DOI: 10.1214/EJP.v16-888

Electronic Journal of Probability

Vol. 16 (2011), Paper no. 31, pages 929–961.

Journal URL
http://www.math.washington.edu/~ejpecp/

The self-similar dynamics of renewal processes

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Abstract

We prove an almost sure invariance principle in log density for renewal processes with gaps in the domain of attraction of an α-stable law. There are three different types of behavior: attraction to a Mittag-Leffler process for $0 < \alpha < 1$, to a centered Cauchy process for $\alpha = 1$ and to a stable process for $1 < \alpha \le 2$. Equivalently, in dynamical terms, almost every renewal path is, upon centering and up to a regularly varying coordinate change of order one, and after removing a set of times of Cesáro density zero, in the stable manifold of a self-similar path for the scaling flow. As a corollary we have pathwise functional and central limit theorems.

Key words: stable process, renewal process, Mittag-Leffler process, Cauchy process, almost-sure invariance principle in log density, pathwise Central Limit Theorem.

AMS 2010 Subject Classification: Primary 37A50, 60F17, 60K05; Secondary: 60G18, 60G52.

Submitted to EJP on June 21, 2010, final version accepted April 20, 2011.

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1 Introduction

Given a self-similar process Z of exponent $\gamma > 0$, a second process Y satisfies an *almost sure invariance principle* (or *asip*) of order $\beta(\cdot)$ if Z and Y can be redefined to be on the same probability space so as to satisfy:

$$||Y - Z||^{\infty}_{[0,T]} = o(\beta(T)), \quad a.s. \tag{1.1}$$

where $||f||^{\infty}_{[0,T]} = \sup_{0\le s\le T} |f(s)|$.

Supposing now for simplicity that both processes have continuous paths, and that $\beta(t) = O(t^{\gamma})$, this equation has a natural dynamical interpretation. Defining for each $t \in \mathbb{R}$ a map τ_t of index γ on the space of continuous real-valued functions $\mathscr{C}(\mathbb{R}^+)$ by

$$(\tau_t f)(x) = \frac{f(e^t x)}{e^{\gamma t}} \tag{1.2}$$

then this collection of maps forms a *flow*, since $\tau_{t+s} = \tau_t \circ \tau_s$, called the *scaling flow* of index γ, and the γ-self-similarity of the process means exactly that the measure on path space for the process Z is flow-invariant. Recalling that a *joining* (or coupling) of two measure spaces is a measure on the product space which projects to the two measures, then (1.1) is equivalent to: there exists a joining of the two path spaces so that, for almost every pair (Y, Z), writing $d^u_1(f, g) = ||f - g||^{\infty}_{[0,1]}$, the paths Y and Z are forward asymptotic under the action of τ_t, for:

$$d^u_1(\tau_t Y, \tau_t Z) \to 0 \text{ as } t \to \infty. \tag{1.3}$$

In the language of dynamical systems, Y is in the d^u_1*-stable manifold* of Z, written $W^{s,d^u_1}(Z)$.

For a first example, when $(X_i)_{i\ge 0}$ is an i.i.d. sequence of random variables of mean zero with finite $(2 + \delta)$-moment, then for $S_n = X_0 + \cdots + X_{n-1}$, we know from Breiman [Bre67] that one has a discrete-time *asip* with a bound stronger than $\beta(n) = \sqrt{n}$. This estimate extends to continuous time for the polygonally interpolated random walk path $(S(t))$, and so the dynamical statement (1.3) holds. For finite second moment however, something striking happens. Despite having the Central Limit Theorem and functional CLT in this case, due to counterexamples of Breiman and Major [Bre67], [Maj76a], see also p. 93 of [CR81], the *asip* of order $\sqrt{n}$ now *fails*, and the best one can get is Strassen's original bound of $\sqrt{n \log\log n}$ [Str64], [Str65]. Though this bound still allows one to pass certain asymptotic information from Brownian motion to the random walk, such as the upper and lower bounds given by the law of the iterated logarithm, it does not give the flow statement of (1.3).

This raises the question as to whether one can nevertheless find an appropriate dynamical statement for finite second moment. What we showed in [Fis] is that the *asip* still holds if one is allowed to discard a set of times of density zero. Thus, defining the Cesáro stable manifold of Z to be $W^{s,d^u_1}_{Ces}(Z) = \{Y : d^u_1(\tau_t Y, \tau_t Z) \to 0 \text{ (Cesáro)}\}$, where *(Cesáro)* means except for a set of times of Cesáro density zero, we proved that there exists a joining of S with B such that for a.e. pair (S, B), S belongs to $W^{s,d^u_1}_{Ces}(B)$. This provides just enough control that the pathwise (or almost-sure) CLT and pathwise functional CLT for Brownian paths [Fis87] carry over to the random walk path S.

An exponential change of variables in the corresponding equation $d^u_1(\tau_t B, \tau_t S) \to 0$ (Cesáro) yields the equivalent statement

$$||S - B||_{[0,T]} = o(\sqrt{T}) \text{ (log)}, \tag{1.4}$$

where (log) means this holds off a set of logarithmic density zero, i.e. except for a set of times $\mathscr{B} \subseteq \mathbb{R}$ with $\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \chi_{\mathscr{B}}(t)\mathrm{d}t/t = 0$. We call this type of statement an *asip in log density* or *asip* (log).

Berkes and Dehling in [BD93] proved a statement like (1.4) for i.i.d. X_i with distribution in the domain of attraction of an $\alpha-$stable law for $\alpha \in (0,2]$, with discrete time. Their work lacked however the full strength of an *asip* (log) as described above, in two respects: first, being given for discrete time, it said nothing about closeness of the continuous-time paths (the stable paths are discontinuous on a dense set of times for $\alpha < 2$); secondly, it involved a spatial rescaling of each stable increment, which bore no clear relation to either the continuous-time stable process nor to the dynamics of the scaling flow.

In [FT11a], building on the work of [BD93], we proved a dynamical *asip* (log) for random walk paths. Our purpose in the present paper is to prove such theorems for renewal processes. We build on results and methods of [FT11a]; as in that paper, the dynamical viewpoint naturally enters in the methods of proof, as well as in the statement of the theorems.

We first review the parts of [FT11a] needed below regarding random walks, and then move on to the renewal processes which are the focus of this paper.

Using the same notational conventions as in [FT11a], a random variable X has a stable law if there are parameters $\alpha \in (0,2], \xi \in [-1,1], b \in \mathbb{R}, c > 0$ such that its characteristic function has the following form:

$$E(e^{itX}) = \begin{cases} \exp\left(ibt + c\cdot\frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}|t|^\alpha\big(\cos\frac{\pi\alpha}{2} - sign(t)i\xi\sin\frac{\pi\alpha}{2}\big)\right) & \text{for } \alpha \neq 1, \\ \exp\left(ibt - c\cdot|t|\big(\frac{\pi}{2} + sign(t)i\xi\log|t|\big)\right) & \text{for } \alpha = 1, \end{cases}$$

where $sign(t) = t/|t|$ with the convention $sign(0) = 0$. The parameters α, ξ, c and b are called the *exponent* or *index*, *symmetry* (or *skewness*), the *scaling* and the *centering* parameters respectively. We write $G_{\alpha,\xi,c,b}$ for the *distribution function* of X, shortening this to $G_{\alpha,\xi}$ for $G_{\alpha,\xi,1,0}$.

Let $(X_i)_{i\geq 0}$ be a sequence of i.i.d. random variables with distribution function F in the domain of attraction of $G_{\alpha,\xi}$ an α-stable law $(0 < \alpha \leq 2)$ of skewness $\xi \in [-1,1]$. That is, there exists a positive normalizing sequence (a_n), and a centering sequence (b_n), such that

$$\frac{S_n - b_n}{a_n} \xrightarrow{law} G_{\alpha,\xi} \quad n \to \infty \tag{1.5}$$

where $\stackrel{law}{\to}$ stands for convergence in law; one knows that then (a_n) is regularly varying of order $1/\alpha$. Writing $\overline{S}$ for the step-path random walk with these increments, so $\overline{S}(t) = S_{[t]}$ where $S_0 = 0$, $S_n = X_0 + \cdots + X_{n-1}$ for $n > 0$, we proved *asips* (log) with attraction to a self-similar process. This is the (α,ξ)-stable process Z for $\alpha \neq 1$. For $\alpha = 1$, one has the Cauchy distribution $G_{1,\xi}$; this case as we shall see is in all respects more subtle, and needs a different approach since (unless $\xi = 0$) the $(1,\xi)$-stable process is not self-similar. We then replace the Cauchy process Z by $\check{Z}(t) = Z(t) - \xi t\log t$, which (while no longer a Lévy process) is 1-self-similar.

For $0 < \alpha < 2$ the stable paths are discontinuous with a dense set of jumps so we replace the space $\mathscr{C}(\mathbb{R}^+)$ used in (1.3) by the Skorokhod space D, and replace the uniform pseudometric d_1^u

by a suitable pseudometric for the Skorokhod topology. Thus $D = D_{\mathbb{R}^+}$ denotes the set of functions from $\mathbb{R}^+$ to $\mathbb{R}$ which are càdlàg (continuous from the right with left limits), and we let μ denote the measure on Skorokhod path space D given by the step-path random walk $\overline{S}$ with increments X_i defined above.

Writing ν (respectively $\check{\nu}$) for the law on D of the $(\alpha,\xi)-$stable process Z (resp. of $\check{Z}$ in the Cauchy case), then self-similarity of Z (resp. $\check{Z}$) translates into the invariance of ν (respectively $\check{\nu}$) for τ_t, the scaling flow of index $1/\alpha$ acting on D by:

$$(\tau_t f)(x) = \frac{f(e^t x)}{e^{t/\alpha}}.$$

The flow τ_t on D with the measure ν (resp. $\check{\nu}$) is a Bernoulli flow of infinite entropy (Lemma 3.3 of [FT11a]), so in particular is ergodic; this is a key ingredient at various points in the proof of our results.

We use the following notation: given a sequence $0 = x_0 < x_1 < \cdots$ going to infinity, letting $\mathscr{P}$ denote the partition $\{[x_n, x_{n+1})\}_{n\geq 0}$ of $\mathbb{R}^+$, then for $Z \in D$,

$$\overline{Z}_{\mathscr{P}}(t) = Z(x_n) \quad \text{for } x_n \leq t < x_{n+1} \tag{1.6}$$

defines the step path version of Z over the partition $\mathscr{P}$.

Writing now d_1 for Billingsley's complete metric of [Bil68] for the Skorokhod J_1 topology on $D_{[0,1]}$, here is the statement we prove in [FT11a]: there exists a *normalizing function* $a(\cdot)$ and *centering function* $\varrho(\cdot)$ explicitly defined from F, which have properties described shortly, and a joining $\widehat{\nu}$ of μ, the law of $\overline{S}$, with ν, such that for $\widehat{\nu}$-almost every pair $(\overline{S}, Z)$ we have for the partition $\mathscr{Q} \equiv \{[n, n+1)\}_{n\geq 0}$:

$$\lim_{t\to\infty} d_1(\tau_t(\,\overline{(S-\varrho)}_{\mathscr{Q}} \circ (a^\alpha)^{-1}), \tau_t Z\,) = 0 \text{ (Cesáro)}, \tag{1.7}$$

with Z replaced by $\check{Z}(t)$ in the Cauchy case. As in (1.4), we can then reformulate (1.7) in terms more familiar to probability theory as follows. Defining d_A to be a rescaling of d_1 to the interval $[0, A]$, with

$$d_A(f,g) = A^{1/\alpha}\, d_1(\Delta_A(f), \Delta_A(g)), \;\; f,g \in D \tag{1.8}$$

where Δ_t is the multiplicative version of the scaling flow of index $1/\alpha$, defined by $\Delta_t = \tau_{\log t}$, statement (1.7) is then equivalent to: the processes $\overline{S}, Z$ *can be redefined so as to live on a (larger) common probability space* (Ω, P) such that for a.e. ω in the underlying space Ω, the pair $(\overline{S}(\omega), Z(\omega))$ satisfies

$$d_T(\overline{(S-\varrho)}_{\mathscr{Q}} \circ (a^\alpha)^{-1}, Z) = o(T^{\frac{1}{\alpha}}) \;\; (\log) \tag{1.9}$$

(with $\check{Z}$ in the Cauchy case), meaning this is $o(T^{\frac{1}{\alpha}})$ off of a set $\mathscr{B} = \mathscr{B}_\omega \subseteq \mathbb{R}^+$ of log density zero.

Next we integrate d_A, giving a complete metric d_∞ on D, with:

$$d_\infty(f,g) = \int_0^\infty e^{-A}\, \frac{d_A(f,g)}{1+d_A(f,g)}\, dA. \tag{1.10}$$

As shown in Lemma 3.6 of [FT11a], equation (1.7) then implies the corresponding statement with the metric d_∞ in place of the pseudometric d_1.

From this we can conclude that, as for Brownian motion, the time-changed random walk path is in the Cesáro stable manifold $W^{s,d_\infty}_{Ces}(\cdot)$ of the stable path Z (respectively $\check{Z}$) with respect to the scaling flow of index $1/\alpha$. This dynamical statement using d_∞ leads to the proof, as a corollary, of the pathwise functional CLT and pathwise CLT, in §6 of [FT11a].

We note that the normalizing and centering functions $a(\cdot)$ and $\varrho(\cdot)$ in (1.7) are constructed to have the following properties; for the precise definition of $a(\cdot)$ see [FT11a] and of $\varrho(\cdot)$ see (4.1) below. When sampled discretely, $a_n = a(n)$ does give a normalizing sequence in the sense of equation (1.5); for its continuous time version, $a^\alpha(t) \sim tL(a(t))$ where $L(t)$ is the slowly varying function equal to $t^{\alpha-2}\int_{-t}^{t} x^2 \mathrm{d}F(x)$. Moreover $a(\cdot)$ is a C^1 increasing, regularly varying function of order $1/\alpha$ with regularly varying derivative. These last facts have a special importance in the proofs, as log averages are preserved by the resulting parameter change. The centering function $\varrho(\cdot)$ is in fact zero for $\alpha \in (0,1)$ and linear (proportional to the mean) for $(1,2]$, and is only nonlinear for the Cauchy case $\alpha = 1$, see below.

This describes the results of [FT11a]; now we move on to the setting of renewal processes.

The i.i.d. gaps X_i of the renewal process are assumed a.s. strictly positive; in other words, for the distribution function F of the law of X_i we assume that $F(0) = 0$. The limiting law $G_{\alpha,\xi}$ in (1.5) is then completely asymmetric stable, with $\xi = 1$; we write G_α for $G_{\alpha,1}$.

With $S(\cdot) \in D$ denoting the polygonal interpolation extension between S_n and S_{n+1}, the assumption of a.s. positive gaps implies that $S(\cdot)$ is a.s. increasing; we let N denote the inverse of S. The step path $\overline{S}$ is nondecreasing a.s.; our step-path renewal process $\overline{N}$ is defined to be the generalized inverse of $\overline{S}$, that is

$$\overline{N}(t) = \inf\{s : \overline{S}(s) > t\}.$$

Noting that $\overline{N}(0) = 1$ (while $N(0) = 0$), $\overline{N}(t)$ gives number of events up to time t plus one; the more usual definition for renewal process is $\overline{N} - 1 = \overline{N}_{S(\mathcal{Q})}$, the step process of the polygonal path N over the partition $S(\mathcal{Q})$. We work with $\overline{N}$ rather than $\overline{N} - 1$ for notational convenience, noting that all results stated for $\overline{N}$ will hold for $\overline{N} - 1$ as well.

The limiting behavior of renewal processes (in the discrete setting) was identified by Feller in [Fel49]. A special feature for renewal processes as compared to random walks is that this behavior passes through three distinct "phases", depending on the parameter $r > 0$ which gives the exponent of the tail behavior of F, the gap distribution. For the phase $r \geq 2$ one has convergence to a stable distribution of index $\alpha = 2$, that is, the Gaussian distribution; after r moves past that mark each limiting law is distinct, indexed by $\alpha = r$, with a completely asymmetric α-stable distribution G_α for $r \in (1,2)$, and a Mittag-Leffler distribution of parameter α for $r \in (0,1)$.

Corresponding functional CLTs were then given by Billingsley for the Gaussian case [Bil68] p. 148, also see §5 of [Ver72], by Bingham for the Mittag-Leffler regime, see Prop. 1a of [Bin71], this last work building on the completely different Darling-Kac approach to the one-dimensional case for this regime [DK57], and by Whitt for the stable regime in his book [Whi02], pp. 235- 238 (where all ranges are treated).

These functional theorems describe convergence respectively to Brownian motion for $r \geq 2$, then to a completely asymmetric α-stable process when $r = \alpha \in (1,2)$, shifting to a Mittag-Leffler process of parameter $r = \alpha \in (0,1)$. In the same spirit we prove in this paper dynamical *asips* (log) for all parameter values. At the "critical point" $r = 1$ the limiting 1-self-similar process we encounter is

$\widetilde{Z}(t)=Z(t)-t\log t+t=\check{Z}(t)+t$, with Z the completely asymmetric Cauchy process; the reason for this choice of nonlinear centering, which has better properties than that used for the random walks, will be seen below. Then as a corollary of our main theorem, as in [FT11a], we derive pathwise functional CLTs and CLTs for all cases. As will be seen from the proofs, passage to the corollary becomes especially clear in the dynamical formulation of the *asip*s (log) as a convergence along stable manifolds of the flow.

Concerning the critical points $\alpha = 1$ and $\alpha = 2$, despite a comment of Feller (for $\alpha = 1$) that "This case is rather uninteresting and ... requires lengthy computations which are here omitted"– we find the study of these points to be interesting both from a technical and conceptual viewpoint. Indeed, here the mean and variance respectively exhibit a phase transition, passing from finite to infinite. Moreover at both these points one has mixed regimes, with the Cauchy case $\alpha = 1$ including laws F of both finite and infinite mean, while the domain of attraction of G_2 contains distributions of both finite and infinite variance. The extra technical challenge is echoed in the need to find the appropriate formulation for the theorems: while for $r > 2$ one has an actual *asip*, at the critical point $r = 2$ this is replaced by an *asip* (log), for the finite variance case, accompanied by a nonlinear time change for the infinite variance part of the domain; for $r = 1$ we precede this by an appropriate nonlinear coordinate change.

We mention a basic example of a renewal process, the occupation time of a state in a recurrent countable state Markov chain; from the ergodic theory point of view one studies the dynamics of the shift map on path space for the corresponding shift-invariant measure. Here the critical point $\alpha = 1$ plays a special role, as it marks the transition from finite to infinite invariant measure, see [FT11b]. In later work [FLT11] we build on these results to study further examples of measure-preserving transformations which exhibit the same changes of phase: a class of doubling maps of the interval with an indifferent fixed point.

We now present our results, beginning with the case $\alpha < 1$.

For this range of α, the stable process Z has increasing paths, with a dense set of (positive) jumps. Its generalized inverse $\widehat{Z}$ is continuous; by definition this is the Mittag-Leffler process of index α.

Theorem 1.1. (Mittag-Leffler case) *Let $(X_i)_{i\geq 0}$ be an i.i.d sequence of a.s. positive random variables of common distribution function F in the domain of attraction of a (completely asymmetric) $\alpha-$stable law with $0 < \alpha < 1$. Then there exist a C^1, increasing, regularly varying function $a(\cdot)$ of order $1/\alpha$ with regularly varying derivative, and a joining of $\overline{N}$ with a Mittag-Leffler process $\widehat{Z}$ of index α, so that for almost every pair $(\overline{N}, \widehat{Z})$,*

$$\lim_{t\to\infty} ||\widehat{\tau}_t(a^\alpha \circ \overline{N}) - \widehat{\tau}_t(\widehat{Z})||^\infty_{[0,1]} = 0 \ \textit{(Cesáro)}, \tag{1.11}$$

where $\widehat{\tau}_t$ denotes the scaling flow of index α. This also holds for the metric d^u_∞ constructed from the uniform pseudometrics $d^u_T(f,g) = ||f-g||^\infty_{[0,T]}$ by integration as in (1.10).

Equivalently to (1.11)*, we have the asip (log):*

$$||a^\alpha \circ \overline{N} - \widehat{Z}||^\infty_{[0,T]} = o(T^\alpha) \text{ (log)}. \tag{1.12}$$

Furthermore we have,

$$\left|\left|\frac{\overline{N}(T\cdot)}{a^{-1}(T)} - \frac{\widehat{Z}(T\cdot)}{T^\alpha}\right|\right|_{[0,1]} \longrightarrow 0 \ \ a.s. \text{ (log)}. \tag{1.13}$$

Defining maps $(\widehat{\tau}_t^a)_{t\in\mathbb{R}}$ *by* $(\widehat{\tau}_t^a f)(x) = f(e^t x)/a^{-1}(e^t)$*, this implies*

$$d_\infty^u\big(\widehat{\tau}_t^a(\overline{N}), \widehat{\tau}_t(\widehat{Z})\big) \to 0, \ \ \textit{a.s. (Cesáro)}. \tag{1.14}$$

All the above results hold also for the polygonal path $N(\cdot)$ *and for the actual renewal process* $\overline{N}-1$*. In particular, therefore, the path* $(a^\alpha \circ N)$ *is an element of* $W_{Ces}^{s,d_\infty^u}(\widehat{Z})$*.*

Note that $\widehat{\tau}_t^a$ in (1.14) is a nonstationary dynamical system; it is not a stationary dynamical system, i.e. a flow, unless $a(t) = t^{1/\alpha}$.

We proceed to the case $\alpha \in (1,2]$, where the X_i's have finite mean $\mu > 0$. In this regime and also for the transition point $\alpha = 1$, the completely asymmetric distribution G_α has support on all of $\mathbb{R}$ and so a.s. the $\alpha-$stable process Z is no longer monotone. As explained above, for $\alpha \neq 2$, the attraction for these renewal processes will be considered in the Skorokhod path space D. A difference to the random walk case is that here and for the Cauchy case convergence will be stated with respect to a *noncomplete* pseudometric d_T^0 and its integrated version, the noncomplete metric d_∞^0; this is due to interesting technical reasons, addressed after the statement of the theorem.

A simple yet insightful idea introduced in [Fel49] gives the key for passing results from $S(\cdot)$ to $N(\cdot)$ for this region: since $S(\cdot)$ is increasing, from the equation $S(t) - \mu t = -\mu(N(u) - u/\mu)$ where $u = S(t)$, information regarding the random walk can be passed over to renewal process.

A version of this idea, written out for step paths, is behind our proof as well. First, with $\mathscr{I}$ denoting the identity map, the two centered processes $S - \mu\mathscr{I}$ and $N - \frac{1}{\mu}\mathscr{I}$ are related in the following way. Defining

$$\breve{N} = -\mu\,\overline{(N(\cdot) - \mathscr{I}/\mu)}_{S(\mathscr{Q})}, \tag{1.15}$$

the above relation becomes, recalling from above that $\mathscr{Q} \equiv \{[n, n+1)\}_{n\geq 0}$,

$$\overline{(S(\cdot) - \mu\,\mathscr{I})}_{\mathscr{Q}} = \breve{N} \circ S. \tag{1.16}$$

The pathwise asymptotic behavior of this centered step path process $\breve{N}$ is described in the following:

Theorem 1.2. (Stable non-Gaussian and infinite variance Gaussian case) *Under the assumptions of Theorem 1.1 but with* $1 < \alpha \leq 2$*, in the case* $\alpha = 2$ *assuming infinite variance, then for* $\breve{N}$ *as defined in* (1.15) *there exist a* C^1*, increasing, regularly varying normalizing function* $a(\cdot)$ *of order* $1/\alpha$ *with regularly varying derivative and a joining of* $\breve{N}$ *with an* $\alpha-$*stable process* Z *so that for almost every pair* $(\breve{N}, Z)$*, we have the asip (log):*

$$d_T^0(\breve{N} \circ \mu \cdot (a^\alpha)^{-1}, Z) = o(T^{1/\alpha}) \ \ (\log) \tag{1.17}$$

and the equivalent dynamical statement: for τ_t *the scaling flow of index* $1/\alpha$*, then*

$$\lim_{t\to\infty} d_1^0(\tau_t(\breve{N} \circ \mu \cdot (a^\alpha)^{-1}), \tau_t(Z)) = 0 \ \textit{(Cesáro)}. \tag{1.18}$$

This also holds for the metric d_∞^0*; that is, the path* $(\breve{N} \circ \mu(a^\alpha)^{-1})$ *is an element of* $W_{Ces}^{s,d_\infty^0}(Z)$*.*
In the case $\alpha = 2$*,* d_T^0 *and* d_∞^0 *reduce to* $||\cdot||_{[0,T]}^\infty$ *and* d_∞^u *respectively.*

We comment on the different metrics which appear in the statements and proofs. Besides the complete metric d_∞ on D, we also employ a noncomplete metric d^0_∞, constructed as follows. Denoting by d^0_1 Billingsley's noncomplete metric on $D_{[0,1]}$ [Bil68], then for any $T > 0$, we extend d^0_1 to a metric d^0_T on the time interval $[0, T]$ exactly as done in (1.8) for the complete metric d_T.

Lifted to $f, g \in D$, this gives the pseudometric

$$d^0_T(f,g) = \inf\{\varepsilon : \exists \lambda_T \in \Lambda_T \text{ with } ||\lambda_T - \mathscr{I}||^\infty_{[0,T]} \le \varepsilon T^{1-1/\alpha} \text{ and } ||f - g\circ\lambda_T||^\infty_{[0,T]} \le \varepsilon\} \qquad (1.19)$$

where Λ_T is the collection of increasing continuous maps of $[0, T]$ onto itself. We define from this a noncomplete metric d^0_∞ on D by integration of d^0_T as in (1.10).

Although d^0_∞ is not complete, it defines the same topology as the complete metric d_∞, see [Whi80] and Lemma 8.1 of [FT11a]; this is the same "J_1 topology" on $D = D_{\mathbb{R}^+}$ defined in a different way by Stone [Sto63]. In the proofs we make use as well of a second noncomplete metric $\widetilde{d}^0_\infty$, more closely related to Stone's original definition, and which also gives that topology, see Proposition 8.4 of [FT11a]; this plays a key role in deriving Corollary 1.1. The fact that there exists a complete equivalent metric shows in particular that D with Stone's topology is a Polish space. That is an important point for the application of ergodic theory methods, see Lemma 2.2 and also Remark 1.1.

We next move to the case $\alpha = 1$. For Z completely asymmetric Cauchy (so the distribution of $Z(1)$ is G_1), we recall that $\widetilde{Z}$ denotes the centered Cauchy process $\widetilde{Z}(t) = Z(t) - t\log t + t$, which is 1-self-similar. We define

$$\widetilde{N} = -\overline{(\widetilde{\varrho}\circ N - \mathscr{I})}_{S(\mathscr{Q})},$$

with $\widetilde{\varrho}(t) \stackrel{def}{=} t\widetilde{v}(a(t))$ where $\widetilde{v}(t) = \int_0^t V(x)/x^2 dx$ and where $V(x) = \int_0^x s^2\, dF(s)$ is the truncated variance of F up to time x; the function $\widetilde{\varrho}(\cdot)$ now has the same properties as $a(\cdot)$, as stated in the theorem to follow. Note that $\widetilde{\varrho}\circ a^{-1} = \widetilde{v}\, a^{-1}$ so, for finite mean, this has the same form as $\mu(a^\alpha)^{-1}$, the time change for the case $\alpha > 1$. We remark that the centering function used here has nicer properties than that we used for random walks in [FT11a]; the present choice has to do with the technically subtle aspects of the Cauchy renewal case, as seen in the proofs.

Here we show:

Theorem 1.3. (Cauchy case) *Under the assumptions of Theorem 1.1, but now with $\alpha = 1$, there exist C^1, increasing, regularly varying functions of order 1 with regularly varying derivative, $a(\cdot)$ and $\widetilde{\varrho}(\cdot)$, and a joining of $\widetilde{N} = -\overline{(\widetilde{\varrho}\circ N - \mathscr{I})}_{S(\mathscr{Q})}$ and $\widetilde{Z}$ such that for almost every pair $(\widetilde{N}, \widetilde{Z})$ we have:*

$$d^0_T(\widetilde{N}\circ\widetilde{\varrho}\circ a^{-1}, \widetilde{Z}) = o(T) \ \ (\log). \qquad (1.20)$$

This implies, for τ_t the scaling flow of order 1:

$$\lim_{t\to\infty} d^0_\infty(\tau_t(\widetilde{N}\circ\widetilde{\varrho}\circ a^{-1}), \tau_t(\widetilde{Z})) = 0 \text{ (Cesáro)}; \qquad (1.21)$$

that is, the path $(\widetilde{N}\circ\widetilde{\varrho}\circ a^{-1})$ belongs to $W^{s,d^0_\infty}_{Ces}(\widetilde{Z})$.

We turn to the remaining part of the Gaussian regime $\alpha = 2$. The first part of (i), assuming finite moments higher than two, is due to Horvath (with discrete-time scaling) and is an actual *asip*, for there is no exceptional set of log density zero to discard. For completeness we give a proof in §5. In fact as Horvath shows this statement is sharp; in contrast to the random walks (see [CR81] pp. 107 and 108), for renewal processes one can do no better than $o(T^{\frac{1}{4}})$. See Theorem 2.1 of [Hor84] for the precise bounds. For part (ii), with finite second moment only, we make use of the *asip*(log) for random walks proved in [Fis]. We emphasize that, just as in the random walk case (see the explanation in the proof of Proposition 5.1 of [FT11a]), the infinite variance Gaussian case is treated separately as the method of proof is completely different, having more in common with the case $\alpha \in (1,2)$.

Theorem 1.4. (Finite variance Gaussian case) *Let (X_i) be an i.i.d sequence of a.s. positive random variables with common mean μ and distribution function F lying in the domain of attraction of the Gaussian.*

(i) Assuming that F has finite $r > 2$ moment (and variance σ^2), then there exists a joining of the polygonal process N and a standard Brownian motion B such that for a.e. pair (N,B),

$$||(\mu N - \mathscr{I})(\frac{\mu}{\sigma^2}\cdot) - B||^{\infty}_{[0,T]} = \begin{cases} o(T^{\frac{1}{r}}) & \text{for } r \in (2,4) \\ o(T^{\gamma}) \text{ for any } \gamma > 1/4, & \text{for } r \geq 4; \end{cases}$$

equivalently, for τ_t the scaling flow of index 1/2:

$$||\tau_t\big((\mu N - \mathscr{I})(\frac{\mu}{\sigma^2}\cdot)\big) - \tau_t(B)||^{\infty}_{[0,1]} = \begin{cases} o(e^{(1/r-1/2)t}), & \text{for } r \in (2,4) \\ o(e^{(\gamma-1/2)t}) \text{ for any } \gamma > 1/4, & \text{for } r \geq 4. \end{cases}$$

As a consequence, the path $(\mu N - \mathscr{I})(\frac{\mu}{\sigma^2}\cdot)$ belongs to $W^{s,d^u_\infty}_{Ces}(B)$.

(ii) There exists a distribution function F with finite variance (and with all higher than second moments infinite) such that no joining exists with the bound of order $o(T^{\frac{1}{2}})$. However, there exists a joining of N and B such that for a.e. pair (N,B),

$$||(\mu N - \mathscr{I})(\frac{\mu}{\sigma^2}\cdot) - B||^{\infty}_{[0,T]} = o(\sqrt{T})\ (\log) \tag{1.22}$$

and equivalently:

$$||\tau_t\big((\mu N - \mathscr{I})(\frac{\mu}{\sigma^2}\cdot)\big) - \tau_t(B)||^{\infty}_{[0,1]} \to 0 \text{ (Cesáro)}.$$

All the above statements hold for the step paths $\overline{N}(\cdot)$ as well.

Lastly we arrive at the limit theorems proved as corollaries of our *asips* (log), recalling first this definition from dynamical systems.

For Ω a Polish space with a metric d, acted on by an ergodic flow τ_t preserving a probability measure ν, an element x of Ω is said to be a *generic point* for the flow iff for any $\varphi \in CB(\Omega,d)$, the collection of real observables bounded and continuous for the topology defined from d, we have

$$\frac{1}{T}\int_0^T \varphi(\tau_t x)\mathrm{d}t \to \int \varphi\ \mathrm{d}\nu, \text{ as } T \to +\infty. \tag{1.23}$$

Fomin's theorem [Fom43] tells us that in this setting ν-a.e. x is a generic point.

Corollary 1.1. (Generic points; pathwise functional CLTs and pathwise CLTs) *Under the assumptions of the above theorems, keeping the same notation, we have the following:*

(i) For $\alpha < 1$, almost every path $(a^\alpha \circ N)$ is a generic point for the scaling flow $\widehat{\tau}_t$ of index α of the Mittag-Leffler process $\widehat{Z}$, and denoting by $\mathscr{C}(\mathbb{R}^+)$ the continuous real-valued functions on $\mathbb{R}^+$, then for any $\varphi \in CB(\mathscr{C}(\mathbb{R}^+), d_\infty^u)$,

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \varphi\left(\frac{\overline{N}(t\cdot)}{a^{-1}(t)}\right) \frac{dt}{t} = \int \varphi \, d\widehat{\nu}.$$

(ii) For $\alpha \in (1,2)$, almost every path $(\check{N} \circ \mu(a^\alpha)^{-1})$ is a generic point for the scaling flow τ_t of index $1/\alpha$ of the α-stable process Z, and for any $\varphi \in CB(D, d_\infty)$ we have:

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \varphi\left(\frac{\check{N}(t\cdot)}{a(t/\mu)}\right) \frac{dt}{t} = \int \varphi \, d\nu. \tag{1.24}$$

For $\alpha = 2$ in the infinite variance case, this holds for any $\varphi \in CB(\mathscr{C}(\mathbb{R}^+), d_\infty^u)$.

(iii) For $\alpha = 1$, almost every path $(\widetilde{N} \circ \widetilde{\varrho} \circ a^{-1})$ is a generic point for the scaling flow τ_t of index one of the centered Cauchy process $\widetilde{Z}$ with law $\widetilde{\nu}$, and for any $\varphi \in CB(D, d_\infty)$:

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \varphi\left(\frac{\widetilde{N}(t\cdot)}{a \circ \widetilde{\varrho}^{-1}(t)}\right) \frac{dt}{t} = \int \varphi \, d\widetilde{\nu}.$$

(iv) For $\alpha = 2$, in the finite variance case, assuming for simplicity $\sigma = 1$, almost every polygonal path $(\mu N - \mathscr{I})(\mu\cdot)$ is a generic point for the scaling flow τ_t of index $1/2$ of Brownian motion, and for any $\varphi \in CB(\mathscr{C}(\mathbb{R}^+), d_\infty^u)$,

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \varphi\left(\frac{(\mu N - \mathscr{I})(\mu t\cdot)}{\sqrt{t}}\right) \frac{dt}{t} = \int \varphi \, d\nu.$$

Furthermore, setting $\cdot = 1$ and $\varphi \in CB(\mathbb{R})$ in all the above pathwise functional CLTs, we have the corresponding pathwise CLTs.

Remark 1.1. The proof of the pathwise functional CLTs follows the same reasoning as in [FT11a]: first, the similar statements hold for the corresponding self-similar flow by Fomin's theorem. The asymptotic convergence of paths given by the *asip* (log) then allows these results to pass over from the self-similar processes to the coordinate-changed random walk, by Lemma 6.1 of [FT11a]; we note that this step makes use of the fact that we are working with Lebesgue spaces. This gives the generic point statements of the corollary, in all cases.

Passage to the statement for the renewal path itself is direct for the Mittag-Leffler and Brownian regimes. For the case $\alpha \in [1,2)$ and $\alpha = 2$ with infinite variance, the argument is more subtle, involving an interplay of the different metrics mentioned above for the Skorokhod topology.

For this range of α, deriving the a.s. CLTs from the functional results is also not so straightforward as for the Mittag-Leffler and Gaussian cases, since the time-t projections in D are measurable but not continuous. We circumvent this difficulty as we did in [FT11a], by convolving along the flow.

We remark that the limit in (i) can equivalently be written as:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(\widehat{\tau}_t^a(\overline{N}))\mathrm{d}t = \int \varphi \, \mathrm{d}\widehat{\nu},$$

and that the limits of the other parts of the corollary can be expressed in terms of nonstationary dynamical systems in a similar way.

We mention further that in the above statements, as in the preceding theorems, one can allow the alternative normalizing functions of Proposition 1.2 of [FT11a]: any $\widetilde{a}(\cdot)$ which is asymptotically equivalent to $a(\cdot)$ and which shares its properties of being C^1, increasing, and with a regularly varying derivative.

The outline of the paper is as follows. In §2 we study the case $\alpha \in (0,1)$. The finite mean framework $\alpha \in (1,2)$ and $\alpha = 2$ in the infinite variance case is addressed in §3. We move on to the Cauchy case $\alpha = 1$ in §4. Lastly, we turn to the finite variance Gaussian case $\alpha = 2$ and then sketch the proof of the corollary, in §5.

Acknowledgements: We thank an anonymous referee for their very careful reading of the manuscript and for helpful remarks and suggestions.

2 The Mittag-Leffler case: $\alpha < 1$

On the road to the proof of Theorem 1.1, we present a lemma which concerns the relationship between Skorokhod's J_1 and M_1 topologies, see [Sko57].

Let D_+ be the subset of D of nondecreasing functions which go to ∞ as t goes to $+\infty$ and let D_{0+} denote the subset of D_+ of functions vanishing at zero.

We endow D with the J_1 topology; as mentioned before, the inherited topology on $\mathscr{C}$, the subspsace of continuous functions, is then the topology of uniform convergence on compact sets.

For f be an element of D_+, the generalized inverse f^{-1} of f is by definition

$$f^{-1}(t) = \inf\{s : f(s) > t\}.$$

We note that for $f \in D_{0+}$, then $f^{-1} \in D_+$, and the generalized inverse of f^{-1} is f.

When studying the relationship between f and its generalized inverse, it is natural to replace Skorokhod's J_1 with his M_1 topology. Though we do not need a formal definition of M_1 metric, we use a similar basic idea.

Let f be an element of D_+. The completed graph Γ_f of f is defined to be the subset of $\mathbb{R}^2$ which is the graph of f union the vertical segments which fill in the jump discontinuities, i.e. which connect $(x, f(x^-))$ and $(x, f(x^+)) \equiv (x, f(x))$ for each $x \geq 0$, where $f(x^-)$ denotes the limit from the left at x.

In other words,

$$\Gamma_f \equiv \{(\eta, u) \in \mathbb{R}^+ \times \mathbb{R}^+ : f(\eta^-) \leq u \leq f(\eta)\}.$$

The inverse Γ^{-1} of $\Gamma = \Gamma_f$ is $\{(u,t): (t,u) \in \Gamma\}$; we just flip it with respect to the diagonal; we note that $\Gamma_{f^{-1}} = (\Gamma_f)^{-1}$.

A *parametrization* of a complete graph Γ_f is a continuous onto function $\gamma_f : \mathbb{R}^+ \to \Gamma_f \subset \mathbb{R}^2$ such that the path γ_f is nondecreasing with respect to the natural order along the graph.

For $x, y \in \mathbb{R}$, we write $||(x,y)||^{\mathbb{R}^2} = |x| + |y|$. The next lemma in fact shows that convergence for the J_1 topology gives convergence for the M_1 topology.

Lemma 2.1. *Let f and g be two elements of D_{0+} and suppose $A, \varepsilon > 0$. Assuming that $d_A^0(f,g) < \varepsilon$, then there exist two parametrizations $\gamma_f = (\eta, u)$, $\gamma_g = (\eta', u')$ of the completed graphs Γ_f and Γ_g such that $||\gamma_f - \gamma_g||_A^{\mathbb{R}^2,\infty} \stackrel{def}{=} \sup_{\{t:\, \eta(t), \eta'(t) \in [0,A]\}} ||\gamma_f(t) - \gamma_g(t)||^{\mathbb{R}^2} < 8\varepsilon$.*

Proof. We first prove the lemma for $A = 1$. By definition of the d_1^0 distance (see (1.19)), if $d_1^0(f,g) < \varepsilon$ then there exists a continuous and increasing function λ of $[0,1]$ onto itself such that:

$$||\lambda - \mathscr{I}||_{[0,1]}^{\infty} < \varepsilon \text{ and } ||f - g \circ \lambda||_{[0,1]}^{\infty} < \varepsilon. \tag{2.1}$$

We take $\gamma_f(t) = (\eta(t), u(t))$ and $\gamma_g(t) = (\rho(t), v(t))$ with η, u, ρ, v nondecreasing and continuous, "tuned" to each other in such a way that $||\gamma_f - \gamma_g||_1^{\mathbb{R}^2,\infty} = \sup_{\{t:\, \eta(t) \in [0,1]\}} ||\gamma_f(t) - \gamma_g(t)||^{\mathbb{R}^2} < 8\varepsilon$.

Given $\eta(\cdot)$, we shall define from this $\rho = \lambda \circ \eta$. That already gives good control for the first coordinate, since by (2.1), then $\sup_{\{t:\eta(t)\in[0,1]\}} |\lambda \circ \eta(t) - \eta(t)| = ||\lambda - \mathscr{I}||_{[0,1]}^{\infty} < \varepsilon$. Note that for t such that $\eta(t)$ is a continuity point of f, respectively of $g \circ \lambda$, then $u(t)$, respectively $v(t)$ are defined from this by applying f and g to the first coordinate. The other values of t will serve to fill in the completed graphs where one of f or $g \circ \lambda$ has a jump. The idea is the following: if the jump is small for one it must be small for the other, and (say for f) $u(t)$ is defined to move continuously along the vertical segment of γ_f while $\eta(t)$ remains constant. If the jump for f is large, then the jump for $g \circ \lambda$ is of comparable size, and u and v are chosen to accompany each other along this vertical segment. To do this, extra time intervals are inserted for each vertical segment. There are a countable number of discontinuities for f and g; enumerating these, the parametrizations are constucted as a limit where these jump segments are sucessively filled in. Since paths in D_+ are nondecreasing, the total length of jumps and hence of the extra time added is finite. For simplicity we do not write down any of the many possible explicit parametrizations, providing instead the estimates which work for all such choices.

Thus, defining the *jump* $\mathscr{J}_l(t)$ of an element l of D at time t by:

$$\mathscr{J}_l(t) = l(t) - l(t^-),$$

and writing $c = \eta(t)$, then

$$\begin{aligned} \mathscr{J}_{g\circ\lambda}(c) &= g \circ \lambda(c) - g \circ \lambda(c^-) = (g \circ \lambda - f + f)(c) + (-f + f - g \circ \lambda)(c^-) \\ &= (g \circ \lambda - f)(c) + (f - g \circ \lambda)(c^-) + \mathscr{J}_f(c). \end{aligned}$$

Thus

$$|\mathscr{J}_{g\circ\lambda}(c) - \mathscr{J}_f(c)| \leq 2||f - g \circ \lambda||_{[0,1]}^{\infty},$$

and hence, from (2.1), we know that

$$|\mathscr{J}_{g\circ\lambda}(c) - \mathscr{J}_f(c)| \le 2\varepsilon. \tag{2.2}$$

Suppose first that f and $g\circ\lambda$ are continuous at $c = \eta(t)$. Then by (2.1)

$$|u(t) - v(t)| = |f(\eta(t)) - g(\lambda(\eta(t)))| = |f(c) - g\circ\lambda(c)| < \varepsilon.$$

Assume next that least one of $f, g\circ\lambda$ has a jump at c and the jump of f is *small*: $\mathscr{J}_f(c) \le 2\varepsilon$. In this case we will add a small time interval to bridge the vertical gaps.

Supposing first that f is continuous at c while $g\circ\lambda$ has a jump there, so $g\circ\lambda(c^-) < g\circ\lambda(c)$, with $c = \eta(t_0)$, then we insert a small time interval $[t_0, t_1]$ so that $\eta(t) \equiv c$ on that interval, (hence $\rho = \lambda\circ\eta$ also remains constant there) while $v(t)$ increases continuously from $g\circ\lambda(c^-)$ to $g\circ\lambda(c)$, so that for these times, $g\circ\lambda(c^-) \le v(t) \le g\circ\lambda(c)$.

Now if $0 < \mathscr{J}_f(c) \le 2\varepsilon$, then we insert a small time interval on which $u(t), v(t)$ traverse respectively the jumps of f and of g; if g is continuous then the curve γ_g is constant along this time interval.

For any of theses cases, using (2.2),

$$|u(t) - v(t)| \le \mathscr{J}_f(c) + \varepsilon + \mathscr{J}_{g\circ\lambda}(c) \le 2\mathscr{J}_f(c) + 3\varepsilon \le 7\varepsilon.$$

Let us now suppose that the jump is *big*, i.e. $\mathscr{J}_f(c) > 2\varepsilon$. Since $||f - g\circ\lambda||^{\infty}_{[0,1]} < \varepsilon$, by the first inequality in (2.2), $\mathscr{J}_{g\circ\lambda}(c) > 0$ and so $g\circ\lambda$ must jump at c.

In fact the jump intervals of f and $g\circ\lambda$ contain a common interval $[f(c^-)+\varepsilon, f(c)-\varepsilon]$. For $u(t)$ with values in this interval, we define $v(t) = u(t)$; for $u(t) \in [f(c^-), f(c^-)+\varepsilon]$, we define $v(t)$ to be any nondecreasing continuous function onto the interval $[g\circ\lambda(c^-), f(c^-)+\varepsilon]$, and for $u(t) \in [f(c)-\varepsilon, f(c)]$ we define $v(t)$ to be any nondecreasing continuous function onto $[f(c)-\varepsilon, g\circ\lambda(c)]$.

So for $u(t)$ and $v(t)$ in the common interval, $|u(t) - v(t)| = 0$; and if they are above or below then $|u(t) - v(t)| \le 2\varepsilon$. Hence for all these t, $|u(t) - v(t)| \le 2\varepsilon$.

Choosing the functions u and v in this way, where we have carried this out for each discontinuity point, we have therefore that

$$\sup_{\{t:\,\eta(t)\in[0,1]\}} ||\gamma_f(t) - \gamma_g(t)||^{\mathbb{R}^2} = \sup_{\{t:\eta(t)\in[0,1]\}} \big(|\lambda\circ\eta - \eta| + |u - v|\big) < 8\varepsilon,$$

as claimed.

This completes the proof for $A = 1$; from this we deduce the result for any $A > 0$. Since $d^0_A(f,g) = A^{1/\alpha} d^0_1(\Delta_A f, \Delta_A g)$, if $d^0_A(f,g) < \varepsilon$, there exist two parametrizations of the completed graphs of the rescaled functions, $\gamma_{\Delta_A f} = (\eta_A, u_A)$ and $\gamma_{\Delta_A g} = (\widehat{\eta}_A, \widehat{u}_A)$, such that $\widehat{\eta}_A = \lambda\circ\eta_A$ for λ a continuous increasing map of $[0,1]$ onto itself with

$$\sup_{\{t:\eta_A(t)\in[0,1]\}} \big(|\eta_A - \widehat{\eta}_A| + |u_A - \widehat{u}_A|\big) < 8\varepsilon/A^{1/\alpha}.$$

Setting $\gamma_f = (A\,\eta_A, A^{1/\alpha} u_A)$ and $\gamma_g = (A\,\widehat{\eta}_A, A^{1/\alpha}\widehat{u}_A)$, we have two parametrizations for Γ_f and Γ_g such that $||\gamma_f - \gamma_g||^{\mathbb{R}^2,\infty}_A = \sup_{\{t:\,\eta(t)\in[0,A]\}} ||\gamma_f(t) - \gamma_g(t)||^{\mathbb{R}^2} < 8\varepsilon$, as in this case $\alpha < 1$ and so $A < A^{1/\alpha}$. This finishes the proof of the lemma. □

Proving Theorem 1.1 hinges upon the next three results; we write $h = a^\alpha$ and $\widehat{h} = h^{-1}$.

Theorem A: (see [FT11a]) *Under the assumptions and notation of Theorem 1.1, there exists a joining of $\overline{S}$ and an $\alpha-$stable process Z such that*

$$d_T(\overline{S} \circ \widehat{h}, Z) = o(T^{1/\alpha}) \; a.s. \; (\log)$$

and equivalently, with τ_t the scaling flow of index $1/\alpha$:

$$d_1(\tau_t(\overline{S} \circ \widehat{h}), \tau_t Z) = o(1) \; a.s. \; (Cesáro).$$

This last statement also holds for d_∞.

Remark 2.1. Here as in [FT11a], d_T is the rescaled version of Billingsley's complete metric on $[0,1]$ defined as in (1.8). We recall that for $\varepsilon > 0$ small enough, if $d_1(f,g) \le \varepsilon$ then $d_1^0(f,g) \le 2\varepsilon$; see [Bil68] p. 113.

Lemma A: (see Theorem 1.20 of [Wal82]) *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a locally integrable function. Then these are equivalent:*

(i) $\forall \varepsilon > 0$, $\{t : |f(t)| > \varepsilon\}$ *has Cesáro density zero,*

(ii) *there exists a set $B \subseteq \mathbb{R}$ of Cesáro density* 0 *such that* $\lim_{t\to\infty, t\notin B} f(t) = 0$.

Let us write τ_t and $\widehat{\tau}_t$ for the scaling flows of index α and $1/\alpha$, respectively.

Lemma 2.2. *For $\alpha \in (0,1)$, the map $f \mapsto f^{-1}$ on D_{0+} is an isomorphism between τ_t and $\widehat{\tau}_{t/\alpha}$. The flow $\widehat{\tau}_t$ is ergodic and is in fact a Bernoulli flow of infinite entropy.*

Proof. The scaling flow τ_t for $f \in D_{0+}$ maps the graph of the path $f(\cdot)$ by action of this matrix on column vectors in $\mathbb{R}^2$: $\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t/\alpha} \end{bmatrix}$. Via the correspondence $f \mapsto f^{-1}$, which is Borel measurable on D_{0+}, this action is by $\begin{bmatrix} e^{-t/\alpha} & 0 \\ 0 & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-s} & 0 \\ 0 & e^{-\alpha s} \end{bmatrix}$ which is that of $\widehat{\tau}_s$ on f^{-1}, for $s = t/\alpha$. In other words,

$$(\tau_t Z)^{-1} = \widehat{\tau}_{t/\alpha} \widehat{Z}.$$

The last part of the proof follows from Lemma 3.3 of [FT11a]; we note that this makes use of the fact that since D_{0+} with the Skorokhod topology is Polish, we have a Lebesgue space. □

This brings us to:

Proof of Theorem 1.1. We start by proving (1.11); here τ_t and $\widehat{\tau}_t$ denote the scaling flows of orders $1/\alpha$ and α respectively. So as not to overburden the notation, as above we abbreviate a^α by h and h^{-1} by $\widehat{h}$, and then write $f_t \equiv \tau_t(\overline{S} \circ \widehat{h})$ and $g_t \equiv \tau_t Z$. By definition both f_t and g_t are nondecreasing, and their generalized inverses are $f_t^{-1} = \widehat{\tau}_{t/\alpha}(h \circ \overline{N})$ and $g_t^{-1} = \widehat{\tau}_{t/\alpha}(\widehat{Z})$.

We want to prove that $||f_t^{-1} - g_t^{-1}||_1^\infty \to 0$, a.s. (Cesáro); from Lemma A this is equivalent to showing that for all $\varepsilon > 0$, for the limit of the Cesáro density up to time T (C_T for short) of the set

$$\mathscr{W} \equiv \{t > 0 \; : ||f_t^{-1} - g_t^{-1}||_1^\infty > \varepsilon\}$$

we have

$$\lim_{T\to\infty} C_T(\mathscr{W}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_{\mathscr{W}}(t)\,\mathrm{d}t = 0, \text{ a.s.}$$

We say a continuous function φ defined on an interval J is (ε,δ)- continuous iff for all $x, y \in J$, if $|x-y| < \delta$ then $|\varphi(x) - \varphi(y)| < \varepsilon$.

Now, for any $\overline{\varepsilon}, \delta, \overline{\delta} > 0, A > 0$ let us write:

$$\begin{aligned}
\mathscr{U}_A &\equiv \left\{t\ : f_t(A) \wedge g_t(A) > 1\right\},\\
\mathscr{V}_{\overline{\varepsilon},\overline{\delta}} &\equiv \left\{t\ : g_t^{-1} \text{ is } (\overline{\varepsilon},\overline{\delta}) - \text{continuous on } [0,3/2]\right\},\\
\mathscr{X}_\delta &\equiv \{t\ : d_A^0(f_t, g_t) \le \delta\}.
\end{aligned}$$

Thus, for any $\varepsilon, A, \delta, \overline{\varepsilon}, \overline{\delta} > 0$ we have that

$$\begin{aligned}
\mathscr{W} &\subseteq (\mathscr{W}\cap\mathscr{X}_\delta)\cup\mathscr{X}_\delta^c\\
&= (\mathscr{W}\cap\mathscr{X}_\delta\cap\mathscr{U}_A\cap\mathscr{V}_{\overline{\varepsilon},\overline{\delta}})\cup(\mathscr{W}\cap\mathscr{X}_\delta\cap\mathscr{U}_A^c)\cup(\mathscr{W}\cap\mathscr{X}_\delta\cap\mathscr{V}^c_{\overline{\varepsilon},\overline{\delta}})\cup\mathscr{X}_\delta^c.
\end{aligned}$$

So

$$\begin{aligned}
C_T(\mathscr{W}) &\le C_T(\mathscr{W}\cap\mathscr{X}_\delta\cap\mathscr{U}_A\cap\mathscr{V}_{\overline{\varepsilon},\overline{\delta}}) + C_T(\mathscr{X}_\delta\cap\mathscr{U}_A^c) + C_T(\mathscr{V}^c_{\overline{\varepsilon},\overline{\delta}}) + C_T(\mathscr{X}_\delta^c)\\
&\equiv \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}. \qquad (2.3)
\end{aligned}$$

In [FT11a], we proved that as $t\to\infty$, $d_1(f_t, g_t)\to 0$ a.s. (Cesáro). For all fixed $A > 0$, from (1.8) for d_T^0 we have $A^{-1/\alpha} d_A^0(f_t, g_t) = d_1^0(\Delta_A(f_t), \Delta_A(g_t))$ which from Remark 2.1 goes a fortiori a.s. to 0 (Cesáro) as t goes to ∞. This implies that for all $\delta > 0$, (IV) goes to 0 as $T\to\infty$.

Now whenever $d_A^0(f_t, g_t) \le \delta$, from (1.19) we have $|f_t(A) - g_t(A)| \le \delta$, so that if in addition $g_t(A)\wedge f_t(A) \le 1$, then $g_t(A) \le 1 + \delta$. Thus, (II) is less than or equal to

$$C_T\left(\{t : g_t(A) \le 1 + \delta\}\right) = \frac{1}{T}\int_0^T \chi_{\{Z\in D: Z(A)\le 1+\delta\}}(\tau_t Z)\,\mathrm{d}t,$$

which converges to $\nu(\{Z\in D : Z(A) \le 1+\delta\}) = \mathbb{P}(Z(A) \le 1+\delta)$ as $T\to\infty$ by the Birkhoff Ergodic Theorem, where ν is the $\alpha-$stable measure on D. (We generally use the first way of writing the measure of this set; though it is natural from the path space point of view we take in this paper, it can be confusing from the probability perspective). By the scaling property, this implies that

$$\limsup_{T\to\infty} \text{(II)} \le \nu\left(\{Z\in D : Z(1) \in (0, \frac{1+\delta}{A^{1/\alpha}})\}\right).$$

In the same fashion, by the Ergodic Theorem, now applied to $\widehat{\tau}_t$, it follows that:

$$\lim_{T\to\infty} \text{(III)} = \widehat{\nu}\left(\{\widehat{Z} : \widehat{Z} \text{ is } (\overline{\varepsilon},\overline{\delta}) - \text{continuous on } [0,3/2]\}^c\right).$$

We shall now prove that for suitably chosen $\varepsilon, \overline{\varepsilon}, \delta, \overline{\delta}$, then we have (I) $\equiv 0$, by showing that if $d_A^0(f_t, g_t) \le \delta$, $g_t(A)\wedge f_t(A) > 1$ and $t\in\mathscr{V}_{\overline{\varepsilon},\overline{\delta}}$, then $||f_t^{-1} - g_t^{-1}||_1^\infty \le \varepsilon$.

From Lemma 2.1, there exist two parametrizations of the completed graphs Γ_{f_t} and Γ_{g_t} of f_t and g_t, written as $\gamma_{f_t} = (\eta_t, u_t)$ and $\gamma_{g_t} = (\hat{\eta}_t, \hat{u}_t)$, such that $||\gamma_{f_t} - \gamma_{g_t}||_A^{\mathbb{R}^2,\infty} \leq 8\delta$. We choose the parametrizations constructed in the proof of Lemma 2.1.

A natural way of parametrizing the completed graphs of the generalized inverses of f_t and g_t comes from interchanging the coordinates, with $\gamma_{f_t^{-1}} = (u_t, \eta_t)$ and $\gamma_{g_t^{-1}} = (\hat{u}_t, \hat{\eta}_t)$. Since $g_t(A) \wedge f_t(A) > 1$, $f_t([0,A])$ and $g_t([0,A])$ contain $[0,1]$, and so for any $x \in [0,1]$ there exist a and b such that:

$$\begin{aligned}
|f_t^{-1}(x) - g_t^{-1}(x)| &= ||(x, f_t^{-1}(x)) - (x, g_t^{-1}(x))||^{\mathbb{R}^2} = ||\gamma_{f_t^{-1}}(a) - \gamma_{g_t^{-1}}(b)||^{\mathbb{R}^2} \\
&\leq ||\gamma_{f_t^{-1}}(a) - \gamma_{g_t^{-1}}(a)||^{\mathbb{R}^2} + ||\gamma_{g_t^{-1}}(a) - \gamma_{g_t^{-1}}(b)||^{\mathbb{R}^2} \\
&\leq ||\gamma_{f_t} - \gamma_{g_t}||_A^{\mathbb{R}^2,\infty} + |\hat{u}_t(b) - \hat{u}_t(a)| + |\hat{\eta}_t(b) - \hat{\eta}_t(a)| \\
&\leq 8\delta + |u_t(a) - \hat{u}_t(a)| + |g_t^{-1}(u_t(a)) - g_t^{-1}(\hat{u}_t(a))|;
\end{aligned}$$

we have used the fact that $f_t^{-1}(1) \leq A$ in deriving the second inequality, then $u_t(a) = \hat{u}_t(b)$ and the fact that g_t^{-1} is continuous in deriving the last one.

If $8\delta < \overline{\delta}$ then the last quantity above is less than or equal to $8\delta + 8\delta + \overline{\varepsilon}$, and so choosing $8\delta < \overline{\delta} < \overline{\varepsilon} = \frac{\varepsilon}{3}$ yields (I) $\equiv 0$.

In (2.3), for chosen $\varepsilon > 0$, let $T \to \infty$ for $\delta, A, \overline{\delta}$ fixed and related as just described. Next, let A increase to $+\infty$, then finally let $\overline{\delta}$ decrease to 0. The Cesáro density of $\mathscr{W} = \mathscr{W}_\varepsilon$ is therefore zero for any $\varepsilon > 0$; this finishes the proof of (1.11). The uniform convergence over the unit interval in Cesáro density implies uniform convergence in Cesáro density over compact intervals of $\mathbb{R}^+$ in the place of $[0,1]$, and hence convergence for d_∞^u, following the proof of Lemma 3.6 of [FT11a].

The equivalence of (1.11) and (1.12) is immediate from the definition of the scaling flow $\widehat{\tau}_t$. We now show that (1.12) holds true for the polygonal path $N(\cdot)$ as well. For this we shall prove that indeed

$$||h(N(t)) - h(\overline{N}(t))||_{[0,T]}^{\infty} = o(T^\alpha), \;\; a.s. \tag{2.4}$$

Here one can replace $[0,T]$ by $[T_0,T]$ with T_0 large.

From the definitions of N and $\overline{N}$, for all $t \geq 0$ we have that $0 \leq \overline{N}(t) - N(t) \leq 1$. Since $h(\cdot)$ is a regularly varying function of order one with a regularly varying derivative, we know from Karamata's theorem (see [BGT87], Proposition 1.5.8) that $sh'(s) \sim h(s)$ and so using the facts that $h(s+1)/h(s) \to 1$ and that h is increasing, we have for any $c_0 > 1$ that for t large enough,

$$0 \leq h(\overline{N}(t)) - h(N(t)) = \int_{N(t)}^{\overline{N}(t)} h'(s)\,ds \leq \int_{N(t)}^{N(t)+1} s\frac{h'(s)}{h(s)}\,\frac{h(s)}{s}\,\mathrm{d}s \leq c_0\,\frac{h(N(t))}{N(t)}, a.s.$$

Hence,

$$\frac{1}{T^\alpha}||h(N) - h(\overline{N})||_{[T_0,T]}^{\infty} \leq \frac{c_0}{T^\alpha} \sup_{T_0 \leq t \leq T} \frac{h(N(t))}{N(t)} \leq c_0 \sup_{T_0 \leq t \leq T} \frac{h(N(t))}{t^\alpha N(t)}, \;\; a.s.$$

We shall show that $h(N(t))/N(t) = o(t^\alpha)$ a.s. Indeed, writing $h(x) = xl(x)$ with l slowly varying, for all $\varepsilon > 0$ we have $h(N(t))/N(t) = l(N(t)) = o((N(t))^\varepsilon)$. By the law of large numbers, S_n/n a.s. goes to infinity so $N(t)/t$ a.s. goes to 0 as $t \to \infty$. Thus for all $\varepsilon > 0$, $h(N(t))/N(t) = o(t^\varepsilon)$ a.s.

This finishes the proof of (2.4), and hence the proof of (1.12) for the polygonal path $N(\cdot)$.

Proving (1.13). We start by demonstrating that

$$||\overline{N} - \widehat{h} \circ \widehat{Z}||^{\infty}_{[0,T]} = o(\widehat{h}(T^{\alpha})) \text{ a.s. (log)}. \tag{2.5}$$

From the definition of $h(\cdot)$, $\widehat{h}(t^{\alpha}) = a^{-1}(t)$. We set $p_T(x) \equiv h(\overline{N}(Tx))/T^{\alpha}$ and $q_T(x) \equiv \widehat{Z}(Tx)/T^{\alpha}$. Statement (2.5) says exactly:

$$\frac{1}{\widehat{h}(T^{\alpha})}||\widehat{h}(T^{\alpha}p_T) - \widehat{h}(T^{\alpha}q_T)||^{\infty}_{[0,1]} \longrightarrow 0 \;\; a.s. \text{ (log)}. \tag{2.6}$$

By Lemma A it is then equivalent to show that for all $\varepsilon \in (0,1)$,

$$\lim_{T\to\infty} \frac{1}{\log T} \int_0^T \chi_{\{t:\, ||\widehat{h}(t^{\alpha}p_t) - \widehat{h}(t^{\alpha}q_t)||^{\infty}_{[0,1]} > \varepsilon \widehat{h}(t^{\alpha})\}} \frac{dt}{t} = 0 \quad a.s. \tag{2.7}$$

We shall show that it will be enough to prove (2.7) with $[0,1]$ replaced by any interval $[\varepsilon',1]$ for $0 < \varepsilon' < 1$. For this, we claim that ε' can be chosen so that the log density in (2.7) for the supremum taken over the interval $[0,\varepsilon']$ is as small as desired. But since $p_T(\cdot)$, $q_T(\cdot)$ and h are nondecreasing, it will be sufficient to check that the sets where $\widehat{h}(T^{\alpha}p_T(\varepsilon'))$ and $\widehat{h}(T^{\alpha}q_T(\varepsilon'))$ are $> \varepsilon\widehat{h}(T^{\alpha})$ each a.s. have arbitrarily small log density.

To show this, note that from the definition of p_t, while recalling that $\overline{S}(s) \le S(s)$, the set $\{t : \widehat{h}(t^{\alpha}p_t(\varepsilon')) \ge \varepsilon\widehat{h}(t^{\alpha})\}$ is contained in $\{t :\ \overline{S}(\varepsilon\widehat{h}(t^{\alpha})) \le t\varepsilon'\}$. Since $a(\cdot)$ is regularly varying, after making the change of variables $x = \varepsilon\widehat{h}(t^{\alpha})$ we get from the pathwise CLT, Corollary 1.4 (iii) of [FT11a], that the log density of the last set is less than or equal to that of $\{x : \overline{S}(x) \le 2\varepsilon'\,\varepsilon^{-1/\alpha}\,a(x)\}$, which a.s. equals $\nu(\{Z :\ Z(1) \le 2\varepsilon'\,\varepsilon^{-1/\alpha}\})$, where ν is the $\alpha-$stable measure.

For q_t, we use the ergodicity of the scaling flow $\widehat{\tau}_t$ for the Mittag-Leffler process $\widehat{Z}$. Letting $\widehat{\Delta}_t$ denote $\widehat{\tau}_{\log t}$, Birkhoff's Ergodic Theorem implies that the log density of $\{t :\ \widehat{h}(t^{\alpha}q_t(\varepsilon')) > \varepsilon\widehat{h}(t^{\alpha})\}$ is less than or equal to the log density of $\{t : \widehat{\Delta}_t\widehat{Z}(\varepsilon') > \varepsilon/2\}$ which equals $\widehat{\nu}(\{\widehat{Z} :\ \widehat{Z}(1) > \varepsilon/2\varepsilon'^{\alpha}\})$.

Now $\nu(\{Z :\ Z(1) < 2\varepsilon'\varepsilon^{-1/\alpha}\})$ and $\widehat{\nu}(\{\widehat{Z} :\ \widehat{Z}(1) > \varepsilon/2\varepsilon'^{\alpha}\})$ approach 0 as ε' goes to zero. So by choosing ε' small, the log density in (2.7) for the sup taken over the interval $[0,\varepsilon']$ is a.s. as small as we want. Therefore once we show that the log density with $[\varepsilon',1]$ replacing $[0,1]$ is a.s. zero, equation (2.7) will be proved.

Since ($a(\cdot)$ hence) $\widehat{h}$ is of slowly varying derivative, we know from Karamata's theorem as before that $t\widehat{h}'(t) \sim \widehat{h}(t)$, so that for all $x \in [\varepsilon',1]$ we have:

$$\begin{aligned}\frac{1}{\widehat{h}(t^{\alpha})}|\widehat{h}(t^{\alpha}p_t(x)) - \widehat{h}(t^{\alpha}q_t(x))| &\le 1^+ \left|\int_{t^{\alpha}p_t(x)}^{t^{\alpha}q_t(x)} \frac{\widehat{h}(s)}{\widehat{h}(t^{\alpha})}\frac{ds}{s}\right| = 1^+ \left|\int_{p_t(x)}^{q_t(x)} \frac{L(st^{\alpha})}{L(t^{\alpha})}\,\mathrm{ds}\right|,\\ &\le 1^+ \left|\int_{p_t(x)}^{q_t(x)} \left(\frac{L(st^{\alpha})}{L(t^{\alpha})} - 1\right) ds\right| + 1^+\,|q_t(x) - p_t(x)|\end{aligned}$$

where 1^+ is a constant > 1 and $L(t) \equiv \widehat{h}(t)/t$ (and so is slowly varying).

Recalling from (1.11) in Theorem 1.1 that $||p_t - q_t||^{\infty}_{[0,1]} \to 0$ a.s. (log), all we have to check is that

$$\text{the log density of } \left\{ t : \sup_{x\in[\varepsilon',1]} \left| \int_{p_t(x)}^{q_t(x)} \left(\frac{L(st^\alpha)}{L(t^\alpha)} - 1 \right) ds \right| > \varepsilon \right\} \text{ is a.s. zero.}$$

Now from Birkhoff's Ergodic Theorem we get that for all $M > 0$ the log density of $\{t : q_t(1) > M\}$ is $\widehat{\nu}(\{\widehat{Z} : \widehat{Z}(1) > M\})$, while for all $\varepsilon > 0$ the log density of $\{t : q_t(\varepsilon') \le 2\varepsilon\,\varepsilon'\}$ is $\widehat{\nu}(\{\widehat{Z} : \widehat{Z}(1) \le 2\varepsilon\,\varepsilon'^{1-\alpha}\})$ (here $\alpha < 1$), so both log densities can be made small by choosing ε' small and M large.

As a result, all that remains to show is that for all $\varepsilon', M > 0$, the log density of the set $\mathscr{T}_{\varepsilon',M}$ defined as:

$$\left\{ t : \sup_{x\in[\varepsilon',1]} \left| \int_{p_t(x)}^{q_t(x)} \left(\frac{L(st^\alpha)}{L(t^\alpha)} - 1 \right) ds \right| > \varepsilon ; ||q_t - p_t||^{\infty}_{[\varepsilon',1]} \le \varepsilon\,\varepsilon' ; q_t(1) \le M ; q_t(\varepsilon') > 2\varepsilon\,\varepsilon' \right\}$$

can be rendered as small as desired.

Since $q_t(\cdot)$ is nondecreasing, then for any $t \in \mathscr{T}_{\varepsilon',M}$ and any $x \in [\varepsilon', 1]$, one has:

$$\begin{aligned} \left| \int_{p_t(x)}^{q_t(x)} \left(\frac{L(st^\alpha)}{L(t^\alpha)} - 1 \right) ds \right| &\le \int_{q_t(x)-\varepsilon\varepsilon'}^{q_t(x)+\varepsilon\varepsilon'} \left| \frac{L(st^\alpha)}{L(t^\alpha)} - 1 \right| ds \\ &\le \int_{\varepsilon\varepsilon'}^{M+\varepsilon\varepsilon'} \left| \frac{L(st^\alpha)}{L(t^\alpha)} - 1 \right| ds \le M \sup_{s\in[\varepsilon\varepsilon',M+1]} \left| \frac{L(st^\alpha)}{L(t^\alpha)} - 1 \right|. \end{aligned}$$

This means that, at fixed $\varepsilon', M > 0$, the log density of $\mathscr{T}_{\varepsilon',M}$ is bounded from above by the log density of $\{t : \sup_{s\in[\varepsilon\varepsilon',M+1]} \left| \frac{L(st^\alpha)}{L(t^\alpha)} - 1 \right| > \varepsilon/M\}$ which is 0 from a well known property of slowly varying functions; see for instance [BGT87] p. 22.

This completes the proof of (2.7) hence that of (2.5), which in turn is equivalent to:

$$\left|\left| \frac{\overline{N}(T\cdot)}{\widehat{h}(T^\alpha)} - \frac{\widehat{h}(T^\alpha(\widehat{\Delta}_T\widehat{Z}))}{\widehat{h}(T^\alpha)} \right|\right|^{\infty}_{[0,1]} \to 0 \;\; a.s. \text{ (log)}.$$

Hence, in order to prove (1.13), we only need to guarantee the following: that

$$||\mathscr{K}_T(\widehat{\Delta}_T\widehat{Z})||^{\infty}_{[0,1]} \equiv \left|\left| \frac{\widehat{h}(T^\alpha(\widehat{\Delta}_T\widehat{Z}))}{\widehat{h}(T^\alpha)} - \widehat{\Delta}_T\widehat{Z} \right|\right|^{\infty}_{[0,1]} \to 0 \;\; a.s. \text{ (log)}. \tag{2.8}$$

As we will see, this is true for f any regularly varying function of order 1 in the place of $\widehat{h}$. From Lemma A, this reduces to proving that the log density of $\{t : ||\mathscr{K}_t(\widehat{\Delta}_t\widehat{Z})||^{\infty}_{[0,1]} > \varepsilon\}$ is zero for any $\varepsilon > 0$. But $\widehat{Z}$ is nondecreasing and moreover for any $M > 0$, either $(\widehat{\Delta}_t\widehat{Z})(1) > M$ or $\le M$, thus

$$\{t : ||\mathscr{K}_t(\widehat{\Delta}_t\widehat{Z})||^{\infty}_{[0,1]} > \varepsilon\} \subseteq \{t : ||\mathscr{K}_t(\cdot)||^{\infty}_{[0,M]} > \varepsilon\} \cup \{t : (\widehat{\Delta}_t\widehat{Z})(1) > M\}.$$

Now the log density of the rightmost set above is, by the Ergodic Theorem, $\widehat{\nu}(\{\widehat{Z} : \widehat{Z}(1) > M\})$; we turn to the set preceding that. Since $\widehat{h}$ is regularly varying of order 1, from [BGT87] p. 22 we have that for any $0 < \delta < M$, $||\mathscr{K}_t(\cdot)||^{\infty}_{[\delta,M]}$ converges to 0 as $t \to \infty$. Moreover since $\widehat{h}(\cdot)$ is increasing, $||\mathscr{K}_t(\cdot)||^{\infty}_{[0,\delta]} \leq \widehat{h}(t^{\alpha}\delta)/\widehat{h}(t^{\alpha}) + \delta$. So for fixed $\delta > 0$, as $t \to \infty$, $||\mathscr{K}_t(\cdot)||^{\infty}_{[0,\delta]} \leq 3\delta$, by the regular variation of $\widehat{h}$. Putting all that together yields that $||\mathscr{K}_t(\cdot)||^{\infty}_{[0,M]} \to 0$ as $t \to \infty$, and so the log density of the first set in the previous union is a fortiori 0.

This finishes the proof of (1.13). From Lemma 3.6 of [FT11a] this then extends to $[0,A]$ for any $A > 0$, and thus after integration holds for the metric d^u_∞ as well, proving (1.14).

The proof of Theorem 1.1 is now complete. □

3 The stable case: $\alpha \in (1,2]$

We begin by recalling two results from [FT11a]. Here as before $\mathscr{Q}$ denotes the partition $\mathscr{Q} \equiv \{[n,n+1)\}_{n\geq 0}$.

Theorem B: *Under the assumptions and notation of Theorem 1.2, there exist a C^1, increasing, regularly varying function $h(\cdot)$ of order 1 with regularly varying derivative, and a joining of $\overline{S}$ and an $\alpha-$stable process Z such that for almost every pair $(\overline{S}, Z)$,*

$$||\overline{(S-\mu\mathscr{I})}_{\mathscr{Q}} \circ h^{-1} - \overline{Z}_{h(\mathscr{Q})}||^{\infty}_{[0,T]} = o(T^{1/\alpha}), a.s. \text{ (log)}.$$

Lemma B: *Let Z be an ergodic self-similar process of index $\beta > 0$ with paths in D; equivalently, assume we are given a probability measure ν on D such that the scaling flow τ_t of index β is ergodic for ν. Let η be a positive, increasing and regularly varying function of index 1. Then for ν-a.e. $Z \in D$,*

$$d_1(\tau_t(\overline{Z}_{\eta(\mathscr{Q})}), \tau_t(Z)) \to 0 \textit{ (Cesáro)},$$

and equivalently,

$$d_T(\overline{Z}_{\eta(\mathscr{Q})}, Z) = o(T^{\beta}) \text{ (log)}.$$

Proof of Theorem 1.2. We start by showing (1.17). As in the previous proofs, we set for simplicity $h(\cdot) = a^{\alpha}(\cdot)$ anf $\widehat{h} = h^{-1}$.

Recalling the definitions of d^0_T, Billingsley's noncomplete pseudo-metric for D on the interval $[0,T]$, and of $\check{N}$, keeping identity (1.16) in mind and writing $N^{\mu} = h \circ N \circ \mu\widehat{h}$, we have:

$$\begin{aligned} d^0_T\left(\check{N} \circ \mu\widehat{h}, Z\right) &= d^0_T\left(\overline{(S-\mu\mathscr{I})}_{\mathscr{Q}} \circ N \circ \mu\widehat{h}, Z\right), \\ &= d^0_T\left(\overline{(S-\mu\mathscr{I})}_{\mathscr{Q}} \circ \widehat{h} \circ N^{\mu}, Z\right). \end{aligned}$$

By the triangle inequality, this is

$$\begin{aligned} &\leq d^0_T\left(\overline{(S-\mu\mathscr{I})}_{\mathscr{Q}} \circ \widehat{h} \circ N^{\mu}, \overline{Z}_{h(\mathscr{Q})} \circ N^{\mu}\right) + d^0_T\left(\overline{Z}_{h(\mathscr{Q})} \circ N^{\mu}, \overline{Z}_{h(\mathscr{Q})}\right) + d^0_T(\overline{Z}_{h(\mathscr{Q})}, Z), \\ &\overset{def}{=} \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned}$$

We note first of all that since $h(\cdot)$ is positive, increasing and regularly varying of exponent 1, applying Lemma B we get $T^{-1/\alpha}d_T(\overline{Z}_{h(\mathcal{Q})}, Z) = d_1(\Delta_T(\overline{Z}_{h(\mathcal{Q})}), \Delta_T(Z)) = o(1)$ a.s. (log), but then from Remark 2.1 and the fact that d_T^0 and d_T scale in the same way, the same holds true with d_1^0, d_T^0 replacing d_1, d_T. This shows that (III) $= o(T^{1/\alpha})$ a.s. (log).

Next we turn to (I). Since $d_T^0(f,g) \le ||f-g||^{\infty}_{[0,T]}$, we have:

$$\begin{aligned} \text{(I)} &\le ||\overline{(S-\mu\mathcal{I})}_{\mathcal{Q}} \circ \widehat{h} \circ N^{\mu} - \overline{Z}_{h(\mathcal{Q})} \circ N^{\mu}||^{\infty}_{[0,T]} \\ &= ||\overline{(S-\mu\mathcal{I})}_{\mathcal{Q}} \circ \widehat{h} - \overline{Z}_{h(\mathcal{Q})}||^{\infty}_{[0,N^{\mu}(T)]} \end{aligned} \tag{3.1}$$

Now as $t \to \infty$, $S(t)/t\mu$ thus equivalently $\mu N(t)/t$ tends to 1 a.s. Since $h(\cdot)$ is regularly varying of exponent 1, it follows that as $t \to \infty$, $h(tx)/h(t)$ goes to x uniformly on a compact interval around $x = 1$. Then taking $s = \mu\widehat{h}(t)$, we have:

$$\frac{N^{\mu}(t)}{t} = \frac{h(N(s))}{h(s/\mu)} = \frac{h(\frac{\mu N(s)}{s}\frac{s}{\mu})}{h(s/\mu)} \to 1,\ a.s.$$

Let Ω denote a common probability space for the two processes, guaranteed by Theorem B. By the above, there exists therefore a set $\Omega_1 \subseteq \Omega$ of full probability such that for all ω in Ω_1 and all $\varepsilon > 0$, we have $N^{\mu}(T)(\omega) \le (1+\varepsilon)T$ for large T $(T \ge T_0(\omega))$.

On the other hand, now using the joining, there exists a set of full probability $\Omega_2 \subseteq \Omega$ such that for all ω in Ω_2 there is a set $\mathcal{B}_{\omega}$ of logarithmic density zero such that, with $(S,Z) = (S,Z)(\omega)$:

$$||\overline{(S-\mu\mathcal{I})}_{\mathcal{Q}} \circ \widehat{h} - \overline{Z}_{h(\mathcal{Q})}||^{\infty}_{[0,M]}(\omega) = o(M^{1/\alpha}), \quad M \notin \mathcal{B}_{\omega}.$$

Thus, for any ω in $\Omega_1 \cap \Omega_2$, and any fixed $\varepsilon > 0$, taking $\widetilde{\mathcal{B}_{\omega}}$ to be $\mathcal{B}_{\omega}/(1+\varepsilon) \cup [0, T_0(\omega)]$ we have:

$$||\overline{(S-\mu\mathcal{I})}_{\mathcal{Q}} \circ \widehat{h} - \overline{Z}_{h(\mathcal{Q})}||^{\infty}_{[0,N^{\mu}(T)]} \le ||\overline{(S-\mu\mathcal{I})}_{\mathcal{Q}} \circ \widehat{h} - \overline{Z}_{h(\mathcal{Q})}||^{\infty}_{[0,(1+\varepsilon)T]} = o(T^{1/\alpha}),\ T \notin \widetilde{\mathcal{B}_{\omega}}.$$

Since multiplication by a positive factor preserves log density, $\widetilde{\mathcal{B}_{\omega}}$ is of zero log density and we have proved that (I) $= o(T^{1/\alpha})$ a.s. (log).

So all that is left to show is: (II) $= o(T^{1/\alpha})$ a.s. (log).

Recalling from (1.6) the definition of a step path of a process over a partition, one checks that:

$$\begin{aligned} \text{(II)} &\equiv d_T^0\left(\overline{Z}_{h(\mathcal{Q})} \circ N^{\mu}, \overline{Z}_{h(\mathcal{Q})}\right) \\ &= T^{1/\alpha} d_1^0\left(\overline{\Delta_T Z}_{\mathcal{Q}_T} \circ N^{\mu}_T, \overline{\Delta_T Z}_{\mathcal{Q}_T}\right) \stackrel{def}{=} T^{1/\alpha} \times \text{(IV)}, \end{aligned}$$

with $N^{\mu}_T(x) \equiv N^{\mu}(Tx)/T$ for x in $[0,1]$ and $\mathcal{Q}_t \stackrel{def}{=} h(\mathcal{Q})/t$, the rescaled partition.

We are done if we show that (IV) $= o(1)$ a.s. (log); the first part of the proof is like that of Proposition 4.2 of [FT11a]. By Lemma A this is equivalent to saying that for all $\varepsilon > 0$,

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \chi_{\{Z:\, d_1^0(\overline{Z}_{\mathcal{Q}_t} \circ N^{\mu}_t, \overline{Z}_{\mathcal{Q}_t}) < \varepsilon\}}(\Delta_t Z)\, \frac{\mathrm{d}t}{t} = 1\ a.s, \tag{3.2}$$

where χ_A denotes the indicator function of the set A.

Since $h(\cdot)$ is regularly varying, $h(x+1) \sim h(x)$ so the mesh $|\mathcal{Q}_t|$ of $\mathcal{Q}_t$ tends to 0 as $t \to \infty$.

Now since N^μ is positive and increasing and satisfies $N^\mu(t) \sim t$ a.s., the same holds for its (true) inverse S^μ. Hence both $N_t^\mu(\cdot)$ and $S_t^\mu(\cdot)$ converge uniformly to the identity on any compact interval of $\mathbb{R}^+$, as t goes to infinity.

For all $\delta' > 0$, let $\mathtt{C}^\uparrow_{\delta'}$ denote the set of continuous, increasing functions, vanishing at 0 and which are δ'-close to the identity uniformly on $[0,2]$. Now for any $\varepsilon > 0$, for any $\delta, \delta' > 0$ small enough, and for t large enough, the event $\{Z : d_1^0\left(\overline{Z}_{\mathcal{Q}_t} \circ N_t^\mu, \overline{Z}_{\mathcal{Q}_t}\right) < \varepsilon\}$ contains the set

$$G^\varepsilon_{\delta,\delta'} \equiv \{Z : \forall\, \mathscr{P} \text{ partition of } [0,2] \text{ with } |\mathscr{P}| < \delta, \text{and } \forall \eta \in \mathtt{C}^\uparrow_{\delta'} \text{ then } d_1^0(\overline{Z}_{\mathscr{P}} \circ \eta, \overline{Z}_{\mathscr{P}}) < \varepsilon\}.$$

By the ergodicity of the scaling flow, the log density of $G^\varepsilon_{\delta,\delta'}$ is $\nu(G^\varepsilon_{\delta,\delta'})$ with ν the law of the $\alpha-$stable process Z. So the log density in (3.2) is bounded from below by $\nu(G^\varepsilon_{\delta,\delta'})$.

Therefore all that is left to prove is that for fixed $\varepsilon > 0$, $\nu(G^\varepsilon_{\delta,\delta'})$ increases to 1 as both δ and δ' decrease to 0.

As in the proof of Lemma 4.1 of [FT11a], from the σ-continuity of ν together with the fact that the set $D_* \subset D$ of paths that are continuous at all positive integers has ν-full measure, we need only prove the following pointwise statement: for all $\varepsilon > 0$ (with $\varepsilon < 1$) and all $Z \in D_*$ there exist $\delta = \delta(\varepsilon, Z)$ and $\delta' = \delta'(\varepsilon, Z)$ such that for any partition $\mathscr{P}$ of mesh less than δ and any $\eta \in \mathtt{C}^\uparrow_{\delta'}$ we have $d_1^0(\overline{Z}_{\mathscr{P}} \circ \eta, \overline{Z}_{\mathscr{P}}) < \varepsilon$.

To this end we construct an increasing, continuous function λ of $[0,1]$ onto itself such that

$$||\lambda - \mathscr{I}||^\infty_{[0,1]} < \varepsilon \text{ and } ||\overline{Z}_{\mathscr{P}} \circ \lambda - \overline{Z}_{\mathscr{P}} \circ \eta||^\infty_{[0,1]} < \varepsilon. \tag{3.3}$$

Here we assume that $\eta(1) \neq 1$; if $\eta(1) = 1$, taking λ to be the restriction of η to $[0,1]$ delivers (3.3).

Now since Z has no jump at 1, there exists a positive $\delta_1 = \delta_1(\varepsilon, Z) < \varepsilon$ such that

$$\sup_{1-\delta_1 \le s,t \le 1+\delta_1} |Z(s) - Z(t)| < \varepsilon/3. \tag{3.4}$$

We define $n = n(\varepsilon, Z) \ge 1$ to be the integer part of ε/δ_1 (so $\varepsilon/(n+1) < \delta_1 \le \varepsilon/n$).

We proved in Lemma 4.1 of [FT11a] that for all $\varepsilon > 0$ and all Z continuous at 1 there exists $\delta_{00} = \delta_{00}(\varepsilon, Z) > 0$ such that for any partition $\mathscr{P}$ of $\mathbb{R}^+$ of mesh less than δ_{00}, $d_1(\overline{Z}_{\mathscr{P}}, Z) < \varepsilon$. Recalling Remark 2.1, the same holds a fortiori for d_1^0 and hence for d_2^0, since one can replace $[0,1]$ by any compact interval, for instance $[0,2]$.

It follows that there exists an increasing, continuous function $\overline{\lambda}$ of $[0,2]$ onto itself (depending on $\mathscr{P}$, ε and Z) such that:

$$||\overline{\lambda} - \mathscr{I}||^\infty_{[0,2]} < \varepsilon/(n+2) \text{ and } ||\overline{Z}_{\mathscr{P}} \circ \overline{\lambda} - Z||^\infty_{[0,2]} < \varepsilon/(n+2).$$

We set

$$\delta \stackrel{def}{=} \left(\delta_{00} \wedge \frac{(n+1)\delta_1 - \varepsilon}{3(n+2)}\right) \text{ and } \delta' \text{ is any positive number} < \delta,$$

writing $x \wedge y$ for $\inf(x,y)$ and $x \vee y$ for $\sup(x,y)$.

Let $\mathscr{P}$ be any partition of the positive reals, $\mathscr{P} = \{[x_i, x_{i+1})\}_{i\geq 0}$ with $x_0 = 0$, which has mesh less than δ, and let η be any element of $\mathtt{C}^{\uparrow}_{\delta'}$ satisfying $||\eta - \mathscr{I}||^{\infty}_{[0,2]} \vee ||\eta^{-1} - \mathscr{I}||^{\infty}_{[0,2]} < \delta'$.

Since $\delta' < \delta$, the interval $(1-2\delta, \eta(1) \wedge \eta^{-1}(1))$ contains at least one boundary point of $\mathscr{P}$; letting x_{p+1} be the largest of these, we then have $x_{p+1} < \eta(1) \wedge \eta^{-1}(1) < 1$.

We define the function λ to be equal to η on $[0, x_{p+1}]$ and linear on $[x_{p+1}, 1]$, so that $\eta(x_{p+1}) = \lambda(x_{p+1})$ and $\lambda(1) = 1$. Accordingly, $||\lambda - \mathscr{I}||^{\infty}_{[x_{p+1},1]} = |\eta(x_{p+1}) - x_{p+1}|$, so

$$||\lambda - \mathscr{I}||^{\infty}_{[0,1]} = ||\eta - \mathscr{I}||^{\infty}_{[0,x_{p+1}]} < \delta' < \varepsilon.$$

It follows that

$$\begin{aligned}
&||\overline{Z}_{\mathscr{P}} \circ \lambda - \overline{Z}_{\mathscr{P}} \circ \eta||^{\infty}_{[0,1]} = ||\overline{Z}_{\mathscr{P}} \circ \lambda - \overline{Z}_{\mathscr{P}} \circ \eta||^{\infty}_{[x_{p+1},1]} \\
\leq\; & ||\overline{Z}_{\mathscr{P}} \circ \lambda - Z \circ \overline{\lambda}^{-1} \circ \lambda||^{\infty}_{[x_{p+1},1]} + ||Z \circ \overline{\lambda}^{-1} \circ \lambda - Z \circ \overline{\lambda}^{-1} \circ \eta||^{\infty}_{[x_{p+1},1]} + ||Z \circ \overline{\lambda}^{-1} \circ \eta - \overline{Z}_{\mathscr{P}} \circ \eta||^{\infty}_{[x_{p+1},1]} \\
\leq\; & ||\overline{Z}_{\mathscr{P}} - Z \circ \overline{\lambda}^{-1}||^{\infty}_{[0,1]} + \sup_{\overline{\lambda}^{-1}(\eta(x_{p+1})) \leq u,v \leq \overline{\lambda}^{-1}(1 \vee \eta(1))} |Z(u) - Z(v)| + ||Z \circ \overline{\lambda}^{-1} - \overline{Z}_{\mathscr{P}}||^{\infty}_{[0,\eta(1)]} \\
\leq\; & 2\, ||\overline{Z}_{\mathscr{P}} \circ \overline{\lambda} - Z||^{\infty}_{[0,\overline{\lambda}^{-1}(1 \vee \eta(1))]} + \sup_{\overline{\lambda}^{-1}(\eta(x_{p+1})) \leq u,v \leq \overline{\lambda}^{-1}(1 \vee \eta(1))} |Z(u) - Z(v)|. \qquad (3.5)
\end{aligned}$$

Since Z is continuous at 1, from the construction of $\overline{\lambda}$ in [FT11a], one sees that $\overline{\lambda}(1) = \overline{\lambda}^{-1}(1) = 1$ and so $\overline{\lambda}^{-1}(1 \vee \eta(1)) = 1 \vee (\overline{\lambda}^{-1}(\eta(1)))$.
Thus,

$$\begin{aligned}
|1 - \overline{\lambda}^{-1}(1 \vee \eta(1))| &\leq |1 - \overline{\lambda}^{-1}(\eta(1))| \\
&\leq |1 - \eta(1)| + |\eta(1) - \overline{\lambda}^{-1}(\eta(1))| \\
&\leq ||\eta - \mathscr{I}||^{\infty}_{[0,1]} + ||\overline{\lambda} - \mathscr{I}||^{\infty}_{[0,2]} \\
&< \delta' + \frac{\varepsilon}{n+2} < \delta_1 \qquad (3.6)
\end{aligned}$$

where we have used $\eta(1) < 1 + \delta' < 2$ in deriving the third inequality above.
Furthermore,

$$\begin{aligned}
|1 - \overline{\lambda}^{-1}(\eta(x_{p+1}))| &\leq (1 - x_{p+1}) + |x_{p+1} - \eta(x_{p+1})| + |\eta(x_{p+1}) - \overline{\lambda}^{-1}(\eta(x_{p+1}))| \\
&< 2\delta + ||\eta - \mathscr{I}||^{\infty}_{[0,1]} + ||\overline{\lambda}^{-1} - \mathscr{I}||^{\infty}_{[0,1]} \\
&< 2\delta + \delta' + \frac{\varepsilon}{n+2} < \delta_1 \qquad (3.7)
\end{aligned}$$

from the definition of δ.

Now (3.6) and (3.7) imply that the second sup in (3.5) is less than the sup in (3.4) which is $< \varepsilon/3$. Therefore, putting all the previous results together, we arrive at:

$$||\overline{Z}_{\mathscr{P}} \circ \lambda - \overline{Z}_{\mathscr{P}} \circ \eta||^{\infty}_{[0,1]} < 2\,\frac{\varepsilon}{n+2} + \frac{\varepsilon}{3} < \varepsilon.$$

We have just shown that $d_1^0(\overline{Z}_{\mathscr{P}} \circ \eta, \overline{Z}_{\mathscr{P}}) < \varepsilon$, and hence using the observation right before (3.3), for all $\varepsilon > 0$, we have that $\nu(G^{\varepsilon}_{\delta,\delta'})$ increases to 1 as δ and δ' decrease to 0.

This proves (3.2) and hence (II), finishing the proof of Theorem 1.2. □

4 The Cauchy case: $\alpha = 1$

As for the previous two regimes, we build on the corresponding results for the random walk:

Theorem C: ([FT11a]) *Under the assumptions and notation of Theorem 1.3, there exists a joining of the step random walk* $\overline{S}$ *and the centered Cauchy process* $\check{Z}$ *(*$\check{Z}(t) \equiv Z(t) - t\log t$*) such that*

$$d_T(\,\overline{(S-\varrho)}_{\mathscr{Q}} \circ a^{-1}, \check{Z}\,) = o(T)\ a.s.\ (\log) \tag{4.1}$$

where $\varrho(t) = t\nu(a(t))$ *and* $\nu(t) = \int_0^t x\,dF(x)$ *is the truncated mean for* F *up to time* t*.*

We start with two lemmas.

Lemma 4.1. *Under the assumptions of Theorem 1.3, we have*

$$||S - \widetilde{\varrho}||^{\infty}_{[0,T]} = o(\widetilde{\varrho}(T))\ a.s.\ (\log), \tag{4.2}$$

where $\widetilde{\varrho}(\cdot)$ *is the function of regularly varying derivative* $\widetilde{\varrho}(t) = t\widetilde{\nu}(a(t))$*, with* $\widetilde{\nu}(t) = \int_0^t V(x)/x^2 dx$ *and with* $V(x) = \int_0^x s^2\,dF(s)$ *the truncated variance of* F *up to time* x*.*

We need a further result:

Lemma 4.2. *Let* g *be a nondecreasing function on the positive reals satisfying*

$$||g - \mathscr{I}||^{\infty}_{[0,T]} = o(T)\ (\log). \tag{4.3}$$

Then for any f *continuous, nondecreasing, and regularly varying of order* 1 *we have:*

$$||f \circ g - f||^{\infty}_{[0,T]} = o(f(T))\ (\log). \tag{4.4}$$

Remark 4.1. We recall from Proposition 2.5 of [FT11a] that if f is increasing, differentiable and with regularly varying derivative, then f preserves zero log density sets.

Proof of Lemma 4.1. We begin with the time change $a(\cdot)$ of [FT11a] defined using the distribution function of F. Now the centering function $\varrho(\cdot)$ of (4.1) does *not* necessarily have a regularly varying derivative; as a first step, then, we shall produce a better behaved centering, as follows.

A crucial step from [FT11a] in the proof of (4.1) was to first show that

$$\frac{S_n - n\nu(a(n))}{a(n)} - \frac{\check{Z}(a(n))}{a(n)} \xrightarrow{\mathbb{P}} 0, \tag{4.5}$$

where $\xrightarrow{\mathbb{P}}$ stands for convergence in probability and $\nu(\cdot)$ is the truncated mean for F.

Recalling that F has support on $\mathbb{R}^+$ (a subtle point is that we continue to work with the original F rather than with the smoothed distribution used to produce the normalizing function $a(\cdot)$), then after performing an integration by parts we have for $t > 0$:

$$\nu(t) \equiv \int_0^t x dF(x) = \frac{V(t)}{t} + \int_0^t \frac{V(x)}{x} \frac{dx}{x}.$$

We have used the fact that for all $t > 0$,

$$0 \le \frac{V(t)}{t} = t \int_0^t \frac{x^2}{t^2} \, dF(x) \le t,$$

so that $V(t)/t$ goes to 0 as $t \to 0$ and $V(t)/t^2$ is bounded hence integrable at 0.

Hence setting $\widehat{\nu}(t) \equiv V(t)/t$, we rewrite $\nu(t)$ as:

$$\nu(t) = \widehat{\nu}(t) + \int_0^t \frac{\widehat{\nu}(x)}{x} \, dx \stackrel{def}{=} \widehat{\nu}(t) + \widetilde{\nu}(t). \tag{4.6}$$

Now as F lies in the domain of attraction of a Cauchy law, $V(t) \sim tL(t)$ and $a(t) \sim tL(a(t))$ for L some slowly varying function.

As a result $\widehat{\nu}$ is slowly varying. From p. 26 of [BGT87], we see that $\widetilde{\nu}$ is slowly varying as well, and that $\widehat{\nu}(t) = o(\widetilde{\nu}(t))$; equation (4.6) shows that the truncated mean $\nu(\cdot)$ is a so-called de Haan function.

We note that F has a finite mean iff $\widetilde{\nu}$ converges to a finite constant as $t \to +\infty$, in other words iff $\widehat{\nu}(x)/x$ is integrable on $\mathbb{R}^+$. In particular, $\widehat{\nu}(t) = o(\widetilde{\nu}(t))$ means that $\widehat{\nu}$, hence equivalently L, goes to 0 as $t \to \infty$.

Now plugging the expression for $\nu(t)$ given by (4.6) into (4.5), while keeping in mind that $n\widehat{\nu}(a(n)) = nV(a(n))/a(n) \sim nL(a(n)) \sim a(n)$, then (4.5) can be rewritten, where $\widetilde{Z}(t) = \check{Z}(t) + t$, as:

$$\frac{S_n - n\widetilde{\nu}(a(n))}{a(n)} - \frac{\widetilde{Z}(a(n))}{a(n)} \xrightarrow{\mathbb{P}} 0. \tag{4.7}$$

Therefore upon replacing $\check{Z}$ by $\widetilde{Z}$, we can trade $\varrho(\cdot)$ for $\widetilde{\varrho}(t) \equiv t\widetilde{\nu}(a(t))$ in (4.1).

We now show that this new centering $\widetilde{\varrho}(\cdot)$ is increasing, differentiable, regularly varying of order 1 and has a slowly varying derivative.

Now from its definition, $\widetilde{v}(\cdot)$ is increasing and differentiable; this is also the case for $a(\cdot)$ thus also for $\widetilde{\varrho}(\cdot)$. On the other hand, since the derivative of $a(\cdot)$ is slowly varying it follows that

$$(\widetilde{v} \circ a)'(t) = \frac{\widehat{v}(a(t))}{a(t)}\, a'(t) \sim \frac{a'(t)}{t}$$

which is regularly varying of order -1. Thus $\widetilde{v} \circ a$ is slowly varying and hence $\widetilde{\varrho}(\cdot)$ is regularly varying of order 1 (with a slowly varying derivative), as claimed.

Next, we know from (4.7) that

$$\frac{S_n - \widetilde{\varrho}(n)}{a(n)} = \frac{\widetilde{\varrho}(n)}{a(n)} \left(\frac{S_n}{\widetilde{\varrho}(n)} - 1 \right) \xrightarrow{law} 1 + G_1.$$

Bearing in mind that $\widehat{v}(t) = o(\widetilde{v}(t))$, it follows that $\widetilde{\varrho}(n)/a(n) \sim \widetilde{v}(a(n))/\widehat{v}(a(n))$ goes to infinity as $n \to \infty$. As a result, as $n \to \infty$, $S_n/\widetilde{\varrho}(n)$ converges to 1 in probability. Equivalently,

$$\frac{1}{\widetilde{\varrho}(n)} \sum_{0 \le i \le n-1} \big(X_i - (\widetilde{\varrho}(i+1) - \widetilde{\varrho}(i))\big) \xrightarrow{\mathbb{P}} 0.$$

This is a sum of independent random variables normalized by a regularly varying sequence of order 1. Therefore Corollary 4 of [BD93] will imply that

$$\sup_{0 \le k \le n} |S_k - \widetilde{\varrho}(k)| = o(\widetilde{\varrho}(n))\; a.s.\; (\log) \tag{4.8}$$

provided we are able to show that for some $p > 0$

$$\mathbb{E}\left| \frac{S_n - \widetilde{\varrho}(n)}{\widetilde{\varrho}(n)} \right|^p = O(1). \tag{4.9}$$

But for all $p < \alpha = 1$, from (1.5) together with Theorem 6.1 of [dAG79], we have that

$$\mathbb{E}\left| \frac{S_n - \widetilde{\varrho}(n)}{a(n)} \right|^p \longrightarrow \int (1+x)^p dG_1(x) < \infty,$$

as $n \to \infty$; recall that $1 - G_1(x) \sim x^{-1}$. This together with the fact that $a(n)/\widetilde{\varrho}(n) \to 0$ yields (4.9) which guarantees (4.8).

As we shall see, proving the desired (4.2) follows from (4.8) together with the fact that $\widetilde{\varrho}$ is regularly varying.

We start with

$$||S - \widetilde{\varrho}||^{\infty}_{[0,T]} \le \sup_{0 \le n \le [T]} \Big(\sup_{n \le t < n+1} |S(t) - \widetilde{\varrho}(t)| \Big).$$

Since S is the polygonal extension of S_n, it is (a.s.) increasing on each $n \le t < n+1$, so

$$|S(t) - \widetilde{\varrho}(t)| \le |S_n - \widetilde{\varrho}(n)| + |S_{n+1} - \widetilde{\varrho}(n+1)| + \big(\widetilde{\varrho}(n+1) - \widetilde{\varrho}(n)\big),$$

as $\widetilde{\varrho}(\cdot)$ is increasing. It follows that

$$||S-\widetilde{\varrho}||^{\infty}_{[0,T]} \leq 2\left(\sup_{0\leq n\leq [T]+1} |S_n - \widetilde{\varrho}(n)| + \sup_{n\leq [T]} (\widetilde{\varrho}(n+1)-\widetilde{\varrho}(n))\right). \tag{4.10}$$

Now let $\mathscr{B}$ be the integer set of zero log density from (4.8) and define $\widetilde{\mathscr{B}}$ to be the set of t such that $[t]+1$ belongs to $\mathscr{B}$. Then $\widetilde{\mathscr{B}}$ is a set of zero log density in the reals. Accordingly, the first supremum of (4.10) is $o(\widetilde{\varrho}(T))$ for $T \notin \widetilde{\mathscr{B}}$. On the other hand, since the sequence $(\widetilde{\varrho}(n))$ is regularly varying we have $\widetilde{\varrho}(n+1) \sim \widetilde{\varrho}(n)$. This is equivalent to $\sup_{n\leq[T]}(\widetilde{\varrho}(n+1)-\widetilde{\varrho}(n)) = o(\widetilde{\varrho}(T))$. Putting these facts together yields (4.2). □

We now move to:

Proof of Lemma 4.2. The proof is akin to that of (2.8). It is equivalent to show:

$$\left|\left|\frac{f(T\Delta_T g)}{f(T)} - \frac{f(T\cdot)}{f(T)}\right|\right|^{\infty}_{[0,1]} \longrightarrow 0 \text{ (log)}, \text{ as } T\to\infty \tag{4.11}$$

where Δ_T is the scaling transformation of order 1.

By the triangle inequality, the left-hand side of (4.11) is

$$\leq \left|\left|\frac{f(T\Delta_T g)}{f(T)} - \Delta_T g\right|\right|^{\infty}_{[0,1]} + ||\Delta_T g - \mathscr{I}||^{\infty}_{[0,1]} + \left|\left|\frac{f(T\cdot)}{f(T)} - \mathscr{I}\right|\right|^{\infty}_{[0,1]}. \tag{4.12}$$

By definition of Δ_T, (4.3) is equivalent to $||\Delta_T g - \mathscr{I}||^{\infty}_{[0,1]} \to 0$ (log). Now since f is regularly varying of order 1, just as in the proof of (2.8) the last term of (4.12) approaches 0 as $T\to\infty$.

Since g is nondecreasing, the first term of (4.12) is bounded from above by

$$\mathscr{U}_T \equiv ||f(T\cdot)/f(T) - \mathscr{I}||^{\infty}_{[0,\Delta_T g(1)]}.$$

Now for all $\delta > 0$ and all $\delta' > 0$, we have:

$$\{t : \mathscr{U}_t > \delta\} \subseteq \{t : ||f(t\cdot)/f(t) - \mathscr{I}||^{\infty}_{[0,1+\delta']} > \delta\} \cup \{t : ||\Delta_t g - \mathscr{I}||^{\infty}_{[0,1]} > \delta'\}.$$

And since the log density of these last two sets is zero, by Lemma A we have proved that the first term in (4.12) goes to 0 (log). This finishes the proof of (4.11) and hence that of (4.4). □

Proof of Theorem 1.3. We start off just as in the proof of Theorem 1.2. By definition of $\widetilde{N}$, we have:

$$d^0_T(\widetilde{N}\circ\widetilde{\varrho}\circ a^{-1}, \widetilde{Z}) = d^0_T(\overline{(S-\widetilde{\varrho})_{\mathscr{Q}}}\circ N\circ\widetilde{\varrho}\circ a^{-1}, \widetilde{Z})$$

which is less than or equal to the following sum:

$$||\overline{(S-\widetilde{\varrho})_{\mathscr{Q}}}\circ N\circ\widetilde{\varrho}\circ a^{-1} - \overline{\widetilde{Z}}_{a(\mathscr{Q})}\circ a\circ N\circ\widetilde{\varrho}\circ a^{-1}||^{\infty}_{[0,T]} + d^0_T(\overline{\widetilde{Z}}_{a(\mathscr{Q})}\circ a\circ N\circ\widetilde{\varrho}\circ a^{-1}, \overline{\widetilde{Z}}_{a(\mathscr{Q})}) + d^0_T(\overline{\widetilde{Z}}_{a(\mathscr{Q})}, \widetilde{Z})$$

$$\stackrel{def}{=} (\text{I}) + (\text{II}) + (\text{III}).$$

That (III) $= o(T)$ a.s. (log) follows from Lemma B. Indeed, since $\widetilde{Z}$ is 1-self-similar while the corresponding scaling flow (of order 1) is ergodic, we have that $d_1(\Delta_T(\overline{\widetilde{Z}}_{a(\mathscr{Q})}), \Delta_T(\widetilde{Z})) = o(1)$ a.s. (log) and a fortiori for d_1^0 from Remark 2.1. This together with (1.8) for d_T^0 yields (III) $= o(T)$, as claimed.

Proving that both (I) and (II) are $o(T)$ (log) requires a crucial step we shall show next, that:

$$||a \circ N \circ \widetilde{\varrho} \circ a^{-1} - \mathscr{I}||_{[0,T]}^{\infty} = o(T)\, a.s.\, (\text{log}). \tag{4.13}$$

We showed above that $\widetilde{\varrho}$ is of regularly varying derivative and so (recalling Remark 4.1) it preserves sets of log density zero.

Accordingly, since $\widetilde{\varrho}$ is increasing, a change of variables in (4.2) leads to:

$$||S \circ \widetilde{\varrho}^{-1} - \mathscr{I}||_{[0,T]}^{\infty} = o(T)\, a.s.\ (\text{log}).$$

Applying Lemma 4.2 with $f \equiv a \circ \widetilde{\varrho}^{-1}$ continuous, increasing, regularly varying of order 1 and with $g \equiv S \circ \widetilde{\varrho}^{-1}$ increasing gives

$$||a \circ \widetilde{\varrho}^{-1} S \circ \widetilde{\varrho}^{-1} - a \circ \widetilde{\varrho}^{-1}||_{[0,T]}^{\infty} = o(a \circ \widetilde{\varrho}^{-1}(T))\, a.s.\, (\text{log}).$$

Once again, since $\widetilde{\varrho}$ is regularly varying of order 1 with regularly varying derivative, then so is its inverse and therefore so is $a \circ \widetilde{\varrho}^{-1}$; as a result $a \circ \widetilde{\varrho}^{-1}$ preserves sets of zero log density. So we can change variables to get:

$$||a \circ \widetilde{\varrho}^{-1} \circ S \circ a^{-1} - \mathscr{I}||_{[0,T]}^{\infty} = o(T)\, a.s.\, (\text{log}).$$

Setting for simplicity $f \equiv a \circ \widetilde{\varrho}^{-1} \circ S \circ a^{-1}$, this is equivalent to

$$||\Delta_T f - \mathscr{I}||_{[0,1]}^{\infty} = o(1)\, a.s.\, (\text{log});$$

moreover, we can replace the interval $[0,1]$ here by the interval $[0,a]$ for any $a > 0$.

We shall prove that

$$||(\Delta_T f)^{-1} - \mathscr{I}||_{[0,1]}^{\infty} = o(1)\, a.s.\, (\text{log}),$$

which is exactly (4.13), since $(\Delta_T f)^{-1} = \Delta_T f^{-1}$.

To this end, all we need is the increasingness of f. Indeed, $||(\Delta_T f)^{-1} - \mathscr{I}||_{[0,1]}^{\infty} = ||\Delta_T f - \mathscr{I}||_{[0,(\Delta_T f)^{-1}(1)]}^{\infty}$ and for any $\varepsilon > 0$ and $\delta > 0$, we have:

$$\begin{aligned}\{t : ||(\Delta_t f)^{-1} - \mathscr{I}||_{[0,1]}^{\infty} > \varepsilon\} &\subseteq \{t : ||(\Delta_t f) - \mathscr{I}||_{[0,1+\delta]}^{\infty} > \varepsilon\} \cup \{t : (\Delta_t f)^{-1}(1) > 1 + \delta\} \\ &\subseteq \{t : ||(\Delta_t f) - \mathscr{I}||_{[0,1+\delta]}^{\infty} > \varepsilon\} \cup \{t : ||(\Delta_t f) - \mathscr{I}||_{[0,1+\delta]}^{\infty} > 1 + \delta\}\end{aligned}$$

each of which has zero log density by Lemma A. This finishes the proof of (4.13).

Turning to (I), after a change of variables we have:

$$(\text{I}) = ||\overline{(S-\widetilde{\varrho})}_{\mathcal{Q}} \circ a^{-1} - \overline{\widetilde{Z}}_{a(\mathcal{Q})}||^{\infty}_{[0,\, a\circ N\circ\widetilde{\varrho}\circ a^{-1}(T)]} \stackrel{def}{=} \mathscr{V}(a\circ N\circ\widetilde{\varrho}\circ a^{-1}(T)).$$

From (4.7), exactly as we proved (4.8), one can derive that $\sup_{0\le k\le n}|S_k - \widetilde{\varrho}(k) - \widetilde{Z}(a(k))| = o(a(n))$ a.s. (log) and hence that $\mathscr{V}(T) = o(T)$ $a.s.$ (log); see Proposition 5.2 of [FT11a] for this last step.

We prove that (I) $= o(T)$ $a.s.$(log) following the above reasoning. For all $\varepsilon, \varepsilon^{'} > 0$, the set of times $\{t : \mathscr{V}(a\circ N\circ\widetilde{\varrho}\circ a^{-1}(t)) > \varepsilon t\}$ is contained in the disjoint union

$$\{t : \mathscr{V}(t(1+\varepsilon')) > \varepsilon t\} \cup \{t : ||a\circ N\circ\widetilde{\varrho}\circ a^{-1} - \mathscr{I}||^{\infty}_{[0,t]} > \varepsilon^{'} t\},$$

each of which has zero log density, which delivers (I) $= o(T)$ $a.s.$(log).

Lastly, proving (II) $= o(T)$ $a.s.$(log) goes the same way as for the proof that (II) $= o(T)$ $a.s.$(log) in Theorem 1.2. Indeed, we write:

$$T^{-1} \times (\text{II}) = d_1^0(\Delta_T(\,\overline{\widetilde{Z}}_{a(\mathcal{Q})}) \circ \eta_T\ ,\ \Delta_T(\,\overline{\widetilde{Z}}_{a(\mathcal{Q})})\,)$$

where $\eta_T \equiv \Delta_T(a\circ N\circ\widetilde{\varrho}\circ a^{-1})$. From (4.13), for any $b > 0$, $||\eta_T - \mathscr{I}||^{\infty}_{[0,b]} \to 0$ (log) as $T\to\infty$. This reduces to proving (3.2) with $\widetilde{Z}$ replacing Z and η_t replacing N_t^{μ}. We split the log density into two pieces depending on whether $||\eta_T - \mathscr{I}||^{\infty}_{[0,1]} \le \delta'$ or $> \delta'$. Since the log density of $\{t : ||\eta_t - \mathscr{I}||^{\infty}_{[0,2]} > \delta'\}$ is zero, all that is left to treat is the part where η_t is δ'-close to the identity, uniformly on $[0,2]$ and we can borrow that argument from the proof of Theorem 1.2.

This completes the proof of Theorem 1.3. □

5 The finite variance Gaussian case

Upon dividing the X_i's by σ, one can assume that $\sigma \equiv 1$; we give the proof for this simpler case.

Proving (i). This is Horvath's *asip*, Theorem 2.1 of [Hor84]. From [KMT76], and Major [Maj76b], see also [CR81] pp. 107 and 108, there exists a joining of S and a normalized Brownian motion B such that for almost every pair (S, B),

$$||(S-\mu\mathscr{I}) - B||^{\infty}_{[0,T]} = o(T^{1/r}). \tag{5.1}$$

Note that in the statement of (i) we can replace B by $-B$ since these processes have the same law. Remembering that S is the inverse of N, performing a change of variables and using the triangle inequality, we have

$$\begin{aligned} ||(\mu N - \mathscr{I})(\mu\cdot) + B||^{\infty}_{[0,T]} &= ||(S-\mu\mathscr{I})\circ N(\mu\cdot) - B||^{\infty}_{[0,T]} \\ &\le ||(S-\mu\mathscr{I}) - B||^{\infty}_{[0,N(\mu T)]} + ||B - B(S(\cdot)/\mu)||^{\infty}_{[0,N(\mu T)]}. \end{aligned} \tag{5.2}$$

From the law of large numbers, $N(\mu t)/t$ goes a.s. to 1. So as in (3.1) one can replace $[0, N(\mu T)]$ in the inequality above by $[0, 2T]$. Consequently, (5.1) implies that the first term of (5.2) a.s. is $o(T^{1/r})$.

As for the second term, we claim that for any $\gamma > 1/4$, we have

$$||B - B(S(\cdot)/\mu)||^{\infty}_{[0,T]} = o(T^{\gamma})\ a.s.$$

For this, from the law of the iterated logarithm for $\overline{S}$ (see for instance [Dur91] p 394), we have for $S(\cdot)$ that

$$\limsup_{t\to\infty} \frac{S(t) - \mu t}{\sqrt{2t \log\log t}} = 1,\ a.s.$$

So almost surely, for all $\varepsilon > 0$ and t large enough, $|S(t) - \mu t| \le (\varepsilon \mu \sqrt{t}) \log\log t$. Accordingly, for T_0 large enough and all $\delta > 0$, we have

$$\begin{aligned}
\mathbb{P}(||B - B(S(\cdot)/\mu)||^{\infty}_{[T_0,T]} > \delta T^{\gamma}) &\le \mathbb{P}\left(\sup_{T_0 \le s \le T} \sup_{|u| \le \varepsilon\sqrt{s} \log\log s} |B(s) - B(s+u)| > \delta T^{\gamma}\right) \\
&\le \mathbb{P}\left(\sup_{s,t \le 2T;\, |t-s| \le \varepsilon \sqrt{T} \log\log T} |B(s) - B(t)| > \delta T^{\gamma}\right) \\
&= \mathbb{P}\left(\sup_{0 \le s \le t \le 1,\, |t-s| \le (\varepsilon \log\log T)/2\sqrt{T}} |B(s) - B(t)| > 2^{-1/2}\delta\ T^{\gamma - 1/2}\right) \\
&\le c\, \frac{\sqrt{T}}{\log\log T} \exp\left(-c' \frac{T^{2\gamma - 1/2}}{\log\log T}\right),
\end{aligned}$$

for some constants c and c', where we have used a Brownian scaling in deriving the equality above followed by an estimate for the Brownian oscillation, see for instance [CR81], p. 24. Hence, since $\gamma > 1/4$, our claim now follows by the Borel-Cantelli Lemma.

For $2 < r < 4$, we take $\gamma = 1/r > 1/4$, in which case the order of the sum (5.2) is that of the first term, namely $o(T^{1/r})$; for $r \ge 4$ the order of the sum is $o(T^{\gamma})$, with $\gamma > 1/4$. (See the second part of Theorem 2.1 (c) of [Hor84] for a proof that this last result is in fact sharp.)

Lastly, the proof that the path $(\mu N - \mathscr{I})(\frac{\mu}{\sigma^2}\cdot)$ belongs to $W^{s,d^u_\infty}_{Ces}(B)$ follows from the corresponding statement for d^u_1 via the Lebesgue Dominated Convergence Theorem; see Lemma 3.6 of [FT11a].

Proving (ii): We know from a counterexample of Breiman [Bre67], see [CR81] p. 93, that there exists a distribution F with mean μ and with variance one such that for any Brownian motion B

$$\limsup_{n\to\infty} \frac{|S_n - \mu n - B(n)|}{\sqrt{n}} = +\infty\ \text{a.s.};$$

this a fortiori holds for continuous time t as well.

Accordingly, as above replacing $-B$ by $+B$, since

$$(\mu N - \mathscr{I})(\mu t) + B(t) = -\big((S - \mu\mathscr{I})N(\mu t) - B(N(\mu t))\big) + (B(N(\mu t)) - B(t)),$$

then the lim sup of the absolute value of the first term on the right-hand side, divided by $\sqrt{t}$, is $+\infty$ (here we again use the fact that $N(\mu t)/t \to 1$). And we proved in (i) that $||B(N(\mu\cdot)) - B||^{\infty}_{[0,T]} =$

$o(T^\gamma)$ a.s. for any $\gamma > 1/4$, in particular, for $\gamma = 1/2$. Thus we are done by dividing the identity above by $\sqrt{t}$, then taking the lim sup.

Hence no *asip* of order $\sqrt{T}$ is possible for F with finite variance and higher moments infinite. However in that case, we proved in [Fis] that there exists a joining of S and a standard Bownian motion B such that for almost every (S,B),

$$||(S - \mu\mathscr{I}) - B||^{\infty}_{[0,T]} = o(\sqrt{T})\,(\log) \tag{5.3}$$

In this light, running the proof of (i) we arrive at (5.2); the second term is still $o(\sqrt{T})$ whereas the first one is only $o(\sqrt{T})$ (log), a.s.

Thus we are done so long as we clarify one point, which we previously encountered while studying (I) in the case where $\alpha > 1$: we can replace $[0, N(\mu T)]$ by $[0, T]$ in (5.3) without altering the *asip* (log) since by the law of large numbers $N(\mu T) \leq (1+\varepsilon)T$ for any $\varepsilon > 0$ (T large), while multiplying by a constant does not alter the log density of a set.

This completes the proof of (1.22) and so of Theorem 1.4.

6 Proving Corollary 1.1

We only give a sketch here as the proof follows similar reasoning as for Proposition 1.3 in §6 of [FT11a].

Proof of (i). Let $\mathscr{C}$ denote the space of continuous functions defined on $\mathbb{R}^+$. For $\alpha < 1$, the law $\widehat{\nu}$ of the Mittag-Leffler process $\widehat{Z}$ is a $\widehat{\tau}_t$-invariant ergodic probability measure on $\mathscr{C}$. By Fomin's theorem, we know that for $\widehat{\nu}$-a.e. path $\widehat{Z}$ we have for any $\varphi \in CB(\mathscr{C}, d^u_\infty)$:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(\widehat{\tau}_t \widehat{Z})\,\mathrm{d}t = \int \varphi\,\mathrm{d}\widehat{\nu}. \tag{6.1}$$

This says exactly that the occupation measures $\frac{1}{T}\int_0^T \delta_{\widehat{\tau}_t\widehat{Z}}\mathrm{d}t$ converge weakly to $\widehat{\nu}$. Note that by p. 12 of [Bil68], it is equivalent to state (6.1) for $\varphi \in UCB(\mathscr{C}, d^u_\infty)$, the *uniformly* continuous and bounded functions.

From (1.11) written for d^u_∞, there exists a joining $\widehat{\nu}$ of N and $\widehat{Z}$ such that for a.e. pair $(N, \widehat{Z})$, the paths $a^\alpha \circ N$ and $\widehat{Z}$ are d^u_∞-forward asymptotic for the flow $\widehat{\tau}_t$; that is, $a^\alpha \circ N$ is in the Cesáro stable manifold $W^{s,d^u_\infty}_{Ces}(\widehat{Z})$. Accordingly, for any $\varphi \in UCB(\mathscr{C}, d^u_\infty)$, (6.1) also holds with $a^\alpha \circ N$ in the place of $\widehat{Z}$; this then passes back to $CB(\mathscr{C}, d^u_\infty)$, telling us that $a^\alpha \circ N$ is a $\widehat{\tau}_t$-generic point for $\widehat{\nu}$.

From (1.13), in the same fashion, (6.1) passes over to $\overline{N}(e^t\cdot)/a^{-1}(e^t)$ and the change of variables $s = e^t$ finishes the proof of (i).

Proof of (ii). We begin as for part (i), with d_∞ replacing d^u_∞. We set $h = a^\alpha$ to simplify the notation. From Theorem 1.2 we have that for any $\varphi \in CB(D, d^0_\infty) = CB(D, d_\infty)$,

$$\lim_{T\to\infty} \frac{1}{\log T}\int_1^T \varphi\left(\frac{\breve{N}(\mu h^{-1}(s\cdot))}{s^{1/\alpha}}\right)\frac{ds}{s} = \int \varphi\,\mathrm{d}\nu, \tag{6.2}$$

which is not quite the equation that we want, (1.24). Making the change of variables $s = h(t/\mu)$, we have $dt/t = (h^{-1})'(s)/h^{-1}(s)ds \sim ds/s$ by Karamata's Theorem, so (1.24) is equivalent to:

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \varphi\left(\frac{\check{N}(\mu h^{-1}(s)\cdot)}{s^{1/\alpha}}\right) \frac{ds}{s} = \int \varphi \, d\nu. \tag{6.3}$$

Thus we want to compare (6.2) and (6.3). Now in the process of proving Proposition 1.3 (ii) of [FT11a], we encountered a similar situation; we use the same argument, borrowing notation from that proof.

First, for any $f, g \in D$, and λ defined on $\mathbb{R}^+$, continuous, increasing and onto, we define

$$t_\lambda = t_{\lambda,f,g} = \sup\{t \geq 0 : ||f - g \circ \lambda||_{[0,t]}^{\infty} \leq \frac{1}{t} \text{ and } ||\lambda(x) - x||_{[0,t]}^{\infty} \leq \frac{1}{t}\} \tag{6.4}$$

and then write

$$\rho(f, g) = \min\{\frac{1}{2}; (\sup_{\lambda\in\Lambda_\infty} t_\lambda)^{-1}\}.$$

Lastly we set

$$\widetilde{d}_\infty^0(f, g) = \rho(f, g) + \rho(g, f).$$

As we showed in [FT11a], $\widetilde{d}_\infty^0$ defines a metric on D which gives the same topology as does d_∞.

Now setting

$$\lambda_s(x) \equiv \frac{h^{-1}(sx)}{h^{-1}(s)},$$

this gives a continuous, increasing and onto function on $\mathbb{R}^+$. Since $h(\cdot)$ and hence $h^{-1}(\cdot)$ are regularly varying of order 1 and increasing, for all $A > 0$ we have that $||\lambda_s(x) - x||_{[0,A]}^{\infty} \to 0$ as $s \to \infty$. On the other hand, defining

$$f_s(x) = \frac{\check{N}(\mu h^{-1}(s)x)}{s^{1/\alpha}}, \text{ and } g_s(x) = \frac{\check{N}(\mu h^{-1}(sx))}{s^{1/\alpha}},$$

we have, for any $s > 0$:

$$f_s \circ \lambda_s - g_s \equiv 0.$$

It follows that $t_{\lambda_s,f_s,g_s} \to \infty$ as $s \to \infty$, which tells us that $\rho(f_s, g_s)$, $\rho(g_s, f_s)$ and hence $\widetilde{d}_\infty^0(f_s, g_s)$ go to 0 as $s \to \infty$.

Accordingly, for any $\varphi \in UCB(D, \widetilde{d}_\infty^0)$ and hence again by p. 12 of [Bil68] for any $\varphi \in CB(D, \widetilde{d}_\infty^0) = CB(D, d_\infty)$ as well, the log average of $\varphi(f_s)$ is the same as that of $\varphi(g_s)$. But these are exactly the log averages appearing in (6.3) and (6.2) respectively, finishing the proof of (1.24) and hence of (ii).

The proofs of (iii) and (iv), which we omit, follow the same pattern.

Finally, deriving pathwise CLTs from the pathwise functional CLTs for the cases where $\alpha < 1$ and $\alpha = 2$ is immediate, as the projection in $\mathscr{C}$ to the time-one coordinate is continuous.

For the remaining cases, the proof follows the pattern we used in [FT11a], by convolving along the flow. This completes the proof of the corollary.

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