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STRONG APPROXIMATION OF THE EMPIRICAL PROCESS OF GARCH SEQUENCES

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We obtain a strong approximation for the empirical process of n observed elements of a GARCH sequence. The weak convergence of the empirical process and the law of the iterated logarithm are immediate consequences.

1. Introduction and results. Over the last years several models have been suggested to serve as models for financial data. Many of these models have the property that the conditional variance (or conditional scaling) depends on past observations. Empirical work has confirmed the applicability of these models to analyze financial time series. One of the well-known examples is the autoregressive conditionally heteroskedastic (ARCH) process introduced by Engle (1982). It is used to model exchange rates, stock prices and so on. The ARCH model has been investigated and generalized by several authors; see, for example, Bollerslev (1986) and Gouriéroux (1997). In this paper we investigate the generalized autoregressive conditionally heteroskedastic (GARCH) process introduced by Bollerslev (1986). A GARCH (p, q) process is defined by the equations

$$y_k = \sigma_k \varepsilon_k \tag{1.1}$$

and

$$\sigma_k^2 = \delta + \sum_{1\le i\le p} \beta_i \sigma_{k-i}^2 + \sum_{1\le j\le q} \alpha_j y_{k-j}^2, \tag{1.2}$$

where $\delta > 0$, β_i, $1 \le i \le p$ and α_j, $1 \le j \le q$ are nonnegative constants. Throughout this paper we assume that $\{\varepsilon_i, -\infty < i < \infty\}$ are independent, identically distributed random variables with distribution function H. The main purpose of this paper is to prove limit theorems for the empirical distribution function of $y_1, y_2, \dots, y_n$ assuming that (1.1) and (1.2) have a unique stationary solution.

Nelson (1990) found a necessary and sufficient condition for the stationarity and ergodicity of the GARCH $(1,1)$ process. He showed that in the case of $p = q = 1$, (1.1) and (1.2) have a unique stationary solution if and only if $E\log(\beta_1 + \alpha_1\varepsilon_0^2) < 0$. A necessary and sufficient condition for the existence of

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a unique stationary solution of (1.1) and (1.2) in the general case was given by Bougerol and Picard (1992a, b). To state this condition, let

$$\tau_n = (\beta_1 + \alpha_1 \varepsilon_n^2, \beta_2, \ldots, \beta_{p-1}) \in R^{p-1},$$

$$\xi_n = (\varepsilon_n^2, 0, \ldots, 0) \in R^{p-1}$$

and

$$\alpha = (\alpha_2, \ldots, \alpha_{q-1}) \in R^{q-1}.$$

(Clearly, without loss of generality we may and shall assume $p \geq 2$ and $q \geq 2$.) Define the $(p+q-1) \times (p+q-1)$ matrix A_n, written in block form, by

$$A_n = \begin{bmatrix} \tau_n & \beta_p & \alpha & \alpha_q \\ I_{p-1} & 0 & 0 & 0 \\ \xi_n & 0 & 0 & 0 \\ 0 & 0 & I_{q-2} & 0 \end{bmatrix},$$

where I_{p-1} and I_{q-2} are the identity matrices of size $p-1$ and $q-2$, respectively. We have assumed that the innovations $\{\varepsilon_i,\ -\infty < i < \infty\}$ are independent, identically distributed random variables and therefore the random matrices $\{A_n,\ -\infty < n < \infty\}$ are independent and identically distributed. Assume that $E(\log^+ \|A_0\|) < \infty$, where for any $d \times d$ matrix M, $\|M\|$ denotes the matrix norm defined by

$$\|M\| = \sup\bigl\{\|Mx\|_d / \|x\|_d\colon\ x \in R^d,\ x \neq 0\bigr\},$$

where $\|\cdot\|$ is the usual (Euclidean) norm in R^d. The top Lyapunov exponent γ associated with the sequence $\{A_n,\ -\infty < n < \infty\}$ is

$$\gamma = \inf_{1 \leq n < \infty} \frac{1}{n} E\ \log \| \ A_0 A_{-1} \cdots A_{-n} \ \| \ .$$

The condition $E(\log^+ \| \ A_0 \ \|) < \infty$ and the subadditive ergodic theorem [cf. Kingman (1973)] imply

$$\lim_{n\to\infty} \frac{1}{n} \log \| \ A_0 A_{-1} \cdots A_{-n} \ \| = \gamma \quad \text{a.s.} \tag{1.3}$$

Bougerol and Picard (1992a, b) showed that (1.1) and (1.2) have a unique stationary solution if and only if

$$\gamma < 0. \tag{1.4}$$

In this paper we investigate the asymptotic properties of the empirical process

$$R(s, t) = \sum_{1 \leq i \leq t} (I\{y_i \leq s\} - F(s)),$$

where F denotes the distribution function of y_0. We make stronger assumptions than $E(\log^+ \| \ A_0 \ \|) < \infty$ and (1.4), so we can assume that we are using the stationary solution of (1.1) and (1.2). Let

$$Y_k(s) = I\{y_k \leq s\} - F(s).$$

THEOREM 1.1. *We assume that*

$$|H(t) - H(s)| \le C|t - s|^{\theta} \quad \text{with some } 0 < C < \infty \text{ and } 0 < \theta \le 1, \tag{1.5}$$

$$E(\log^+ \| A_0 \|)^{\mu} < \infty \quad \text{with some } \mu > 2 + 16/\theta \tag{1.6}$$

and

$$\gamma_1 = E(\log \| A_0 \|) < 0. \tag{1.7}$$

Then the series

$$\Gamma(s, s') = EY_0(s)Y_0(s') + \sum_{1 \le n < \infty} EY_0(s)Y_n(s') + \sum_{1 \le n < \infty} EY_0(s')Y_n(s) \tag{1.8}$$

is absolutely convergent for any $-\infty < s, s' < \infty$ *and there is a Gaussian process* $K(s, t)$ *with* $EK(s, t) = 0$, $EK(s, t)K(s', t') = \min(t, t')\Gamma(s, s')$, *such that*

$$\sup_{0 \le t \le T} \sup_{-\infty < s < \infty} |R(s, t) - K(s, t)| = o\big(T^{1/2}(\log T)^{-\lambda}\big) \quad a.s. \tag{1.9}$$

with some $\lambda > 0$.

REMARK 1.1. In GARCH (1, 1), conditions (1.6) and (1.7) are satisfied if and only if

$$E\big(\log^+\big(\beta_1 + \alpha_1 \varepsilon_0^2\big)\big)^{\mu} < \infty \quad \text{with some } \mu > 2 + 16/\theta \tag{1.10}$$

and

$$E \log\big(\beta_1 + \alpha_1 \varepsilon_0^2\big) < 0. \tag{1.11}$$

We note again that by Nelson (1990), (1.11) is necessary and sufficient for the existence of the stationary GARCH (1, 1). For general GARCH sequences, (1.7) is more restrictive than (1.4).

REMARK 1.2. If $E(\log^+ |\varepsilon_0|)^{\mu} < \infty$ with some $\mu > 2 + 16/\theta$, then (1.6) holds in any GARCH (p, q) model.

REMARK 1.3. Davis, Mikosch and Basrak (1999) showed that if (1.4) holds and $E|\varepsilon_0|^{\mu} < \infty$ with some $\mu > 0$, then $\{y_n^2, -\infty < n < \infty\}$ is strongly mixing with a geometric rate. Combining the mixing property of $\{y_n^2\}$ with the main result in Philipp and Pinzur (1980) and Philipp (1984) one could get strong approximations for $\sum_{1 \le i \le t}(I\{y_i^2 \le s\} - F^*(s))$, where $F^*(s)$ is the distribution function of y_0^2. However, the method used by Davis, Mikosch and Basrak (1999) does not give a similar result for $R(s, t)$. Also, to get the strong mixing of $\{y_n^2\}$, Davis, Mikosch and Basrak (1999) assume that $E|\varepsilon_0|^{\mu} < \infty$ with some $\mu > 0$, which is stronger than (1.6).

In the proof of our theorem we will use a totally different approach and establish a new structural property of GARCH sequences which is easier to verify than strong mixing and is much more convenient in applications.

(See Lemma 2.4.) Indeed, we prove that if (1.6), (1.7) hold then there is a sequence $\{y'_n\}$ which is close to $\{y_n\}$ (in the sense that $y_n - y'_n \to 0$ rapidly) and for each $n \geq 1$ the finite sequence $\{y'_1, \dots, y'_n\}$ is not only mixing, but in fact is m-dependent with $m = n^\rho$ for some $0 < \rho < 1$; that is, terms of this sequence with indices differing at least n^ρ are independent. This property does not imply strong mixing for $\{y_n\}$ but it is more convenient to use. In fact, it permits us to deduce asymptotic properties of $\{y_n\}$ directly from known results for independent random variables via standard blocking techniques. We note also that this approach requires weaker moment conditions on $\{\varepsilon_n\}$ than the strong mixing technique. It seems likely that the same method will be applicable in many other situations.

The weak convergence of the empirical process of $y_1, \dots, y_n$ is a simple consequence of Theorem 1.1. If $\widehat{K}(s)$ is a Gaussian process with $E\widehat{K}(s) = 0$ and $E\widehat{K}(s)\widehat{K}(s') = \Gamma(s, s')$, then

$$n^{1/2}\left(\frac{1}{n}\sum_{1\le i\le n} I\{y_i \le s\} - F(s)\right) \overset{\mathscr{D}[-\infty,\infty]}{\longrightarrow} \widehat{K}(s),$$

as $n \to \infty$.

Similarly, the law of the iterated logarithm also follows from Theorem 1.1. It is enough to observe that $K(s, n)$ is a partial sum of independent, identically distributed Gaussian processes, so by Ledoux and Talagrand (1991) the law of the iterated logarithm holds for $K(s, n)$. Hence

$$\limsup_{n\to\infty}\left(\frac{n}{2\log\log n}\right)^{1/2} \sup_{-\infty<s<\infty}\left|\frac{1}{n}\sum_{1\le i\le n} I\{y_i \le s\} - F(s)\right| = c \quad \text{a.s.}$$

with some $0 < c < \infty$.

2. Proofs. Let

$$X_n = \left(\sigma^2_{n+1}, \dots, \sigma^2_{n-p+2}, y^2_n, \dots, y^2_{n-q+2}\right) \in R^{p+q-1}$$

and

$$B = (\delta, 0, \dots, 0) \in R^{p+q-1}.$$

Bougerol and Picard (1992a, b) showed that

$$X_n = B + \sum_{0\le k<\infty} A_n A_{n-1} \cdots A_{n-k} B. \tag{2.1}$$

We start with two elementary lemmas.

LEMMA 2.1. *If* (1.7) *holds and*

$$E(\log^+ \| A_0 \|)^\mu < \infty \quad \textit{with some } \mu > 2, \tag{2.2}$$

then

$$P\left\{\| A_0A_{-1}\cdots A_{-k} \|\geq \exp\left(-\frac{|\gamma_1|}{2}k\right)\right\} \leq Ck^{-\mu/2} \tag{2.3}$$

for all $1 \leq k < \infty$ *with some* $0 < C < \infty$.

PROOF. Clearly,

$$\begin{aligned}
&P\left\{\| A_0A_{-1}\cdots A_{-k} \|\geq \exp\left(-\frac{|\gamma_1|}{2}k\right)\right\}\\
&\quad\leq P\left\{\prod_{0\leq j\leq k}\| A_{-j}\|\geq \exp\left(-\frac{|\gamma_1|}{2}k\right)\right\}\\
&\quad= P\left\{\sum_{0\leq j\leq k}\log\| A_{-j}\|\geq -\frac{|\gamma_1|}{2}k\right\}\\
&\quad\leq P\left\{\sum_{0\leq j\leq k}(\log\| A_{-j}\|+|\gamma_1|)\geq \frac{|\gamma_1|}{2}k\right\}\\
&\quad\leq \left(\frac{2}{|\gamma_1|}\right)^{\mu}k^{-\mu}E\left|\sum_{0\leq j\leq k}(\log\| A_{-j}\|+|\gamma_1|)\right|^{\mu}\\
&\quad\leq Ck^{-\mu/2}
\end{aligned}$$

with an application of the Rosenthal inequality [cf. Petrov (1995), page 59].

Let

$$Z_0 = \sum_{0\leq k<\infty} A_0A_{-1}\cdots A_{-k}B.$$

LEMMA 2.2. *If* (1.7) *and* (2.2) *hold*, *then*

$$P\{\| Z_0 \|\geq t\} \leq C(\log t)^{-(\mu-2)/2} \tag{2.4}$$

for all $t_0 \leq t < \infty$.

PROOF. Choose $0 < \rho' < 1$ so close to 1 that $|\log\rho'| < |\gamma_1|/2$ and let $c = \rho'/(1-\rho')$. Then using again the Rosenthal inequality [cf. Petrov (1995), page 59], we obtain for $t \geq t_0$,

$$\begin{aligned}
P\{\| Z_0\|\geq t\} &\leq \sum_{1\leq k<\infty} P\{\| A_0A_{-1}\cdots A_{-k}B\|\geq c(\rho')^k t\}\\
&\leq \sum_{1\leq k<\infty} P\left\{\| B\|\prod_{0\leq j\leq k}\| A_{-j}\|\geq c(\rho')^k t\right\}\\
&\leq \sum_{1\leq k<\infty} P\left\{|\log\delta|+\sum_{0\leq j\leq k}\log\| A_{-j}\|\geq \log c + k\log\rho' + \log t\right\}
\end{aligned}$$

$$= \sum_{1\le k<\infty} P\Big\{ \sum_{1\le j\le k} \big(\log \| A_{-j} \| + |\gamma_1|\big) \ge \log t + \big(|\gamma_1| - |\log \rho'|\big) k$$
$$+ \log c - |\log \delta| \Big\}$$
$$\le \sum_{1\le k<\infty} P\Big\{ \sum_{1\le j\le k} (\log \| A_{-j} \| + |\gamma_1|) > \frac{1}{2}\Big(\log t + \frac{|\gamma_1|}{2} k\Big)\Big\}$$
$$\le \sum_{1\le k<\infty} c_1 \Big(\log t + \frac{|\gamma_1|}{2} k\Big)^{-\mu} k^{\mu/2}$$
$$\le \sum_{1\le k<\infty} c_2 \Big(\log t + \frac{|\gamma_1|}{2} k\Big)^{-\mu/2}$$
$$\le c_2 \int_1^\infty \Big(\log t + \frac{|\gamma_1|}{2} x\Big)^{-\mu/2} dx + c_2 \Big(\log t + \frac{|\gamma_1|}{2}\Big)^{-\mu/2}$$
$$= \frac{2c_2}{\mu-2} \Big(\log t + \frac{|\gamma_1|}{2}\Big)^{1-\mu/2} + c_2 \Big(\log t + \frac{|\gamma_1|}{2}\Big)^{-\mu/2},$$

completing the proof of (2.4). □

Let

$$X_n' = B + \sum_{0\le k\le [n^\rho]} A_n A_{n-1} \cdots A_{n-k} B$$

with some $0 < \rho < 1$. We show that X_n and X_n' are near to each other.

LEMMA 2.3. *If* (1.7) *and* (2.2) *hold, then*

$$P\big\{\| X_n - X_n' \| > C n^{-\rho(\mu-2)/2}\big\} \le C n^{-\rho(\mu-2)/2} \tag{2.5}$$

with some $C > 0$.

PROOF. We prove the somewhat stronger inequality

$$P\{\| X_n - X_n' \| > \exp(-c_1 n^\rho)\} \le C n^{-\rho(\mu-2)/2} \tag{2.6}$$

with some c_1. First we write

$$X_n - X_n' = \sum_{[n^\rho]<k<\infty} A_n A_{n-1} \cdots A_{n-k} B = A_n A_{n-1} \cdots A_{n-[n^\rho]} \widehat{Z}_N, \tag{2.7}$$

where $N = n - [n^\rho] - 1$ and

$$\widehat{Z}_N = \sum_{0\le j<\infty} A_N A_{N-1} \cdots A_{N-j} B.$$

Since $\{A_n, -\infty < n < \infty\}$ are independent and identically distributed, by Lemma 2.1 we have

$$P\left\{\| A_n A_{n-1} \dots A_{n-[n^\rho]} \| \geq \exp\left(-\frac{|\gamma_1|}{2}[n^\rho]\right)\right\} \leq c_2 n^{-\mu\rho/2} \tag{2.8}$$
$$\leq c_2 n^{-\rho(\mu-2)/2}.$$

Since $\widehat{Z}_N \stackrel{\mathscr{D}}{=} Z_0$ for any N, Lemma 2.2 yields

$$P\left\{\| \widehat{Z}_N \| \geq \exp\left(\frac{|\gamma_1|}{4}[n^\rho]\right)\right\} \leq c_3 n^{-\rho(\mu-2)/2}. \tag{2.9}$$

Putting together (2.7)–(2.9) we conclude

$$P\left\{\| X_n - X'_n \| \leq \exp\left(-\frac{|\gamma_1|}{2}[n^\rho]\right)\exp\left(\frac{|\gamma_1|}{4}[n^\rho]\right)\right\}$$
$$\geq 1 - (c_2 + c_3) n^{-\rho(\mu-2)/2},$$

completing the proof of (2.6). □

LEMMA 2.4. *We assume that* (1.7), (2.2) *hold and* $0 < \rho < 1$. *Then there exist measurable functions* f_n: $R^{[n^\rho]} \to R$ $(n = 1, 2, \dots)$ *such that setting*

$$y'_n = f_n(\varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_{n-[n^\rho]}),$$

we have

$$P\{|y_n - y'_n| > C n^{-\rho(\mu-2)/4}\} \leq C n^{-\rho(\mu-2)/4} \tag{2.10}$$

with some $C > 0$.

PROOF. Define y'_n in such a way that $(y'_n)^2$ is identical with the $(p+1)$th coordinate of X'_n and y'_n has the sign of y_n (i.e., the sign of ε_n). Since X'_n is a measurable function of $\varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_{n-[n^\rho]}$, the same holds for y'_n. Also, (2.5) implies that

$$P\{|y_n^2 - (y'_n)^2| > c_1 n^{-\rho(\mu-2)/2}\} \leq c_1 n^{-\rho(\mu-2)/2}. \tag{2.11}$$

We consider the event when $|y_n^2 - (y'_n)^2| \leq c_1 n^{-\rho(\mu-2)/2}$. The variables y_n and y'_n have the same sign, so we have

$$|y_n - y'_n| = \frac{|y_n^2 - (y'_n)^2|}{|y_n + y'_n|} \leq \frac{|y_n^2 - (y'_n)^2|}{|y_n|}. \tag{2.12}$$

On this event, (2.12) yields that

$$|y_n - y'_n| = |y_n - y'_n| I\{|y_n| > n^{-\rho(\mu-2)/4}\} + |y_n - y'_n| I\{|y_n| \leq n^{-\rho(\mu-2)/4}\}$$
$$\leq c_1 n^{-\rho(\mu-2)/4} + \left((c_1+1)^{1/2} + 1\right) n^{-\rho(\mu-2)/4}$$

and therefore (2.10) follows from (2.11). □

Next we note that condition (1.5) implies that F is also Lipshitz continuous of order θ. Indeed, it follows from the definition of y_0 that ε_0 and σ_0 are independent. Hence by (1.5) we have

$$\begin{aligned} |F(x) - F(y)| &= |P\{y_0 \le x\} - P\{y_0 \le y\}| \\ &\le E|H(x/\sigma_0) - H(y/\sigma_0)| \\ &\le C\delta^{-\theta/2}|x-y|^{\theta}, \end{aligned} \tag{2.13}$$

since $\sigma_0 \ge \delta^{1/2}$ by (1.2). Let

$$Y'_n(s) = I\{y'_n \le s\} - F(s).$$

LEMMA 2.5. *If* (1.5), (1.7) *and* (2.2) *hold*, $0 < \rho < 1$, *then for any* $n \ge 2$ *and* $-\infty < t, s < \infty$ *we have*

$$E|Y_0(s)Y_n(t)| \le Cn^{-\rho\theta(\mu-2)/4}$$

with some $C > 0$.

PROOF. By Lemma 2.4 there is c_1 such that $P(A^c) \le c_1 n^{-\rho(\mu-2)/4}$, where

$$A = \{|y_n - y'_n| \le c_1 n^{-\rho(\mu-2)/4}\}.$$

The event $\{Y_n(s) \ne Y'_n(s)\} \cap A$ implies that y_n and y'_n are on different sides of s, their distance is less than $c_1 n^{-\rho(\mu-2)/4}$ and therefore both of them are in the interval $(s - c_1 n^{-\rho(\mu-2)/4}, s + c_1 n^{-\rho(\mu-2)/4})$. According to (2.13),

$$P\{s - c_1 n^{-\rho(\mu-2)/4} \le y_n \le s + c_1 n^{-\rho(\mu-2)/4}\} \le c_2 n^{-\rho\theta(\mu-2)/4}$$

and therefore

$$P\{Y_n(s) \ne Y'_n(s)\} \le 2c_3 n^{-\rho\theta(\mu-2)/4}. \tag{2.14}$$

By (2.14) we also have

$$E|Y_n(s) - Y'_n(s)| \le c_3 n^{-\rho\theta(\mu-2)/4}. \tag{2.15}$$

Since $|Y_0(s)| \le 1$, using (2.15) we get

$$|EY_0(s)Y_n(t) - EY_0(s)Y'_n(t)| \le E|Y_n(t) - Y'_n(t)| \le 2c_3 n^{-\rho\theta(\mu-2)/4}. \tag{2.16}$$

On the other hand, Lemma 2.4 shows that if $n \ge 2$, then y_0 and y'_n are independent, which implies that $Y_0(s)$ and $Y'_n(t)$ are independent. Since $EY_0(s) = 0$ we get

$$EY_0(s)Y'_n(t) = EY_0(s)EY'_n(t) = 0,$$

and therefore Lemma 2.5 follows from (2.16). $\square$

REMARK 2.1. Under the condition $\mu > 2 + 16/\theta$ of Theorem 1.1 we can choose $0 < \rho < 1$ such that $\rho\theta(\mu - 2)/4 > 1$, establishing the absolute convergence in (1.8).

For any $-\infty < s < t < \infty$ let

$$\overline{Y}_n(s,t) = Y_n(t) - Y_n(s) = I\{s < y_n \le t\} - (F(t) - F(s))$$

and

$$\overline{Y}'_n(s,t) = Y'_n(t) - Y'_n(s) = I\{s < y'_n \le t\} - (F(t) - F(s)).$$

LEMMA 2.6. *If* (1.5), (1.6) *and* (1.7) *hold, then there is a* $\tau > 0$ *such that for any* $-\infty < s \le t < \infty$,

$$|E\overline{Y}_0(s,t)\overline{Y}_n(s,t)| \le \frac{C}{n^{2+\tau}}(F(t) - F(s))^\tau$$

with some $C > 0$.

PROOF. Following the proof of Lemma 2.5 one can show that for any $0 < \rho < 1$ there is a constant $c_1 > 0$ such that

$$|E\overline{Y}_0(s,t)\overline{Y}_n(s,t)| \le c_1 n^{-\rho\theta(\mu-2)/4}. \tag{2.17}$$

Observing that $E\overline{Y}_n^2(s,t) = (F(t) - F(s))(1 - (F(t) - F(s))) \le F(t) - F(s)$ for any $n \ge 0$, by the Cauchy–Schwarz inequality we have

$$|E\overline{Y}_0(s,t)\overline{Y}_n(s,t)| \le F(t) - F(s). \tag{2.18}$$

Choosing $0 < \rho < 1/2$ close enough to $1/2$ we can have that $\rho\theta(\mu - 2)/4 > 2$ and therefore Lemma 2.6 follows from (2.17) and (2.18). □

LEMMA 2.7. *If* (1.5), (1.6) *and* (1.7) *hold, then there is a* $\tau > 0$ *such that for any* $-\infty < s < t < \infty$,

$$E\biggl(\sum_{1\le k\le N} \overline{Y}_k(s,t)\biggr)^2 = \sigma^2 N + O((F(t) - F(s))^\tau) \quad \textit{as } N \to \infty,$$

uniformly in s *and* t, *where*

$$\sigma^2 = \sigma^2(s,t) = E\overline{Y}_1^2(s,t) + 2\sum_{2\le k<\infty} E\overline{Y}_1(s,t)\overline{Y}_k(s,t). \tag{2.19}$$

PROOF. For notational simplicity we write $\overline{Y}_k$ instead of $\overline{Y}_k(s,t)$ and l stands for $F(t) - F(s)$. First we note that by Lemma 2.6 we have

$$|E\overline{Y}_0\overline{Y}_k| \le \frac{c_1}{k^{2+\tau}} l^\tau, \qquad 0 \le k < \infty, \tag{2.20}$$

with some constants $\tau > 0$ and $c_1 > 0$. Thus the series in (2.19) is absolute convergent and

$$\sigma^2(s,t) \le c_2(F(t) - F(s))^\tau. \tag{2.21}$$

By stationarity and (2.20) we conclude

$$
\begin{aligned}
E\biggl(\sum_{1\le k\le N}\overline{Y}_k\biggr)^2 &= NE\overline{Y}_1^2 + 2\sum_{1\le k\le N-1}(N-k)E\overline{Y}_1\overline{Y}_{k+1}\\
&= N\sigma^2 - 2N\sum_{N\le k<\infty}E\overline{Y}_1\overline{Y}_{k+1} - 2\sum_{1\le k\le N-1}kE\overline{Y}_1\overline{Y}_{k+1}\\
&= N\sigma^2 + O(l^\tau) \quad \text{as } N\to\infty,
\end{aligned}
$$

and Lemma 2.7 is proved. □

The previous arguments also show that

$$2\Gamma(s,t) = \sigma^2(0,s) + \sigma^2(0,t) - \sigma^2(s,t) \tag{2.22}$$

and

$$\sigma^2(s,t) = \Gamma(s,s) + \Gamma(t,t) - 2\Gamma(s,t). \tag{2.23}$$

LEMMA 2.8. *If* (1.5), (1.6) *and* (1.7) *hold, then there exist constants* $\tau > 0$, $0 < \rho < 1/2$ *such that for any* $x > 0$,

$$
\begin{aligned}
P\biggl\{\biggl|\sum_{1\le k\le N}\overline{Y}_k(s,t)\biggr| > x\biggr\} &\le c_1\exp\bigl(-c_2x^2/(N|F(t)-F(s)|^\tau)\bigr)\\
&\qquad + c_3\exp(-c_4x/N^\rho) + c_5x^{-(2+\tau)},
\end{aligned} \tag{2.24}
$$

where $c_1,\dots,c_5$ *are positive constants.*

PROOF. To simplify the notation, we write again $\overline{Y}_k$, $\overline{Y}'_k$ instead of $\overline{Y}_k(s,t)$ and $\overline{Y}'_k(s,t)$. Assume that $s\le t$ and write $l = F(t) - F(s)$,

$$S_N = \sum_{1\le k\le N}\overline{Y}_k \quad \text{and} \quad S'_N = \sum_{1\le k\le N}\overline{Y}'_k.$$

Choose $0 < \rho < 1/2$ so that $\rho\theta(\mu-2)/4 > 2$ and let τ be the constant in Lemma 2.7. By (2.14) we have

$$
\begin{aligned}
P\{\overline{Y}_k \ne \overline{Y}'_k\} &\le c_6k^{-\rho\theta(\mu-2)/4}\\
&\le c_6k^{-(2+\delta)}
\end{aligned}
$$

with some $0<\delta<1$. Since $|\overline{Y}_k|\le 1$ and $|\overline{Y}'_k|\le 1$, we conclude that

$$E|\overline{Y}_k - \overline{Y}'_k|^p \le 8c_6k^{-(2+\delta)},$$

where $p = 2+\delta/2$ and therefore by the Minkowski inequality we have

$$
\begin{aligned}
\bigl(E|S_N - S'_N|^p\bigr)^{1/p} &\le \sum_{1\le k\le N}\bigl(E|\overline{Y}_k - \overline{Y}'_k|^p\bigr)^{1/p}\\
&\le c_7\sum_{1\le k\le N}k^{-\nu} \le c_8,
\end{aligned}
$$

where $\nu = (2+\delta)/(2+\delta/2) > 1$. Hence using the Markov inequality we get that

$$P\{|S_N - S'_N| > x\} \le c_9 x^{-(2+\delta/2)}. \tag{2.25}$$

According to (2.25), the inequality in (2.24) will be proved if we show that

$$P\{|S'_N| > x\} \le c_1 \exp(-c_2 x^2/(Nl^\tau)) + c_3 \exp(-c_4 x/N^\rho). \tag{2.26}$$

Let us split the interval $[1, N]$ into blocks $I_1, J_1, I_2, J_2, \dots, I_M, J_M$ with equal length $[N^\rho]$. (The blocks I_M, J_M can be incomplete.) Clearly, M is proportional to $N^{1-\rho}$. Set

$$T_r^{(1)} = \sum_{k \in I_r} \overline{Y}'_k \quad \text{and} \quad T_r^{(2)} = \sum_{k \in J_r} \overline{Y}'_k.$$

We note that

$$S'_N = S_N^{(1)} + S_N^{(2)}$$

with

$$S_N^{(1)} = \sum_{1 \le r \le M} T_r^{(1)} \quad \text{and} \quad S_N^{(2)} = \sum_{1 \le r \le M} T_r^{(2)}.$$

By the construction of y'_k (cf. Lemma 2.4), the random variables $T_r^{(1)}, r = 1, 2, \dots, M$ are independent and

$$E(T_r^{(1)})^2 \le c_7 N^\rho l^\tau$$

[cf. Lemma 2.7 and (2.21)], and therefore

$$E(S_N^{(1)})^2 \le c_8 N l^\tau.$$

Also,

$$\max_{1 \le r \le M} |T_r^{(1)}| \le N^\rho.$$

Hence by Kolmogorov's exponential bounds [cf. Petrov (1975), page 293] we obtain

$$P\{|S_N^{(1)}| > x\} \le c_9 \exp(-c_{10} x^2/(Nl^\tau)) + c_{11} \exp(-c_{12} x/N^\rho).$$

A similar inequality holds for $S_N^{(2)}$ and therefore (2.26) is proved. □

Next we need an estimate in the central limit theorem for sums of independent, identically distributed random vectors. Let $||\cdot||$ denote the Euclidean norm of vectors and matrices.

LEMMA 2.9. *Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_n$ be independent, identically distributed random vectors in R^d satisfying $E\boldsymbol{\xi}_1 = \mathbf{0}$, $\operatorname{cov}(\boldsymbol{\xi}_1) = \boldsymbol{\Sigma}$ and $m_4 = E||\boldsymbol{\xi}_1||^4 < \infty$. Let Q_n denote the distribution function of $(\boldsymbol{\xi}_1 + \cdots + \boldsymbol{\xi}_n)/n^{1/2}$ and let $\Phi_{\boldsymbol{\Sigma}}$ be the multivariate normal distribution function with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$. Let $h(\mathbf{x})$ be a (real or complex valued) Borel function satisfying $|h(\mathbf{x})| \le L$ and*

$|h(\mathbf{x}) - h(\mathbf{y})| \leq L||\mathbf{x} - \mathbf{y}||$ *for any* $\mathbf{x}, \mathbf{y} \in R^d$ *with some* $L > 0$. *Then there are absolute constants* c_1, c_2 *and* c_3 *such that*

$$\left|\int_{R^d} h\, dQ_n - \int_{R^d} h\, d\Phi_{\mathbf{\Sigma}}\right|$$
$$\leq L^2 c_1\Big(||\mathbf{\Sigma}|| + 1\Big)\Big\{e^{c_2 d}||\mathbf{\Sigma}||^2 m_4^4 (\log n)^{d/2} n^{-1/2} + e^{-c_3 n} d^{d/2}\Big\}.$$

PROOF. We can assume without loss of generality that

$$\boldsymbol{\xi}_i = \mathbf{\Sigma}^{1/2}\boldsymbol{\xi}_i^*,$$

where $\boldsymbol{\xi}_1^*, \dots, \boldsymbol{\xi}_n^*$ are independent, identically distributed random vectors with $E\boldsymbol{\xi}_i^* = \mathbf{0}$ and $\operatorname{cov}(\boldsymbol{\xi}_i^*) = I_d$, the identity matrix of R^d. We also note that $E||\boldsymbol{\xi}_1'||^4 \leq ||\mathbf{\Sigma}||^2 m_4$. Let Q_n^* denote the distribution function of $(\boldsymbol{\xi}_1^* + \cdots + \boldsymbol{\xi}_n^*)/n^{1/2}$. We assume that $|h^*(\mathbf{x})| \leq L^*$ and $|h^*(\mathbf{x}) - h^*(\mathbf{y})| \leq L^*||\mathbf{x} - \mathbf{y}||$. By Theorem 3.2 in Bhattacharya and Rao [(1976), page 113] we have

$$\left|\int_{R^d} h^* dQ_n^* - \int_{R^d} h^* d\Phi_{I_d}\right|$$
$$\leq (L^*)^2 c_1\Big\{e^{c_2 d}(E||\boldsymbol{\xi}_1'||^4)^4 (\log n)^{d/2} n^{-1/2} + e^{-c_3 n} d^{d/2}\Big\}. \tag{2.27}$$

We use (2.27) with $h^*(\mathbf{x}) = h(\mathbf{\Sigma}^{1/2}\mathbf{x})$. Since

$$\big|h(\mathbf{\Sigma}^{1/2}\mathbf{x}) - h(\mathbf{\Sigma}^{1/2}\mathbf{y})\big| \leq L\big|\big|\mathbf{\Sigma}^{1/2}(\mathbf{x} - \mathbf{y})\big|\big| \leq L||\mathbf{\Sigma}||^{1/2}\, ||\mathbf{x} - \mathbf{y}||,$$

Lemma 2.9 is proved. □

The inner product of vectors will be denoted by $\langle \cdot, \cdot \rangle$.

LEMMA 2.10. *We assume that the conditions of Lemma* 2.8 *are satisfied. Let* $-\infty < s_1 < s_2 \cdots < s_d < \infty$ *and*

$$\boldsymbol{\xi}_k = (Y_k(s_1), Y_k(s_2), \dots, Y_k(s_d)), \qquad 1 \leq k < \infty.$$

Then for any $\mathbf{u} \in R^d$ *we have*

$$\left|E\exp\Big(i\Big\langle \mathbf{u}, N^{-1/2}\sum_{1\leq k\leq N}\boldsymbol{\xi}_k\Big\rangle\Big) - \exp\Big(-\frac{1}{2}\langle \mathbf{u}, \Gamma_d \mathbf{u}\rangle\Big)\right|$$
$$\leq C_1||\mathbf{u}||^2\{dN^{-\tilde{\rho}} + \exp(C_2 d)N^{-1/4}(\log N)^{d/2}$$
$$+ d^{d/2}\exp(-C_3 N^{1/2})\}, \tag{2.28}$$

where Γ_d *is the matrix* $(\Gamma(s_i, s_j),\ 1 \leq i,\ j \leq d)$, C_1, C_2 *and* C_3 *are absolute constants and* $\tilde{\rho} > 0$ *is a constant depending on* ρ *in Lemma* 2.8.

PROOF. In our argument we will repeatedly use the observation that if $\boldsymbol{\xi}, \boldsymbol{\xi}'$ are random variables in R^d, then

$$E\|\boldsymbol{\xi}-\boldsymbol{\xi}'\| \le \lambda \quad \text{implies } |E\exp(i\langle \mathbf{u}, \boldsymbol{\xi}\rangle) - E\exp(i\langle \mathbf{u}, \boldsymbol{\xi}'\rangle)| \le \lambda\|\mathbf{u}\| \quad \text{for any } \mathbf{u}\in R^d. \tag{2.29}$$

This is clear from the inequality $|e^{ix}-e^{iy}| \le |x-y|$, valid for any real x and y.

Similarly to $\boldsymbol{\xi}_k$, we introduce

$$\boldsymbol{\xi}'_k = (Y'_k(s_1), Y'_k(s_2), \dots, Y'_k(s_d)), \qquad 1 \le k < \infty.$$

Let ρ be the constant in Lemma 2.8. By (2.14) we have

$$E|Y_k(s) - Y'_k(s)|^2 \le c_1 k^{-\rho\theta(\mu-2)/4} \quad \text{for any } -\infty < s < \infty$$

and therefore

$$E\|\boldsymbol{\xi}_k - \boldsymbol{\xi}'_k\| \le (c_1 d)^{1/2} k^{-\rho\theta(\mu-2)/8}, \qquad 1 \le k < \infty. \tag{2.30}$$

Since $\rho\theta(\mu-2)/8 > 1$, we get from (2.30) that

$$E\left\|\sum_{1\le k\le N} (\boldsymbol{\xi}_k - \boldsymbol{\xi}'_k)\right\| \le c_2 d^{1/2}, \qquad 1 \le N < \infty, \tag{2.31}$$

with some absolute constant c_2. Putting together (2.29) and (2.31) we conclude

$$\left|E\exp\left(i\left\langle \mathbf{u}, N^{-1/2}\sum_{1\le k\le N}\boldsymbol{\xi}_k\right\rangle\right) - E\exp\left(i\left\langle \mathbf{u}, N^{-1/2}\sum_{1\le k\le N}\boldsymbol{\xi}'_k\right\rangle\right)\right| \le c_2\|\mathbf{u}\| d^{1/2} N^{-1/2} \tag{2.32}$$

for any $\mathbf{u} \in R^d$.

Let us split the interval $[1, N]$ into blocks $I_1, J_1, I_2, J_2, \dots, I_M, J_M$ so that the length of I_r is $[N^{\rho^*}]$ and the length of J_r is $[N^\rho]$, $1 \le r \le M$ (the last block may be incomplete) with some ρ^* satisfying $\rho < \rho^* < 1/2$. Then M is proportional to $N^{1-\rho^*}$. Introduce

$$T_r = \sum_{i\in I_r} \boldsymbol{\xi}_i, \quad T'_r = \sum_{i\in I_r} \boldsymbol{\xi}'_i \quad \text{and} \quad T''_r = \sum_{i \in J_r} \boldsymbol{\xi}'_i.$$

Then

$$\sum_{1\le i\le N} \boldsymbol{\xi}'_i = \sum_{1\le r\le M} T'_r + \sum_{1\le r\le M} T''_r. \tag{2.33}$$

The separation between the terms of the sums $T'_1, \dots, T'_M$ is at least $[N^\rho]$, and thus the random vectors $T'_1, \dots, T'_M$ are independent and similarly $T''_1, \dots, T''_M$ are independent. Let $\mathscr{L}_r$ denote the joint law of T_r and T'_r. By (2.30) we have that

$$E\|T_r - T'_r\| \le c_1 d^{1/2} \sum_{k\in I_r} k^{-\rho\theta(\mu-2)/8}. \tag{2.34}$$

Since $T'_1, T'_2, \ldots, T'_M$ are independent, we can construct independent random vectors $\widetilde{T}_1, \widetilde{T}_2, \ldots, \widetilde{T}_M$ such that the joint law of $(\widetilde{T}_r, T'_r)$ is $\mathscr{L}_r$ for each $1 \le r \le M$. By (2.33) we have

$$\sum_{1\le i\le N} \boldsymbol{\xi}'_i = S_M^{(1)} + S_M^{(2)} + S_M^{(3)}, \tag{2.35}$$

where

$$S_M^{(1)} = \sum_{1\le r\le M} \widetilde{T}_r,\ \ S_M^{(2)} = \sum_{1\le r\le M} T''_r \quad \text{and} \quad S_M^{(3)} = \sum_{1\le r\le M} (T'_r - \widetilde{T}_r).$$

The vectors (T_r, T'_r) and $(\widetilde{T}_r, T'_r)$ have the same distribution, so (2.34) yields

$$E||T'_r - \widetilde{T}_r|| \le c_1 d^{1/2} \sum_{k\in I_r} k^{-\rho\theta(\mu-2)/8}$$

and therefore

$$E||S_M^{(3)}|| \le c_2 d^{1/2}, \tag{2.36}$$

where c_2 depends only on ρ, θ and μ. On the other hand, by (2.30) and Lemma 2.7 we have

$$\begin{aligned} E||T''_r|| &\le E\left|\left|\sum_{i\in J_r} \boldsymbol{\xi}_i\right|\right| + E\left|\left|\sum_{i\in J_r} (\boldsymbol{\xi}_i - \boldsymbol{\xi}'_i)\right|\right| \\ &\le c_3 d N^{\rho/2} + c_4 d^{1/2} \\ &\le c_5 d N^{\rho/2} \end{aligned}$$

and since $T''_1, T''_2, \ldots, T''_M$ are independent we get that

$$E\left|\left|S_M^{(2)}\right|\right|^2 \le c_5^2 d^2 N^\rho M \le c_6 d^2 N^{\rho+1-\rho^*},$$

that is,

$$E\left|\left|S_M^{(2)}\right|\right| \le c_6^{1/2} d N^{(\rho+1-\rho^*)/2}. \tag{2.37}$$

Putting together (2.35)–(2.37) and (2.29) we conclude that

$$\begin{aligned} &\left| E\exp\left(i\left\langle \mathbf{u}, N^{-1/2}\sum_{1\le k\le N} \boldsymbol{\xi}'_k\right\rangle\right) - E\exp\left(i\left\langle \mathbf{u}, N^{-1/2}\sum_{1\le r\le M} \widetilde{T}_r\right\rangle\right)\right| \\ &\qquad \le c_7 ||\mathbf{u}|| d N^{(\rho+1-\rho^*)/2} N^{-1/2}. \end{aligned} \tag{2.38}$$

Next we write

$$\frac{1}{N^{1/2}} \sum_{1\le r\le M} \widetilde{T}_r = \frac{1}{M^{1/2}} \sum_{1\le r\le M} \widetilde{T}_r/(N/M)^{1/2}.$$

We note that $\widetilde{T}_r/(N/M)^{1/2}$ are independent identically distributed random vectors. Since $\Gamma(s,t)$ is a bounded function, there is a constant c_8 such that

$$||\mathrm{cov}(\widetilde{T}_r/(N/M)^{1/2})|| \le c_8 d,$$

and therefore Lemma 2.9 yields

$$\begin{aligned}&\left|E\exp\left(i\langle\mathbf{u},\frac{1}{N^{1/2}}\sum_{1\le r\le M}\widetilde{T}_r\rangle\right)-\exp\left(-\frac{1}{2}\langle\mathbf{u},\Gamma_d\mathbf{u}\rangle\right)\right|\\ &\qquad\le c_9d\{d^2e^{c_{10}d}(\log M)^{d/2}d^4M^{-1/2}+e^{-c_{11}M}d^{d/2}\}||\mathbf{u}||^2\\ &\qquad\le c_{12}\{N^{-(1-\rho^*)/2}(\log N)^{d/2}\exp(c_{13}d)+d^{d/2}\exp(-c_{14}N^{1-\rho^*})\}||\mathbf{u}||^2.\end{aligned}\tag{2.39}$$

Combining (2.32), (2.38) and (2.39) we get Lemma 2.10. □

The following lemma will be used to estimate the increments of the approximating Gaussian process.

LEMMA 2.11. *We assume that the conditions of Lemma* 2.8 *are satisfied. Let* $\{K(s,t), -\infty < s < \infty,\ 0\le t<\infty\}$ *be a Gaussian process with mean zero and covariance* $EK(s,t)K(s',t')=\min(t,t')\Gamma(s,s')$. *For any* $-\infty<a<a'<\infty$ *and* $0\le b<b'<\infty$ *let* $Z([a,a']\times[b,b'])$ *denote the maximal fluctuation of* K *over the rectangle* $[a,a']\times[b,b']$. *Then for any* $x\ge C_1$ *we have*

$$\begin{aligned}P\{Z([a,a']\times[b,b'])>C_2x((b'-b)^{1/2}+(b')^{1/2}(F(a')-F(a))^{\tau/2})\}\\ \le C_3e^{-x^2/2}\end{aligned}\tag{2.40}$$

where C_1, C_2, C_3 *are absolute constants and* $\tau>0$ *is from Lemma* 2.6.

PROOF. Using (2.21)–(2.23), one can easily verify that

$$E(K(s,t_2)-K(s,t_1))^2=\Gamma(s,s)(t_2-t_1)\le c_1(t_2-t_1)\tag{2.41}$$

and

$$\begin{aligned}E(K(s_2,t)-K(s_1,t))^2&=t(\Gamma(s_1,s_1)+\Gamma(s_2,s_2)-2\Gamma(s_1,s_2))\\&=t\sigma^2(s_1,s_2)\le c_2t(F(s_2)-F(s_1))^\tau.\end{aligned}\tag{2.42}$$

Let

$$\widehat{Z}(u,v)=K(a+u(a'-a),\ b+v(b'-b))-K(a,b),\quad 0\le u,\ v\le 1.$$

The estimates in (2.41) and (2.42) imply that the conditions of Fernique's inequality are satisfied [cf. Lai (1974)] and therefore an upper bound for the tail of the distribution of $\sup_{0\le u,\,v\le 1}|\widehat{Z}(u,v)|$ can be easily obtained. The proof of (2.40) is complete now. □

LEMMA 2.12. *For any* $T\ge 1$, $\lambda\ge T^{1/2}$ *and any* $-\infty<a<b<\infty$ *we have*

$$P\left\{\sup_{a\le s<s'\le b,\,0\le t\le T}\left|\sum_{k\le t}\overline{Y}_k(s,s')\right|\ge\lambda\right\}\le C_1\exp\left(-\frac{C_2\lambda^2}{T(b-a)^\delta}\right)+\frac{C_3}{T^\delta},$$

where C_1, C_2, C_3 *and* δ *are positive constants.*

PROOF. Since $\overline{Y}_k(s,s') = \overline{Y}_k(0,s') - \overline{Y}_k(0,s)$, it suffices to prove that

$$P\left\{\sup_{0\le s\le b,\,0\le t\le T}\left|\sum_{k\le t}\overline{Y}_k(0,s)\right| \ge \lambda\right\} \le \widetilde{C}_1\exp\left(-\frac{\widetilde{C}_2\lambda^2}{Tb^\delta}\right)+\frac{\widetilde{C}_3}{T^\delta}. \tag{2.43}$$

Let, for any integers $u \ge 1$ and $v \ge 1$,

$$M_{u,v} = \max_{0\le i<2^u,\,0\le j<2^v}\left|\sum_{Tj2^{-v}\le k\le T(j+1)2^{-v}}\overline{Y}_k(bi2^{-u}, b(i+1)2^{-u})\right|.$$

It is easy to see that for any $0 \le s \le b$, $0 \le t \le T$ and any integer $L \ge 1$ we have

$$\left|\sum_{k\le t}\overline{Y}_k(0,s)\right| \le \sum_{1\le u,\,v\le L} M_{u,v} + \frac{2T}{2^L}. \tag{2.44}$$

Let $\varepsilon > 0$ be a small positive number to be chosen later. If $\lambda \ge T^{1/2}$, $1 \le u$, $v \le L$, then we have, using Lemma 2.8 and (2.13),

$$\begin{aligned}
&P\{M_{u,v} \ge \lambda 2^{-\varepsilon(u+v)}\}\\
&\quad\le 2^{u+v}[c_1\exp(-c_2\lambda^2 2^{-2\varepsilon(u+v)}/(T2^{-v}(b2^{-u})^{\tau\theta}))\\
&\qquad + c_3\exp(-c_4\lambda 2^{-\varepsilon(u+v)}/(T2^{-v})^{\rho}) + c_5(\lambda 2^{-\varepsilon(u+v)})^{-(2+\tau)}]\\
&\quad\le 2^{u+v}[c_1\exp(-c_2\lambda^2 2^{\tau\theta(u+v)/2}/(Tb^{\tau\theta}))\\
&\qquad + c_3\exp(-c_4T^{1/2-\rho}2^{-\varepsilon u} + c_5 2^{8\varepsilon L}T^{-(1+\tau/2)}]\\
&\quad\le 2^{u+v}[c_1\exp(-c_2\lambda^2 2^{\tau\theta(u+v)/2}/(Tb^{\tau\theta}))\\
&\qquad + c_6 2^{c_7L\varepsilon}T^{-2} + c_5 2^{8L\varepsilon}T^{-(1+\tau/2)}]\\
&\quad\le 2^{u+v}[c_1\exp(-c_2\lambda^2 2^{\tau\theta(u+v)/2}/(Tb^{\tau\theta})) + c_8 2^{c_9L\varepsilon}T^{-(1+\tau/2}],
\end{aligned}$$

provided that $\varepsilon < \min(1/4, \tau\theta/4)$. We used here the fact that $\exp(-x) \le c_r x^{-r}$ for any $r > 0$ and $x > 0$, where c_r is a positive constant depending on r. Thus setting

$$A = \{M_{u,v} \ge \lambda 2^{-\varepsilon(u+v)} \text{ for some } 1 \le u,\ v \le L\}$$

we have

$$\begin{aligned}
P\{A\} &\le \sum_{1\le u,\,v\le L} 2^{u+v}c_1\exp(-c_2\lambda^2 2^{\tau\theta(u+v)/2}/(Tb^{\tau\theta})) + c_8 2^{c_9L\varepsilon+2L+2}T^{-(1+\tau/2)}\\
&\le \sum_{1\le u,\,v\le L} 2^{u+v}c_1\exp(-c_2\lambda^2(2^{\tau\theta u/2}+2^{\tau\theta v/2})/(Tb^{\tau\theta}))\\
&\qquad + c_8 2^{c_9L\varepsilon+2L+2}T^{-(1+\tau/2)}\\
&= \left(\sum_{1\le u\le L} 2^u c_1^{1/2}\exp(-c_2\lambda^2 2^{\tau\theta u/2}/(Tb^{\tau\theta}))\right)^2 + c_8 2^{c_9L\varepsilon+2L+2}T^{-(1+\tau/2)}\\
&\le c_{10}\exp(-c_{11}\lambda^2/(Tb^{\tau\theta})) + c_8 2^{c_9L\varepsilon+2L+2}T^{-(1+\tau/2)},
\end{aligned}$$

since the terms of the last sum decrease at least exponentially for $u \geq u_0$. On the set A^c we clearly have

$$\sum_{1 \leq u,\, v \leq L} M_{u,v} \leq \lambda \sum_{1 \leq u,\, v \leq L} 2^{-\varepsilon(u+v)} \leq c_{12}\lambda$$

and thus choosing $L \geq 1$ so that $T^{1/2} \leq 2^L \leq 2T^{1/2}$, it follows that the right-hand side of (2.44) is not greater than $c_{12}\lambda + 2T^{1/2} \leq c_{13}\lambda$ with the exception of a set of probability not greater than $c_{10}\exp(-c_{11}\lambda^2/(Tb^{\tau\theta})) + c_{14}T^{-\tau/4}$ provided that $\varepsilon > 0$ is so small that $c_9\varepsilon \leq \tau/2$. Thus (2.43) is proved. □

Now we are ready to construct the approximating process. We approximate the increments of $R(s,t)$ with normal random variables and from these random variables we construct a suitable $K(s,t)$. Let

$$t_k = [\exp(k^{1-\varepsilon})], \;\; p_k = 2[t_k^{\rho}], \;\; q_k = [\log k/\log 4], \qquad d_k = 2^{q_k},$$

where $\varepsilon > 0$ is a small number to be specified later and ρ is taken from Lemma 2.6. Set

$$M_k = t_{k+1} - t_k - p_k$$

and

$$\begin{aligned} s_i = s_{k,\,i} &= (i-1)/d_k, \qquad 1 \leq i \leq d_k,\\ \eta_{k,\,i} &= R(s_i, t_{k+1}) - R(s_i, t_k + p_k), \qquad 1 \leq i \leq d_k \end{aligned}$$

and

$$\boldsymbol{\eta}_k = (\eta_{k,\,1}, \ldots, \eta_{k,\,d_k}).$$

Clearly $\eta_{k,\,i}$ can be written as

$$\eta_{k,\,i} = \sum_{j=t_k+p_{k+1}}^{t_{k+1}} Y_j(s_i),$$

so Lemma 2.10 can be used to estimate the difference between the distribution functions of $M_k^{-1/2}\boldsymbol{\eta}_k$ and a normal random variable $N(\mathbf{0}, \Gamma_k)$, where $\Gamma_k = (\Gamma(s_i, s_j), 1 \leq i, j \leq d_k)$. The Prohorov–Lévy distance between the distributions of $M_k^{-1/2}\boldsymbol{\eta}_k$ and $N(\mathbf{0}, \Gamma_k)$ will be denoted by $\Psi_{\mathrm{PL}}(M_k^{-1/2}\boldsymbol{\eta}_k, N(\mathbf{0}, \Gamma_k))$.

LEMMA 2.13. *We assume that the conditions of Lemma* 2.8 *are satisfied. Then*

$$\Psi_{\mathrm{PL}}(M_k^{-1/2}\boldsymbol{\eta}_k, N(\mathbf{0}, \Gamma_k)) \leq C_1 \exp(-C_2 k^{\varepsilon})$$

with any $0 < \varepsilon < 1/4$, *where* C_1 *and* C_2 *are absolute constants.*

PROOF. In the argument that follows, $c_1, c_2, \dots$ will be absolute constants. By Lemma 2.10 we have

$$\begin{aligned}
&\left|E\exp\left(i\left\langle \mathbf{u}, M_k^{-1/2}\boldsymbol{\eta}_k\right\rangle\right) - \exp\left(-\frac{1}{2}\langle \mathbf{u}, \Gamma_k \mathbf{u}\rangle\right)\right| \\
&\quad \le c_1\|\mathbf{u}\|^2\{d_k M_k^{-(\rho^*-\rho)/2} + \exp(c_2 d_k) M_k^{-(1-\rho^*)/2}(\log M_k)^{d_k/2} \\
&\qquad\qquad + d_k^{d_k/2}\exp(-c_3 M_k^{1-\rho^*})\} \\
&\quad \le \|\mathbf{u}\|^2[c_4 k^{1/2}\exp(-c_5 k^{1-\varepsilon}) \\
&\qquad + c_6\exp(c_2(k^{1/2} - c_7 k^{1-\varepsilon} + c_8 k^{1/2}(1-\varepsilon)\log k)) \\
&\qquad + c_9\exp(c_{10}k^{1/2}\log k - c_{11}k^{1-\varepsilon})] \\
&\quad \le c_{12}\|\mathbf{u}\|^2\exp(-c_{13}k^{1-\varepsilon}),
\end{aligned} \tag{2.45}$$

assuming that $\varepsilon < 1/2$. Using an analogue of the Berry–Esseen inequality [cf. Berkes and Philipp (1979), Lemma 2.2] we have that

$$\begin{aligned}
&\Psi_{\mathrm{PL}}(M_k^{-1/2}\boldsymbol{\eta}_k, N(\mathbf{0}, \Gamma_k)) \\
&\quad \le \frac{16 d_k}{T}\log T \\
&\qquad + T^{d_k}\int_{\|\mathbf{u}\|\le T}\left|E\exp\left(i\left\langle \mathbf{u}, M_k^{-1/2}\boldsymbol{\eta}_k\right\rangle\right) - \exp\left(-\frac{1}{2}\langle \mathbf{u}, \Gamma_k\mathbf{u}\rangle\right)\right| d\mathbf{u} \\
&\qquad + P\{\|N(\mathbf{0}, \Gamma_k)\| > T/2\}
\end{aligned} \tag{2.46}$$

for any $T > 0$. By (2.45) we have

$$\begin{aligned}
&\int_{\|\mathbf{u}\|\le T}\left|\exp\left(i\left\langle \mathbf{u}, M_k^{-1/2}\boldsymbol{\eta}_k\right\rangle\right) - \exp\left(-\frac{1}{2}\langle \mathbf{u}, \Gamma_k\mathbf{u}\rangle\right)\right| d\mathbf{u} \\
&\qquad \le c_{12}T^2(2T)^{d_k}\exp(-c_{13}k^{1-\varepsilon}).
\end{aligned} \tag{2.47}$$

If $\|N(\mathbf{0}, \Gamma_k)\| > T$, then the absolute value of at least one of the coordinates of $N(\mathbf{0}, \Gamma_k)$ is larger than $T/d_k^{1/2}$. Since the function $\Gamma(s, t)$ is bounded, we get that

$$\begin{aligned}
P\{\|N(\mathbf{0}, \Gamma_k)\| > T\} &\le c_{15} d_k\exp(-c_{17}T^2/d_k) \\
&\le c_{15}k^{1/2}\exp(-c_{17}T^2 k^{-1/2}).
\end{aligned} \tag{2.48}$$

Choosing $T = \exp(k^{\varepsilon})$ with any $0 < \varepsilon < 1/4$ in (2.46)–(2.48), the proof of Lemma 2.13 is complete. □

Now we can return to the construction. Similarly to $\eta_{k,i}$ we define

$$\eta'_{k,i} = \sum_{j=t_k+p_k+1}^{t_{k+1}} Y'_j(s_i), \qquad 1 \le i \le d_k,$$

and let $\boldsymbol{\eta}'_k = (\eta'_{k,1}, \dots, \eta'_{k,d_k})$. Using (2.31) and Lemma 2.12 we get that

$$\Psi_{\mathrm{PL}}(M_k^{-1/2}\boldsymbol{\eta}'_k, N(\mathbf{0}, \Gamma_k)) \le c_1\exp(-c_2 k^{\varepsilon}) \tag{2.49}$$

for any $0 < \varepsilon < 1/4$ with some constants c_1 and c_2. The vector $\boldsymbol{\eta}'_k$ depends only on the random variables $\{y'_j, t_k+p_k+1 \le j \le t_{k+1}\}$. Thus there is a separation

$p_k = 2[t_k^\rho]$ between the sets of random variables defining the vectors $\boldsymbol{\eta}'_{k-1}$ and $\boldsymbol{\eta}'_k$ and therefore $\boldsymbol{\eta}'_1, \boldsymbol{\eta}'_2, \ldots$ are independent. By (2.49) and the Strassen (1964)–Dudley (1976) representation theorem, in light of the independence of $\boldsymbol{\eta}'_1, \boldsymbol{\eta}'_2, \ldots$ we can define independent normal vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ such that $\boldsymbol{\xi}_k / M_k^{1/2}$ is $N(\mathbf{0}, \Gamma_k)$ and

$$P\{\|\boldsymbol{\eta}'_k / M_k^{1/2} - \boldsymbol{\xi}_k / M_k^{1/2}\| \geq c_1 \exp(-c_2 k^\varepsilon)\} \\ \leq c_1 \exp(-c_2 k^\varepsilon). \tag{2.50}$$

Putting together (2.31) and (2.50) we get that

$$P\{\|\boldsymbol{\eta}_k / M_k^{1/2} - \boldsymbol{\xi}_k / M_k^{1/2}\| \geq \delta_k\} \leq \delta_k, \qquad k = 1, 2, \ldots, \tag{2.51}$$

where $\delta_k = c_3 \exp(-c_4 k^\varepsilon)$ for any $0 < \varepsilon < 1/4$ with some c_3 and c_4. The joint distribution of $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ matches the joint distribution of the increments of a Gaussian process with zero mean and covariance $\min(t, t')\Gamma(s, s')$. Hence there is a Gaussian process $\{K(s, t), -\infty < s < \infty,\ 0 \leq t < \infty\}$ with mean 0 and the above covariance such that

$$\xi_{k,i} = K(s_i, t_{k+1}) - K(s_i, t_k + p_k), \qquad i \leq i \leq d_k.$$

Now (2.51) and the Borel–Cantelli lemma imply that there is a random variable $k_0 = k_0(\omega)$ such that

$$|\eta_{k,i} - \xi_{k,i}| \leq c_5 t_k^{1/2} \exp(-c_4 k^\varepsilon), \qquad 1 \leq i \leq d_k \tag{2.52}$$

for all $k \geq k_0$. Thus suitable vertical increments of R and K are close to each other. So Theorem 1.1 will be proved if we can control the oscillations of $R(s, t)$ and $K(s, t)$.

Modifying slightly the definitions of $\eta_{k,i}$ and $\xi_{k,i}$ we introduce

$$\eta^*_{k,i} = R(s_i, t_{k+1}) - R(s_i, t_k), \qquad 1 \leq i \leq d_k$$

and

$$\xi^*_{k,i} = K(s_i, t_{k+1}) - K(s_i, t_k), \qquad 1 \leq i \leq d_k.$$

Next we show that (2.52) implies

$$|\eta^*_{k,i} - \xi^*_{k,i}| \leq c_6 t_k^{1/2} \exp(-c_7 k^\varepsilon), \qquad 1 \leq i \leq d_k \tag{2.53}$$

for all $k \geq k_0$, where $0 < \varepsilon < 1/4$ and c_6, c_7 are constants. Indeed, the difference between $\eta_{k,i}$ and $\eta^*_{k,i}$ is bounded by $p_k \leq 2t_k^\rho \leq c_8 t_k^{1/2} \exp(-c_9 k^\varepsilon)$. On the other hand, $E(\xi_{k,i} - \xi^*_{k,i})^2 \leq c_{10} p_k$ and therefore

$$P\left\{\max_{1 \leq i \leq d_k} |\xi_{k,i} - \xi^*_{k,i}| > k p_k^{1/2}\right\} \leq d_k \exp(-c_{11} k^2) \leq \frac{c_{12}}{k^2},$$

so the Borel–Cantelli lemma gives

$$\max_{1 \leq i \leq d_k} |\xi_{k,i} - \xi^*_{k,i}| \leq c_{13} k p_k^{1/2} \leq c_{14} t_k^{1/2} \exp(-c_{15} k^\varepsilon),$$

if $k \geq k_0 = k_0(\omega)$. The proof of (2.53) is now complete. □

Let $\widehat{R}_{i,k}$ denote the maximal oscillation of $R(s,t)$ over the rectangle $[s_i, s_{i+1}] \times [t_k, t_{k+1}]$ and let $\widehat{K}_{i,k}$ denote the same increment with respect to K. We claim that there are constants ε' and c_{16} and a random variable $k_0(\omega)$ such that

$$\max_{1\le i\le d_k} \widehat{R}_{i,k} \le c_{16} t_k^{1/2} (\log t_k)^{-\varepsilon'} \tag{2.54}$$

and

$$\max_{1\le i\le d_k} \widehat{K}_{i,k} \le c_{16} t_k^{1/2} (\log t_k)^{-\varepsilon'}, \tag{2.55}$$

if $k \ge k_0$. By Lemma 2.11 and (2.13) we have

$$P\left\{\max_{1\le i\le d_k} \widehat{K}_{i,k} > c_{17} \log k\big((t_{k+1}-t_k)^{1/2} + t_{k+1}^{1/2} d_k^{-\theta\tau/2}\big)\right\} \le c_{18}\frac{1}{k^2}$$

and thus the Borel–Cantelli lemma gives

$$\begin{aligned}\max_{1\le i\le d_k} \widehat{K}_{i,k} &\le 2c_{17} \log k\big((t_{k+1}-t_k)^{1/2} + t_{k+1}^{1/2} d_k^{-\theta\tau/2}\big)\\ &\le c_{18}(t_k^{1/2} k^{-\varepsilon/2} + t_k^{1/2} k^{-\theta\tau/8}) \log k\\ &\le c_{19} t_k^{1/2} (\log t_k)^{-\varepsilon'}\end{aligned}$$

with some ε' small enough, if $k \ge k_0$. Hence (2.55) is proved. Replacing Lemma 2.11 with Lemma 2.12, similar arguments give (2.54). □

Now a geometrical picture shows that

$$R(s_i, t_k) = \sum_{1\le l\le k-1} \eta^*_{l,i_l} + \sum_{1\le l\le k-1} \big(R(s_{i_l}, t_l) - R(s_{i_{l-1}}, t_l)\big)$$

and

$$K(s_i, t_k) = \sum_{1\le l\le k-1} \xi^*_{l,i_l} + \sum_{1\le l\le k-1} \big(K(s_{i_l}, t_l) - K(s_{i_{l-1}}, t_l)\big),$$

where $i_1, i_2, \ldots$ are suitably chosen integers satisfying $|i_l - i_{l-1}| \le 1$. Thus using (2.53)–(2.55) we get for all $1 \le i \le d_k$ and $k \ge k_0$,

$$\begin{aligned}|R(s_i,t_k) - K(s_i,t_k)| &\le c_6 \sum_{1\le l\le k-1} t_l^{1/2} \exp(-c_7 l^{\varepsilon})\\ &\quad + \sum_{1\le l\le k-1} \max_{1\le i\le d_l} \widehat{R}_{i,l} + \sum_{1\le l\le k-1} \max_{1\le i\le d_l} \widehat{K}_{i,l}\\ &\le c_{20} k t_k^{1/2} \exp(-c_7 k^{\varepsilon}) + c_{16} k t_k^{1/2} (\log t_k)^{-\varepsilon'}\\ &\quad + c_{16} k t_k^{1/2} (\log t_k)^{-\varepsilon'}\\ &\le c_{21} t_k^{1/2} (\log t_k)^{-\varepsilon'/2}\end{aligned} \tag{2.56}$$

if $k \ge k_0$. The approximation in Theorem 1.1 follows immediately from (2.54)–(2.56). □

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