\mathscr{P}^n(\alpha) = \big(\mathscr{P}^{(n)}(\alpha; y, y');\ y, y' \in E\big)$, $n \geq 0$, is the nth iterate of the matrix $\mathscr{P}(\alpha)$:

$$\mathscr{P}^{(n)}(\alpha; y, y') = \mathbb{E}_{(0,y)}\left(\exp\{\langle \alpha, X(n)\rangle\}\, \mathbb{1}_{\{Y(n)=y'\}}\right), \qquad y, y' \in E.$$

This definition does not depend on the choice of $y, y' \in E$ (see Seneta [20], for example).

ASSUMPTION 2. We will suppose moreover, that there exist $y_0 \in E$ and a strictly positive function $\gamma(\alpha, \rho)$ defined for all $\alpha \in \mathbb{R}^N$ and $0 < \rho < \rho(\alpha)$ such that for every $y \in E$,

$$\sum_{n=1}^{\infty} \mathscr{P}^{(n)}(\alpha; y, y_0)\rho^n \geq (\gamma(\alpha, \rho))^{\operatorname{dist}(y, y_0)}.$$

The last assumption is not very restrictive. For example, the inequality

$$\mathscr{P}^{(n)}(\alpha; y, y_0) \geq p^{(n)}((0, y), (0, y_0))$$

being verified for all $\alpha \in \mathbb{R}^N$, $y, y_0 \in E$ and $n \geq 0$, the above assumption holds if for every $y \in E$ there exists $n \geq 1$ such that $n \leq C \operatorname{dist}(y, y_0)$ and

$$p^{(n)}((0, y), (0, y_0)) \geq \gamma^n$$

with some constants $C > 0$ and $\gamma > 0$ not depending on $y \in E$. This condition is always satisfied when the communication condition of Dupuis and Ellis [12, 11] holds.

DEFINITION 2. Define the function $\lambda(\alpha)$, $\alpha \in \mathbb{R}^N$, by setting $\lambda(\alpha) = -\log \rho(\alpha)$ and let $\lambda^*(\cdot)$ be the convex conjugate of the function $\lambda(\cdot)$:

$$\lambda^*(v) = \sup_{\alpha \in \mathbb{R}^N} \{\langle \alpha, v\rangle - \lambda(\alpha)\}.$$

The main result of our paper is the following theorem.

THEOREM 1. *Under Assumption* 1, *the lower large deviation estimate*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}(|X(t)-vt|+\operatorname{dist}(Y(t), y_0))<\delta n\right) \geq -\tau\lambda^*(v) \tag{2.1}$$

holds for all $v \in \mathbb{R}^N$, *where* $[[0, n\tau]] = [0, n\tau] \cap \mathbb{Z}$. *Suppose moreover that Assumption* 2 *is verified*, *then the upper large deviation estimate*

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}(|X(t)-vt|+\operatorname{dist}(Y(t),y_0))<\delta n\right) \le -\tau\lambda^*(v) \tag{2.2}$$

holds also for all $v \in \mathbb{R}^N$.

To prove this theorem, we use the following fundamental properties of convergence parameter (see Seneta [20] for example):

(a) $\lambda(\alpha) = -\log\rho(\alpha)$ is the infimum over all λ for which there exists a non-negative function f_λ on E such that $f_\lambda \not\equiv 0$ and

$$\mathscr{P}(\alpha)f_\lambda \le e^\lambda f_\lambda.$$

For $\lambda > -\log\rho(\alpha)$, the above inequality holds with

$$f_\lambda(y) = \sum_{t=0}^{\infty}\mathscr{P}^{(t)}(\alpha, y, y_0)e^{-\lambda t}, \quad y \in E,$$

where $y_0 \in E$ is fixed. A function f_λ satisfying this inequality, is usually called a e^λ-superharmonic function relative to $\mathscr{P}(\alpha)$.

(b) If the set E is finite, $\rho^{-1}(\alpha)$ is the Perron-Frobenius (i.e., maximal real) eigenvalue of the matrix $\mathscr{P}(\alpha)$; otherwise, $\rho^{-1}(\alpha)$ is the supremum of the Perron-Frobenius eigenvalues of the finite irreducible truncations of $\mathscr{P}(\alpha)$.

To prove the upper estimate (2.2), we use the change of measure associated with the e^λ-superharmonic functions f_λ. The traditional exponential change of measure is ineffective because of our general framework.

The proof of the lower estimate (2.1) uses the second property (b). When the set E is finite and $N = 1$, the lower estimate (2.1) is a consequence of a result by Mogulskii [16]. We extend it to the case of sub-stochastic Markov-additive processes on $\mathbb{Z}^N \times K$ for finite subsets $K \subset E$. Using this extension for finite irreducible truncations of the Markov chain $(Y(t))$ we obtain the lower estimate (2.1) for all $v \in \mathbb{R}^N$ which belongs to the relative interior of the set $\{v : \lambda^*(v) < +\infty\}$. To extend this estimate for an arbitrary $v \in \mathbb{R}^N$, we use the upper semi-continuity of the left hand side in (2.1) with respect to v.

Section 3 is devoted to the properties of the convergence parameter and the function $\lambda(\cdot)$. Theorem 1 is proved in Section 4. In Section 5, we consider the case when $E \subset \mathbb{Z}^k$ and we give a general rough upper estimate $\hat{\lambda}(\cdot)$ for $\lambda(\cdot)$. This upper estimate corresponds to the general upper large deviation bound proved by Dupuis et al. in [8].

In Section 6, we apply Theorem 1 to identify the sample path large deviation rate function for a random walk on $\mathbb{Z}^N \times \mathbb{Z}_+$. The dynamic of this process is discontinuous at the boundary set $\{(x, y) \in \mathbb{R}^N \times \mathbb{R}_+ : y = 0\}$. The explicit expression of the rate function for this random walk was obtained in [15] by using a careful analysis of the related fluid limits. We give another very simple proof of this result by showing that in this case, the rough upper estimate $\hat{\lambda}(\cdot)$ is tight, i.e the identity $\lambda(\cdot) = \hat{\lambda}(\cdot)$ holds.

In Section 7, we use Theorem 1 to evaluate the sample path large deviation rate function for a finite perturbation of a homogeneous random walk on $\mathbb{Z}^k$. Here, the discontinuity occurs only in a neighborhood of $0 \in \mathbb{R}^k$. The results of this section show that the general large deviation bound [8] is sensitive to local perturbations: while this upper bound is clearly tight for homogeneous random walk on $\mathbb{Z}^k$, this is not necessarily true after a slight local perturbation.

3. The properties of the convergence parameter. In this section we recall some properties of the convergence parameters and we deduce from them the properties of the function $\lambda(\cdot)$.

We begin with the definition of the convergence parameter $\rho(\alpha)$.

Recall that the Markov chain $(Y(t))$ is irreducible by assumption. This implies that the transform matrix $\mathscr{P}(\alpha)$ is also irreducible: for every $y, y' \in E$ there exists $t \in \mathbb{N}$ such that $\mathscr{P}^{(t)}(\alpha, y, y') > 0$. In this case, the power series

$$\sum_{k \geq 0} \mathscr{P}^{(k)}(\alpha, y, y') z^k \tag{3.1}$$

either converge or diverge simultaneously for all $y, y' \in E$ (see Theorem 6.1 in Seneta [20] for example). The common convergence radius $\rho(\alpha)$ of the the power series (3.1) is usually called a convergence parameter of the matrix $\mathscr{P}(\alpha)$.

The value $\lambda(\alpha) = -\log \rho(\alpha)$ can be defined by setting

$$\lambda(\alpha) = \limsup_{t \to \infty} \frac{1}{t} \log \mathscr{P}^{(t)}(\alpha, y, y'). \tag{3.2}$$

The limit at the right hand side of the above relation does not depend on $y, y' \in E$.

There is another characterization of the function $\lambda(\alpha) = -\log \rho(\alpha)$ in terms of ρ-superharmonic functions and ρ-superharmonic measures relative to $\mathscr{P}(\alpha)$.

A function $f : E \to \mathbb{R}$ is called a ρ-superharmonic function relative to $\mathscr{P}(\alpha)$ if

$$\mathscr{P}(\alpha) f(y) \leq \rho f(y) \qquad \forall y \in E. \tag{3.3}$$

Similarly, a measure ν on E is called a ρ-superharmonic measure relative to $\mathscr{P}(\alpha)$ if

$$\nu \mathscr{P}(\alpha)(y) \leq \rho \nu(y) \qquad \forall y \in E.$$

Given $\lambda \in \mathbb{R}$ denote by $C_\lambda^r(\alpha)$ the set of all non-negative e^λ-superharmonic functions relative to $\mathscr{P}(\alpha)$ and let $C_\lambda^l(\alpha)$ be the set of all non-negative e^λ-superharmonic measures relative to $\mathscr{P}(\alpha)$.

PROPOSITION 1.

$$\lambda(\alpha) = \inf \left\{ \lambda : C_\lambda^l(\alpha) \neq \{0\} \right\} = \inf \left\{ \lambda : C_\lambda^r(\alpha) \neq \{0\} \right\}.$$

PROOF. The first identity follows from parts (c) and (d) of Theorem 6.3 in the book of Seneta [20], and the second identity can be proved similarly.

In particular, for $\lambda > \lambda(\alpha)$, the function

$$f(y) = \sum_{n=0}^{\infty} \mathscr{P}^{(n)}(\alpha; y, y_0) e^{-\lambda n}, \qquad y \in E,$$

is a non-negative e^{λ}-superharmonic function relative to $\mathscr{P}(\alpha)$ for every $y_0 \in E$. We will use this property to prove the upper estimate (2.2) and also to evaluate the function $\lambda(\alpha)$ in Sections 6 and 7.

Another possible characterization of the function $\lambda(\alpha) = -\log \rho(\alpha)$ can be given by using the Perron-Frobenius eigenvalues of finite irreducible truncations of the matrix $\mathscr{P}(\alpha)$.

When the set E is finite, the relation (3.2) together with Perron-Frobenius theorem implies that $\rho^{-1}(\alpha) = \exp \lambda(\alpha)$ is the Perron-Frobenius (i.e., maximal real) eigenvalue of the matrix $\mathscr{P}(\alpha)$. When the set E is infinite, the value $\rho^{-1}(\alpha) = \exp \lambda(\alpha)$ can be approximated by the Perron-Frobenius eigenvalues as follows.

Denote by $\mathscr{K}$ the collection of the all finite subsets K of E for which the restriction of the Markov chain $(Y(t))$ on K is irreducible. For $K \in \mathscr{K}$, the matrix

$$\mathscr{P}_K(\alpha) = (\mathscr{P}(\alpha, y, y'),\, y, y' \in K)$$

is irreducible for all $\alpha \in \mathbb{R}^N$. Denote by $\rho_K(\alpha)$ its convergence parameter ($\rho_K^{-1}(\alpha)$ is the Perron-Frobenius eigenvalue of $\mathscr{P}_K(\alpha)$) and let $\lambda_K(\alpha) = -\log \rho_K(\alpha)$.

PROPOSITION 2. *The collection of the functions* $\{\lambda_K(\cdot), K \in \mathscr{K}\}$ *is increasing with respect to* K *and for every* $\alpha \in \mathbb{R}^N$,

$$\lambda(\alpha) = \sup_{K \in \mathscr{K}} \lambda_K(\alpha). \tag{3.4}$$

PROOF. Under some additional assumptions on the transform matrix $\mathscr{P}(\alpha)$, this proposition is a consequence of Theorem 6.3 from the book of Seneta [20]. In general case, the same arguments as in [20] show that the collection of the functions $\{\lambda_K(\cdot), K \in \mathscr{K}\}$ is increasing with respect to K and the identity (3.4) can be verified by the following way.

For $K \in \mathscr{K}$, Perron-Frobenius theorem proves that

$$\lambda_K(\alpha) = \limsup_{n \to \infty} \frac{1}{n} \log \mathscr{P}_K^{(n)}(\alpha; y, y') \qquad \forall y, y' \in K \tag{3.5}$$

where

$$\mathscr{P}_K^{(t)}(\alpha; y, y') = \mathbb{E}_{(0,y)} \left(\exp\{\langle \alpha, X(t)\rangle\} \mathbb{1}_{\{Y(t)=y' \text{ and } Y(s)\in K\ \forall s \le t\}}\right).$$

Using the identity (3.5) for $y = y' \in K$ together with the inequality

$$\mathscr{P}_K^{(nt)}(\alpha; y, y) \ge \left(\mathscr{P}_K^{(t)}(\alpha; y, y)\right)^n$$

we obtain

$$\lambda_K(\alpha) \geq \limsup_{n\to\infty} \frac{1}{nt} \log \mathscr{P}_K^{(tn)}(\alpha; y, y) \geq \frac{1}{t} \log \mathscr{P}_K^{(t)}(\alpha; y, y) \tag{3.6}$$

for all $t \geq 0$. Consider now an increasing sequence (K_n) of finite subsets of E such that $y \in K_0$, $K_n \in \mathscr{K}$ for all n and $\cup_n K_n = E$. Then by monotone convergence theorem, $\mathscr{P}_{K_n}^{(t)}(\alpha; y, y) \to \mathscr{P}^{(t)}(\alpha; y, y)$ as $n \to \infty$ for every $t \geq 0$. This proves that for every $t \geq 0$,

$$\sup_{K \in \mathscr{K}} \mathscr{P}_K^{(t)}(\alpha; y, y) = \mathscr{P}^{(t)}(\alpha; y, y)$$

and hence, the last inequality in (3.6) implies that

$$\sup_{K \in \mathscr{K}} \lambda_K(\alpha) \geq \frac{1}{t} \log \mathscr{P}^{(t)}(\alpha; y, y).$$

Comparison of the above relation with (3.2) yields that $\sup_{K\in\mathscr{K}} \lambda_K(\alpha) \geq \lambda(\alpha)$. Recall now that $\lambda(\alpha) \geq \lambda_K(\alpha)$ for every $K \in \mathscr{K}$ and hence, relation (3.4) is verified.

Notice finally that the above sequence (K_n) exists because the Markov chain $(Y(t))$ is irreducible. Indeed, consider the transition probabilities of the Markov chain $(Y(t))$

$$p(y, y') = \sum_{x \in \mathbb{Z}^N} p\left((0, y), (x, y')\right), \qquad y, y' \in E.$$

Given $y_0, y \in E$, let us choose a sequence of points $V(y) = \{y_0, \ldots, y_m\}$ with $y_{m+1} = y_0$ such that $p(y_l, y_{l+1}) > 0$ for every $l = 0, \ldots, m$ and $y_l = y$ for some $1 \leq l \leq m$. Now, let (K'_n) be an arbitrary increasing sequence of finite subsets of E such that $\cup_n K'_n = E$. Then the sequence of sets

$$K_n = \bigcup_{y \in K'_n} V(y), \qquad n \geq 0,$$

is also increasing, $\cup_n K_n = E$ and by construction, the matrix $(p(y, y');\ y, y' \in K_n)$ is irreducible, that is, $K_n \in \mathscr{K}$ for every $n \geq 0$. $\square$

Before formulating the next property of the function $\lambda(\cdot)$ we recall some properties of the functions $\lambda_K(\alpha)$.

LEMMA 1. *For every $K \in \mathscr{K}$, the function $\lambda_K(\cdot)$ is convex and infinitely differentiable on $\mathbb{R}^N$.*

PROOF. Indeed, let $K \in \mathscr{K}$. Then the matrix $\mathscr{P}_K(\alpha)$ is irreducible and hence, by Perron-Frobenius theorem, its maximal real eigenvalue $\rho_K^{-1}(\alpha)$ is finite and strictly positive for all $\alpha \in \mathbb{R}^N$. This implies that the function $\lambda_K(\cdot) = -\log \rho_K(\alpha)$ is finite everywhere on $\mathbb{R}^N$. Furthermore, the function $\alpha \to \mathscr{P}_K^{(t)}(\alpha; y, y')$ being convex for every $t \in \mathbb{N}$, the function $\lambda(\cdot)$ is also convex as a limit of convex functions (3.5). Finally, the Perron-Frobenius eigenvalue

$\rho_K^{-1}(\alpha)$ of the matrix $\mathscr{P}_K(\alpha)$ is a simple root of its characteristic equation for all $\alpha \in \mathbb{R}^N$, and under Assumption 1, the coefficients of the matrix $\mathscr{P}_K(\alpha)$ are analytic with respect to α everywhere on $\mathbb{C}^N$. Using therefore, the implicit function theorem for analytic functions we conclude that the function $\rho_K(\cdot)$ is infinitely differentiable on $\mathbb{R}^N$. Since $\rho_K(\cdot)$ is strictly positive, this proves that the function $\lambda_K(\cdot) = \log \rho_K(\cdot)$ is also infinitely differentiable on $\mathbb{R}^N$. □

Lemma 1 and Proposition 2 imply the following statement.

LEMMA 2. *The function $\lambda(\cdot)$ is a closed convex proper function on $\mathbb{R}^N$.*

PROOF. Indeed, $\lambda(\cdot)$ convex and closed on $\mathbb{R}^N$ as a supremum of closed convex functions $\lambda_K(\cdot)$, $K \in \mathscr{K}$. To show that it is proper it is sufficient to notice that $\lambda(\cdot) \not\equiv +\infty$ because $\lambda(0) \le 0$, and $\lambda(\alpha) \ge \lambda_K(\alpha) > -\infty$ for all $\alpha \in \mathbb{R}^N$. □

Consider now the convex conjugates of the functions $\lambda(\cdot)$ and $\lambda_K(\cdot)$:

$$\lambda^*(v) = \sup_{\alpha \in \mathbb{R}^N} \{\langle \alpha, v\rangle - \lambda(\alpha)\} \quad \text{and} \quad \lambda_K^*(v) = \sup_{\alpha \in \mathbb{R}^N} \{\langle \alpha, v\rangle - \lambda_K(\alpha)\}.$$

The following lemma relates the functions $\lambda_K^*(\cdot)$ and $\lambda^*(\cdot)$.

LEMMA 3. *For every $v \in \mathbb{R}^N$ which belongs to the relative interior* $\mathrm{ri}(\mathrm{dom}\,\lambda^*)$ *of the set* $\mathrm{dom}\,\lambda^* = \{v \in \mathbb{R}^N : \lambda^*(v) < +\infty\}$,

$$\lambda^*(v) = \inf_{K \in \mathscr{K}} \lambda_K^*(v).$$

PROOF. We begin our proof by showing that the function $\overline{\lambda}(v) = \inf_{K\in\mathscr{K}} \lambda_K^*(v)$ is convex and proper.

The collection of the functions $\lambda_K(\cdot)$ being increasing with respect to K, the collection of the functions $\lambda_K^*(\cdot)$ is decreasing with respect to K. Consider an increasing sequence of finite subsets K_n of E such that $\cup_n K_n = E$ and $K_n \in \mathscr{K}$ for all n (the existence of such a sequence was verified in the proof of Proposition 2). Then

$$\overline{\lambda}(\cdot) = \lim_{n\to\infty} \lambda_{K_n}^*(\cdot)$$

and hence the function $\overline{\lambda}(\cdot)$ is convex as a limit of convex functions.

The function $\overline{\lambda}(\cdot)$ is proper because $\lambda^*(\cdot) \le \overline{\lambda}(\cdot) \le \lambda_K^*(\cdot)$ for any $K \in \mathscr{K}$, and the functions $\lambda^*(\cdot)$ and $\lambda_K^*(\cdot)$ are proper.

Notice now that the convex conjugate of the function $\overline{\lambda}(\cdot)$ is identical to $\lambda(\cdot)$. Indeed,

$$\begin{aligned}(\overline{\lambda})^*(\alpha) &= \sup_{v\in\mathbb{R}^N} \sup_{K\in\mathscr{K}} \{\langle \alpha, v\rangle - \lambda_K^*(v)\} \\ &= \sup_{K\in\mathscr{K}} \sup_{v\in\mathbb{R}^N} \{\langle \alpha, v\rangle - \lambda_K^*(v)\} = \sup_{K\in\mathscr{K}} \lambda_K^{**}(\alpha) = \sup_{K\in\mathscr{K}} \lambda_K(\alpha)\end{aligned} \tag{3.7}$$

for all $\alpha \in \mathbb{R}^N$. The last identity of (3.7) is verified because for every $K \in \mathscr{K}$, the function $\lambda_K(\cdot)$ is convex and finite everywhere on $\mathbb{R}^N$ (see Theorem 12.2 and Corollary 7.4.2 in the book of Rockafellar [19]). Using relation (3.7) together with Proposition 2 we obtain the identity

$$(\overline{\lambda})^*(\cdot) = \lambda(\cdot).$$

The function $\overline{\lambda}(\cdot)$ being convex and proper, the last identity implies that $\lambda^*(\cdot) = (\overline{\lambda})^{**}(\cdot) = \mathrm{cl}(\overline{\lambda})(\cdot)$ (see again Theorem 12.2 in [19]) and hence, using Theorem 7.4 from the book of Rockafellar [19] we conclude that $\lambda^*(v) = \overline{\lambda}(v)$ for every $v \in \mathrm{ri}(\mathrm{dom}\,\lambda^*)$. $\square$

4. Proof of the large deviation estimates. In this section we prove Theorem 1. Throughout this section, we will assume that $X(0) = 0$ and $Y(0) = y_0 \in E$ are given.

4.1. *Upper large deviation estimate.*

PROPOSITION 3. *Under Assumption* 1 *and Assumption* 2, *the upper large deviation estimate*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[0,n\tau]}(|X(t)-vt|+\mathrm{dist}(Y(t),y_0))<\delta n\right) \le -\tau\lambda^*(v) \tag{4.1}$$

is verified for all $v \in \mathbb{R}^N$.

PROOF. Given $\delta > 0$, $n \in \mathbb{N}$ and $v \in \mathbb{R}^N$, denote

$$A_{n\delta}(v) = \left\{\sup_{t\in[0,n\tau]}(|X(t)-vt|+\mathrm{dist}(Y(t),y_0))<\delta n,\right\}.$$

To prove this proposition we will show that for every $\alpha \in \mathbb{R}^N$ such that $\lambda(\alpha) < +\infty$, and for every $\lambda > \lambda(\alpha)$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(A_{n\delta}(v)\right) \le -\tau\left(\langle\alpha,v\rangle-\lambda\right) \tag{4.2}$$

from which the upper bound (4.1) will follow.

Let $\alpha \in \mathbb{R}^N$ and $\lambda > \lambda(\alpha)$. Define the function $f : E \to \mathbb{R}_+$ by setting

$$f(y) = \sum_{k=0}^{\infty}\mathscr{P}^{(k)}(\alpha, y, y_0)e^{-\lambda k}, \qquad y \in E.$$

According to the definition of $\lambda(\alpha)$, the above series converge and moreover, for all $y \in E$, $f(y) > 0$ because the Markov chain $(Y(t))$ is irreducible.

Consider a new Markov chain on $\mathbb{Z}^N \times E$ with initial state $(0, y_0)$ and transition probabilities

$$\tilde{p}\left((x,y),(x',y')\right) = p\left((x,y),(x',y')\right)e^{\langle\alpha,x'-x\rangle}\frac{f(y')}{f(y)R_y(\alpha)}$$

where

$$R_y(\alpha) = \sum_{x', y'} p\left((x, y), (x', y')\right) e^{\langle \alpha, x'-x \rangle} \frac{f(y')}{f(y)}$$
$$= \frac{1}{f(y)} \sum_{y'} \mathscr{P}(\alpha, y, y') f(y').$$

Let $\tilde{\mathbb{P}}$ be the distribution of this new Markov chain and let $\tilde{\mathbb{E}}$ denote an expectation with respect to $\tilde{\mathbb{P}}$. Then the standard arguments of the change of measure give the following relation:

$$\begin{aligned} \mathbb{P}\left(A_{n\delta}(v)\right) = \tilde{\mathbb{E}}\Bigg(& \mathbb{1}_{A_{n\delta}(v)} \frac{f(y_0)}{f(Y([n\tau]))} \\ & \times \exp\left\{ -\langle \alpha, X([n\tau]) \rangle + \sum_{t=0}^{[n\tau]-1} \log R_{Y(t)}(\alpha) \right\} \Bigg). \end{aligned} \tag{4.3}$$

One can easily verify that the function f satisfies the inequality

$$\mathscr{P}(\alpha) f \leq e^{\lambda} f$$

and consequently, $R_y(\alpha) \leq \exp \lambda$ for all $y \in E$. Using this inequality in the right hand side of (4.3) we get

$$\mathbb{P}\left(A_{n\delta}(v)\right) \leq \tilde{\mathbb{E}}\left(\mathbb{1}_{A_{n\delta}(v)} \frac{f(y_0)}{f(Y([n\tau]))} \exp\left\{ -\langle \alpha, X([n\tau]) \rangle + [n\tau]\lambda \right\} \right). \tag{4.4}$$

Furthermore, because of Assumption 2 there exist $\gamma = \gamma(\alpha, \lambda) > 0$ such that $f(y) \geq \gamma^{\mathrm{dist}(y, y_0)}$ for all $y \in E$ and hence, inequality (4.4) yields the following relation:

$$\begin{aligned} \mathbb{P}(A_{n\delta}(v)) \leq f(y_0)\, \tilde{\mathbb{E}}(\mathbb{1}_{A_{n\delta}(v)} \exp\{ & -\langle \alpha, X([n\tau]) \rangle \\ & + |\log \gamma| \ \mathrm{dist}(Y([n\tau]), y_0) + [n\tau]\lambda \}). \end{aligned} \tag{4.5}$$

Since on $A_{n\delta}(v)$,

$$-\langle \alpha, X([n\tau]) \rangle = -n\tau \langle \alpha, v \rangle + \langle \alpha, n\tau v - X([n\tau]) \rangle \leq -n\tau \langle \alpha, v \rangle + |\alpha| \delta n,$$

and $\mathrm{dist}(Y([n\tau]), y_0) \leq \delta n$, inequality (4.5) implies that

$$\log \mathbb{P}\left(A_{n\delta}(v)\right) \leq -n\tau\left(\langle \alpha, v \rangle - \lambda\right) + \log f(y_0) + \left(|\alpha| + |\log \gamma|\right) \delta n.$$

The last relation proves (4.2).

Using the upper estimate (4.2) we obtain

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(A_{n\delta}(v)\right) \leq -\tau\left(\langle \alpha, v \rangle - \lambda(\alpha)\right)$$

for all $\alpha \in \mathrm{dom}(\lambda) = \left\{ \alpha \in \mathbb{R}^N : \lambda(\alpha) < +\infty \right\}$, and consequently,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(A_{n\delta}(v)\right) \leq -\tau \sup_{\alpha \in \mathrm{dom}\,\lambda} \left(\langle \alpha, v \rangle - \lambda(\alpha)\right).$$

But it is known that

$$\lambda^*(v) = \sup_{\alpha \in \operatorname{dom} \lambda} (\langle \alpha, v\rangle - \lambda(\alpha)) \qquad \forall\, v \in \mathbb{R}^N$$

(see [19] Corollary 12.2.2 of Theorem 12.2) and hence, the last inequality proves the upper estimate (4.1). $\square$

4.2. *Lower large deviation estimate.* To prove the lower estimate (2.1) we will use the following proposition.

PROPOSITION 4. *Let K be a finite subset of E, $X(0) = 0$ and $Y(0) = y_0 \in K$. Suppose that the restriction of the Markov chain $(Y(t))$ on K is irreducible, let* $\exp \lambda_K(\alpha)$ *be the Perron-Frobenius eigenvalue of the matrix*

$$\mathscr{P}_K(\alpha) = (\mathscr{P}(\alpha; y, y');\ y, y' \in K), \qquad \alpha \in \mathbb{R}^N, \tag{4.6}$$

and let $\mathscr{T}_K$ be the first exit time of the process $(Y(t))$ from the set K. Then for all $\tau > 0$ and $v \in \mathbb{R}^N$, the following inequality holds

$$\begin{aligned}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right| < \delta n \text{ and } \mathscr{T}_{\mathrm{K}} > [\mathrm{n}\tau]\right)& \\ \geq -\tau\lambda_K^*(v),&\end{aligned} \tag{4.7}$$

where $\lambda_K^*(\cdot)$ is the convex conjugate of the function $\lambda_K(\cdot)$.

This is an extension of the theorem due to Mogulskii [16] to the sub-stochastic Markov-additive process. It can be proved by using the same method as in [16] where instead of Theorem 1 from [16] one has to apply an extension of Theorem 3.1.2 from Dembo and Zeitouni [6] to random functions and to sub-stochastic processes. In Appendix A we propose another straightforward proof of this proposition which uses the martingale method.

PROPOSITION 5. *Under Assumption* 1, *the lower large deviation estimate*

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[0,n\tau]}(|X(t)-vt|+\operatorname{dist}(Y(t),y_0))<\delta n\right)\geq -\tau\lambda^*(v) \tag{4.8}$$

holds for all $v \in \mathbb{R}^N$.

PROOF. Notice that for any finite subset K of E such that $y_0 \in K$ and for all n large enough,

$$\begin{aligned}&\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}(|X(t)-vt| + d\,(Y(t),y_0)) < 2\delta n\right)\\ &\qquad \geq \mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right| < \delta n \ \text{ and } \ \mathscr{T}_K > [n\tau]\right)\end{aligned}$$

and consequently,

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left(\left|X(t)-vt\right|+d\left(Y(t),y_0\right)\right)<\delta n\right)$$
$$\geq \lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right|<\delta n \text{ and } \mathscr{T}_K>[n\tau]\right).$$

If the restriction of the Markov chain $(Y(t))$ on K is irreducible, the last inequality together with Proposition 4 implies that

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left(\left|X(t)-vt\right|+d\left(Y(t),y_0\right)\right)<\delta n\right)\geq -\tau\lambda^*_K(v).$$

Consider now collection $\mathscr{K}$ of all finite subsets K of E for which the restriction of the Markov chain $(Y(t))$ on K is irreducible. Using the last inequality we obtain

$$\begin{aligned}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}&\left(\sup_{t\in[[0,n\tau]]}\left(\left|X(t)-vt\right|+d\left(Y(t),y_0\right)\right)<\delta n\right)\\ &\geq -\tau\inf_{K\in\mathscr{K}}\lambda^*_K(v).\end{aligned}\tag{4.9}$$

By Lemma 2, $\lambda^*(v)=\inf_{K\in\mathscr{K}}\lambda^*_K(v)$ for all $v\in\mathbb{R}^N$ which belongs to the relative interior $\mathrm{ri}(\mathrm{dom}\,\lambda^*)$ of the set $\mathrm{dom}\,\lambda^*=\left\{v\in\mathbb{R}^N:\lambda^*(v)<+\infty\right\}$, and hence, inequality (4.9) proves the lower bound (4.8) for $v\in\mathrm{ri}(\mathrm{dom}\,\lambda^*)$.

To extend this result for an arbitrary $v\in\mathbb{R}^N$ we will use the upper semi-continuity of the function

$$w(v)=\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left(\left|X(t)-vt\right|+d\left(Y(t),y_0\right)\right)<\delta n\right).$$

Let us verify that this function is upper semi-continuous. Indeed, for $\delta'<\delta/2$ and for $v,v'\in\mathbb{R}^N$ such that $|v-v'|<\delta/(2\tau)$, the inequality

$$\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left(|X(t)-v't|+d\left(Y(t),y_0\right)\right)<\delta' n\right)$$
$$\leq\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}\left(|X(t)-vt|+d\left(Y(t),y_0\right)\right)<\delta n\right)$$

holds. Letting first $\delta'\to 0$ in the left hand side and then letting $\delta\to 0$ in the right hand side we obtain

$$\limsup_{v'\to v} w(v')\leq w(v),$$

and since $v\in\mathbb{R}^N$ is arbitrary, we conclude that the function $w(\cdot)$ is upper semi-continuous.

Now, we are ready to extend inequality (4.8) for an arbitrary $v\in\mathbb{R}^N$.

When $\lambda^*(v) = +\infty$, the lower bound (4.8) is trivial and hence, it is sufficient to consider the case where $v \in \operatorname{dom} \lambda^* \setminus \operatorname{ri}(\operatorname{dom} \lambda^*)$. But in this case, there exists a sequence of points $v_k \in \operatorname{ri}(\operatorname{dom} \lambda^*)$ such that $v_k \to v$ and $\lambda^*(v_k) \to \lambda^*(v)$ as $k \to +\infty$, because $\lambda^*(\cdot)$ is a closed proper convex function on $\mathbb{R}^N$. Indeed, the set $\operatorname{ri}(\operatorname{dom} \lambda^*)$ is nonempty by Theorem 6.2 in [19]. Choosing $x_0 \in \operatorname{ri}(\operatorname{dom} \lambda^*)$ and letting $v_k = v_0/k + (1 - 1/k)v$ for $k \geq 1$, we get $v_k \in \operatorname{ri}(\operatorname{dom} \lambda^*)$ for all $k \geq 1$, by Theorem 6.1 in [19] and $\lambda^*(v_k) \to \lambda^*(v)$ as $k \to +\infty$ by Theorem 7.5 in [19].

Using the lower bound (4.8) for $v_k \in \operatorname{ri}(\operatorname{dom} \lambda^*)$, we obtain

$$w(v_k) \geq -\tau \lambda^*(v_k) \qquad \forall k \geq 1,$$

and using then the upper semi-continuity of the function $w(\cdot)$ we conclude that

$$w(v) \geq \limsup_{n \to \infty} w(v_n) \geq \limsup_{n \to \infty} \lambda^*(v_n).$$

Since $\lambda^*(v_k) \to \lambda^*(v)$ as $k \to +\infty$, the last inequality implies the lower bound (4.8) for $v \in \operatorname{dom} \lambda^* \setminus \operatorname{ri}(\operatorname{dom} \lambda^*)$ and therefore, Proposition 5 is proved.

This proposition completes the proof of Theorem 1. $\square$

5. A rough upper estimate for $\boldsymbol{\lambda}(\cdot)$. It is clear that there is no an explicit expression of the function $\lambda(\cdot)$ in general. In this section we consider the case when $E \subset \mathbb{Z}^k$ and we give a rough upper estimate for $\lambda(\cdot)$. This rough upper estimate is closely related to the general upper large deviation bound proved by Dupuis et al. in [8].

Suppose that $E \subseteq \mathbb{Z}^k$ and let for each $y \in E$, the function

$$R_y(\alpha, \beta) = \log \mathbb{E}_{(0,y)}(\exp\{\langle \alpha, X(1)\rangle + \langle \beta, Y(1) - y\rangle\})$$

be finite everywhere on $\mathbb{R}^N \times \mathbb{R}^k$. Then according to the definition of the transform matrix $\mathscr{P}(\alpha)$, we have

$$R_y(\alpha, \beta) = \sum_{y' \in E} \mathscr{P}(\alpha; y, y') e^{\langle \beta, y' - y\rangle} \qquad \forall\, y \in E.$$

The above relation shows that the exponential function $f(y) = e^{\langle \beta, y\rangle}$, $y \in E$, satisfies the inequality

$$\mathscr{P}(\alpha) f \leq e^{\lambda} f$$

with $\lambda = \sup_{y \in E} \log R_y(\alpha, \beta)$ and hence, using Proposition 1 we get the following rough upper estimate for $\lambda(\alpha)$.

PROPOSITION 6. *For every $\alpha \in \mathbb{R}^N$,*

$$\lambda(\alpha) \leq \inf_{\beta \in \mathbb{R}^k} \sup_{y \in \mathbb{E}} \log R_y(\alpha, \beta). \tag{5.1}$$

Given $\alpha \in \mathbb{R}^N$, we will denote

$$\hat{\lambda}(\alpha) = \inf_{\beta \in \mathbb{R}^k} \sup_{y \in \mathbb{E}} \log R_y(\alpha, \beta).$$

The convex conjugate of the function $\hat{\lambda}(\cdot)$ satisfies the following relation

$$\hat{\lambda}^*(v) = \sup_{\alpha \in \mathbb{R}^N, \beta \in \mathbb{R}^k} \left\{ \langle \alpha, v \rangle - \sup_{y \in E} \log R_y(\alpha, \beta) \right\} = \left(\sup_{y \in E} \log R_y \right)^* (v, 0)$$

where $(\sup_y \log R_y)^*$ denotes the convex conjugate of the function $\sup_y \log R_y$. Theorem 16.5 from [19] shows that the convex conjugate of the function $\sup_y \log R_y$ is the closure of the convex hull of the collection of functions $\{(\log R_y)^* | y \in E\}$, and therefore, the last relation implies that

$$\hat{\lambda}^*(v) = \mathrm{cl} \left(\inf \left\{ \sum_y \theta_y (\log R_y)^*(v_y, u_y) \right\} \right),$$

where the infimum is taken over all representation of $(v, 0) \in \mathbb{R}^N \times \mathbb{R}^k$ as a convex combination of elements $(v_y, u_y) \in \mathbb{R}^N \times \mathbb{R}^k$ such that only finitely many coefficients θ_y are nonzero. The closure operation can be omitted from the right hand side of the above identity if the collection $\{R_y | \ y \in E\}$ contain finitely many different functions.

Observe now that $\lambda^*(\cdot) \geq \hat{\lambda}^*(\cdot)$ because $\lambda(\cdot) \leq \hat{\lambda}(\cdot)$ and hence, using Theorem 1 we get the following rough upper large deviation estimate.

PROPOSITION 7. *Under Assumption* 1 *and Assumption* 2,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{(0,0)} \left(\sup_{t \in [0, n\tau]} (|X(t) - vt| + |Y(t)|) \leq \delta n \right) \leq -\tau \hat{\lambda}^*(v) \tag{5.2}$$

Notice moreover, that the identity $\lambda^*(\cdot) = \hat{\lambda}^*(\cdot)$ holds if and only if $\lambda(\cdot)$ is the closure of the convex hull of the function $\hat{\lambda}(\cdot)$ because the function $\lambda(\cdot)$ is convex, proper and closed. Hence, using again Theorem 1 we get the following statement.

PROPOSITION 8. *Under Assumption* 1 *and Assumption* 2, *the upper estimate* (5.2) *is tight*, *that is*,

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{(0,0)} \left(\sup_{t \in [0, n\tau]} (|X(t) - vt| + |Y(t)|) \leq \delta n \right) \geq -\tau \hat{\lambda}^*(v) \tag{5.3}$$

if and only if the upper bound (5.1) *is tight*, *that is*, *the identity* $\lambda(\cdot) = \mathrm{cl}(\mathrm{conv}\, \hat{\lambda})(\cdot)$ *holds*.

The estimate (5.2) corresponds to the upper large deviation bound proved by Dupuis et al. in [8]. The well known classical example where the identity (5.3) is verified and $\lambda(\cdot) = \hat{\lambda}(\cdot)$, is a homogeneous random walk on $\mathbb{Z}^N \times \mathbb{Z}^k$ (see the example considered below). Another nontrivial example will be considered in the section 6.

EXAMPLE [A homogeneous random walk on $\mathbb{Z}^N \times \mathbb{Z}^k$]. Let $E = \mathbb{Z}^k$, and let the transition probabilities $p((x, y), (x + x', y + y'))$ of our Markov chain $(X(t), Y(t))$ do not depend on (x, y) for all $(x, y), (x', y') \in \mathbb{Z}^N \times \mathbb{Z}^k$. Suppose that this Markov chain is irreducible on $\mathbb{Z}^N \times \mathbb{Z}^k$ and let the function

$$R(\alpha, \beta) = \sum_{y \in \mathbb{Z}^k} \mathscr{P}(\alpha; 0, y) e^{\langle \beta, y \rangle} = \sum_{(x,y) \in \mathbb{Z}^N \times \mathbb{Z}^k} p((0, 0), (x, y)) e^{\langle \alpha, x \rangle + \langle \beta, y \rangle}$$

be finite for all $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}^k$.

In this case, the function $\hat{\lambda}(\alpha) = \inf_{\beta \in \mathbb{R}^k} \log R(\alpha, \beta)$ is finite and convex everywhere on $\mathbb{R}^N$ which implies that $\mathrm{cl}(\mathrm{conv}\, \hat{\lambda})(\cdot) = \hat{\lambda}(\cdot)$. The convex conjugate of $\hat{\lambda}(\cdot)$ is $\hat{\lambda}^*(v) = (\log R)^* (v, 0)$, and the sample path large deviation principle for homogeneous random walk implies that for every $v \in \mathbb{R}^N$,

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{(0,0)} \left(\sup_{t \in [0, n\tau]} (|X(t) - vt| + |Y(t)|) \le \delta n \right) \ge -\tau (\log R)^* (v, 0)$$

and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{(0,0)} \left(\sup_{t \in [0, n\tau]} (|X(t) - vt| + |Y(t)|) \le \delta n \right) \le -\tau (\log R)^* (v, 0).$$

Hence, for this example, the upper bound (5.2) is tight and using Proposition 8 it follows that

$$\lambda(\alpha) = \inf_{\beta \in \mathbb{R}^k} \log R(\alpha, \beta) \qquad \forall \alpha \in \mathbb{R}^N. \tag{5.4}$$

The last relation can be easily proved in a straightforward way as follows.

Indeed, in this case, we have $\mathscr{P}(\alpha; y, y') = \mathscr{P}(\alpha; 0, y' - y)$ for all $y, y' \in \mathbb{Z}^k$, and $R_y(\alpha, \beta) = R(\alpha, \beta)$ for all $y \in \mathbb{Z}^k$ and $\beta \in \mathbb{R}^k$, which implies that the matrix

$$\left(R^{-1}(\alpha, \beta) \mathscr{P}(\alpha; y, y') \exp\{\langle \beta, y' - y \rangle\};\ y, y' \in \mathbb{Z}^k \right)$$

is stochastic and for any $\beta \in \mathbb{R}^k$, and the transition probabilities

$$\tilde{p}(y, y') = R^{-1}(\alpha, \beta) \mathscr{P}(\alpha; y, y') \exp\{\langle \beta, y' - y \rangle\}, \qquad y, y' \in \mathbb{Z}^k, \tag{5.5}$$

satisfy

$$\tilde{p}(y, y') = \tilde{p}(0, y' - y) \qquad \forall\ y, y' \in \mathbb{R}^k.$$

Given $\alpha \in \mathbb{R}^N$, let $\beta_0(\alpha)$ achieve the minimum of the function $\beta \to R(\alpha, \beta)$ in $\mathbb{R}^k$ (notice that $\beta_0(\alpha)$ exists because this function has the compact level sets by Lemma 8). Consider a homogeneous random walk $(S(t))$ on $\mathbb{Z}^k$ having the transition probabilities (5.5) with $\beta = \beta_0(\alpha)$. This random walk has the mean zero because

$$\mathbb{E}\left(S(t+1) - S(t)\right) = \sum_{y\in\mathbb{Z}^k} \tilde{p}(0, y)y = \nabla_\beta R(\alpha, \beta)\big|_{\beta=\beta_0(\alpha)=0} = 0$$

where $\nabla_\beta R(\alpha, \beta)$ denotes the gradient of the function $\beta \to R(\alpha, \beta)$ for given α. Moreover, the second moments if this random walk are finite because for each $\beta \in \mathbb{R}^k$,

$$\begin{aligned}\mathbb{E}\left(\exp\{\langle\beta, S(t+1) - S(t)\rangle\}\right) &= R^{-1}\left(\alpha, \beta_0(\alpha)\right) \sum_{y\in\mathbb{Z}^k} \mathscr{P}(\alpha; 0, y)e^{\langle\beta+\beta_0(\alpha), y\rangle} \\ &= R^{-1}\left(\alpha, \beta_0(\alpha)\right) R\left(\alpha, \beta_0(\alpha) + \beta\right) < +\infty.\end{aligned}$$

This implies that the t-time transition probabilities of $(S(t))$ satisfy

$$\frac{1}{t}\log \tilde{p}^{(t)}(y, y) \to 0 \qquad \text{as } t \to \infty$$

for all $y \in \mathbb{Z}^k$ (see [22]) and consequently,

$$\begin{aligned}\lambda(\alpha) = \limsup_{t\to\infty} \frac{1}{t}\log \mathscr{P}^{(t)}(\alpha; y, y) &= \log R(\alpha, \beta_0(\alpha)) + \limsup_{t\to\infty} \frac{1}{t}\log \tilde{p}^{(t)}(y, y) \\ &= \log R(\alpha, \beta_0(\alpha))\end{aligned}$$

as required.

6. Reflected random walks in $\mathbb{Z}^N \times \mathbb{Z}_+$. In this section we show how Theorem 1 can be used to identify the sample path large deviation rate function for a random walk on $\mathbb{Z}^N \times \mathbb{Z}_+$. The results of this section are known (see [15, 21]).

Consider a Markov chain $(X(t), Y(t))$ on $\mathbb{Z}^N \times \mathbb{Z}_+$ with transition probabilities $p((x, y), (x', y'))$. Suppose that for every pair $(x, y), (x', y') \in \mathbb{Z}^N \times \mathbb{Z}_+$,

$$p((x, y), (x', y')) = \begin{cases}\mu(x' - x, y' - y), & \text{if } y \neq 0,\\ \mu_0(x' - x, y' - y), & \text{if } y = 0,\end{cases}$$

where μ and μ_0 are two different probability measures on $\mathbb{Z}^N \times \mathbb{Z}$ such that for any $x \in \mathbb{Z}^N$, $\mu(x, y) = 0$ if $y < -1$, and $\mu_0(x, y) = 0$ if $y < 0$.

We will assume that the Markov chain $(X(t), Y(t))$ is irreducible on $\mathbb{Z}^N \times \mathbb{Z}_+$ as well as the homogeneous random walk on $\mathbb{Z}^{N+1}$ with transition probabilities

$$p_h((x, y), (x', y')) = \mu(x' - x, y' - y). \tag{6.1}$$

We will suppose moreover, that there exists a constant $C > 0$ such that for every $x \in \mathbb{Z}^k$, $\mu(x, y) = 0$ and $\mu_0(x, y) = 0$ whenever $|x| + |y| > C$.

Define for $(x, y) \in \mathbb{R}^N \times \mathbb{R}_+$ and $t \in [0, \tau]$, the renormalized process $Z_n(t, x, y)$ by setting

$$Z_n(t, x, y) = \frac{1}{n}\left(X([nt]), Y([nt])\right), \quad t \in [0, T]$$

given that $X(0) = [nx]$ and $Y(0) = [ny]$.

Under the above assumptions, the sequence of processes $Z_n(t, x, y)$ satisfy the sample path large deviation principle (see [15, 21]) with the good rate function

$$I_{x,y,\tau}(\varphi) = \begin{cases} \int_0^T L(\varphi(t), \dot\varphi(t))dt, & \text{if } \varphi \text{ is absolutely continuous} \\ & \text{and } \varphi(0) = (x, y), \\ +\infty, & \text{otherwise.} \end{cases} \tag{6.2}$$

For $(x, y) \in \mathbb{R}^N \times \mathbb{R}_+$ with $y > 0$, the function $(v, u) \to L((x, y), (v, u))$ is the convex conjugate of the function $(\alpha, \beta) \to \log R(\alpha, \beta)$:

$$L((x, y), (v, u)) = (\log R)^*(v, u) = \sup_{\alpha\in\mathbb{R}^N, \beta\in\mathbb{R}} \{\langle\alpha, v\rangle + \beta u - \log R(\alpha, \beta)\}, \tag{6.3}$$

where

$$R(\alpha, \beta) = \sum_{x\in\mathbb{Z}^N,\ y\in\mathbb{Z}} \mu(x, y) \exp\{\langle\alpha, x\rangle + \beta y\}.$$

For $y = 0$, the function $v \to L((x, 0), (v, 0))$ is the convex conjugate of the function

$$\alpha \to \inf_{\beta\in\mathbb{R}} \max\{\log R_0(\alpha, \beta), \log R(\alpha, \beta)\}$$

where

$$R_0(\alpha, \beta) = \sum_{x\in\mathbb{Z}^N,\ y\in\mathbb{Z}} \mu_0(x, y) \exp\{\langle\alpha, x\rangle + \beta y\},$$

or equivalently (see Section 5),

$$L((x, 0), (v, 0)) = \inf\{\theta_1(\log R)^*(v_1, u_1) + \theta_2(\log R_0)^*(v_2, u_2)\} \tag{6.4}$$

where the infimum is taken over all representations of $(v, 0)$ as a convex combinations of the elements (v_1, u_1) and (v_2, u_2) in $\mathbb{R}^N \times \mathbb{R}$.

The rate function $I(\cdot)$ is completely determined by the equalities (6.3) and (6.4) because for any absolutely continuous path

$$\varphi = (\varphi_1, \varphi_2) : [0, T] \to \mathbb{R}^N \times \mathbb{R}_+$$

with $\varphi_1 : [0, T] \to \mathbb{R}^N$ and $\varphi_2 : [0, T] \to \mathbb{R}_+$, the Lebesgue measure of the set

$$\{t \in [0, T] : \varphi_2(t) = 0 \text{ and } \dot\varphi_2(t) \neq 0\}$$

is zero. Indeed, when $\varphi_2(t) = 0$, the inequality $\varphi_2(s) \geq \varphi_2(t)$ holds for all $s \in [0, T]$ and hence, $\dot\varphi_2(t)$ must be zero if $\dot\varphi_2(t)$ exists and $t \in (0, T)$.

The proof of the identity (6.3) is simple because for $y > 0$, the local large deviation behavior of the renormalized process in a neighborhood of (x, y) do not depend on the boundary conditions.

For $y = 0$, the influence of the boundary occurs and the proof of the identity (6.4) is more difficult. In this section, we show how this identity can be obtained by using Theorem 1.

To apply Theorem 1, we notice that $(X(t), Y(t))$ is a Markov-additive process with an additive part $X(t)$ on $\mathbb{Z}^N$ and with a Markovian part $Y(t)$ on $\mathbb{Z}_+$. The transform matrix $\mathscr{P}(\alpha)$ of this Markov-additive process is defined by

$$\mathscr{P}(\alpha; y, y') = \mathbb{E}_{(0,y)}\left(e^{\langle \alpha, X(1)\rangle}\,\mathbb{1}_{\{Y(1)=y'\}}\right), \qquad y, y' \in \mathbb{Z}_+.$$

Let $\rho(\alpha)$ be the convergence parameter of $\mathscr{P}(\alpha)$ and let $\lambda(\alpha) = -\log \rho(\alpha)$.

Recall that the function $L((x, 0), (v, 0))$ is defined by the limits

$$\lim_{\tau\to 0}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n\tau}\log\mathbb{P}_{([nx],0)}\left(\sup_{t\in[[0,n\tau]]}(|X(t)-vt-nx|+|Y(t)|)<\delta n\right)$$

$$=\lim_{\tau\to 0}\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{1}{n\tau}\log\mathbb{P}_{([nx],0)}\left(\sup_{t\in[[0,n\tau]]}(|X(t)-vt-nx|+|Y(t)|)<\delta n\right).$$

It is clear that $L((x, 0), (v, 0)) = L((0, 0), (v, 0))$ for all $x, v \in \mathbb{R}^N$, because the transition probabilities of the Markov chain $(X(t), Y(t))$ are invariant with respect to the translations on $x \in \mathbb{Z}^N$. Using Theorem 1 we get therefore

$$L((x, 0), (v, 0)) = \lambda^*(v) \qquad \forall x, v \in \mathbb{R}^N, \tag{6.5}$$

where $\lambda^*(\cdot)$ is the convex conjugate of the function $\lambda(\cdot)$. Hence, to get the identity (6.4), it is sufficient to prove the following proposition.

PROPOSITION 9. *For every* $\alpha \in \mathbb{R}^N$,

$$\lambda(\alpha) = \inf_{\beta\in\mathbb{R}} \max\{\log R(\alpha, \beta), \log R_0(\alpha, \beta)\}. \tag{6.6}$$

Before proving Proposition 9, let us rewrite the right hand side of (6.6) in a more explicit form. For this, let us first notice that the function $(\alpha, \beta) \to R(\alpha, \beta)$ is strictly convex everywhere on $\mathbb{R}^N$ and it has the compact level sets, because the infinite matrix

$$\left(\mu(x' - x, y' - y);\ (x, y), (x', y') \in \mathbb{Z}^{N+1}\right)$$

is irreducible by assumption (see Lemma 8 in Appendix B for more details). This implies that for any $\alpha \in \mathbb{R}^N$, there exists a unique $\beta_0(\alpha) \in \mathbb{R}$ which achieves the minimum of the function $\beta \to R(\alpha, \beta)$ in $\mathbb{R}$, and it is clear that the right hand side of (6.6) is equal to $\log R(\alpha, \beta_0(\alpha))$ whenever $R(\alpha, \beta_0(\alpha)) \ge R_0(\alpha, \beta_0(\alpha))$.

Suppose now that $R(\alpha, \beta_0(\alpha)) < R_0(\alpha, \beta_0(\alpha))$ and notice that for the given $\alpha \in \mathbb{R}^N$:

(i) the function $\beta \to R(\alpha, \beta)$ is strictly decreasing on the interval $(-\infty, \beta_0(\alpha)]$ because the function $(\alpha, \beta) \to R(\alpha, \beta)$ is strictly convex;

(ii) $R(\alpha, \beta) \to +\infty$ as $\beta \to -\infty$, because the function $(\alpha, \beta) \to R(\alpha, \beta)$ has the compact level sets;

(iii) the function $\beta \to R_0(\alpha, \beta)$ is strictly decreasing on $\mathbb{R}$ because $\mu_0(x, y) > 0$ for some x and $y > 0$, and $\mu_0(x, y) = 0$ for all x if $y < 0$.

This proves that there exists a unique $\beta_1(\alpha) < \beta_0(\alpha)$ such that

$$R(\alpha, \beta_1(\alpha)) = R_0(\alpha, \beta_1(\alpha))$$

and it is clear that in this case, the right hand side of (6.6) is equal to $\log R(\alpha, \beta_1(\alpha))$.

We conclude therefore, that Proposition 9 is equivalent to the following one.

PROPOSITION 10. *Given $\alpha \in \mathbb{R}^N$, let $\beta_0(\alpha)$ achieve the minimum of the function $\beta \to R(\alpha, \beta)$ in $\mathbb{R}$.*

1. *If $R(\alpha, \beta_0(\alpha)) \geq R_0(\alpha, \beta_0(\alpha))$, then $\lambda(\alpha) = \log R(\alpha, \beta_0(\alpha))$.*
2. *Otherwise, for given α, the equation $R(\alpha, \beta) = R_0(\alpha, \beta)$ has a unique solution $\beta_1(\alpha)$ in the interval $(-\infty, \beta_0(\alpha))$ and $\lambda(\alpha) = \log R(\alpha, \beta_1(\alpha))$.*

We are ready now to prove this proposition.

PROOF. The upper bound

$$\lambda(\alpha) \leq \inf_{\beta \in \mathbb{R}} \max\{\log R(\alpha, \beta), \log R_0(\alpha, \beta)\} \tag{6.7}$$

immediately follows from estimate (5.1).

To prove the lower bound

$$\lambda(\alpha) \geq \inf_{\beta \in \mathbb{R}} \max\{\log R(\alpha, \beta), \log R_0(\alpha, \beta)\} \tag{6.8}$$

let us first verify that

$$\lambda(\alpha) \geq \log R(\alpha, \beta_0(\alpha)). \tag{6.9}$$

Indeed, consider a homogeneous random walk $(X_h(t), Y_h(t))$ on $\mathbb{Z}^N \times \mathbb{Z}$ with transition probabilities (6.1). Let $\rho_h(\alpha)$ be the convergence parameter of the matrix $\mathscr{P}_h(\alpha) = (\mathscr{P}_h(\alpha; y, y');\ y, y' \in \mathbb{Z}_+)$ where

$$\mathscr{P}_h(\alpha; y, y') = \mathbb{E}_{(0,y)}\left(\exp\{\langle \alpha, X_h(1)\rangle\}\mathbb{1}_{\{Y_h(1)=y'\}}\right),$$

and let $\lambda_h(\alpha) = -\log \rho_h(\alpha)$. The example considered in the previous section shows that

$$\lambda_h(\alpha) = \log R(\alpha, \beta_0(\alpha))$$

and hence, to prove inequality (6.9) it is sufficient to show that $\lambda(\alpha) \geq \lambda_h(\alpha)$. For this, we will use Proposition 2.

Proposition 2 implies that

$$\lambda(\alpha) = \sup_{K \subset \mathbb{Z}_+} \lambda_K(\alpha). \tag{6.10}$$

The supremum is taken here over all finite subsets K of $\mathbb{Z}_+$ and for every K, $\exp \lambda_K(\alpha)$ the maximal real eigenvalue of the matrix

$$\mathscr{P}_K(\alpha) = (\mathscr{P}(\alpha; y, y');\ y, y' \in K).$$

Similarly,

$$\lambda_h(\alpha) = \sup_{K\subset\mathbb{Z}} \lambda_{h,K}(\alpha) \tag{6.11}$$

where the supremum is taken over all finite subsets K of $\mathbb{Z}$ and for every K, $\exp \lambda_{h,K}(\alpha)$ the maximal real eigenvalue of the matrix

$$\mathscr{P}_{h,K}(\alpha) = (\mathscr{P}_h(\alpha; y, y');\ y, y' \in K).$$

Since for any finite subset K of $\mathbb{Z}$ there exists $y \in \mathbb{Z}_+$ such that $0 \notin K + y \subset \mathbb{Z}_+$ and the matrices $\mathscr{P}_{h,K}(\alpha)$ and $\mathscr{P}_{h,K+y}(\alpha)$ are identical, the relation (6.11) can be rewritten as follows:

$$\lambda_h(\alpha) = \sup_{K\subset\mathbb{Z}_+:\, 0\notin K} \lambda_{h,K}(\alpha).$$

Furthermore, for any finite subset K of $\mathbb{Z}_+$ such that $0 \notin K$, the matrices $\mathscr{P}_K(\alpha)$ and $\mathscr{P}_{h,K}(\alpha)$ are also identical and hence, comparison of the last relation with (6.10) proves that $\lambda(\alpha) \geq \lambda_h(\alpha)$. Inequality (6.9) is therefore verified.

Relation (6.9) proves the lower bound (6.8) when $R(\alpha, \beta_0(\alpha)) \geq R_0(\alpha, \beta_0(\alpha))$.

Suppose now that $R(\alpha, \beta_0(\alpha)) < R_0(\alpha, \beta_0(\alpha))$ and let $\beta_1(\alpha)$ be a unique solution of the equation $R(\alpha, \beta) = R_0(\alpha, \beta)$ in the interval $(0, \beta_0(\alpha))$. To complete the proof of our proposition we have to show that, in this case,

$$\lambda(\alpha) = R(\alpha, \beta_1(\alpha)). \tag{6.12}$$

Consider the matrix $(\tilde{p}(y, y'); y, y' \in \mathbb{Z}_+)$ with

$$\tilde{p}(y, y') = R^{-1}(\alpha, \beta_1(\alpha)) e^{\beta_1(\alpha)(y'-y)} \mathscr{P}(\alpha; y, y'), \qquad y, y' \in \mathbb{Z}_+.$$

This matrix is stochastic because for every $y \in \mathbb{Z}_+$,

$$\sum_{y'\in\mathbb{Z}_+} \tilde{p}(y, y') = R^{-1}(\alpha, \beta_1(\alpha)) \mathbb{E}_{(0,y)}(\exp\{\langle\alpha, X(1)\rangle + \beta_1(\alpha)(Y(1) - y)\}) = 1,$$

and the Markov chain $\left(\tilde{Y}(t)\right)$ on $\mathbb{Z}_+$ with transition probabilities $\tilde{p}(y, y')$ satisfies the following relation

$$\mathbb{E}_y\left(\tilde{Y}(1)\right) - y = \sum_{y'\in\mathbb{Z}} (y' - y)\tilde{p}(y, y') = \nabla_\beta R(\alpha, \beta)|_{\beta=\beta_1(\alpha)} < 0$$

for all $y \in \mathbb{Z}_+$ such that $y \neq 0$. Using therefore, the general criteria of ergodicity for countable Markov chains due to Foster (see [13] for example) with the test function $f(y) = y$, $y \in \mathbb{Z}_+$, we conclude that the Markov chain $\left(\tilde{Y}(t)\right)$ is ergodic. This proves that the t-time transition probabilities of this Markov chain satisfy

$$\lim_{t\to\infty} \frac{1}{t} \log \tilde{p}^{(t)}(y, y') = 0$$

for all $y, y' \in \mathbb{Z}_+$. Using the last relation together with the definition of $\tilde{p}(y, y')$ we get

$$\begin{aligned}\lambda(\alpha) = \limsup_{t\to\infty} \frac{1}{t} \log \mathscr{P}^{(t)}(\alpha; y, y) &= \log R(\alpha, \beta_1(\alpha)) + \lim_{t\to\infty} \frac{1}{t} \log \tilde{p}^{(t)}(y, y') \\ &= \log R(\alpha, \beta_1(\alpha))\end{aligned}$$

and hence the identity (6.12) holds. □

7. Finite perturbations of random walks in $\mathbb{Z}^k$.

In this section we apply Theorem 1 to identify the local rate function for a random walk $(Y(t))$ on $\mathbb{Z}^k$ with transition probabilities $p(x, x+y)$ which do not depend on x for all but finitely many $x \in \mathbb{Z}^k$. Such a random walk can be considered as a local perturbation of a homogeneous random walk on $\mathbb{Z}^k$.

7.1. *General statements.* Consider a Markov process $(Y(t))$ on $\mathbb{Z}^k$ with transition probabilities $p(x, y)$ such that for all $x, y \in \mathbb{Z}^k$,

$$p(x, y) = p(y - x) \qquad \text{if } x \notin A, \tag{7.1}$$

where A is a finite subset of $\mathbb{Z}^k$. We will assume that the Markov chain $(Y(t))$ is irreducible as well as a homogeneous random walk $(S(t))$ on $\mathbb{Z}^k$ with transition probabilities

$$p_h(x, y) = p(y - x), \qquad x, y \in \mathbb{Z}^k.$$

We will suppose moreover, that there exists a constant $C > 0$ such that for every $x \in \mathbb{Z}^k$, $p(x, x+y) = 0$ whenever $|y| > C$.

The same arguments as in [12] show that the sequence of processes

$$Z_n(t, x) = \frac{Y([nt])}{n}, \qquad t \in [0, \tau]$$

with $Y(0) = [nx]$, satisfies the sample path large deviation principle with a good rate function $I_{\tau,x}(\cdot)$ which is the lower semi-continuous regularization of the local rate function

$$\tilde{I}_{\tau,x}(\varphi) = \begin{cases} \int_0^\tau L(\varphi(t), \dot{\varphi}(t))\, dt, & \text{if } \varphi(0) = x, \\ +\infty, & \text{otherwise,} \end{cases}$$

defined on the set of piece-wise linear functions $\varphi(\cdot) : [0, \tau] \to \mathbb{R}^k$. The function $L(\cdot, \cdot)$ is defined by the following limits:

$$\begin{aligned} L(x, v) &= -\lim_{\tau\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{\tau n} \log \mathbb{P}_{[nx]} \left(\sup_{t\in[[0,n\tau]]} |Y(t) - vt - nx| < \delta n \right) \\ &= -\lim_{\tau\to 0} \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{\tau n} \log \mathbb{P}_{[nx]} \left(\sup_{t\in[[0,n\tau]]} |Y(t) - vt - nx| < \delta n \right). \end{aligned}$$

When $x \neq 0$, the above limits do not depend on the transition probabilities $p(y, y')$ for $y \in A$. Using therefore the sample path large deviation bounds for homogeneous random walk, we conclude that for $x \neq 0$,

$$L(x, v) = \sup_{\alpha \in \mathbb{R}^k} \left(\langle \alpha, v \rangle - \log R(\alpha) \right),$$

where

$$R(\alpha) = \sum_{y \in \mathbb{Z}^k} p(y) e^{\langle \alpha, y \rangle}.$$

For $x = 0$, Theorem 1 yields that

$$L(0, 0) = -\limsup_{n \to \infty} \frac{1}{n} \log p^{(n)}(0, 0)$$

where $p^{(n)}(0, 0)$ is a transition probability of the Markov chain $(Y(t))$ to go from 0 to 0 in time n, or equivalently,

$$L(0, 0) = -\lambda_0$$

where $e^{-\lambda_0}$ is a convergence parameter of the transition matrix $(p(x, y); x, y \in \mathbb{Z}^k)$.

REMARK. Notice moreover, that for any absolutely continuous function $\varphi = (\varphi_1, \dots, \varphi_k) : [0, \tau] \to \mathbb{R}^k$ the set

$$D = \{t : \varphi(t) = 0 \text{ and there exists } \dot{\varphi}(t) \neq 0\}$$

is at most countable. Indeed, consider for $j = 1, \dots, k$, the set

$$D_j = \left\{t : \varphi_j(t) = 0 \text{ and there exists } \dot{\varphi}_j(t) \neq 0\right\}.$$

It is clear that $D = \cup_{j=1}^k D_j$ and hence, it is sufficient to show that the set D_j is at most countable for all $j = 1, \dots, k$. Observe furthermore, that for each $t \in D_j$ there exists $\varepsilon(t) > 0$ such that $\varphi_j(s) \neq 0$ for all $t < s < t + \varepsilon(t)$. This implies in particular that the intervals $(t, t + \varepsilon(t))$, $t \in D_j$ do not contain the points from D_j and consequently, they are disjoint. But it is known that a collection of disjoint open intervals is at most countable and hence, the set D_j is also at most countable as required.

In view of the above remark, the rate function $I(\varphi)$ does not depend on the values $L(0, v)$ for $v \neq 0$ an thus, to identify the rate function $I(\varphi)$ we have to identify only the value λ_0.

Remark that the general upper large deviation bound of Dupuis Ellis and Weiss is tight in this case if and only if

$$\lambda_0 = \inf_{\alpha \in \mathbb{R}^k} \max_x \log R_x(\alpha) \tag{7.2}$$

where

$$R_x(\alpha) = \sum_{y \in \mathbb{Z}^k} p(x, y) e^{\langle \alpha, y \rangle}.$$

We will see that the identity (7.2) can be wrong even for a slight perturbation of a homogeneous random walk [see, e.g., part (c) in Corollary 1 below].

To identify the value λ_0, we need to introduce some definitions.

Given $x \in \mathbb{Z}^k$ and $y \in A$, let $Q^{(n)}(x, y)$ be the probability for the Markov chain $(Y(t))$ to go from x to y in time n without visiting the set A in a meantime. Denote by $Q(z; x, y)$ the generating function of the sequence $Q^{(n)}(x, y)$:

$$Q(z; x, y) = \sum_{n \geq 1} Q^{(n)}(x, y) z^n. \tag{7.3}$$

It is clear that for $z \in \mathbb{R}_+$, the matrix

$$(Q(z; x, y);\ x, y \in A) \tag{7.4}$$

is positive and irreducible because the Markov chain $(Y(t))$ is irreducible. Let $\Lambda(z)$ be Perron-Frobenius eigenvalue of this matrix. We say that $\Lambda(z) = +\infty$ if at least one of the series (7.3) diverges.

Furthermore, let α_0 achieve the minimum of the function $R(\alpha)$ in $\mathbb{R}^k$ and let $\Lambda_0 = \Lambda(R^{-1}(\alpha_0))$.

REMARK. The infinite matrix $\big(p(y - x);\ x, y \in \mathbb{Z}^k\big)$ being irreducible by assumption, the function $R(\cdot)$ is strictly convex and the level sets of that are compact (see Lemma 8 in Appendix B). This implies that there exists a unique point α_0 which achieves the minimum of the function $R(\cdot)$ in $\mathbb{R}^k$. It is clear moreover, that $R(\alpha_0) > 0$ because $R(\alpha) > 0$ for all α, and $R(\alpha_0) \leq 1$ because $R(0) = 1$.

We begin the analysis of λ_0 with the following theorem.

THEOREM 2. *If* $\Lambda_0 \leq 1$ *then* $\lambda_0 = \log R(\alpha_0)$. *Otherwise, the equation* $\Lambda(z) = 1$ *has a unique solution* z_0 *in the open interval* $(1, R^{-1}(\alpha_0))$ *and* $\lambda_0 = -\log z_0$.

It is useful to notice that $\Lambda_0 \leq 1$ if $R(\alpha_0) = 1$ because for $z = 1$, the matrix (7.4) is sub-stochastic. This implies that $R(\alpha_0) < 1$ and the open interval $(1, R^{-1}(\alpha_0))$ is nonempty whenever $\Lambda_0 > 1$.

The proof of this theorem will be given at the end of the section.

The value Λ_0 is crucial for identification of λ_0. To get more information about Λ_0, let us consider a new Markov chain $\big(\tilde{Y}(t)\big)$ on $\mathbb{Z}^k$ with transition probabilities

$$\tilde{p}(x, y) = p(x, y) \exp\{\langle \alpha_0, y - x\rangle - \log R_x(\alpha_0)\}, \tag{7.5}$$

where

$$R_x(\alpha) = \sum_{y \in \mathbb{Z}^k} p(x, y) e^{\langle \alpha, y\rangle}.$$

Denote by $\tilde{Q}^{(n)}(x, y)$ the probability that this new Markov chain goes from x to y in time n without visiting the set A in the meantime and let

$$\tilde{Q}(x, y) = \sum_{n \ge 1} \tilde{Q}^{(n)}(x, y). \tag{7.6}$$

The following proposition expresses the value Λ_0 in terms of the new Markov chain $(\tilde{Y}(t))$ and gives the lower and upper bounds for that.

PROPOSITION 11.

1. Λ_0 *is the Perron-Frobenius eigenvalue of the matrix*

$$\Big(R_x(\alpha_0)R^{-1}(\alpha_0)\tilde{Q}(x, y);\ x, y \in A\Big); \tag{7.7}$$

2. *for* $k \le 2$, *the value* Λ_0 *satisfies the inequalities*

$$\min_{x \in A}\ R_x(\alpha_0)R^{-1}(\alpha_0) \le \Lambda_0 \le \max_{x \in A}\ R_x(\alpha_0)R^{-1}(\alpha_0),$$

where equality in either side implies equality throughout;

3. *and for* $k \ge 3$,

$$\Lambda_0 < \max_{x \in A} R_x(\alpha_0)R^{-1}(\alpha_0).$$

PROOF. Indeed, using the definition of the transition probabilities (7.5) we get

$$Q^{(n)}(x, y) = e^{-\langle \alpha_0, y - x\rangle} R_x(\alpha_0) R^{(n-1)}(\alpha_0)\tilde{Q}^{(n)}(x, y) \qquad \forall n \in \mathbb{N}$$

for all $x, y \in A$, which implies that

$$Q\left(R^{-1}(\alpha_0); x, y\right) = e^{\langle \alpha_0, x - y\rangle} \frac{R_x(\alpha_0)}{R(\alpha_0)}\ \tilde{Q}(x, y) \qquad \forall x, y \in A,$$

and consequently, Λ_0 is the Perron-Frobenius eigenvalue of the matrix

$$\Big(e^{\langle \alpha_0, x - y\rangle} R_x(\alpha_0)R^{-1}(\alpha_0)\tilde{Q}(x, y);\ x, y \in A\Big).$$

But the above matrix has the same eigenvalues as the matrix (7.7) and hence, the first statement of Proposition 11 is verified.

Using now Theorem 1.5 (Corollary 1) from the book by Seneta [20] we conclude that

$$\min_{x \in A}\ R_x(\alpha_0)R^{-1}(\alpha_0) \sum_{y \in A}\ \tilde{Q}(x, y) \le \Lambda_0 \le \max_{x \in A}\ R_x(\alpha_0)R^{-1}(\alpha_0) \sum_{y \in A}\ \tilde{Q}(x, y),$$

where equality in either side implies equality throughout. When the matrix

$$\Big(\tilde{Q}(x, y);\ x, y \in A\Big)$$

is stochastic, the above relation can be rewritten as follows:

$$\min_{x \in A}\ R_x(\alpha_0)R^{-1}(\alpha_0) \le \Lambda_0 \le \max_{x \in A}\ R_x(\alpha_0)R^{-1}(\alpha_0).$$

Otherwise, we get

$$\Lambda_0 < \max_{x\in A} R_x(\alpha_0)R^{-1}(\alpha_0).$$

Thus, to complete the proof of Proposition 11 it is sufficient to show that the matrix $(\tilde{Q}(x, y);\ x, y \in A)$ is stochastic if and only if $d \le 2$.

The matrix $(\tilde{Q}(x, y);\ x, y \in A)$ is stochastic if and only if the new Markov chain $(\tilde{Y}(t))$ is recurrent and it is clear that this Markov chain is recurrent if and only if the homogeneous random walk $\tilde{S}(t)$ on $\mathbb{Z}^k$ with transition probabilities

$$\tilde{p}(x, y) = \tilde{p}(y - x) = p(y - x)e^{\langle\alpha_0, y-x\rangle}R^{-1}(\alpha_0)$$

is recurrent. Since α_0 achieves the minimum of the function $R(\cdot)$,

$$\sum_{y\in\mathbb{Z}^k} y\tilde{p}(y) = \nabla R(\alpha)|_{\alpha=\alpha_0} = 0,$$

which proves that the homogeneous random walk $\tilde{S}(t)$ is recurrent if $k \le 2$, and it is transient whenever $k \ge 3$ (see [22]). Proposition 11 is therefore verified. $\square$

Before we prove Theorem 2 let us rewrite the equation $\Lambda(z) = 1$ in a more explicit form for the case where the set A consists of a single point 0.

7.2. *Single point perturbation.* Suppose that $A = \{0\}$. In this case we have, obviously,

$$\Lambda(z) = Q(z; 0, 0) = \sum_{n\ge1} Q^{(n)}(0, 0)z^n \tag{7.8}$$

where $Q^{(n)}(0, 0)$ is the probability that the first return to 0 of the Markov chain $(Y(t))$ starting from 0 occurs at time n.

To rewrite the equation $\Lambda(z) = 1$ in a more explicit form let us consider a homogeneous random walk $S(t)$ on $\mathbb{Z}^k$ with transition probabilities

$$p_h(x, y) = p(y - x), \qquad x \in \mathbb{Z}^k.$$

Let $p_h^{(n)}(x, y)$, $x, y \in \mathbb{Z}^k$ be the n-time transition probabilities of this random walk, denote by $Q_h^{(n)}(x, 0)$ the probability that the above random walk goes from x to 0 without visiting the point 0 in the meantime, and let

$$Q_h(z; x, 0) = \sum_{n\ge1} Q_h^{(n)}(x, 0)z^n.$$

Then for any $n > 1$,

$$Q^{(n)}(0, 0) = \sum_{x\ne0} p(0, x)Q_h^{(n-1)}(x, 0)$$

because for $x \neq 0$, the probability to go from x to 0 without visiting the point 0 in the meantime for the Markov chain $(Y(t))$ is the same as for the homogeneous random walk $S(t)$. Using the last relation we get

$$Q(z;0,0) = p(0,0) + z\sum_{x\neq 0} p(0,x)Q_h(z;x,0). \tag{7.9}$$

Moreover, the example in Section 5 shows that

$$\limsup_{n\to\infty}\frac{1}{n}\log p_h^{(n)}(x,y) = \log R(\alpha_0)$$

which implies that for $0 < z < R^{-1}(\alpha_0)$, the series

$$G_h(z;x,y) = \sum_{n\geq 0} p_h^{(n)}(x,y)z^n$$

converge. Using therefore the identity $Q_h(z;x,0) = G_h(z;x,0)/G_h(z;0,0)$ together with (7.8) and (7.9) we conclude that for $0 < z < R^{-1}(\alpha_0)$, the equation $\Lambda(z) = 1$ is verified if and only if

$$\sum_{x\in\mathbb{Z}^k} zp(0,x)G_h(z;x,0) = G_h(z;0,0). \tag{7.10}$$

Let us rewrite the above equality in term of the generating function

$$\Psi(z,w) = \sum_{y\in\mathbb{Z}^k}\sum_{n\geq 0} z^n w^y p_h^{(n)}(0,y),$$

where for $w = (w_1,\dots,w_d) \in \mathbb{C}^k$ and $y = (y_1,\dots,y_d) \in \mathbb{Z}^k$ we denote

$$w^y = w_1^{y_1}\cdots w_d^{y_d}.$$

Given $w = (w_1,\dots,w_d) \in \mathbb{C}^k$, let $\log|w|$ denote the point $(\log|w_1|,\dots,\log|w_d|)$ in $\mathbb{R}^k$ and let

$$\Omega_z = \left\{w\in\mathbb{C}^k : zR(\log|w|) < 1\right\} \quad\text{and}\quad C^k = \left\{w\in\mathbb{C}^k : \log|w| = \alpha_0\right\}.$$

Notice that for any $0 < z < R^{-1}(\alpha_0)$, the set Ω_z is open, nonempty and $C^k \subset \Omega_z$. Moreover, the function $\Psi(z,w)$ is analytic with respect to w in Ω_z and

$$\Psi(z,w) = \sum_{y\in\mathbb{Z}^k} w^y G_h(z;0,y) = \sum_{n\geq 0}(z\psi(w))^n = (1-z\psi(w))^{-1}$$

where

$$\psi(w) = \sum_{y\in\mathbb{Z}^k} p(y)w^y, \qquad w\in\mathbb{C}^k.$$

Using Cauchy's formulae we get

$$G_h(z;y,0) = G_h(z;0,-y) = (2\pi i)^{-d}\int_{C^k}\frac{dw}{w^{-y+1}(1-z\psi(w))}, \qquad y\in\mathbb{Z}^k,$$

and therefore, the equation (7.10) can be rewritten as follows:

$$\int_{C^k} \frac{1 - z\psi_0(w)}{w(1 - z\psi(w))} dw = 0 \tag{7.11}$$

where

$$\psi_0(w) = \sum_{y\in\mathbb{Z}^k} p(0, y)w^y.$$

Thus, for the case when $A = \{0\}$, Theorem 2 and Proposition 11 imply the following statement.

COROLLARY 1. *Suppose that the set A consists of a single point 0 and let $\tilde{Q}$ be the probability that the Markov chain on $\mathbb{Z}^k$ with transition probabilities* (7.5) *returns to* 0 *starting from* 0. *Then*:

(a) $\tilde{Q} = 1$ *if* $k \le 2$, *and* $\tilde{Q} < 1$ *if* $k > 2$;
(b) *when* $R_0(\alpha_0)\tilde{Q} > R(\alpha_0)$, *equation* (7.11) *has a unique solution* z_0 *in the open interval* $\big(1, R^{-1}(\alpha_0)\big)$;
(c) $\lambda_0 = \log R(\alpha_0)$ *if* $R_0(\alpha_0)\tilde{Q} \le R(\alpha_0)$, *and* $\lambda_0 = -\log z_0$ *otherwise*.

For $k = 1$, equation (7.11) can be rewritten explicitly by using the residue method.

EXAMPLE. Let $(Y(t))$ be the Markov chain on $\mathbb{Z}$ with transition probabilities $p(x, y)$, $x, y \in \mathbb{Z}$ such that for all $x \neq 0$,

$$p(x, y) = \begin{cases} p, & \text{if } y = x + 1, \\ q, & \text{if } y = x - 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $0 < p < 1$ and $p + q = 1$. As above, we suppose that the Markov chain $(Y(t))$ is irreducible and the function

$$R_0(\alpha) = \sum_{y\in\mathbb{Z}} p(0, y)e^{\alpha y}$$

is finite everywhere on $\mathbb{R}$.

In this particular case, $R(\alpha) = pe^{\alpha} + qe^{-\alpha}$ and it is clear that $\alpha_0 = \log\sqrt{q/p}$ and $R(\alpha_0) = \sqrt{4pq}$. The functions $\psi(\cdot)$ and $\psi_0(\cdot)$ are given here by

$$\psi(w) = pw + qw^{-1} \quad \text{and} \quad \psi_0(w) = \sum_{y\in\mathbb{Z}} p(0, y)w^y,$$

and $C^1 = \{w \in \mathbb{C} : |w| = \sqrt{q/p}\}$.

Suppose that $0 < z < 1/\sqrt{4pq}$ and let us rewrite the integral

$$\int_{C^1} \frac{1 - z\psi_0(w)}{w(1 - z\psi(w))} dw$$

explicitly by using the residue method. The equation $w(1 - z\psi(w)) = 0$ has two simple zeros

$$w_1(z) = \frac{1 - \sqrt{1 - 4pqz^2}}{2pz} \quad \text{and} \quad w_2(z) = \frac{1 + \sqrt{1 - 4pqz^2}}{2pz}.$$

It is clear that $w_1(z)$ and $w_2(z)$ are real and $0 < w_1(z) < \sqrt{q/p} < w_2(z)$ whenever z is real and $0 < z < \sqrt{4pq}$. The Residue Theorem applied for the disk $\{|w| \le \sqrt{q/p}\}$ yields that, for $m \ge 0$,

$$(2\pi i)^{-1} \int_{C^1} \frac{w^m}{w(1 - z\psi(w))} dw = \operatorname{Res}_{w_1} \left(\frac{w^m}{w(1 - z\psi(w))} \right) = \frac{w_1^m(z)}{pz(w_2(z) - w_1(z))}.$$

For $m < 0$, the residue at infinity of the function $w^{m-1}/(1 - z\psi(w))$ is zero and hence, the Residue Theorem applied for the outside of the disk $\{|w| \le \sqrt{q/p}\}$ gives

$$(2\pi i)^{-1} \int_{C^1} \frac{w^m}{w(1 - z\psi(w))} dw = -\operatorname{Res}_{w_2} \left(\frac{w^m}{w(1 - z\psi(w))} \right) = \frac{w_2^m(z)}{pz(w_2(z) - w_1(z))}.$$

We conclude that

$$\begin{aligned}
&\int_{C^1} \frac{1 - z\psi_0(w)}{w(1 - z\psi(w))} dw \\
&\quad = \frac{1}{pz(w_2(z) - w_1(z))} \left\{ 1 - z \sum_{m<0} p(0, m) w_2^m(z) - z \sum_{m \ge 0} p(0, m) w_1^m(z) \right\}
\end{aligned}$$

and hence, for this particular case, Corollary 1 proves that $\lambda_0 = \log \sqrt{4pq}$ if

$$\sum_{y \in \mathbb{Z}} p(0, y) \left(\sqrt{q/p} \right)^y > \sqrt{4pq}$$

and otherwise, the equation

$$z \sum_{m<0} p(0, m) w_2^m(z) + z \sum_{m \ge 0} p(0, m) w_1^m(z) = 1$$

has a unique solution z_0 in the open interval $(1, 1/\sqrt{4pq})$ and $\lambda_0 = \log z_0$.

7.3. *Proof of Theorem* 2. We start the proof of Theorem 2 with the following lemma.

LEMMA 4. $\lambda_0 \ge \log R(\alpha_0)$.

The proof of this lemma is quite similar to the proof of inequality (6.9) in the section 6.

The following lemma completes the proof of the first part of Theorem 2.

LEMMA 5. *Suppose that* $\Lambda_0 \le 1$, *then* $\lambda_0 = \log R(\alpha_0)$.

PROOF. In view of Lemma 4 it is sufficient to verify that $\lambda_0 \le \log R(\alpha_0)$. To prove the last inequality it is sufficient to show (see Proposition 1) that there exists a non-zero positive function $g : \mathbb{Z}^k \to \mathbb{R}_+$ such that for any $x \in \mathbb{Z}^k$,

$$Pg\,(x) = \sum_{y \in \mathbb{Z}^k} p(x, y) g(y) \le R(\alpha_0) g(x). \tag{7.12}$$

Let $(f(y))_{y \in A}$ be a strictly positive eigenvector of the matrix (7.4) with $z = R^{-1}(\alpha_0)$, corresponding to $\Lambda_0 = \Lambda(R^{-1}(\alpha_0))$ (this eigenvector exists by the Perron-Frobenius theorem). Define

$$g(x) = \begin{cases} f(x), & \text{if } x \in A, \\ \sum_{y \in A} Q\left(R^{-1}(\alpha_0); x, y\right) f(y), & \text{if } x \notin A. \end{cases} \tag{7.13}$$

Then

$$\begin{aligned} Pg\,(x) &= \sum_{y \in A} p(x, y) f(y) + \sum_{y \notin A,\, y' \in A} p(x, y) Q\left(R^{-1}(\alpha_0); y, y'\right) f(y') \\ &= R_x(\alpha_0) \sum_{y \in A} Q\left(R^{-1}(\alpha_0); x, y\right) f(y) \end{aligned}$$

which implies that

$$Pg(x) = \begin{cases} R(\alpha_0)\; g(x), & \text{if } x \notin A, \\ R(\alpha_0)\; \Lambda_0\; g(x), & \text{if } x \in A, \end{cases}$$

and hence relation (7.12) is verified whenever $\Lambda_0 \le 1$. Lemma 5 is proved. □

The first part of Theorem 2 is proved. To prove the second part we will use the following lemma.

LEMMA 6. *Let* $\Lambda(z_0) = 1$ *for some* $z_0 \in [1, R^{-1}(\alpha_0)]$, *then* $\lambda_0 = -\log z_0$.

PROOF. The proof of the upper bound

$$\lambda_0 \le -\log z_0 \tag{7.14}$$

is quite similar to that of Lemma 5. Instead of the function (7.13), one has to consider here the function

$$g(x) = \begin{cases} f(x), & \text{if } x \in A, \\ \sum_{y \in A} Q(z_0; x, y) f(y), & \text{if } x \notin A, \end{cases}$$

where $f = (f(y))_{y \in A}$ is a strictly positive right eigenvector of the matrix

$$(Q(z_0; x, y))_{x, y \in A} \tag{7.15}$$

corresponding to $\Lambda(z_0) = 1$. For this function, a straightforward calculation gives $Pg = g/z_0$ and using therefore Proposition 1 we get the upper bound (7.14).

To prove the lower bound

$$\lambda_0 \geq -\log z_0 \tag{7.16}$$

it is sufficient to show that the series

$$G(z_0; x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z_0^n$$

diverge for some $x, y \in \mathbb{Z}^k$. For this, we shall use the following relation

$$p^{(n)}(x, y) = \sum_{k=1}^{n} \sum_{y' \in A} Q^{(k)}(x, y') p^{(n-k)}(y', y)$$

which is verified for all $n \geq 1$ and $x, y \in A$. For generating functions $G(z_0; x, y)$, $x, y \in A$, the above relation gives

$$G(z; x, y) = 1 + \sum_{y' \in A} Q(z; x, y') G(z; y', y). \tag{7.17}$$

Let $\pi = (\pi(x);\ x \in A)$ be a strictly positive left eigenvector of the matrix (7.15) corresponding to $\Lambda(z_0) = 1$. Then the relation (7.17) implies

$$\sum_{x \in A} \pi(x) G(z_0; x, y) = 1 + \sum_{y' \in A} \pi(x) G(z_0; y', y)$$

which proves that

$$\sum_{x \in A} \pi(x) G(z_0; x, y) = +\infty,$$

and hence, there exists $x \in A$ such that

$$G(z_0; x, y) = +\infty. \tag{7.18}$$

The lower bound (7.16) is therefore verified, and Lemma 6 is proved. □

To complete the proof of Theorem 2 it is sufficient now to prove the following lemma.

LEMMA 7. *Let* $\Lambda_0 > 1$, *then there exists a unique* $1 < z_0 < R^{-1}(\alpha_0)$ *such that* $\Lambda(z_0) = 1$.

PROOF. Notice first that:

(i) the matrix (7.4) is irreducible for all $z \in [0, R^{-1}(\alpha_0)]$, and
(ii) for any $x, y \in A$, the generating function $z \to Q(z; x, y)$ is strictly increasing on the interval $[0, R^{-1}(\alpha_0)]$ whenever $Q(z; x, y) \not\equiv 0$.

Using Perron-Frobenius theorem for irreducible matrices (see [20]) we conclude therefore, that the function $\Lambda(z)$ is strictly increasing on the interval $[0, R^{-1}(\alpha_0)]$.

Furthermore, for $z = 1$ the matrix (7.4) is sub-stochastic and hence $\Lambda(1) \leq 1$. But by assumption $\Lambda(R^{-1}(\alpha_0)) = \Lambda_0 > 1$ and the function $\Lambda(z)$ is continuous on the interval $[1, R^{-1}(\alpha_0)]$ because the coefficients of the matrix (7.3) are continuous. This proves that there exists a unique $1 \le z_0 < R^{-1}(\alpha_0)$ for which $\Lambda(z_0) = 1$.

Suppose finally that $z_0 = 1$, that is $\Lambda(1) = 1$. Then the relation (7.18) shows that our Markov chain $(Y(t))$ must be recurrent. It is clear that the Markov chain $(Y(t))$ is recurrent if and only if the homogeneous random walk $S(t)$ with transition probabilities $p_h(x, y) = p(y - x)$, $x, y \in \mathbb{Z}^k$ is recurrent. But it is known that the above homogeneous random walk is recurrent if and only if $d \le 2$ and

$$\nabla R(\alpha)|_{\alpha=0} = \sum_{y \in \mathbb{Z}^k} y p(y) = 0$$

which implies that $\alpha_0 = 0$, $R(\alpha_0) = 1$ and $\Lambda_0 = \Lambda(1) = 1$. We conclude therefore, that $z_0 > 1$ whenever $\Lambda_0 > 1$. Lemma 7 is proved. □

APPENDIX A

This section is devoted to the proof of Proposition 4. We will prove this proposition in two steps. The first one consists in proving the relation (4.7) for $v \in \mathbb{R}^d$ which belongs to the relative interior $\mathrm{ri}(\mathrm{dom}\,\lambda_K^*)$ of the set

$$\mathrm{dom}\,\lambda_K^* = \left\{ v \in \mathbb{R}^d : \lambda_K^*(v) < +\infty \right\},$$

and the second one extends this result for an arbitrary $v \in \mathbb{R}^d$ by using the upper semi-continuity of the function

$$w_K(v) = \lim_{\delta \to 0} \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left(\sup_{t \in [[0, n\tau]]} |X(t) - vt| < \delta n \ \text{ and } \ \mathscr{T}_K > [n\tau] \right).$$

Suppose that $v \in \mathrm{ri}\,(\mathrm{dom}\,\lambda_K^*)$. The function $\lambda_K(\cdot)$ is convex and differentiable on $\mathbb{R}^d$ (see Lemma 1) and hence, by Corollary 26.4.1 from the book of Rockafellar [19], there is $\alpha_v \in \mathbb{R}^d$ such that

$$\lambda_K^*(v) = \langle \alpha_v, v \rangle - \lambda_K(\alpha_v). \tag{A.1}$$

Consider the matrix (4.6) with $\alpha = \alpha_v$ and let $(f(y); y \in K)$ be the strictly positive right eigenvector of this matrix associated with its Perron-Frobenius eigenvalue $\exp \lambda_K(\alpha_v)$. Then the matrix $(\tilde{p}((x, y), (x', y'));\ (x, y), (x', y') \in \mathbb{Z}^N \times K)$ with

$$\tilde{p}((x, y), (x', y')) = p((x, y), (x', y')) \frac{f(y')}{f(y)} \exp(\langle \alpha_v, x' - x \rangle - \lambda_K(\alpha_v)) \tag{A.2}$$

is stochastic because for each $(x, y) \in \mathbb{Z}^N \times K$,

$$\sum_{x' \in \mathbb{Z}^N, y' \in K} p((x, y), (x', y')) = \exp(-\lambda_K(\alpha_v)) \frac{1}{f(y)} \times \sum_{y' \in K} \mathscr{P}(\alpha_v; y, y') f(y') = 1,$$

and we can consider a new Markov chain on $\mathbb{Z}^N \times K$ starting from $(0, y_0)$, with transition probabilities (A.2). Let $\tilde{\mathbb{P}}$ denote the distribution of this new Markov chain. To prove inequality (4.7) we will show that

$$\begin{aligned}\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}_{(0,y_0)}\left(\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right|<\delta n \text{ and } \mathscr{T}_{\mathrm{K}}>[\mathrm{n}\tau]\right)\\ \geq -\tau\lambda_K^*(v)+\lim_{\delta\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\tilde{\mathbb{P}}\left(\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right|<\delta n\right),\end{aligned}\tag{A.3}$$

and we will prove then that

$$\tilde{\mathbb{P}}\left(\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right|<\delta n\right)\to 1 \qquad \text{as } \ n\to\infty,\tag{A.4}$$

from which inequality (4.7) will follow.

We begin with the proof of relation (A.3). Given $n\in\mathbb{N}$, $\delta>0$ and $v\in\mathbb{R}^N$, denote

$$A_{n\delta}(v)=\left\{\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right|<\delta n\right\}.$$

Since $\mathscr{T}_K=+\infty$ almost surely with respect to the new probability measure $\tilde{\mathbb{P}}$, the standard arguments of the change of measures give

$$\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}|X(t)-vt|<\delta n \text{ and } \mathscr{T}_{\mathrm{K}}>[\mathrm{n}\tau]\right)=\tilde{\mathbb{E}}\left(\mathbb{1}_{A_{n\delta}(v)}M^{-1}(t)\right),\tag{A.5}$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$ and

$$M(t)=\exp\left\{\langle\alpha_v, X(t)\rangle-\lambda_K(\alpha_v)t\right\}f(Y(t))f^{-1}(y_0),\qquad t\in\mathbb{N}.$$

Since on the event $A_{n\delta}(v)$, the inequality $\left|X([n\tau])-n\tau v\right|<\delta n$ holds, identity (A.5) implies that

$$\begin{aligned}\log\mathbb{P}&\left(\sup_{t\in[[0,n\tau]]}\left|X(t)-vt\right|<\delta n \text{ and } \mathscr{T}_{\mathrm{K}}>[\mathrm{n}\tau]\right)\\ &\geq -n\tau\langle\alpha_v, v\rangle+[n\tau]\lambda_K(\alpha_v)+\delta n|\alpha_v|+\log f(y_0)\\ &\quad -\max_{y\in K}\log f(y)+\log\tilde{\mathbb{P}}\left(A_{n\delta}(v)\right)\end{aligned}$$

and hence, using (A.1), we obtain inequality (A.3).

Let us prove now (A.4). For this we will use a martingale technique.

Given $\alpha\in\mathbb{R}^d$, consider

$$M(\alpha,t)=\mathbb{1}_{\{\mathscr{T}_K>t\}}\exp\left\{\langle\alpha, X(t)\rangle-\lambda_K(\alpha)t\right\}f_\alpha(Y(t))f_\alpha^{-1}(y_0),\qquad t\in\mathbb{N},$$

where $f_\alpha=(f_\alpha(y);\ y\in K)$ is a strictly positive right eigenvector of the matrix $\mathscr{P}_K(\alpha)$ associated with its Perron-Frobenius eigenvalue $\exp\lambda_K(\alpha)$. The vector

f_α is unique to constant multiples because the matrix $\mathscr{P}_K(\alpha)$ is irreducible by assumption, and it is convenient to choose f_α so that for given $f = f_{\alpha_v}$,

$$f_\alpha(y) \geq f(y) \qquad \text{for all } y \in K. \tag{A.6}$$

Straightforward calculation shows that for $Y(0) = y_0 \in K$, $(M(\alpha, t))$ is a martingale with $\mathbb{E}(M(\alpha, t)) \equiv 1$. Since $M(\alpha_v, t) \equiv M(t)\mathbb{1}_{\{\mathscr{T}_K > t\}}$, we conclude that for $Y(0) = y_0 \in K$,

$$\begin{aligned} M(\alpha, t)M^{-1}(\alpha_v, t) = \exp\{\langle \alpha - \alpha_v, X(t)\rangle - (\lambda_K(\alpha) - \lambda_K(\alpha_v))\, t\} \\ \times f(y_0) f_\alpha^{-1}(y_0) f_\alpha(Y(t)) f^{-1}(Y(t)) \end{aligned} \tag{A.7}$$

is a martingale relative to the new probability measure $\tilde{\mathbb{P}}$ with

$$\tilde{\mathbb{E}}\big(M(\alpha, t)M^{-1}(\alpha_v, t)\big) \equiv 1.$$

The left hand side of (A.7) is defined and equality (A.7) holds almost surely with respect to the new probability measure $\tilde{\mathbb{P}}$ because $\mathscr{T}_K = +\infty$ almost surely with respect to $\tilde{\mathbb{P}}$.

Using (A.1) one can rewrite the right hand side of (A.7) as follows:

$$\begin{aligned} &\exp\{\langle \alpha - \alpha_v, X(t) - vt\rangle - (\lambda_K(\alpha) - \langle \alpha, v\rangle + \lambda_K^*(v))\, t\} \\ &\quad \times f(y_0) f_\alpha^{-1}(y_0) f_\alpha(Y(t)) f^{-1}(Y(t)) \end{aligned}$$

where by Fenchel inequality, $\lambda_K(\alpha) - \langle \alpha, v\rangle + \lambda_K^*(v) \geq 0$ for any $\alpha \in \mathbb{R}^N$. This proves that

$$\begin{aligned} \Xi(t) &= M(\alpha, t)M^{-1}(\alpha_v, t) \times \exp\{(\lambda_K(\alpha) - \langle \alpha, v\rangle + \lambda_K^*(v))\, t\} \times f_\alpha(y_0) f^{-1}(y_0) \\ &= \exp\{\langle \alpha - \alpha_v, X(t) - vt\rangle\} \times f_\alpha(Y(t)) f^{-1}(Y(t)) \mathbb{1}_{\{\mathscr{T}_K > t\}} \end{aligned}$$

is a sub-martingale relative to the new probability measure $\tilde{\mathbb{P}}$ with

$$\tilde{\mathbb{E}}(\Xi(t)) = f_\alpha(y_0) f^{-1}(y_0) \exp\{(\lambda_K(\alpha) - \langle \alpha, v\rangle + \lambda_K^*(v))\, t\}.$$

Using now relation (A.6) together with the sub-martingale inequality we obtain that for any $\gamma > 0$,

$$\begin{aligned} &\tilde{\mathbb{P}}\left(\sup_{t \in [[0, n\tau]]} \langle \alpha - \alpha_v, X(t) - vt\rangle \geq \gamma\right) \\ &\quad \leq \tilde{\mathbb{P}}_{(0, y_0)}\left(\sup_{t \in [[0, n\tau]]} \Xi(t) \geq e^\gamma\right) \\ &\quad \leq f_\alpha(y_0) f^{-1}(y_0) \exp\{-\gamma + (\lambda_K(\alpha) - \langle \alpha, v\rangle + \lambda_K^*(v))\, [n\tau]\}. \end{aligned} \tag{A.8}$$

Consider $\alpha = \alpha_v + \theta\epsilon$, where $\theta > 0$ and $\epsilon \in \mathbb{R}^N$ is a unit vector, and let

$$C = \max_{\alpha : |\alpha - \alpha_v| \leq 1} \left\langle \epsilon, \frac{\partial^2}{\partial \alpha^2} \lambda_K(\alpha) \epsilon \right\rangle$$

where $\partial^2 \lambda_K(\alpha)/\partial \alpha^2$ denotes Hessian matrix of $\lambda_K(\cdot)$. Then

$$0 \leq \lambda_K(\alpha) - \langle \alpha, v\rangle + \lambda_K^*(v) = \lambda_K(\alpha_v + \theta\epsilon) - \theta\langle \epsilon, v\rangle - \lambda_K(\alpha_v) \leq \theta^2 C$$

and letting $\gamma = (C\tau + 1)\theta^2 n$ in (A.8) we obtain

$$\tilde{\mathbb{P}}\left(\sup_{t\in[[0,n\tau]]}\langle\epsilon, X(t) - vt\rangle \geq \theta(C\tau + 1)n\right) \leq f_\alpha(y_0)f^{-1}(y_0)\exp\{-\theta^2 n\}.$$

Finally, the unit vector ϵ being arbitrary, the last inequality yields

$$\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}|X(t) - vt| \geq 2N\theta(C\tau + 1)n\right) \leq 2N\max_{\epsilon} f_{\alpha_v+\theta\epsilon}f^{-1}(y_0)\exp\left\{-\theta^2 n\right\}$$

and letting $\delta = 2N\theta(C\tau + 1)$ we get (A.4).

Relation (A.3) together with (A.4) implies (4.7) and therefore, our lemma is proved for $v \in \mathrm{ri}\,(\mathrm{dom}\,\lambda_K^*)$.

To extend this result for an arbitrary $v \in \mathbb{R}^N$ let us first verify that the function $w_k(\cdot)$ is upper semi-continuous. Indeed, for $\delta' < \delta/2$ and $|v - v'| < \delta/(2\tau)$,

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}&\left(\sup_{t\in[[0,n\tau]]}|X(t) - v't| < \delta' n \text{ and } \mathscr{T}_K > [n\tau]\right)\\ &\leq \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\sup_{t\in[[0,n\tau]]}|X(t) - vt| < \delta n \text{ and } \mathscr{T}_{\mathrm{K}} > [\mathrm{n}\tau]\right).\end{aligned}$$

Letting first $\delta' \to 0$ in the left hand side of the above relation and then letting $\delta \to 0$ in the right hand side we obtain

$$\limsup_{v'\to v} w_K(v') \leq w_K(v).$$

Since $v \in \mathbb{R}^N$ is arbitrary, we conclude that the function $w(\cdot)$ is upper semi-continuous.

We are ready now to extend the inequality (4.7) for an arbitrary $v \in \mathbb{R}^N$. When $v \notin \mathrm{dom}\,\lambda_K^*$, that is, $\lambda_K^*(v) = +\infty$, this inequality is trivial. Suppose that $v \in \mathrm{dom}\,\lambda_K^* \setminus \mathrm{ri}(\mathrm{dom}\,\lambda_K^*)$. In this case, there exists a sequence $v_k \in \mathrm{ri}(\mathrm{dom}\,\lambda^*)$ such that $v_k \to v$ and $\lambda_K^*(v_k) \to \lambda_K^*(v)$ as $k \to \infty$ because $\lambda_K^*(\cdot)$ is closed proper convex function on $\mathbb{R}^N$. Indeed, the set $\mathrm{ri}(\mathrm{dom}\,\lambda_K^*)$ is nonempty by Theorem 6.2 in [19], choosing $x_0 \in \mathrm{ri}(\mathrm{dom}\,\lambda_K^*)$ and letting $v_k = v_0/k + (1-1/k)v$ for $k \geq 1$, we obtain $v_k \in \mathrm{ri}(\mathrm{dom}\,\lambda^*)$ by Theorem 6.1 in [19] and $\lambda^*(v_k) \to \lambda^*(v)$ as $k \to +\infty$ by Theorem 7.5 in [19]. Using finally the upper semi-continuity of the function $w_K(\cdot)$ we get

$$w_K(v) \geq \limsup_{n\to\infty} w_K(v_n) \geq -\tau\lim_{n\to\infty}\lambda_K^*(v_n) = -\tau\lambda_K^*(v),$$

and hence, for $v \in \mathrm{dom}\,\lambda_K^* \setminus \mathrm{ri}(\mathrm{dom}\,\lambda_K^*)$, inequality (4.7) is also verified.

Proposition 4 is proved. □

APPENDIX B

LEMMA 8. *Let $a(x) \geq 0$ for all $x \in \mathbb{Z}^N$, and let the function*

$$H(\alpha) = \sum_{x \in \mathbb{Z}^N} a(x) e^{\langle \alpha, x \rangle}$$

be finite for all $\alpha \in \mathbb{R}^N$. If the infinite matrix $\big(a(x - x'); x, x' \in \mathbb{Z}^N\big)$ is irreducible, the function $H(\cdot)$ is strictly convex everywhere on $\mathbb{R}^N$ and the level sets of this function are compact.

PROOF. Since the infinite matrix $(a(x - x'); x, x' \in \mathbb{Z}^N)$ is irreducible, the set $\{x \in \mathbb{Z}^k : a(x) > 0\}$ contain the basis of the linear space $\mathbb{R}^k$. This implies that for every $v \in \mathbb{R}^k$, there exists $x \in \mathbb{Z}^k$ such that $a(x)\langle v, x\rangle \neq 0$ and consequently,

$$\langle v, \partial_\alpha^2 H(\alpha) v\rangle = \sum_{i,j=1}^N \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} H(\alpha) v_i v_j = \sum_{x \in \mathbb{Z}^N} a(x) e^{\langle \alpha, x\rangle} \langle v, x\rangle^2 > 0 \tag{B.1}$$

for all $\alpha, v \in \mathbb{R}^K$. This proves that the function $H(\cdot)$ is strictly convex.

Furthermore, the function $H(\cdot)$ being a finite convex function on $\mathbb{R}^k$, is continuous on $\mathbb{R}^k$ and hence, to prove that the level sets of this function are compact, it is sufficient to show that they are bounded.

Let us verify that the level sets of the function $H(\cdot)$ are bounded. Since the infinite matrix $(a(x - x'); x, x' \in \mathbb{Z}^N)$ is irreducible, for any $x \in \mathbb{Z}^N$, there exists $n \in \mathbb{N}$ and there exists a sequence $x_0, \dots, x_n \in \mathbb{Z}^k$ such that $a(x_k) > 0$ for all $k = 1, \dots, n$ and $x = x_0 + \dots + x_n$. This implies that

$$\begin{aligned}\langle \alpha, z\rangle - H(\alpha) &\leq \sum_{k=1}^n (\langle \alpha, x_k\rangle - a(x_k) e^{\langle \alpha, x_k\rangle}) \\ &\leq \sum_{k=1}^n \sup_{t \in \mathbb{R}} (t - a(x_k) e^t) \leq \sum_{k=1}^n (-\log a(x_k) - 1).\end{aligned}$$

Consider now the convex conjugate $H^*(\cdot)$ of the function $H(\cdot)$. The last relation shows that $H^*(x) < +\infty$ for all $x \in \mathbb{Z}^K$ and consequently, $H^*(v) < +\infty$ for all $v \in \mathbb{R}^k$ because the function $H^*(\cdot)$ is convex. Using now Fenchel's inequality

$$\langle \alpha, v\rangle \leq H(\alpha) + H^*(v), \qquad \alpha, v \in \mathbb{R}^N,$$

we get

$$\sup_{v \in \mathbb{R}^N : |v| \leq 1} \langle \alpha, v\rangle \leq c + \sup_{v \in \mathbb{R}^N : |v| \leq 1} H^*(v)$$

for all $\alpha \in \mathbb{R}^N$ such that $H(\alpha) \leq c$. The function $H^*(\cdot)$ being finite and convex on $\mathbb{R}^k$, is continuous on $\mathbb{R}^k$. This proves that the right hand side of the above inequality is finite and hence, the set $\{\alpha \in \mathbb{R}^N : H(\alpha) \leq c\}$ is bounded for any $c \in \mathbb{R}_+$. Lemma 8 is proved. $\square$

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