

Some norm estimates for semimartingales

Triet Pham* Jianfeng Zhang†

Abstract

In this paper we introduce a new type of norm for semimartingales. Our norm is defined in the spirit of quasimartingales, and it characterizes square integrable semimartingales. This work is motivated by our study of zero-sum stochastic differential games, whose value process we conjecture to be a semimartingale under a class of probability measures under some conditions. The norm introduced here seems to be the right one to study general square integrable semimartingales, and it is also suitable for studying semimartingales under nonlinear expectation. Using a similar idea, we introduce a new norm for the barriers of doubly reflected BSDEs and establish some a priori estimates for the solutions. Our norm provides an alternative but more tractable characterization for the standard Mokobodski’s condition in the literature.

Keywords: Semimartingale; quasimartingale; G -expectation; second order backward SDEs; doubly reflected backward SDEs; Doob-Meyer decomposition.

AMS MSC 2010: 60G46; 60H10; 60H30.

Submitted to EJP on October 30, 2012, final version accepted on December 10, 2013.

1 Introduction

In recent years, the notion of nonlinear expectation, in particular the G -expectation of Peng [21], has received strong attention in the literature. Roughly speaking, a G -expectation is a nonlinear expectation taking the following form: $\mathbb{E}^G := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}$, where \mathcal{P} is a family of mutually singular probability measures \mathbb{P} and in general the family \mathcal{P} does not have a dominating probability measure. For a random variable ξ , the conditional G -expectation $\mathbb{E}_t^G[\xi]$ can also be defined so that it satisfies the time consistency property, see e.g. Section 4 of this paper. Such conditional G -expectation is called a G -martingale which, by Soner, Touzi and Zhang [27], has the following representation: denoting $Y_t := \mathbb{E}_t^G[\xi]$,

$$Y_t = Y_0 + \int_0^t Z_s dB_s - K_t, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, \quad (1.1)$$

*Department of Mathematics, Rutgers University, USA. E-mail: triet.pham@rutgers.edu. Support: University of Southern California Graduate School Dissertation Completion Fellowship.

†Department of Mathematics, University of Southern California, USA. E-mail: jianfenz@usc.edu. Partial support: NSF grant DMS 1008873.

where B is the canonical process, \mathcal{P} is a class of martingale measures, and K is a nondecreasing process with $K_0 = 0$. This result can be extended to second order BSDEs of [30], and the closely related G -BSDE of [14]. In particular, a G -martingale is a supermartingale under each $\mathbb{P} \in \mathcal{P}$. It is clear that a G -supermartingale is also a supermartingale under each $\mathbb{P} \in \mathcal{P}$.

In Pham and Zhang [24], we studied a zero sum stochastic differential game. Under certain conditions, we show that the game value exists:

$$Y_t := \inf_{v \in \mathcal{V}} \sup_{u \in \mathcal{U}} \mathbb{E}_t^{\mathbb{P}^{u,v}}[\xi] = \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}} \mathbb{E}_t^{\mathbb{P}^{u,v}}[\xi]. \tag{1.2}$$

Here \mathcal{U} and \mathcal{V} are appropriate sets of admissible controls, $\mathbb{P}^{u,v}$ is a probability measure induced by the controls (u, v) , and $\mathbb{E}_t^{\mathbb{P}^{u,v}}$ denotes conditional expectations (abusing the notations slightly here by using \sup and \inf instead of ess sup and ess inf in appropriate sense). The present paper is motivated by our efforts to understand the dynamics of the game value process Y .

Notice that, for any fixed v , $\sup_{u \in \mathcal{U}} \mathbb{E}_t^{\mathbb{P}^{u,v}}[\xi]$ can be viewed roughly as a martingale under a nonlinear expectation, and thus has supermartingale property under each probability measure. However, the additional $\inf_{v \in \mathcal{V}}$ induces submartingale property. Indeed, one can show that Y is a submartingale under the nonlinear expectation induced by $\mathcal{P} := \{\mathbb{P}^{u,v} : (u, v) \in \mathcal{U} \times \mathcal{V}\}$. So our natural question is:

What is the structure of a G -submartingale?

Since the $\sup_{u \in \mathcal{U}}$ and $\inf_{v \in \mathcal{V}}$ induce the supermartingale and submartingale properties respectively, we conjecture that the game value process Y should be a semimartingale under each $\mathbb{P}^{u,v}$. More generally, given a G -submartingale Y , one may expect that $Y = M + L$, where M is a G -martingale (and thus a supermartingale under each probability measure) and L is a nondecreasing process, in the spirit of the Doob-Meyer decomposition, but under nonlinear expectation. Then by (1.1) we expect that

$$Y_t = Y_0 + \int_0^t Z_s dB_s + A_t, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, \tag{1.3}$$

where $A := L - K$ is a semi-martingale under each $\mathbb{P} \in \mathcal{P}$. While the above analysis is intuitively clear, its rigorous proof is by no means easy, because it involves a priori estimates for total variations of A under each $\mathbb{P} \in \mathcal{P}$.

Our first goal of this paper is to introduce a norm which characterizes square integrable semimartingales, under a fixed (linear) probability measure. Our norm is strongly motivated from the definition of quasimartingales. The main feature is that the norm involves only the semimartingale itself, without involving directly its decomposition. This is important in applications because the semimartingale under consideration is typically a value process and thus has a representation, e.g. the process Y in (1.2). We prove that a progressively measurable process is a square integrable semimartingale if and only if it has finite norm in our sense.

We next extend our norm to semimartingales under nonlinear expectations, in particular the G -expectation. We show that, any progressively measurable process with finite norm under G -expectation in our sense has to be a semimartingale under each probability measure. Our long term goal is to apply our norm, or its variations if necessary, to study the structure of general G -semimartingales, and in particular the nonlinear Doob-Meyer decomposition. We remark that the game value process Y in (1.2) is the unique viscosity solution of path dependent Bellman-Isaacs equations, see [24]. Thus the semimartingale property of Y can also be viewed as regularity of viscosity solutions

of path dependent PDEs. For the viscosity theory of path dependent PDEs we refer the readers to [7] for the semi-linear case and [8, 9, 10] for the fully nonlinear case.

Another contribution of this paper is to provide a sufficient condition for the well-posedness of doubly reflected backward SDEs (DRBSDE, for short). There are typically two approaches in the literature. One is to assume the Mokobodski's condition, namely there exists a square integrable semimartingale between the two given barriers, see e.g. Cvitanic and Karatzas [4], Peng and Xu [22] and Crépey and Matoussi [2], and the other is to use local solutions, see e.g. Hamadène and Hassani [12] and Hamadène, Hassani and Ouknine [13]. The latter approach, while easy to verify its conditions, does not yield any norm estimates. We remark that such estimates are important in applications, for example when one considers discretization of DRBSDEs, see e.g. Chassagneux [1]. In the spirit of our semimartingale norm, we introduce a norm for the barriers of DRBSDEs and provide a priori estimates for the solution of DRBSDEs based on our new barrier norm. Such estimates seem to be new in the literature and are important in numerical discretization of DRBSDEs. It turns out that our barrier norm is finite if and only if the Mokobodski's condition is satisfied. In this sense, we provide a necessary and sufficient condition for the Mokobodski's condition. However, we remark that our norm depends on the barriers more explicitly and is (hopefully) easier to verify in practice.

The rest of the paper is organized as follows. In next section we introduce the norm for semimartingales under a fixed probability measure and obtain the estimates. In Section 3 we study DRBSDEs by introducing a norm for the barriers in the same spirit. In Section 4 we extend the norm to the G -framework.

2 Norm Estimates for Semimartingales

Let $T > 0$ be fixed, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space on $[0, T]$, and $\mathbb{D}(\mathbb{F})$ be the space of \mathbb{F} -progressively measurable càdlàg processes. Throughout this section, we shall always assume (without mentioning in all the results):

$$\begin{aligned} \mathbb{F} \text{ is right continuous and its } \mathbb{P}\text{-augmentation } \bar{\mathbb{F}}^{\mathbb{P}} \text{ is a Brownian filtration,} \\ \text{and consequently, any } \mathbb{F}\text{-martingale } M \text{ is continuous, } \mathbb{P}\text{-a.s.} \end{aligned} \quad (2.1)$$

We note that the filtration \mathbb{F} is not necessarily complete under \mathbb{P} . The removal of the completeness requirement will be important in Section 4 below. However, the following simple lemma, see e.g. [28], shows that we may assume all the processes involved in this section are \mathbb{F} -progressively measurable.

Lemma 2.1. *For any $\bar{\mathbb{F}}^{\mathbb{P}}$ -progressively measurable process X , there exists a unique ($dt \times d\mathbb{P}$ -a.s.) \mathbb{F} -progressively measurable process \tilde{X} such that $\tilde{X} = X$, $dt \times d\mathbb{P}$ -a.s. Moreover, if X is càdlàg, \mathbb{P} -a.s., then so is \tilde{X} .*

We recall that a semimartingale $Y \in \mathbb{D}(\mathbb{F})$ has the following decomposition:

$$Y_t = Y_0 + M_t + A_t, \quad (2.2)$$

where M is a local martingale, A has finite variation, and $M_0 = A_0 = 0$. Now given a process $Y \in \mathbb{D}(\mathbb{F})$, we are interested in the following questions:

- (i) Is Y a semimartingale?
- (ii) Do we have appropriate norm estimates for Y , M , and A ?

The first question was answered by Bichteler-Dellacherie, see e.g. [25] for some further discussion. The main goal of this section is to answer the second question. As explained in the Introduction, the latter question is natural and important for our study of semimartingales under nonlinear expectations.

2.1 Some preliminary results

We first note that, when Y is a supermartingale or submartingale, it is well known that Y is a semimartingale and the following norm estimates hold. Since the arguments will be important for our general case, we provide a proof for completeness.

Lemma 2.2. *There exist universal constants $0 < c < C$ such that, for any Y in the form of (2.2) with monotone A , it holds*

$$c\|Y\|_{\mathbb{P},0}^2 \leq \mathbb{E}^{\mathbb{P}} \left[|Y_0|^2 + \langle M \rangle_T + |A_T|^2 \right] \leq C\|Y\|_{\mathbb{P},0}^2; \tag{2.3}$$

where, for any $Y \in \mathbb{D}(\mathbb{F})$,

$$\|Y\|_{\mathbb{P},0}^2 := \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right]. \tag{2.4}$$

Proof. The first inequality is obvious. We shall only prove the second inequality. By otherwise using the standard stopping techniques, we may assume without loss of generality that $\mathbb{E}^{\mathbb{P}} [\sup_{0 \leq t \leq T} |Y_t|^2 + \langle M \rangle_T + |A_T|^2] < \infty$.

Apply Itô's formula and recall (2.1) that M is continuous, we have

$$Y_T^2 = Y_0^2 + \langle M \rangle_T + 2 \int_0^T Y_t dM_t + 2 \int_0^T Y_{t-} dA_t + \sum_{0 < t \leq T} |\Delta Y_t|^2. \tag{2.5}$$

Note that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |Y_t|^2 d\langle M \rangle_t \right)^{\frac{1}{2}} \right] \leq \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t| \langle M \rangle_t^{\frac{1}{2}} \right] \leq \frac{1}{2} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t|^2 + \langle M \rangle_T \right] < \infty.$$

Then $Y_t dM_t$ is a true martingale, and thus, for any $\varepsilon > 0$, it follows from (2.5) and the monotonicity of A that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} [\langle M \rangle_T] &\leq \mathbb{E}^{\mathbb{P}} \left[\langle M \rangle_T + \sum_{0 \leq t \leq T} |\Delta Y_t|^2 \right] = \mathbb{E}^{\mathbb{P}} \left[Y_T^2 - Y_0^2 - 2 \int_0^T Y_{t-} dA_t \right] \\ &\leq \mathbb{E}^{\mathbb{P}} \left[|Y_T|^2 + |Y_0|^2 + 2 \sup_{0 \leq t \leq T} |Y_t| |A_T| \right] \leq C\varepsilon^{-1} \|Y\|_{\mathbb{P},0}^2 + \varepsilon \mathbb{E}^{\mathbb{P}} [|A_T|^2]. \end{aligned} \tag{2.6}$$

Moreover, note that $A_T = Y_T - Y_0 - M_T$. Then (2.6) leads to

$$\mathbb{E}^{\mathbb{P}} [|A_T|^2] \leq C\|Y\|_{\mathbb{P},0}^2 + C\mathbb{E}^{\mathbb{P}} [\langle M \rangle_T] \leq C\varepsilon^{-1} \|Y\|_{\mathbb{P},0}^2 + C\varepsilon \mathbb{E}^{\mathbb{P}} [|A_T|^2].$$

Set $\varepsilon := \frac{1}{2C}$ for the above C , we obtain $\mathbb{E}^{\mathbb{P}} [|A_T|^2] \leq C\|Y\|_{\mathbb{P},0}^2$. This, together with (2.6), proves the second inequality. \square

The next lemma is a discrete version of Lemma 2.2. Since the arguments are very similar, we omit the proof.

Lemma 2.3. *Let $0 = \tau_0 \leq \dots \leq \tau_n = T$ be a sequence of stopping times. In the setting of Lemma 2.2, if $A_{\tau_i} \in \mathcal{F}_{\tau_{i-1}}$, then*

$$c\mathbb{E}^{\mathbb{P}} \left[\max_{0 \leq i \leq n} |Y_{\tau_i}|^2 \right] \leq \mathbb{E}^{\mathbb{P}} \left[|Y_0|^2 + \langle M \rangle_T + |A_T|^2 \right] \leq C\mathbb{E}^{\mathbb{P}} \left[\max_{0 \leq i \leq n} |Y_{\tau_i}|^2 \right]. \tag{2.7}$$

2.2 Square integrable semimartingales

In this subsection we characterize the norm for square integrable semimartingales.

For $0 \leq t_1 < t_2 \leq T$, let $\bigvee_{t_1}^{t_2} A$ denote the total variation of A over the interval $(t_1, t_2]$.

Definition 2.4. We say a semimartingale Y in the form of (2.2) is a square integrable semimartingale if

$$\mathbb{E}^{\mathbb{P}} \left[|Y_0|^2 + \langle M \rangle_T + \left(\bigvee_0^T A \right)^2 \right] < \infty. \quad (2.8)$$

We remark that (2.8) is the norm used in standard literature for semimartingales, see e.g. [25]. Clearly, for a square integrable semimartingale Y , we have $\|Y\|_{\mathbb{P},0} < \infty$. However, when A is not monotone, in general the left side of (2.8) cannot be dominated by $C\|Y\|_{\mathbb{P},0}^2$, as illustrated by the following simple example.

Example 2.5. Let $K \in \mathbb{D}(\mathbb{F})$ be continuous and increasing such that $K_0 = 0$ and $\mathbb{E}^{\mathbb{P}}[K_T^2] = \infty$. Define a sequence of stopping times: $\tau_0 := 0$ and $\tau_n := \inf\{t \geq 0 : K_t = n\} \wedge T$ for $n \geq 1$. Since $K_T < \infty$, $\tau_n = T$ for n large enough, a.s. We now define the process Y_t as follows: $Y_0 := 0$, and for $n \geq 0$,

$$Y_t := \begin{cases} Y_{\tau_{2n}} - K_t + K_{\tau_{2n}}, & t \in (\tau_{2n}, \tau_{2n+1}); \\ Y_{\tau_{2n+1}} + K_t - K_{\tau_{2n+1}}, & t \in (\tau_{2n+1}, \tau_{2n+2}). \end{cases} \quad (2.9)$$

Then $\|Y\|_{\mathbb{P},0} < \infty$ but $\|Y\|_{\mathbb{P}} = \infty$.

Proof. It is easy to check that $-1 \leq Y_t \leq 0$ and $\bigvee_0^T Y = K_T$. Then $\|Y\|_{\mathbb{P},0} \leq 1$ and $\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T Y \right)^2 \right] = \infty$. By Theorem 2.7, we get $\|Y\|_{\mathbb{P}} = \infty$. \square

Our goal is to characterize square integrable semimartingales through the process Y itself, without involving M and A directly. In many applications, we may have a representation formula for the process Y , see e.g. (1.2), but in general it is difficult to obtain representation formulas for M and A . So conditions imposed on Y are more tractable than those on M and A . We introduce the following norm:

$$\|Y\|_{\mathbb{P}}^2 := \|Y\|_{\mathbb{P},0}^2 + \sup_{\pi} \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} |\mathbb{E}_{\tau_i}^{\mathbb{P}}(Y_{\tau_{i+1}}) - Y_{\tau_i}| \right)^2 \right], \text{ for any } Y \in \mathbb{D}(\mathbb{F}), \quad (2.10)$$

where the supremum is over all stopping time partitions $\pi : 0 = \tau_0 \leq \dots \leq \tau_n = T$.

Remark 2.1. (i) Our norm $\|\cdot\|_{\mathbb{P}}$ is strongly motivated by the definition of quasimartingale: a process $Y \in \mathbb{D}(\mathbb{F})$ is a quasimartingale if

$$\text{Var}(Y) := \sup_{\pi} \mathbb{E}^{\mathbb{P}} \left[\sum_{i=0}^{n-1} |\mathbb{E}_{t_i}^{\mathbb{P}}(Y_{t_{i+1}}) - Y_{t_i}| \right] < \infty, \quad (2.11)$$

where the supremum is over all deterministic partition $\pi : 0 = t_0 < \dots < t_n = T$. We note that a process $Y \in \mathbb{D}(\mathbb{F})$ is a quasimartingale if and only if it can be written as the difference of two nonnegative supermartingales, see e.g. Protter [25] Chapter III Theorem 17. We also refer to Rao [26], Dellacherie and Meyer [6], and Meyer and Zheng [16] for the theory of quasimartingales.

(ii) By the Rao's theorem, see e.g. [25] Chapter III Theorem 18, a quasimartingale Y has a unique decomposition $Y = M + A$, where M is a local martingale and A is a predictable process with paths of locally integrable variation and $A_0 = 0$. However, in this case we do not have a priori estimates of $\mathbb{E}[\langle M \rangle]$ and $\mathbb{E}[(\bigvee_0^T A)^2]$ in terms of $\text{Var}(Y)$. Indeed, M and A only have local integrability property. This type of estimates are important in applications and, in order to derive them, our stronger norm $\|\cdot\|_{\mathbb{P}}$ is needed: it is clear that $\|Y\|_{\mathbb{P}} < \infty$ implies $\text{Var}(Y) < \infty$ and thus Y is a quasimartingale, but not vice versa.

The following a priori estimate is the main technical result of this paper.

Theorem 2.6. *There exist universal constants $0 < c < C$ such that, for any square integrable semimartingale $Y_t = Y_0 + M_t + A_t$,*

$$c\|Y\|_{\mathbb{P}}^2 \leq \mathbb{E}^{\mathbb{P}} \left[|Y_0|^2 + \langle M \rangle_T + \left(\bigvee_0^T A \right)^2 \right] \leq C\|Y\|_{\mathbb{P}}^2. \tag{2.12}$$

Proof. (i) We first prove the left inequality. Let $\pi : 0 = \tau_0 \leq \dots \leq \tau_n = T$ be an arbitrary partition, and denote $\Delta A_{\tau_{i+1}} := A_{\tau_{i+1}} - A_{\tau_i}$. Then

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} |\mathbb{E}_{\tau_i}^{\mathbb{P}}(Y_{\tau_{i+1}}) - Y_{\tau_i}| \right)^2 \right] = \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} |\mathbb{E}_{\tau_i}^{\mathbb{P}}(A_{\tau_{i+1}}) - A_{\tau_i}| \right)^2 \right] \\ & \leq \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} \mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|) \right)^2 \right] = \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} [\mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|] + \sum_{i=0}^{n-1} |\Delta A_{\tau_{i+1}}| \right)^2 \right] \\ & \leq C\mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} [\mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|] \right)^2 \right] + C\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right]. \end{aligned} \tag{2.13}$$

Note that $\sum_{i=0}^j [\mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|]$, $j = 0, \dots, n-1$, is a martingale. Then

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} [\mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|] \right)^2 \right] = \mathbb{E}^{\mathbb{P}} \left[\sum_{i=0}^{n-1} [\mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|) - |\Delta A_{\tau_{i+1}}|]^2 \right] \\ & \leq C\mathbb{E}^{\mathbb{P}} \left[\sum_{i=0}^{n-1} [(\mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|))^2 + |\Delta A_{\tau_{i+1}}|^2] \right] \leq C\mathbb{E}^{\mathbb{P}} \left[\sum_{i=0}^{n-1} [\mathbb{E}_{\tau_i}^{\mathbb{P}}(|\Delta A_{\tau_{i+1}}|^2) + |\Delta A_{\tau_{i+1}}|^2] \right] \\ & \leq C\mathbb{E}^{\mathbb{P}} \left[\sum_{i=0}^{n-1} |\Delta A_{\tau_{i+1}}|^2 \right] \leq C\mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} |\Delta A_{\tau_{i+1}}| \right)^2 \right] \leq C\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right]. \end{aligned}$$

This, together with (2.13) and the left inequality of (2.3), proves the left inequality.

(ii) We next prove the right inequality. First, for any $\varepsilon > 0$, following the arguments in Lemma 2.2 one can easily show that

$$\mathbb{E}^{\mathbb{P}} [\langle M \rangle_T] \leq C\varepsilon^{-1}\|Y\|_{\mathbb{P},0}^2 + \varepsilon\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right]. \tag{2.14}$$

We claim that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right] \leq C\|Y\|_{\mathbb{P}}^2 + C\mathbb{E}^{\mathbb{P}} [\langle M \rangle_T]. \tag{2.15}$$

Then, combining (2.14) and by choosing ε small enough, we obtain the right inequality of (2.12) immediately.

We now prove (2.15) in four steps.

Step1. We first show that, for any random partition $\pi : 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_n = T$:

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{i=0}^{n-1} [A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}^{\mathbb{P}}[A_{\tau_{i+1}}]]^2 \right] \leq C\|Y\|_{\mathbb{P}}^2 + C\mathbb{E}^{\mathbb{P}} [\langle M \rangle_T]. \tag{2.16}$$

Indeed, note that $\mathbb{E}_{\tau_i}^{\mathbb{P}}[A_{\tau_{i+1}}] - A_{\tau_i} = \mathbb{E}_{\tau_i}^{\mathbb{P}}[Y_{\tau_{i+1}}] - Y_{\tau_i}$. Then

$$\begin{aligned} \sum_{i=0}^{n-1} [A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}^{\mathbb{P}}[A_{\tau_{i+1}}]] &= A_T - \sum_{i=0}^{n-1} (\mathbb{E}_{\tau_i}^{\mathbb{P}}[A_{\tau_{i+1}}] - A_{\tau_i}) \\ &= Y_T - Y_0 - M_T - \sum_{i=0}^{n-1} (\mathbb{E}_{\tau_i}^{\mathbb{P}}[Y_{\tau_{i+1}}] - Y_{\tau_i}). \end{aligned}$$

By the definition of $\|Y\|_{\mathbb{P}}$, (2.10), we see that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} [A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}^{\mathbb{P}}[A_{\tau_{i+1}}]] \right)^2 \right] \leq C \|Y\|_{\mathbb{P}}^2 + C \mathbb{E}^{\mathbb{P}}[\langle M \rangle_T].$$

This, together with the fact that $\sum_{i=0}^{j-1} [A_{\tau_{i+1}} - \mathbb{E}_{\tau_i}^{\mathbb{P}}[A_{\tau_{i+1}}]]$, $j = 1, \dots, n$, is a martingale, implies (2.16) immediately.

Step 2. In this step we assume $A_t = \int_0^t a_s dK_s$, where K is a continuous nondecreasing process and a is a simple process. That is,

$$a = a_{t_0} \mathbf{1}_{\{t_0\}} + \sum_{i=0}^{n-1} a_{t_i} \mathbf{1}_{(t_i, t_{i+1}]} \quad \text{for some } 0 = t_0 < \dots < t_n = T, \quad a_{t_i} \in \mathcal{F}_{t_i}.$$

Then, denoting $\alpha_i := \text{sgn}(a_{t_i}) \in \mathcal{F}_{t_i}$,

$$\begin{aligned} \bigvee_0^T A &= \int_0^T |a_t| dK_t = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \alpha_i a_t dK_t = \sum_{i=0}^{n-1} \alpha_i [A_{t_{i+1}} - A_{t_i}] \\ &= \sum_{i=0}^{n-1} \alpha_i (A_{t_{i+1}} - \mathbb{E}_{t_i}^{\mathbb{P}}[A_{t_{i+1}}]) + \sum_{i=0}^{n-1} \alpha_i (\mathbb{E}_{t_i}^{\mathbb{P}}[A_{t_{i+1}}] - A_{t_i}). \end{aligned}$$

Note that $\sum_{i=0}^j \alpha_i (A_{t_{i+1}} - \mathbb{E}_{t_i}^{\mathbb{P}}[A_{t_{i+1}}])$, $j = 0, \dots, n-1$, is a martingale. Then

$$\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right] \leq C \mathbb{E}^{\mathbb{P}} \left[\sum_{i=0}^{n-1} |A_{t_{i+1}} - \mathbb{E}_{t_i}^{\mathbb{P}}[A_{t_{i+1}}]|^2 + \left(\sum_{i=0}^{n-1} |\mathbb{E}_{t_i}^{\mathbb{P}}[A_{t_{i+1}}] - A_{t_i}| \right)^2 \right].$$

By (2.16) and the definition of $\|Y\|_{\mathbb{P}}$ (2.10) we obtain (2.15).

Step 3. We now prove (2.15) for general continuous process A . Denote $K_t := \bigvee_0^t A$.

Since A is continuous, K is also continuous. Moreover dA_t is absolutely continuous with respect to dK_t and thus $dA_t = a_t dK_t$ for some a . By [15], Chapter 3 Lemma 2.7, for every $\varepsilon > 0$ there exists a simple process $\{a^\varepsilon\}$ such that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |a_t^\varepsilon - a_t| dK_t \right)^2 \right] \leq \varepsilon. \tag{2.17}$$

Denote

$$A_t^\varepsilon := \int_0^t a_s^\varepsilon dK_s, \quad Y_t^\varepsilon := Y_0 + M_t + A_t^\varepsilon.$$

Then by *Step 2* we see that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A^\varepsilon \right)^2 \right] \leq C \|Y^\varepsilon\|_{\mathbb{P}}^2 + C \mathbb{E}^{\mathbb{P}}[\langle M \rangle_T]. \tag{2.18}$$

Note that

$$\bigvee_0^T A \leq \bigvee_0^T A^\varepsilon + \bigvee_0^T [A^\varepsilon - A] \leq \bigvee_0^T A^\varepsilon + \int_0^T |a_t^\varepsilon - a_t| dK_t.$$

Then

$$\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right] \leq C \mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A^\varepsilon \right)^2 \right] + C\varepsilon. \tag{2.19}$$

On the other hand, apply the left inequality of (2.12) on $Y^\varepsilon - Y = A^\varepsilon - A$, we get

$$\|Y^\varepsilon - Y\|_{\mathbb{P}}^2 \leq C \mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T (A^\varepsilon - A) \right)^2 \right] \leq C \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |a_t^\varepsilon - a_t| dK_t \right)^2 \right] \leq C\varepsilon.$$

Then $\|Y^\varepsilon\|_{\mathbb{P}}^2 \leq C\|Y\|_{\mathbb{P}}^2 + C\varepsilon$. Plug this and (2.19) into (2.18), we get

$$\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right] \leq C\|Y\|_{\mathbb{P}}^2 + C\mathbb{E}^{\mathbb{P}}[\langle M \rangle_T] + C\varepsilon.$$

Since ε is arbitrary, we obtain (2.15).

Step 4. We now prove (2.15) for the general case. Since A has finite variation, we can decompose $A = A^c + A^d$, where A^c is the continuous part and A^d is the part with pure jumps. Since Y is càdlàg and M is continuous, A and A^d are càdlàg. We denote $Y_t^c := Y_0 + M_t + A_t^c$. From *Step 3* we have

$$\mathbb{E}^{\mathbb{P}} \left[|Y_0|^2 + \langle M \rangle_T + \left(\bigvee_0^T A^c \right)^2 \right] \leq C\|Y^c\|_{\mathbb{P}}^2.$$

Note that $\|Y^c\|_{\mathbb{P}} \leq \|Y\|_{\mathbb{P}} + \|A^d\|_{\mathbb{P}}$, $\bigvee_0^T A \leq \bigvee_0^T A^c + \bigvee_0^T A^d$, and it follows from the left inequality of (2.12) (on A^d) that $\|A^d\|_{\mathbb{P}}^2 \leq C \mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A^d \right)^2 \right]$. Then

$$\mathbb{E}^{\mathbb{P}} \left[|Y_0|^2 + \langle M \rangle_T + \left(\bigvee_0^T A \right)^2 \right] \leq C\|Y\|_{\mathbb{P}}^2 + C \mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A^d \right)^2 \right]. \tag{2.20}$$

Note that

$$\bigvee_0^T A^d = \sum_{0 < t \leq T} |\Delta A_t| = \sum_{0 < t \leq T} |\Delta Y_t|. \tag{2.21}$$

Define, for each $n \geq 1$,

$$D_n := \sum_{0 < t \leq T} |\Delta Y_t| \mathbf{1}_{\{|\Delta Y_t| \geq \frac{1}{n}\}},$$

and, $\tau_0^n := 0$, and for $m \geq 0$, by denoting $Y_t := Y_T$ for $t \geq T$,

$$\tau_{m+1}^n := \inf \left\{ t > \tau_m^n : |\Delta Y_t| \geq \frac{1}{n} \right\} \wedge (T + 1).$$

We remark that we use $T+1$ instead of T here so that ΔY_T will not be counted repeatedly at below. By the right continuity of \mathbb{F} we see that τ_i^n are stopping times. It is clear that

$$D_n \uparrow \sum_{0 \leq t \leq T} |\Delta Y_t| \text{ as } n \rightarrow \infty, \text{ and } \sum_{i=1}^m |\Delta Y_{\tau_i^n}| \uparrow D_n \text{ as } m \rightarrow \infty.$$

We claim that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=1}^m |\Delta Y_{\tau_i^n}| \right)^2 \right] \leq \|Y\|_{\mathbb{P}}^2 \quad \text{for all } n, m. \quad (2.22)$$

Then it follows from (2.21) that $\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A^d \right)^2 \right] \leq C \|Y\|_{\mathbb{P}}^2$. This, together with (2.20), proves (2.15).

It remains to prove (2.22). To this end, we fix n, m . Since \mathbb{F} is a Brownian filtration, all \mathbb{F} -stopping times are predictable, see e.g. [23], Corollary 4.5.7. Then for each τ_i^n , there exist $\{\tau_{i,j}^n, j \geq 1\}$ such that $\tau_{i,j}^n < \tau_i^n$ and $\tau_{i,j}^n \uparrow \tau_i^n$ as $j \rightarrow \infty$. By definition of $\|Y\|_{\mathbb{P}}$ (2.10), we have

$$\mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=1}^m \left| \mathbb{E}_{\tau_{i-1}^n \vee \tau_{i,j}^n}^{\mathbb{P}} [Y_{\tau_i^n}] - Y_{\tau_{i-1}^n \vee \tau_{i,j}^n} \right| \right)^2 \right] \leq \|Y\|_{\mathbb{P}}^2. \quad (2.23)$$

Moreover, since all \mathbb{F} -stopping times are predictable, the filtration (\mathcal{F}_t) does not have any discontinuity time, i.e.: $\bigvee_{j=1}^{\infty} \mathcal{F}_{\tau_{i-1}^n \vee \tau_{i,j}^n} = \mathcal{F}_{\tau_i^n}$, see, e.g. [6] Theorem 83, p.217. Send $j \rightarrow \infty$, we obtain

$$\lim_{j \rightarrow \infty} \left[\mathbb{E}_{\tau_{i-1}^n \vee \tau_{i,j}^n}^{\mathbb{P}} [Y_{\tau_i^n}] - Y_{\tau_{i-1}^n \vee \tau_{i,j}^n} \right] = \mathbb{E}_{\tau_i^n}^{\mathbb{P}} [Y_{\tau_i^n}] - Y_{\tau_i^n} = Y_{\tau_i^n} - Y_{\tau_i^n-} = \Delta Y_{\tau_i^n};$$

Then, noting that $\mathbb{E}^{\mathbb{P}} [\sup_{0 \leq t \leq T} |\Delta Y_t|^2] < \infty$ and applying the Dominated Convergence Theorem, we obtain (2.22) from (2.23), and complete the proof. \square

As a direct consequence of the above a priori estimates, we have

Theorem 2.7. *A process $Y \in \mathbb{D}(\mathbb{F})$ is a square integrable semimartingale if and only if $\|Y\|_{\mathbb{P}} < \infty$.*

Proof. By Theorem 2.6, it suffices to prove the if part. Assume $\|Y\|_{\mathbb{P}} < \infty$. By Rao's theorem in Remark 2.1 (ii), we have decomposition $Y = M + A$, where M is a local martingale and A is a predictable process with paths of locally integrable variation. Define the sequence of stopping times τ_n as

$$\tau_n := \inf \{ t \geq 0 : \langle M \rangle_t + \bigvee_0^t A \geq n \} \wedge T.$$

Then $\tau_n \rightarrow T$, \mathbb{P} -a.s. Denoting $Y_t^n := Y_{t \wedge \tau_n}$, $M_t^n := M_{t \wedge \tau_n}$, $A_t^n := A_{t \wedge \tau_n}$, then $\|Y^n\|_{\mathbb{P}} \leq \|Y\|_{\mathbb{P}}$ and Y^n is a square integrable semimartingale. By Theorem 2.6 we have,

$$\mathbb{E}^{\mathbb{P}} \left[|Y_0^n|^2 + \langle M^n \rangle_T + \left(\bigvee_0^T A^n \right)^2 \right] \leq C \|Y^n\|_{\mathbb{P}}^2 \leq C \|Y\|_{\mathbb{P}}^2.$$

Send $n \rightarrow \infty$, by the Dominated Convergence Theorem we have

$$\mathbb{E}^{\mathbb{P}} \left[|Y_0|^2 + \langle M \rangle_T + \left(\bigvee_0^T A \right)^2 \right] \leq C \|Y\|_{\mathbb{P}}^2.$$

Thus Y is a square integrable semimartingale. \square

3 Doubly Reflected BSDEs

In this section we assume \mathbb{F} is generated by a standard Brownian motion B and augmented with all the \mathbb{P} -null sets. We consider the following Doubly Reflected Backward SDE (DRBSDE, for short) with \mathbb{F} -progressively measurable solution (Y, Z, A) :

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s + A_T - A_t; \\ L \leq Y \leq U, \quad [Y_{t-} - L_{t-}] dK_t^+ = [U_{t-} - Y_{t-}] dK_t^- = 0. \end{cases} \quad (3.1)$$

Here $Y \in \mathbb{D}(\mathbb{F})$ and A has finite variation with orthogonal decomposition $A = K^+ - K^-$. We say (Y, Z, A) satisfying (3.1) is a local solution if

$$\sup_{0 \leq t \leq T} |Y_t| + \int_0^T |Z_t|^2 dt + \bigvee_0^T A < \infty, \quad \mathbb{P}\text{-a.s.} \quad (3.2)$$

and a solution if

$$\|(Y, Z, A)\|^2 := \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt + \left(\bigvee_0^T A \right)^2 \right] < \infty. \quad (3.3)$$

Throughout this section, we assume the following standing assumptions:

Assumption 3.1. (i) ξ is \mathcal{F}_T -measurable, f is \mathbb{F} -progressively measurable, and

$$I_0^2 := I_0^2(\xi, f) := \mathbb{E}^{\mathbb{P}} \left[|\xi|^2 + \left(\int_0^T |f(t, 0, 0)| dt \right)^2 \right] < \infty. \quad (3.4)$$

(ii) f is uniformly Lipschitz continuous in (y, z) ;

(iii) $L, U \in \mathbb{D}(\mathbb{F})$; $L \leq U$, $L_T \leq \xi \leq U_T$; and

$$\|(L, U)\|_{\mathbb{F}, 0}^2 := \|L^+\|_{\mathbb{F}, 0}^2 + \|U^-\|_{\mathbb{F}, 0}^2 < \infty. \quad (3.5)$$

Moreover, we shall always denote

$$\hat{L}_t := L_t \vee L_{t-}, \quad \hat{U}_t := U_t \wedge U_{t-}. \quad (3.6)$$

Remark 3.2. In the standard BSDE literature, one requires $\mathbb{E}^{\mathbb{P}} \left[\int_0^T |f(t, 0, 0)|^2 dt \right] < \infty$. Our condition (3.4) is slightly weaker. In fact, most estimates in the BSDE literature can be improved by replacing $\mathbb{E}^{\mathbb{P}} \left[\int_0^T |f(t, 0, 0)|^2 dt \right]$ with $\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |f(t, 0, 0)| dt \right)^2 \right]$, and the arguments are rather standard. We refer to the monograph Cvitanic and Zhang [5] Theorem 9.3.2 for interested readers.

It is well known that Assumption 3.1 does not yield the wellposedness of DRBSDE (3.1). At below is a simple counterexample.

Example 3.1. Let $L = U$ be deterministic, càdlàg, and $\bigvee_0^T L = \infty$. Then DRBSDE (3.1) with $\xi = L_T$ and $f = 0$ has no solution.

In the literature, there are typically two approaches for wellposedness of DRBSDEs. We first report a result from Hamadène, Hassani and Ouknine [13] Theorem 4.1 and its proof:

Lemma 3.2. *Let Assumption 3.1 and the following separation condition hold:*

$$L_t < U_t \quad \text{and} \quad L_{t-} < U_{t-} \quad \text{for all } t. \quad (3.7)$$

Then (3.1) admits a local solution (Y, Z, A) satisfying:

$$\|(Y_{\cdot, \tau_n}, Z \mathbf{1}_{[0, \tau_n]}, A_{\cdot \wedge \tau_n})\| < \infty, \quad \text{for all } n \geq 1, \quad (3.8)$$

where $\tau_0 := 0$ and, for $i \geq 0$,

$$\tau_{2i+1} := \inf\{t \geq \tau_{2i} : Y_t \leq \hat{L}_t\} \wedge T, \quad \tau_{2i+2} := \inf\{t \geq \tau_{2i+1} : Y_t \geq \hat{U}_t\} \wedge T. \quad (3.9)$$

The condition (3.7) is mild and very easy to verify, but it does not yield any a priori estimates. We remark that [13] takes a slightly different form of DRBSDEs. However, one can easily check that a local solution in [13] is a local solution in our sense, so the existence in Lemma 3.2 is valid. Moreover, for the γ_n in the proof of [13] Theorem 4.1, it is clear that

$$\tau_n \leq \gamma_n \leq \tau_{2n}. \quad (3.10)$$

We next report a result from Peng and Xu [22], following the original work Cvitanic and Karatzas [4]:

Lemma 3.3. *Let Assumption 3.1 hold. Assume further the following Mokobodski's type of condition:*

$$\text{there exists a square integrable semimartingale } Y^0 \text{ such that } L_t \leq Y_t^0 \leq U_t. \quad (3.11)$$

Then DRBSDE (3.1) admits a unique solution and the following estimate holds:

$$\|(Y, Z, A)\|^2 \leq C \left[I_0^2 + \|Y^0\|_{\mathbb{P}}^2 \right]. \quad (3.12)$$

We note that, in those works there is no discussion on the sufficient conditions for the existence of such Y^0 . More recently, Crépey and Matoussi [2] provides a priori bound, as well as error estimates for solutions of DRBSDE under the assumption that the barrier L (or U) is a quasimartingale, $\|L\|_{\mathbb{P},0} < \infty$ and L has canonical decomposition

$$L_t = L_0 + M_t + A_t,$$

for a uniformly integrable martingale M and a predictable process of integrable variation A . The assumption on the structure of the barriers here can be seen as a similar approach to the Mokobodski's condition. The advantage of such assumption is that it provides an explicit representation for the structures of K^+, K^- , which in turn helps for the derivation of the estimates.

Our goal in this section is to provide another approach, in the spirit of norm estimates, to impose a sufficient condition on the barriers L and U that would lead to a priori bound and error estimates of the solutions. In light of the norm $\|\cdot\|_{\mathbb{P}}$ (2.10), we introduce the following norm for the barriers (L, U) : recalling \hat{L} and \hat{U} in (3.6),

$$\begin{aligned} \|(L, U)\|_{\mathbb{P}}^2 &:= \|(L, U)\|_{\mathbb{P},0}^2 \\ &+ \sup_{\pi} \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} \left([\mathbb{E}_{\tau_i}^{\mathbb{P}}(\hat{L}_{\tau_{i+1}}) - \hat{U}_{\tau_i}]^+ + [\hat{L}_{\tau_i} - \mathbb{E}_{\tau_i}^{\mathbb{P}}(\hat{U}_{\tau_{i+1}})]^+ \right) \right)^2 \right], \end{aligned} \quad (3.13)$$

where the supremum is again taken over all random partitions $\pi : 0 = \tau_0 \leq \dots \leq \tau_n = T$. Our main result of this section is:

Theorem 3.4. *Let Assumption 3.1 hold. Then the following are equivalent:*

- (i) *The DRBSDE (3.1) admits a unique solution (Y, Z, A) ;*
- (ii) *the Mokobodski condition (3.11) holds;*
- (iii) $\|(L, U)\|_{\mathbb{P}} < \infty$.

Moreover, in this case we have the estimate:

$$\|(Y, Z, A)\|^2 \leq C \left[I_0^2 + \|(L, U)\|_{\mathbb{P}}^2 \right]. \tag{3.14}$$

In addition, we have the following estimates for the difference of two DRBSDEs:

Theorem 3.5. *Assume (ξ_i, f^i, L^i, U^i) , $i = 1, 2$, satisfy all the conditions in Theorem 3.4, and let (Y^i, Z^i, A^i) denote the solution to the corresponding DRBSDE (3.1). Denote $\delta Y := Y^1 - Y^2$, and similarly for the other notations. Then*

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} [|\delta Y_t|^2 + |\delta A_t|^2] + \int_0^T |\delta Z_t|^2 dt \right] \\ & \leq C \mathbb{E}^{\mathbb{P}} \left[|\delta \xi|^2 + \left(\int_0^T |\delta f(t, Y_t^1, Z_t^1)| dt \right)^2 \right] \\ & \quad + C \sum_{i=1}^2 \left[I_0(\xi_i, f^i) + \|(L^i, U^i)\|_{\mathbb{P}} \right] \left(\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} [|\delta L_t|^2 + |\delta U_t|^2] \right] \right)^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

These two theorems will be proved in the rest of this section. We first note that

Remark 3.3. (i) *In the case that there is only one barrier L , we may view it as $U = \infty$. One can check straightforwardly that $\|(L, U)\|_{\mathbb{P}} = \|L^+\|_{\mathbb{P},0}$. Then Theorems 3.4 and 3.5 reduce to standard results for reflected BSDEs with one barrier, see e.g. El Karoui et al [11] and Peng and Xu [22].*

(ii) *In the case $(L^1, U^1) = (L^2, U^2)$, the last term in (3.15) vanishes and [22] has already obtained the estimate.*

3.1 Proof of Theorem 3.5

As usual we start with some a priori estimates.

Lemma 3.6. *Assume (ξ_i, f^i, L^i, U^i) , $i = 1, 2$, satisfy Assumption 3.1. If the corresponding DRBSDE (3.1) has a solution (Y^i, Z^i, A^i) , then*

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} [|\delta Y_t|^2 + |\delta A_t|^2] + \int_0^T |\delta Z_t|^2 dt \right] \leq CI^2, \tag{3.16}$$

where, recalling the norm $\|(Y, Z, A)\|$ defined by (3.3),

$$\begin{aligned} I^2 & := \mathbb{E}^{\mathbb{P}} \left[|\delta \xi|^2 + \left(\int_0^T |\delta f(t, Y_t^1, Z_t^1)| dt \right)^2 \right] \\ & \quad + \sum_{i=1}^2 \|(Y^i, Z^i, A^i)\| \left(\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} [|\delta L_t|^2 + |\delta U_t|^2] \right] \right)^{\frac{1}{2}}. \end{aligned} \tag{3.17}$$

Proof. Let $\lambda > 0$ be a constant which will be specified later. Applying Itô's formula on $e^{\lambda t} |\delta Y_t|^2$ we have

$$\begin{aligned} & e^{\lambda t} |\delta Y_t|^2 + \lambda \int_t^T e^{\lambda s} |\delta Y_s|^2 ds + \int_t^T e^{\lambda s} |\delta Z_s|^2 ds \\ & = e^{\lambda T} |\delta \xi|^2 + 2 \int_t^T e^{\lambda s} \delta Y_s (f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)) ds + 2 \int_t^T e^{\lambda s} \delta Y_{s-} d\delta A_s \\ & \quad - 2 \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s. \end{aligned} \tag{3.18}$$

For any $\varepsilon > 0$, note that

$$\begin{aligned}
 & 2 \int_t^T e^{\lambda s} |\delta Y_s| |f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)| ds \\
 & \leq C \int_t^T e^{\lambda s} |\delta Y_s| [|\delta f(s, Y_s^1, Z_s^1)| + |\delta Y_s| + |\delta Z_s|] ds \\
 & \leq C \left[\sup_{t \leq s \leq T} |\delta Y_s| \int_t^T e^{\lambda s} |\delta f(s, Y_s^1, Z_s^1)| ds + \int_t^T e^{\lambda s} [|\delta Y_s|^2 + |\delta Y_s| |\delta Z_s|] ds \right] \\
 & \leq \varepsilon \sup_{t \leq s \leq T} |\delta Y_s|^2 + \frac{1}{2} \int_t^T e^{\lambda s} |\delta Z_s|^2 ds \\
 & \quad + C \int_t^T e^{\lambda s} |\delta Y_s|^2 ds + C\varepsilon^{-1} \left(\int_t^T e^{\lambda s} |\delta f(s, Y_s^1, Z_s^1)| ds \right)^2;
 \end{aligned} \tag{3.19}$$

and, with the orthogonal decompositions $A^i = K^{i,+} - K^{i,-}$,

$$\begin{aligned}
 & 2 \int_t^T e^{\lambda s} \delta Y_{s-} d\delta A_s \\
 & = 2 \int_t^T e^{\lambda s} (Y_{s-}^1 dK_s^{1,+} - Y_{s-}^1 dK_s^{1,-} - Y_{s-}^2 dK_s^{1,+} + Y_{s-}^2 dK_s^{1,-} \\
 & \quad - Y_{s-}^1 dK_s^{2,+} + Y_{s-}^1 dK_s^{2,-} + Y_{s-}^2 dK_s^{2,+} - Y_{s-}^2 dK_s^{2,-}) \\
 & \leq 2 \int_t^T e^{\lambda s} (L_{s-}^1 dK_s^{1,+} - U_{s-}^1 dK_s^{1,-} - L_{s-}^2 dK_s^{1,+} + U_{s-}^2 dK_s^{1,-} \\
 & \quad - L_{s-}^1 dK_s^{2,+} + U_{s-}^1 dK_s^{2,-} + L_{s-}^2 dK_s^{2,+} - U_{s-}^2 dK_s^{2,-}) \\
 & = 2 \int_t^T e^{\lambda s} (\delta L_{s-} dK_s^{1,+} - \delta U_{s-} dK_s^{1,-} - \delta L_{s-} dK_s^{2,+} + \delta U_{s-} dK_s^{2,-}) \\
 & \leq 2e^{\lambda T} \sup_{0 \leq s \leq T} [|\delta L_s| + |\delta U_s|] \left[\bigvee_t^T A^1 + \bigvee_t^T A^2 \right].
 \end{aligned} \tag{3.20}$$

Plug (3.19) and (3.20) into (3.18), we obtain

$$\begin{aligned}
 & e^{\lambda t} |\delta Y_t|^2 + \lambda \int_t^T e^{\lambda s} |\delta Y_s|^2 ds + \int_t^T e^{\lambda s} |\delta Z_s|^2 ds \\
 & \leq e^{\lambda T} |\delta \xi^2| + \varepsilon \sup_{t \leq s \leq T} |\delta Y_s|^2 + \frac{1}{2} \int_t^T e^{\lambda s} |\delta Z_s|^2 ds \\
 & \quad + C \int_t^T e^{\lambda s} |\delta Y_s|^2 ds + C\varepsilon^{-1} \left(\int_t^T e^{\lambda s} |\delta f(s, Y_s^1, Z_s^1)| ds \right)^2 \\
 & \quad + 2e^{\lambda T} \sup_{0 \leq s \leq T} [|\delta L_s| + |\delta U_s|] \left[\bigvee_t^T A^1 + \bigvee_t^T A^2 \right] - 2 \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s.
 \end{aligned}$$

Set $\lambda = C$ for the above C , we get

$$\begin{aligned}
 & e^{\lambda t} |\delta Y_t|^2 + \frac{1}{2} \int_t^T e^{\lambda s} |\delta Z_s|^2 ds \\
 & \leq e^{\lambda T} |\delta \xi^2| + \varepsilon \sup_{t \leq s \leq T} |\delta Y_s|^2 + C\varepsilon^{-1} \left(\int_t^T e^{\lambda s} |\delta f(s, Y_s^1, Z_s^1)| ds \right)^2 \\
 & \quad + 2e^{\lambda T} \sup_{0 \leq s \leq T} [|\delta L_s| + |\delta U_s|] \left[\bigvee_t^T A^1 + \bigvee_t^T A^2 \right] - 2 \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s.
 \end{aligned} \tag{3.21}$$

Take expectation on both sides, we have

$$\sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{P}}[|\delta Y_t|^2] + \mathbb{E}^{\mathbb{P}} \left[\int_0^T |\delta Z_t|^2 dt \right] \leq C[1 + \varepsilon^{-1}]I^2 + \varepsilon \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right]. \quad (3.22)$$

Moreover, by (3.21) we have

$$\begin{aligned} \sup_{0 \leq t \leq T} e^{\lambda t} |\delta Y_t|^2 &\leq e^{\lambda T} |\delta \xi^2| + \varepsilon \sup_{0 \leq t \leq T} |\delta Y_t|^2 + C\varepsilon^{-1} \left(\int_0^T e^{\lambda t} |\delta f(t, Y_t^1, Z_t^1)| dt \right)^2 \\ &+ 2e^{\lambda T} \sup_{0 \leq t \leq T} [|\delta L_t| + |\delta U_t|] \left[\bigvee_0^T A^1 + \bigvee_0^T A^2 \right] + 2 \sup_{0 \leq t \leq T} \left| \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s \right|. \end{aligned} \quad (3.23)$$

Apply the Burkholder-Davis-Gundy Inequality and note that $\lambda = C$, we get

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} \left| \int_t^T e^{\lambda s} \delta Y_s \delta Z_s dB_s \right| \right] &\leq C \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |\delta Y_t \delta Z_t|^2 dt \right)^{\frac{1}{2}} \right] \\ &\leq C \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t| \left(\int_0^T |\delta Z_t|^2 dt \right)^{\frac{1}{2}} \right] \\ &\leq \sqrt{\varepsilon} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C\varepsilon^{-\frac{1}{2}} \mathbb{E}^{\mathbb{P}} \left[\int_0^T |\delta Z_t|^2 dt \right]. \end{aligned} \quad (3.24)$$

Take expectation on both sides of (3.23), and apply (3.24) and then (3.22), we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] &\leq C[1 + \varepsilon^{-1}]I^2 + C\varepsilon \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] \\ &\quad + C\sqrt{\varepsilon} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C\varepsilon^{-\frac{1}{2}} \mathbb{E}^{\mathbb{P}} \left[\int_0^T |\delta Z_t|^2 dt \right] \\ &\leq C[\sqrt{\varepsilon} + \varepsilon(1 + \varepsilon^{-\frac{1}{2}})] \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C[1 + \varepsilon^{-\frac{1}{2}}][1 + \varepsilon^{-1}]I^2 \\ &\leq C\sqrt{\varepsilon} \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] + C\varepsilon^{-\frac{3}{2}} I^2. \end{aligned}$$

Set $\varepsilon := \frac{1}{4C^2}$ for the above C , and then by (3.22), we have

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |\delta Y_t|^2 \right] \leq CI^2, \quad \mathbb{E}^{\mathbb{P}} \left[\int_0^T |\delta Z_t|^2 dt \right] \leq CI^2.$$

Finally, notice that

$$\delta A_t = \delta Y_0 - \delta Y_t - \int_0^t [f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)] ds + \int_0^t \delta Z_s dB_s.$$

One can easily get the estimate for δA . □

Proof of Theorem 3.5. This is a direct consequence of Lemma 3.6 and Theorem 3.4. □

We emphasize that in next subsection, we shall prove Theorem 3.4 by using Lemma 3.6, but without using Theorem 3.5. So there is no danger of cycle proof.

3.2 Proof of Theorem 3.4

Again, we start with a priori estimate.

Lemma 3.7. *Let Assumption 3.1 and (3.7) hold, and $f = 0$. Then the local solution (Y, Z, A) of DRBSDE (3.1) satisfies (3.14).*

Proof. Without loss of generality, we assume $\|(L, U)\|_{\mathbb{P}} < \infty$. We proceed in three steps.

Step 1. We first assume (Y, Z, A) is a solution of (3.1) and Y is continuous. Then K^+ and K^- are also continuous. Apply Itô's formula on $|Y_t|^2$, by the minimum condition in (3.1) we have,

$$\begin{aligned} d|Y_t|^2 &= 2Y_t Z_t dB_t + |Z_t|^2 dt - 2Y_{t-} dK_t^+ + 2Y_{t-} dK_t^- \\ &= 2Y_t Z_t dB_t + |Z_t|^2 dt - 2L_{t-} dK_t^+ + 2U_{t-} dK_t^-. \end{aligned} \tag{3.25}$$

Then, for any $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}} \left[|Y_t|^2 + \int_t^T |Z_s|^2 ds \right] = \mathbb{E}^{\mathbb{P}} \left[|\xi|^2 + 2 \int_t^T L_{s-} dK_s^+ - 2 \int_t^T U_{s-} dK_s^- \right] \\ &\leq \mathbb{E}^{\mathbb{P}} \left[|\xi|^2 + 2 \sup_{0 \leq s \leq T} L_s^+ K_T^+ + 2 \sup_{0 \leq s \leq T} U_s^- K_T^- \right] \\ &\leq \mathbb{E}^{\mathbb{P}} \left[|\xi|^2 + C\varepsilon^{-1} \sup_{0 \leq s \leq T} [|L_s^+|^2 + |U_s^-|^2] + \varepsilon [K_T^+|^2 + |K_T^-|^2] \right] \\ &\leq \mathbb{E}^{\mathbb{P}} [|\xi|^2] + C\varepsilon^{-1} \|(L, U)\|_{\mathbb{P},0}^2 + \varepsilon \mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right]. \end{aligned}$$

Following standard arguments, in particular by applying the Burkholder-Davis-Gundy Inequality on (3.25), we have

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] \leq C \left[\mathbb{E}^{\mathbb{P}} [|\xi|^2] + \varepsilon^{-1} \|(L, U)\|_{\mathbb{P},0}^2 + \varepsilon \mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right] \right]. \tag{3.26}$$

We claim that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\bigvee_0^T A \right)^2 \right] \leq C \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] + C \|(L, U)\|_{\mathbb{P}}^2. \tag{3.27}$$

Combine (3.26) and (3.27) and set ε small, we prove (3.14) immediately.

To prove (3.27), we recall the sequence of stopping times τ_n defined by (3.9). By the proof of [13] Theorem 4.1 and (3.10),

$$\text{for } \mathbb{P}\text{-a.e. } \omega, \tau_n(\omega) = T \text{ for } n \text{ large enough.} \tag{3.28}$$

By the continuity of K^+, K^- and the minimal condition in (3.1), $dK_t^+ = 0$ on $[\tau_{2i}, \tau_{2i+1}]$ and $dK_t^- = 0$ on $[\tau_{2i+1}, \tau_{2i+2}]$, and thus

$$\begin{aligned} Y_t &= Y_{\tau_{2i+1}} - \int_t^{\tau_{2i+1}} Z_s dB_s - (K_{\tau_{2i+1}}^- - K_t^-), \quad t \in [\tau_{2i}, \tau_{2i+1}]; \\ Y_t &= Y_{\tau_{2i+2}} - \int_t^{\tau_{2i+2}} Z_s dB_s + (K_{\tau_{2i+2}}^+ - K_t^+), \quad t \in [\tau_{2i+1}, \tau_{2i+2}]. \end{aligned} \tag{3.29}$$

Moreover, recalling (3.6) we have

$$Y_{\tau_{2i}} = \hat{U}_{\tau_{2i}} \mathbf{1}_{\{\tau_{2i} < T\}} + \xi \mathbf{1}_{\{\tau_{2i} = T\}}, \quad Y_{\tau_{2i+1}} = \hat{L}_{\tau_{2i+1}} \mathbf{1}_{\{\tau_{2i+1} < T\}} + \xi \mathbf{1}_{\{\tau_{2i+1} = T\}}; \tag{3.30}$$

Indeed, on $\{\tau_{2i} < T\}$ if U is continuous or has a negative jump at τ_{2i} , then \hat{U} is right continuous at τ_{2i} and the first equality of (3.30) holds. If U has a positive jump at τ_{2i} , we must have $Y_{\tau_{2i}} = U_{\tau_{2i}}^-$. By continuity of Y , it follows that $Y_{\tau_{2i}} = \hat{U}_{\tau_{2i}}$. A similar argument holds for the second equality in (3.30).

For each i , by (3.29) and (3.30),

$$\begin{aligned} 0 &\leq \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-] - K_{\tau_{2i}}^- = \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[Y_{\tau_{2i+1}}] - Y_{\tau_{2i}} \\ &= \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}\left[\hat{L}_{\tau_{2i+1}} \mathbf{1}_{\{\tau_{2i+1} < T\}} + \xi \mathbf{1}_{\{\tau_{2i+1} = T\}}\right] - \hat{U}_{\tau_{2i}} \mathbf{1}_{\{\tau_{2i} < T\}} - \xi \mathbf{1}_{\{\tau_{2i} = T\}} \\ &= \left[\mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[\hat{L}_{\tau_{2i+1}}] - \hat{U}_{\tau_{2i}}\right] \mathbf{1}_{\{\tau_{2i} < T\}} + \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}\left[\hat{L}_{\tau_{2i+1}} - \xi\right] \mathbf{1}_{\{\tau_{2i} < T = \tau_{2i+1}\}} \\ &\leq \left[\mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[\hat{L}_{\tau_{2i+1}}] - \hat{U}_{\tau_{2i}}\right]^+. \end{aligned}$$

Then for any n ,

$$\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{i=0}^n \left[\mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-] - K_{\tau_{2i}}^-\right]\right)^2\right] \leq \|(L, U)\|_{\mathbb{P}}^2.$$

Send $n \rightarrow \infty$, we get

$$\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{i \geq 0} \left[\mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-] - K_{\tau_{2i}}^-\right]\right)^2\right] \leq \|(L, U)\|_{\mathbb{P}}^2. \tag{3.31}$$

Similarly,

$$\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{i \geq 0} \left[\mathbb{E}_{\tau_{2i+1}}^{\mathbb{P}}[K_{\tau_{2i+2}}^+] - K_{\tau_{2i+1}}^+\right]\right)^2\right] \leq \|(L, U)\|_{\mathbb{P}}^2. \tag{3.32}$$

Denote

$$\hat{Y}_{\tau_n} := Y_{\tau_n} - \sum_{i \leq \frac{n}{2}} \left[\mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-] - K_{\tau_{2i}}^-\right] + \sum_{i \leq \frac{n-1}{2}} \left[\mathbb{E}_{\tau_{2i+1}}^{\mathbb{P}}[K_{\tau_{2i+2}}^+] - K_{\tau_{2i+1}}^+\right]. \tag{3.33}$$

By (3.31) and (3.32), we have

$$\mathbb{E}^{\mathbb{P}}\left[\max_{n \geq 0} |\hat{Y}_{\tau_n}|^2\right] \leq C \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} |Y_t|^2\right] + C \|(L, U)\|_{\mathbb{P}}^2. \tag{3.34}$$

Note that

$$\hat{Y}_{\tau_n} = Y_0 + \int_0^{\tau_n} Z_s dB_s + \sum_{i \leq \frac{n}{2}} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-]] - \sum_{i \leq \frac{n-1}{2}} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^{\mathbb{P}}[K_{\tau_{2i+2}}^+]]$$

is a martingale. By (3.34), we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left[\sum_{i \leq \frac{n}{2}} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-]]^2 + \sum_{i \leq \frac{n-1}{2}} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^{\mathbb{P}}[K_{\tau_{2i+2}}^+]]^2\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{i \leq \frac{n}{2}} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-]] - \sum_{i \leq \frac{n-1}{2}} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^{\mathbb{P}}[K_{\tau_{2i+2}}^+]]\right)^2\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\hat{Y}_{\tau_n} - Y_0 - \int_0^{\tau_n} Z_s dB_s\right)^2\right] \leq C \mathbb{E}^{\mathbb{P}}\left[\sup_{i \geq 0} |\hat{Y}_{\tau_i}|^2 + \int_0^{\tau_n} |Z_t|^2 dt\right] \\ &\leq C \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt\right] + C \|(L, U)\|_{\mathbb{P}}^2. \end{aligned}$$

Send $n \rightarrow \infty$, we get

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left[\sum_{i \geq 0} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-]]^2 + \sum_{i \geq 0} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^{\mathbb{P}}[K_{\tau_{2i+2}}^+]]^2\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{i \geq 0} [K_{\tau_{2i+1}}^- - \mathbb{E}_{\tau_{2i}}^{\mathbb{P}}[K_{\tau_{2i+1}}^-]] - \sum_{i \geq 0} [K_{\tau_{2i+2}}^+ - \mathbb{E}_{\tau_{2i+1}}^{\mathbb{P}}[K_{\tau_{2i+2}}^+]]\right)^2\right] \\ &\leq C \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt\right] + C \|(L, U)\|_{\mathbb{P}}^2. \end{aligned} \tag{3.35}$$

This, together with (3.28), (3.31) and (3.32), implies further that

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[|K_T^-|^2 + |K_T^+|^2 \right] = \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i \geq 0} [K_{\tau_{2i+1}}^- - K_{\tau_{2i}}^-] \right)^2 + \left(\sum_{i \geq 0} [K_{\tau_{2i+2}}^+ - K_{\tau_{2i+1}}^+] \right)^2 \right] \\ & \leq C \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i \geq 0} [K_{\tau_{2i+1}}^- - \mathbb{E}^{\mathbb{P}}_{\tau_{2i}} [K_{\tau_{2i+1}}^-]] \right)^2 + \left(\sum_{i \geq 0} [\mathbb{E}^{\mathbb{P}}_{\tau_{2i}} [K_{\tau_{2i+1}}^- - K_{\tau_{2i}}^-]] \right)^2 \right] \\ & \quad + C \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i \geq 0} [K_{\tau_{2i+2}}^+ - \mathbb{E}^{\mathbb{P}}_{\tau_{2i+1}} [K_{\tau_{2i+2}}^+]] \right)^2 + \left(\sum_{i \geq 0} [\mathbb{E}^{\mathbb{P}}_{\tau_{2i+1}} [K_{\tau_{2i+2}}^+] - K_{\tau_{2i+1}}^+] \right)^2 \right] \\ & = C \mathbb{E}^{\mathbb{P}} \left[\sum_{i \geq 0} [K_{\tau_{2i+1}}^- - \mathbb{E}^{\mathbb{P}}_{\tau_{2i}} [K_{\tau_{2i+1}}^-]]^2 + \sum_{i \geq 0} [K_{\tau_{2i+2}}^+ - \mathbb{E}^{\mathbb{P}}_{\tau_{2i+1}} [K_{\tau_{2i+2}}^+]]^2 \right] \\ & \quad + C \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i \geq 0} [\mathbb{E}^{\mathbb{P}}_{\tau_{2i}} [K_{\tau_{2i+1}}^- - K_{\tau_{2i}}^-]] \right)^2 + \left(\sum_{i \geq 0} [\mathbb{E}^{\mathbb{P}}_{\tau_{2i+1}} [K_{\tau_{2i+2}}^+] - K_{\tau_{2i+1}}^+] \right)^2 \right] \\ & \leq C \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Z_t|^2 dt \right] + C \|(L, U)\|_{\mathbb{P}}^2. \end{aligned}$$

This proves (3.27) and hence (3.14).

Step 2. We next assume (Y, Z, A) is a local solution but Y is still continuous. Let τ_i be defined by (3.9). Then (3.28)-(3.30) still hold. Thus

$$Y_{\tau_{2i}} \geq -\hat{U}_{\tau_{2i}}^- \mathbf{1}_{\{\tau_{2i} < T\}} - |\xi| \mathbf{1}_{\{\tau_{2i} = T\}} \geq - \left[\sup_{0 \leq t \leq T} U_t^- + |\xi| \right].$$

Moreover, by (3.8), the stochastic integral in (3.29) is a true martingale, and then

$$Y_{\tau_{2i}} \leq \mathbb{E}^{\mathbb{P}}_{\tau_{2i}} [Y_{\tau_{2i+1}}] \leq \mathbb{E}^{\mathbb{P}}_{\tau_{2i}} \left[\hat{L}_{\tau_{2i+1}}^+ \mathbf{1}_{\{\tau_{2i+1} < T\}} + |\xi| \mathbf{1}_{\{\tau_{2i+1} = T\}} \right] \leq \mathbb{E}^{\mathbb{P}}_{\tau_{2i}} \left[\sup_{0 \leq t \leq T} L_t^+ + |\xi| \right].$$

Therefore,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\max_{i \geq 0} |Y_{\tau_{2i}}|^2 \right] & \leq \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} U_t^- + |\xi| \right)^2 \vee \left(\sup_{0 \leq s \leq T} \mathbb{E}^{\mathbb{P}}_s \left[\sup_{0 \leq t \leq T} L_t^+ + |\xi| \right] \right)^2 \right] \quad (3.36) \\ & \leq \mathbb{E}^{\mathbb{P}} \left[\left(\sup_{0 \leq s \leq T} \mathbb{E}^{\mathbb{P}}_s \left[\sup_{0 \leq t \leq T} [L_t^+ + U_t^-] + |\xi| \right] \right)^2 \right] \\ & \leq C \mathbb{E}^{\mathbb{P}} \left[\left(\sup_{0 \leq t \leq T} [L_t^+ + U_t^-] + |\xi| \right)^2 \right] \leq C \mathbb{E}^{\mathbb{P}} [|\xi|^2] + C \|(L, U)\|_{\mathbb{P},0}^2. \end{aligned}$$

Now for any n , define

$$\hat{\tau}_n := \inf \left\{ t : \sup_{0 \leq s \leq t} |Y_s| + \int_0^t |Z_s|^2 ds + \bigvee_0^t A \geq n \right\} \wedge T. \quad (3.37)$$

Then

$$\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t < \hat{\tau}_n} |Y_t|^2 + \int_0^{\hat{\tau}_n} |Z_t|^2 dt + \left(\bigvee_0^{\hat{\tau}_n} A \right)^2 \right] < \infty. \quad (3.38)$$

Define $\tilde{\tau}_n := \inf \{ \tau_{2i} : \tau_{2i} \geq \hat{\tau}_n \}$. Then by (3.1) and (3.36) we have

$$\begin{aligned} Y_t & = Y_{\tilde{\tau}_n} + \int_t^{\tilde{\tau}_n} Z_s dB_s - (K_{\tilde{\tau}_n}^- - K_t^-), \quad Y_t \leq U_t, \quad [U_t - Y_t] dK_t^- = 0, \quad t \in [\hat{\tau}_n, \tilde{\tau}_n]; \\ \mathbb{E}^{\mathbb{P}} [|Y_{\tilde{\tau}_n}|^2] & \leq C \mathbb{E}^{\mathbb{P}} [|\xi|^2] + C \|(L, U)\|_{\mathbb{P},0}^2. \end{aligned}$$

By standard arguments for Reflected BSDEs with one barrier, see e.g. [11],

$$\mathbb{E}^{\mathbb{P}}[|Y_{\hat{\tau}_n}|^2] \leq C\mathbb{E}^{\mathbb{P}}\left[|\xi|^2 + \sup_{0 \leq t \leq T} |U_t^-|^2\right] + C\|(L, U)\|_{\mathbb{P},0}^2 \leq C\mathbb{E}^{\mathbb{P}}[|\xi|^2] + C\|(L, U)\|_{\mathbb{P},0}^2.$$

This, together with (3.38), implies that

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq \hat{\tau}_n} |Y_t|^2 + \int_0^{\hat{\tau}_n} |Z_t|^2 dt + \left(\bigvee_0^{\hat{\tau}_n} A\right)^2\right] < \infty.$$

Then by Step 1, we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[\sup_{0 \leq t \leq \hat{\tau}_n} |Y_t|^2 + \int_0^{\hat{\tau}_n} |Z_t|^2 dt + \left(\bigvee_0^{\hat{\tau}_n} A\right)^2\right] &\leq C\mathbb{E}^{\mathbb{P}}[|Y_{\hat{\tau}_n}|^2] + C\|(L, U)\|_{\mathbb{P}}^2 \\ &\leq C\mathbb{E}^{\mathbb{P}}[|\xi|^2] + C\|(L, U)\|_{\mathbb{P}}^2. \end{aligned}$$

Note that $\hat{\tau}_n = T$ when n is large enough. Send $n \rightarrow \infty$ and apply the Monotone Convergence Theorem, we prove (3.14).

Step 3. Finally we allow Y to be discontinuous. Let

$$J_t := \sum_{0 < s \leq t} \Delta Y_s = - \sum_{0 < s \leq t} \Delta A_s = - \sum_{0 < s \leq t} \Delta K_s^+ + \sum_{0 < s \leq t} \Delta K_s^-.$$

Note that, when $\Delta K_t^+ > 0$, by the minimum condition of (3.1) we see that $Y_{t-} = L_{t-}$. Since K^+ and K^- are orthogonal, we have $\Delta Y_t = -\Delta K_t^+$. Thus $L_t \leq Y_t = Y_{t-} - \Delta K_t^+ = L_{t-} - \Delta K_t^+$. This implies that $\Delta K_t^+ \leq [\Delta L_t]^-$. Similarly we have $\Delta K_t^- \leq [\Delta U_t]^+$. Following the arguments for (2.20), one can easily prove that

$$\mathbb{E}^{\mathbb{P}}\left[\left(\sum_{0 < t \leq T} [[\Delta L_t]^- + [\Delta U_t]^+]\right)^2\right] \leq C\|(L, U)\|_{\mathbb{P}}^2.$$

Thus

$$\mathbb{E}^{\mathbb{P}}\left[\left(\bigvee_0^T J\right)^2\right] = \mathbb{E}^{\mathbb{P}}\left[\left(\sum_{0 < t \leq T} [\Delta K_t^+ + \Delta K_t^-]\right)^2\right] \leq C\|(L, U)\|_{\mathbb{P}}^2. \quad (3.39)$$

Now define

$$\begin{aligned} \bar{Y}_t &:= Y_t - J_t, \quad \bar{K}_t^+ := K_t^+ - \sum_{0 < s \leq t} \Delta K_s^+, \quad \bar{K}_t^- := K_t^- - \sum_{0 < s \leq t} \Delta K_s^-, \\ \bar{A}_t &:= A_t + J_t, \quad \bar{L}_t := L_t - J_t, \quad \bar{U}_t := U_t - J_t, \quad \bar{\xi} := \xi - J_T. \end{aligned}$$

Then it is clear that \bar{Y} is continuous, (\bar{L}, \bar{U}) satisfies (3.7), and (\bar{Y}, Z, \bar{A}) is a local solution to DRBSDE (3.1) with coefficients $(\bar{\xi}, 0, \bar{L}, \bar{U})$. Moreover, by (3.39) we see that (\bar{Y}, Z, \bar{A}) still satisfies the estimate (3.8). By Step 2, we have

$$\|(\bar{Y}, Z, \bar{A})\|^2 \leq C\mathbb{E}^{\mathbb{P}}[|\bar{\xi}|^2] + C\|(\bar{L}, \bar{U})\|_{\mathbb{P}}^2. \quad (3.40)$$

One can check straightforwardly that

$$\begin{aligned} \|(Y, Z, A)\|^2 &\leq C\|(\bar{Y}, Z, \bar{A})\|^2 + C\mathbb{E}^{\mathbb{P}}\left[\left(\bigvee_0^T J\right)^2\right]; \\ \mathbb{E}^{\mathbb{P}}[|\bar{\xi}|^2] &\leq C\mathbb{E}^{\mathbb{P}}[|\xi|^2] + C\mathbb{E}^{\mathbb{P}}\left[\left(\bigvee_0^T J\right)^2\right]; \quad \|(\bar{L}, \bar{U})\|_{\mathbb{P}}^2 \leq \|(L, U)\|_{\mathbb{P}}^2. \end{aligned}$$

This, together with (3.39) and (3.40), implies (3.14) immediately. \square

Proof of Theorem 3.4. First, by Lemma 3.3 we know (ii) implies (i). On the other hand, if (i) holds true, then $Y^0 := Y$ is clearly a square integrable semimartingale between L and U . That is, (i) and (ii) are equivalent.

Next, assume (ii) holds true with decomposition $Y^0 = Y_0^0 + M^0 + A^0$. Since $L \leq Y^0 \leq U$, then

$$L^+ + U^- \leq (Y^0)^+ + (Y^0)^- = |Y^0|.$$

Moreover, for any partition $\pi : 0 = \tau_0 < \dots < \tau_n = T$,

$$\begin{aligned} \left[\mathbb{E}_{\tau_i}^{\mathbb{P}} [\hat{L}_{\tau_{i+1}}] - \hat{U}_{\tau_i} \right]^+ &\leq \left[\mathbb{E}_{\tau_i}^{\mathbb{P}} [Y_{\tau_{i+1}}^0 + |\Delta Y_{\tau_{i+1}}^0|] - Y_{\tau_i}^0 + |\Delta Y_{\tau_i}^0| \right]^+ \\ &\leq \left[\mathbb{E}_{\tau_i}^{\mathbb{P}} [Y_{\tau_{i+1}}^0] - Y_{\tau_i}^0 \right]^+ + \mathbb{E}_{\tau_i}^{\mathbb{P}} [|\Delta Y_{\tau_{i+1}}^0| + |\Delta Y_{\tau_i}^0|]. \end{aligned}$$

Similarly,

$$\left[\hat{L}_{\tau_i} - \mathbb{E}_{\tau_i}^{\mathbb{P}} [\hat{U}_{\tau_{i+1}}] \right]^+ \leq \left[Y_{\tau_i}^0 - \mathbb{E}_{\tau_i}^{\mathbb{P}} [Y_{\tau_{i+1}}^0] \right]^+ + \mathbb{E}_{\tau_i}^{\mathbb{P}} [|\Delta Y_{\tau_{i+1}}^0| + |\Delta Y_{\tau_i}^0|].$$

Note that $\sum_{i=1}^n |\Delta Y_{\tau_i}^0| \leq \sqrt{0}^T A^0$. Then by Theorem (2.6), we may easily show that $\|(L, U)\|_{\mathbb{P}} \leq C\|Y^0\|_{\mathbb{P}}$, and thus (iii) holds.

It remains to prove that (iii) implies (ii). We first assume (3.7) holds. Then it follows from Lemma 3.2 that DRBSDE (3.1) with $f = 0$ admits a local solution (Y^0, Z^0, A^0) . Applying Lemma 3.7 we see that $\|(Y^0, Z^0, A^0)\| \leq C[I_0 + \|(L, U)\|_{\mathbb{P}}]$. This implies (3.11).

In the general case, denote $U^n := U + \frac{1}{n}$. Then (L, U^n) satisfies (3.7). By the above arguments, DRBSDE (3.1) with coefficients $(\xi, 0, L, U^n)$ has a unique solution (Y^n, Z^n, A^n) satisfying

$$\|(Y^n, Z^n, A^n)\|^2 \leq C\mathbb{E}^{\mathbb{P}}[|\xi|^2] + C\|(L, U^n)\|_{\mathbb{P}}^2.$$

It is obvious that $\|(L, U^n)\|_{\mathbb{P}} \leq \|(L, U)\|_{\mathbb{P}}$. Then

$$\|(Y^n, Z^n, A^n)\|^2 \leq C\mathbb{E}^{\mathbb{P}}[|\xi|^2] + C\|(L, U)\|_{\mathbb{P}}^2.$$

Now for $m > n$, applying Lemma 3.6 we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} (|Y_t^n - Y_t^m|^2 + |A_t^n - A_t^m|^2) + \int_0^T |Z_t^n - Z_t^m|^2 dt \right] \\ &\leq C \left[\|(Y^n, Z^n, A^n)\| + \|(Y^m, Z^m, A^m)\| \right] \left[\frac{1}{n} - \frac{1}{m} \right] \\ &\leq \frac{C}{n} \left[\left(\mathbb{E}^{\mathbb{P}}[|\xi|^2] \right)^{\frac{1}{2}} + \|(L, U)\|_{\mathbb{P}} \right]. \end{aligned}$$

Send $n \rightarrow \infty$, we obtain limit processes (Y^0, Z^0, A^0) . Following standard arguments we see that Y^0 satisfies the requirement in (ii). \square

4 Semimartingales under G -expectation

In this section we introduce a nonlinear expectation, which is a variation of the G -expectation proposed by Peng [21], and we shall still call it G -expectation. Let $(\Omega, \mathcal{F}, \mathbb{F})$ be a filtered space such that \mathbb{F} is right continuous and \mathcal{P} be a family of probability measures. For each $\mathbb{P} \in \mathcal{P}$ and \mathbb{F} -stopping time τ , denote

$$\mathcal{P}(\tau, \mathbb{P}) := \{ \mathbb{P}' \in \mathcal{P} : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_{\tau} \}. \tag{4.1}$$

Throughout this section, we shall always assume

Assumption 4.1. (i) (2.1) holds for every $\mathbb{P} \in \mathcal{P}$;

(ii) $\mathcal{N}_{\mathcal{P}} \subset \mathcal{F}_0$, where $\mathcal{N}_{\mathcal{P}}$ is the set of all \mathcal{P} -polar sets, that is, all $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 0$ for all $\mathbb{P} \in \mathcal{P}$.

(iii) For any $\mathbb{P} \in \mathcal{P}$, \mathbb{F} -stopping time τ , $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\tau, \mathbb{P})$, and any partition $E_1, E_2 \in \mathcal{F}_\tau$ of Ω , the probability measure $\bar{\mathbb{P}}$ defined below also belongs to $\mathcal{P}(\tau, \mathbb{P})$:

$$\bar{\mathbb{P}}(E) := \mathbb{P}_1(E \cap E_1) + \mathbb{P}_2(E \cap E_2), \quad \text{for all } E \in \mathcal{F}. \quad (4.2)$$

We provide below an important example for such \mathcal{P} , which induces the G -expectation of Peng [21], and we refer to [28] for more examples.

Example 4.1. Let $\Omega := \{\omega \in C([0, T], \mathbb{R}) : \omega_0 = 0\}$, B the canonical process, \mathbb{F} the right limit of the filtration generated by B , \mathbb{P}_0 the Wiener measure. Let $0 \leq \underline{\sigma} < \bar{\sigma}$ be two constants. For each bounded \mathbb{F} -progressively measurable process σ , denote $X_t^\sigma := \int_0^t \sigma_s dB_s$, \mathbb{P}_0 -a.s. Then the following class \mathcal{P} satisfies Assumption 4.1:

$$\mathcal{P} := \{\mathbb{P}^\sigma : \underline{\sigma} \leq \sigma \leq \bar{\sigma}\} \quad \text{where } \mathbb{P}^\sigma := \mathbb{P}_0 \circ (X^\sigma)^{-1}.$$

4.1 Definitions

We first define

Definition 4.2. We say an \mathbb{F} -progressively measurable process Y is a \mathcal{P} -martingale (resp. \mathcal{P} -supermartingale, \mathcal{P} -submartingale, \mathcal{P} -semimartingale) if it is a \mathbb{P} -martingale (resp. \mathbb{P} -supermartingale, \mathbb{P} -submartingale, \mathbb{P} -semimartingale) for all $\mathbb{P} \in \mathcal{P}$.

We next define the G -expectation and conditional G -expectation. For any \mathcal{F} -measurable random variable ξ such that $\mathbb{E}^\mathbb{P}[|\xi|] < \infty$ for all $\mathbb{P} \in \mathcal{P}$, its G -expectation is defined by

$$\mathbb{E}^G[\xi] := \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P}[\xi]. \quad (4.3)$$

The conditional G -expectation is more involved. For any \mathbb{F} -stopping time τ , denote

$$\mathbb{E}_\tau^{G, \mathbb{P}}[\xi] := \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(\tau, \mathbb{P})} \mathbb{E}_\tau^{\mathbb{P}'}[\xi], \quad \mathbb{P}\text{-a.s.} \quad (4.4)$$

We note that, by Lemma 2.1, we may take the convention that $\mathbb{E}_\tau^{G, \mathbb{P}}[\xi]$ is \mathcal{F}_τ -measurable. When the family $\{\mathbb{E}_\tau^{G, \mathbb{P}}[\xi], \mathbb{P} \in \mathcal{P}\}$ can be aggregated, that is, there exists an \mathcal{F}_τ -measurable random variable, denoted as $\mathbb{E}_\tau^G[\xi]$, such that

$$\mathbb{E}_\tau^G[\xi] = \mathbb{E}_\tau^{G, \mathbb{P}}[\xi], \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}, \quad (4.5)$$

we call $\mathbb{E}_\tau^G[\xi]$ the conditional G -expectation of ξ .

Remark 4.2. Given an σ -field \mathcal{F} , the universal completion of \mathcal{F} is the σ -field $\mathcal{F}^* = \bigcap_{\mathbb{P} \in \mathcal{P}} \mathcal{F}^\mathbb{P}$, where \mathbb{P} ranges over all probability measures on \mathcal{F} and $\mathcal{F}^\mathbb{P}$ is the completion of \mathcal{F} under \mathbb{P} . In a recent work Nutz and van Handel [19] gives a general and beautiful construction of time-consistent sublinear expectations on the space of continuous paths with respect to the filtration $\{\mathcal{F}_t^*\}_{0 \leq t \leq T}$. In particular, by [19] Theorem 2.3, when ξ is Borel measurable, there exists an \mathcal{F}_τ^* -measurable random variable $\mathbb{E}_\tau^G(\xi)$ that satisfies (4.5). Thus if we choose to work under \mathbb{F}^* then the existence of $\mathbb{E}_\tau^G(\xi)$ is guaranteed. Since the aggregation property is not the main focus of this paper, and [19] involves the more abstract analytic sets, in this paper we choose to work with the $\mathbb{E}_\tau^{G, \mathbb{P}}(\xi)$ version of the G -expectation and refer the readers to [19] for more general results.

Following standard arguments, see e.g. [29] Proposition 4.10, we have the following time consistency (or say, dynamic programming principle), whose proof is omitted:

Lemma 4.3. Under Assumption 4.1, for any $\tau_1 \leq \tau_2$ and any $\mathbb{P} \in \mathcal{P}$, we have

$$\mathbb{E}_{\tau_1}^{G,\mathbb{P}}[\xi] = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(\tau_1, \mathbb{P})}^{\mathbb{P}} \mathbb{E}_{\tau_1}^{\mathbb{P}'}[\mathbb{E}_{\tau_2}^{G,\mathbb{P}'}[\xi]], \quad \mathbb{P} - \text{a.s.}$$

We finally define

Definition 4.4. We say an \mathbb{F} -progressively measurable process Y is a G -martingale (resp. G -supermartingale, G -submartingale) if, for any $\mathbb{P} \in \mathcal{P}$ and any \mathbb{F} -stopping times $\tau_1 \leq \tau_2$,

$$Y_{\tau_1} = (\text{resp. } \geq, \leq) \mathbb{E}_{\tau_1}^{G,\mathbb{P}}[Y_{\tau_2}], \quad \mathbb{P}\text{-a.s.}$$

We remark that a \mathcal{P} -martingale is also called a symmetric G -martingale in the literature, see e.g. [31].

4.2 Characterization of \mathcal{P} -semimartingales

The following result is immediate:

Proposition 4.5. Let Assumption 4.1 hold.

(i) A \mathcal{P} -martingale (resp. \mathcal{P} -supermartingale, \mathcal{P} -submartingale) must be a G -martingale (resp. G -supermartingale, G -submartingale).

(ii) If Y is a G -martingale (resp. G -supermartingale, G -submartingale) and M is a \mathcal{P} -martingale, then $Y + M$ is a G -martingale (resp. G -supermartingale, G -submartingale).

(iii) A G -supermartingale is a \mathcal{P} -supermartingale. In particular, a G -martingale is a \mathcal{P} -supermartingale.

Proof. (i) and (ii) are obvious. To prove (iii), let Y be a G -supermartingale. Then for any $\tau_1 \leq \tau_2$ and any $\mathbb{P} \in \mathcal{P}$,

$$Y_{\tau_1} \geq \mathbb{E}_{\tau_1}^{G,\mathbb{P}}[Y_{\tau_2}] \geq \mathbb{E}_{\tau_1}^{\mathbb{P}}[Y_{\tau_2}], \quad \mathbb{P}\text{-a.s.}$$

That is, Y is a \mathbb{P} -supermartingale for all $\mathbb{P} \in \mathcal{P}$, and thus is a \mathcal{P} -supermartingale. \square

We next study \mathcal{P} -semimartingales. In light of Theorem 2.7, we define a new norm:

$$\|Y\|_{\mathcal{P}} := \sup_{\mathbb{P} \in \mathcal{P}} \|Y\|_{\mathbb{P}}. \quad (4.6)$$

The following result is a direct consequence of Theorems 2.6 and 2.7.

Theorem 4.6. Let Assumption 4.1 hold. If $\|Y\|_{\mathcal{P}} < \infty$, then Y is a \mathcal{P} -semimartingale. Moreover, for any $\mathbb{P} \in \mathcal{P}$ and for the decomposition

$$Y_t = Y_0 + M_t^{\mathbb{P}} + A_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.} \quad (4.7)$$

we have

$$\mathbb{E}^{\mathbb{P}} \left[\langle M^{\mathbb{P}} \rangle_T + \left(\bigvee_0^T A^{\mathbb{P}} \right)^2 \right] \leq C \|Y\|_{\mathcal{P}}^2.$$

The norm $\|\cdot\|_{\mathcal{P}}$ is defined through each $\mathbb{P} \in \mathcal{P}$. The following definition relies on the G -expectation directly:

$$\|Y\|_G^2 := \mathbb{E}^G \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] + \sup_{\pi} \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\left(\sum_{i=0}^{n-1} \left| \mathbb{E}_{\tau_i}^{G,\mathbb{P}}(Y_{\tau_{i+1}}) - Y_{\tau_i} \right| \right)^2 \right]. \quad (4.8)$$

Remark 4.3. (i) If the involved conditional G -expectations exist, see e.g. Remark 4.2, then we may simplify the definition of $\|Y\|_G$:

$$\|Y\|_G^2 := \mathbb{E}^G \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] + \sup_{\pi} \mathbb{E}^G \left[\left(\sum_{i=0}^{n-1} \left| \mathbb{E}_{\tau_i}^G(Y_{\tau_{i+1}}) - Y_{\tau_i} \right| \right)^2 \right].$$

(ii) In general $\|\cdot\|_G$ does not satisfy the triangle inequality and thus is not a norm.

(iii) For G -submartingales Y^1, Y^2 , the triangle inequality holds:

$$\|Y^1 + Y^2\|_G \leq \|Y^1\|_G + \|Y^2\|_G.$$

However, in general $Y^1 + Y^2$ may not be a G -submartingale anymore.

Nevertheless, $\|Y\|_G$ involves the process Y only. The following estimate is the main result of this section.

Theorem 4.7. Assume Assumption 4.1 holds. Then there exists a universal constant C such that $\|Y\|_{\mathcal{P}} \leq C\|Y\|_G$.

Proof. Without loss of generality, we assume $\|Y\|_G < \infty$. For any $\mathbb{P} \in \mathcal{P}$ and any partition $\pi : 0 = \tau_0 \leq \dots \leq \tau_n = T$, denote

$$N_{\tau_i} := \sum_{j=0}^{i-1} \left[\mathbb{E}_{\tau_j}^{G, \mathbb{P}}(Y_{\tau_{j+1}}) - Y_{\tau_j} \right].$$

Then

$$\begin{aligned} Y_{\tau_i} - N_{\tau_i} &= Y_0 + \sum_{j=0}^{i-1} \left[Y_{\tau_{j+1}} - \mathbb{E}_{\tau_j}^{G, \mathbb{P}}(Y_{\tau_{j+1}}) \right] \\ &= Y_0 + \sum_{j=0}^{i-1} \left[Y_{\tau_{j+1}} - \mathbb{E}_{\tau_j}^{\mathbb{P}}(Y_{\tau_{j+1}}) \right] - \sum_{j=0}^{i-1} \left[\mathbb{E}_{\tau_j}^{G, \mathbb{P}}(Y_{\tau_{j+1}}) - \mathbb{E}_{\tau_j}^{\mathbb{P}}(Y_{\tau_{j+1}}) \right]. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{j=0}^{i-1} \left[Y_{\tau_{j+1}} - \mathbb{E}_{\tau_j}^{\mathbb{P}}(Y_{\tau_{j+1}}) \right] &\text{ is a } \mathbb{P}\text{-martingale,} \\ \sum_{j=0}^{i-1} \left[\mathbb{E}_{\tau_j}^{G, \mathbb{P}}(Y_{\tau_{j+1}}) - \mathbb{E}_{\tau_j}^{\mathbb{P}}(Y_{\tau_{j+1}}) \right] &\text{ is nondecreasing and is } \mathcal{F}_{\tau_{i-1}}\text{-measurable.} \end{aligned}$$

Applying Lemma 2.3 we obtain

$$\mathbb{E}^{\mathbb{P}} \left[\left(\sum_{j=0}^{n-1} \left[\mathbb{E}_{\tau_j}^{G, \mathbb{P}}(Y_{\tau_{j+1}}) - \mathbb{E}_{\tau_j}^{\mathbb{P}}(Y_{\tau_{j+1}}) \right] \right)^2 \right] \leq C \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq i \leq n} \left[|Y_{\tau_i}|^2 + |N_{\tau_i}|^2 \right] \right] \leq C \|Y\|_G^2.$$

This, together with the definition of $\|\cdot\|_G$, implies that

$$\mathbb{E}^{\mathbb{P}} \left[\left(\sum_{j=0}^{n-1} \left| \mathbb{E}_{\tau_j}^{\mathbb{P}}(Y_{\tau_{j+1}}) - Y_{\tau_j} \right| \right)^2 \right] \leq C \|Y\|_G^2.$$

Since π is arbitrary, we get $\|Y\|_{\mathbb{P}} \leq C\|Y\|_G$. Finally, since $\mathbb{P} \in \mathcal{P}$ is arbitrary, we prove the result. \square

4.3 Doob-Meyer Decomposition for G -submartingales

As a special case of Theorem 4.6, we have the following decomposition for G submartingales.

Proposition 4.8. *Assume Assumption 4.1 holds. If Y is a G -submartingale satisfying $\|Y\|_{\mathcal{P}} < \infty$ (or $\|Y\|_G < \infty$), then all the results in Theorem 4.6 hold.*

Unlike Lemma 2.2, in general we do not have $\|Y\|_{\mathcal{P}} \leq C \sup_{\mathbb{P} \in \mathcal{P}} \|Y\|_{\mathbb{P},0}$ for G -submartingales Y . Indeed, we have the following example:

Example 4.9. *Fix \mathcal{P} . Let K be as in Example 2.5 such that $-K$ is a G martingale and $\mathbb{E}^G[K_T^2] = \infty$, instead of $\mathbb{E}^{\mathbb{P}}[K_T^2] = \infty$. Then the process Y defined in Example 2.5 is a G -submartingale such that $\sup_{\mathbb{P} \in \mathcal{P}} \|Y\|_{\mathbb{P},0} < \infty$, but $\|Y\|_{\mathcal{P}} = \infty$.*

Proof. By the proof of Example 2.5, clearly $\sup_{\mathbb{P} \in \mathcal{P}} \|Y\|_{\mathbb{P},0} < \infty$, but $\|Y\|_{\mathcal{P}} = \infty$. Moreover, on $(\tau_{2n}, \tau_{2n+1}]$, $dY_t = -dK_t$ and thus is a G martingale; and on $(\tau_{2n+1}, \tau_{2n+2}]$, $dY_t = dK_t$, then Y is increasing and thus is a G -submartingale. So Y is a G -submartingale on $[0, T]$. \square

Now let Y be as in Proposition 4.8, and consider its decomposition (4.7). Let $A^{\mathbb{P}} = L^{\mathbb{P}} - K^{\mathbb{P}}$ be the orthogonal decomposition. We have the following conjecture:

Conjecture (Doob-Meyer decomposition) : *The family $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ satisfies the following property:*

$$-K_t^{\mathbb{P}} = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}(t, \mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} \left[-K_T^{\mathbb{P}'} \right]. \quad (4.9)$$

In particular, if the families $\{M^{\mathbb{P}}, K^{\mathbb{P}}, L^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ can be aggregated into $\{M, K, L\}$, then $-K$ is a G -martingale, and we have the following desired Doob-Meyer decomposition for G -submartingales:

$$Y_t = Y_0 + [M_t - K_t] + L_t, \quad (4.10)$$

where $M - K$ is a G -martingale and L is nondecreasing.

Remark 4.4. *Assume each $\mathbb{P} \in \mathcal{P}$ satisfies the martingale representation property. Then, under the additional assumption Continuum Hypothesis which is independent of the axiom of choice, Nutz [18] proved that the family $\{M^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}\}$ can always be aggregated. This implies further that $K^{\mathbb{P}}$ and $L^{\mathbb{P}}$ can also be aggregated, and thus the aggregation will not be an issue (again under those additional assumptions).*

Remark 4.5. *While it seems quite natural, this conjecture is very subtle. Our estimates in this section are rather preliminary. There is a very interesting recent development by Matoussi, Piozin and Possamai [17] in the context of doubly reflected second-order BSDEs, which extends the DRBSDE in Section 3 to the nonlinear expectation framework. Under the condition that the upper barrier U is a semimartingale, they obtain the well-posedness of the equation, whose Y component is by definition a G -semimartingale. We believe their a priori estimates is a first step towards our conjecture, at least for the decomposition of their solution Y , and we hope to address the issue more thoroughly in some future research.*

References

- [1] Chassagneux, J. : A discrete-time approximation for doubly reflected BSDE, *Advances in Applied Probability*, **41**, (2009), 101-130. MR-2514947
- [2] Crépey, S. and Matoussi, A. : Reflected and doubly reflected BSDEs with jumps: a priori estimates and comparison, *Ann. App. Prob.*, **18**, (2008) 2041-2069. MR-2462558
- [3] Cheridito, P., Soner, H.M. and Touzi, N., Victoir, N. : Second order backward stochastic differential equations and fully nonlinear PDEs, *Communications in Pure and Applied Mathematics*, **60**, (2007), 1081-1110. MR-2319056
- [4] Cvitanic, J. and Karatzas, I. : Backward stochastic differential equations with reflection and Dynkin games, *Annals of Probability*, **24**, (1996), 2024-2056. MR-1415239
- [5] Cvitanic, J. and Zhang, J. : *Contract Theory in Continuous Time Models*, (2012), Springer Finance. MR-2963805
- [6] Dellacherie, C. and Meyer, P. A. : *Probabilities and Potential B*, (1982), North-Holland, Amsterdam. MR-0745449
- [7] Ekren, I., Keller, C., Touzi, N., and Zhang, J. : On Viscosity Solutions of Path Dependent PDEs, *Annals of Probability*, to appear, arXiv:1109.5971.
- [8] Ekren, I., Touzi, N., and Zhang, J. : Optimal Stopping under Nonlinear Expectation, preprint, arXiv:1209.6601.
- [9] Ekren, I., Touzi, N., and Zhang, J. : Viscosity Solutions of Fully Nonlinear Parabolic Path Dependent PDEs: Part I, preprint, arXiv:1210.0006.
- [10] Ekren, I., Touzi, N., and Zhang, J. : Viscosity Solutions of Fully Nonlinear Path Parabolic Dependent PDEs: Part II, preprint, arXiv:1210.0007.
- [11] El. Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and Quenez, M. : Reflected Solutions of Backward SDE's, and Related Obstacle Problems for PDE's, *The Annals of Probability*, **25**, (1997), 702-737. MR-1434123
- [12] Hamadène, S. and Hassani, M. : BSDEs with two reflecting barriers: the general result, *Probability Theory and Related Fields*, **132**, (2005), 237-264. MR-2199292
- [13] Hamadène, S., Hassani, M., and Ouknine, Y. : BSDEs with general discontinuous reflecting barriers without Mokobodski's condition, *Bull. Sci. math.*, **134**, (2010), 874-899 ; DOI : 10.1016/j.bulsci.2010.03.001. MR-2737357
- [14] Hu, M., Ji, S., Peng, S., and Song, Y. : Backward Stochastic Differential Equations Driven by G-Brownian Motion, preprint, arXiv:1206.5889.
- [15] Karatzas, I and Shreve, S. : *Brownian Motion and Stochastic Calculus*, 2nd Edition, Springer. MR-1121940
- [16] Meyer, P. and Zheng, W. : Tightness criteria for laws of semimartingales, *Ann. Inst. Henri Poincaré*, **20**, (1984), 353-372. MR-0771895
- [17] Matoussi, A., Piozin, L. and Possamaï, D. : Second-order BSDEs with general reflection and Dynkin games under uncertainty, preprint, arXiv:1212.0476.
- [18] Nutz, M. : Pathwise construction of stochastic integrals, *Elec. Com. Prob.*, **17**, (2012), 1 - 7. MR-2950190
- [19] Nutz, M. and van Handel, R. : Constructing sublinear expectations on path space, *Stochastic Processes and their Applications*, **123**, (2013), 3100-3121. MR-3062438
- [20] Peng, S. : Backward SDE and related g-expectation, *Backward stochastic differential equations*, (N. El Karoui and L. Mazliak, eds.), Pitman Res. Notes Math. Ser., **364**, (1997), Longman, Harlow, 141-159. MR-1752680
- [21] Peng, S. : Nonlinear Expectations and Stochastic Calculus under Uncertainty, preprint, arXiv:1002.4546.
- [22] Peng, S. and Xu, M. : The smallest g -supermartingale and reflected BSDE with single and double L^2 obstacles, *Annales de I.H.P.*, **141**, (2005), 605-630. MR-2139035
- [23] Revuz, D. and Yor, M. (1999) : *Continuous martingales and Brownian motion*, Springer, third edition. MR-1725357

- [24] Pham, T. and Zhang, J. : Two Person Zero-sum Game in Weak Formulation and Path Dependent Bellman-Isaacs Equation, preprint, arXiv:1209.6605.
- [25] Protter, P. *Stochastic Integration and Differential Equations*, (2004), 2nd Edition, Springer. MR-2020294
- [26] Rao, K. M. : Quasi-Martingales, *Math. Scand.*, 24, (1969), 79-92. MR-0275511
- [27] Soner, M., Touzi, N. and Zhang, J. : Martingale representation theorem for the G -expectation, *Stochastic Processes and Their Applications*, **121**, (2011), 265-287. MR-2746175
- [28] Soner, M., Touzi, N. and Zhang, J. : Quasi-sure stochastic analysis through aggregation, *Electronic Journal of Probability*, **16**, (2011), 1844-1879. MR-2842089
- [29] Soner, M., Touzi, N. and Zhang, J. : Dual Formulation of Second Order Target Problems, *Annals of Applied Probability*, **23**, (2013), 308-347. MR-3059237
- [30] Soner, M., Touzi, N. and Zhang, J. : Wellposedness of Second Order BSDEs, *Probability Theory and Related Fields*, **153**, (2012), 149-190. MR-2925572
- [31] Xu, J. and Zhang, B. : Martingale characterization of G -Brownian motion, *Stochastic Processes and their Applications*, **119**, (2009), 232-248. MR-2485026

Electronic Journal of Probability

Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)

Economical model of EJP-ECP

- Low cost, based on free software (OJS¹)
- Non profit, sponsored by IMS², BS³, PKP⁴
- Purely electronic and secure (LOCKSS⁵)

Help keep the journal free and vigorous

- Donate to the IMS open access fund⁶ (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

¹OJS: Open Journal Systems <http://pkp.sfu.ca/ojs/>

²IMS: Institute of Mathematical Statistics <http://www.imstat.org/>

³BS: Bernoulli Society <http://www.bernoulli-society.org/>

⁴PK: Public Knowledge Project <http://pkp.sfu.ca/>

⁵LOCKSS: Lots of Copies Keep Stuff Safe <http://www.lockss.org/>

⁶IMS Open Access Fund: <http://www.imstat.org/publications/open.htm>