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Stochastic nonlinear wave equations in local Sobolev spaces

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Abstract

Existence of weak solutions of stochastic wave equations with nonlinearities of a critical growth driven by spatially homogeneous Wiener processes is established in local Sobolev spaces and local energy estimates for these solutions are proved. A new method to construct weak solutions is employed.

Key words: stochastic wave equation.

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1 Introduction

Nonlinear wave equations

$$u_{tt} = \mathscr{A}u + f(x,u,u_t,\nabla_x u) + g(x,u,u_t,\nabla_x u)\dot{W} \tag{1.1}$$

subject to random excitations have been thoroughly studied recently under various sets of hypotheses (see e.g. [6], [7], [8], [10], [12], [13], [14], [22], [23], [25], [26], [27], [28], [29], [30], [32], [35], [36], [37] and references therein) with possible applications in physics (e.g. in relativistic quantum mechanics or oceanography) in view.

The random perturbation has been usually modelled by a spatially homogeneous Wiener process which corresponds to a centered Gaussian random field $(W(t,x) : t \geq 0,\, x \in \mathbb{R}^d)$ satisfying

$$\mathbb{E}\, W(t,x)W(s,y) = (t \wedge s)\Gamma(x-y), \qquad t,s \geq 0, \quad x,y \in \mathbb{R}^d$$

for some function or even a distribution Γ called *the spatial correlation* of W (see e.g. [37] for details). The operator $\mathscr{A}$ in (1.1) is a second order elliptic differential operator, usually the Laplacian. (More general elliptic operators are considered only in [29]).

Functions f and g dependent only on u are dealt predominantly and their global Lipschitz continuity is assumed in most of the papers cited above. Then the Nemytskii operators associated with f and g are also globally Lipschitzian and existence (and uniqueness) of solutions to (1.1) may be proved for rather general spatial correlations Γ (which may be a distribution, e.g. the standard cylindrical Wiener process is allowed if the space dimension is one, or at least a continuous function unbounded at the origin), the state space (to which the pair (u,u_t) belongs) being $L^2(\mathbb{R}^d) \oplus W^{-1,2}(\mathbb{R}^d)$. If Γ is more regular then solutions live in the so called "energy space" $W^{1,2}(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ (see e.g. [38] for a discussion of the role of the energy space in the deterministic case).

Locally Lipschitz (or even continuous) real functions f and g are considered in the papers [10], [27], [29], [30], [31] and [32].

Various techniques including Lyapunov functions, energy estimates, Sobolev embeddings, Strichartz inequalities or compactness methods - have been employed to show existence of global solutions in this case. These methods require the state space to be the energy space $W^{1,2}(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ and the spatial correlation Γ to be a bounded function, i.e. the *spectral measure* $\mu = (2\pi)^{\frac{d}{2}}\widehat{\Gamma}$ is a finite measure (cf. equality (3.1) below). The assumptions on Γ are relaxed in [27] in the case of the planar domain at a price of assuming g bounded and globally Lipschitz while $f(u) = -u|u|^{p-1}$, $1 \leq p \leq 3$.

Let us survey the available results in the most important case of a wave equation with polynomial nonlinearities

$$u_{tt} = \Delta u - u|u|^{p-1} + |u|^q \dot{W}, \qquad u(0) = u_0, \qquad u_t(0) = v_0 \tag{1.2}$$

according to particular ranges of the exponents $p,q \in (0,\infty)$: It is known that global weak solutions (weak both in the probabilistic and in the PDE sense) exist provided that (u_0,v_0) is an $\mathscr{F}_0$-measurable $[W^{1,2}(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)] \oplus L^2(\mathbb{R}^d)$-valued random variable, W is a spatially homogeneous Wiener process with bounded spectral correlation Γ (i.e. $\mu = (2\pi)^{\frac{d}{2}}\widehat{\Gamma}$ must be a finite measure) and

$$1 \leq q < \frac{p+1}{2} < \infty. \tag{1.3}$$

Under (1.3), paths of (u,u_t) take values in $[W^{1,2}(\mathbb{R}^d)\cap L^{p+1}(\mathbb{R}^d)]\oplus L^2(\mathbb{R}^d)$ and are weakly continuous in $W^{1,2}(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$ (see [29]). In the critical case $q=\frac{p+1}{2}$, existence of solutions was shown if $d\in\{1,2\}$ or $d\geq 3$ and $p\leq \frac{d}{d-2}$ (see [30]). Pathwise uniqueness and pathwise norm continuity of solutions in $W^{1,2}(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$ are known to hold if $d\in\{1,2\}$ or if $d\geq 3$ and $p\leq\frac{d+2}{d-2}$, $q\leq\frac{d+1}{d-2}$, irrespective of (1.3). These results were proved in [29], [30], [31] and [32] in a more general setting (that is, for more general non-linearities). In case $\frac{d}{d-2}<p\leq\frac{d+2}{d-2}$ or $\frac{d}{d-2}<q\leq\frac{d+1}{d-2}$ some small additional assumptions are needed. In the subcritical case $p<\frac{d+2}{d-2}$, $q<\frac{d+1}{d-2}$, these results correspond to the present state of art for the deterministic Cauchy problem

$$u_{tt}=\Delta u-u|u|^{p-1},\qquad u(0)=u_0,\quad u_t(0)=v_0 \tag{1.4}$$

on $\mathbb{R}^d$ (see [19], [41], [43] and [44]) exactly, whereas there are still some open problems in the (stochastic) critical case $p=\frac{d+2}{d-2}$, $q=\frac{d+1}{d-2}$ (see the discussion in [31]).

The aim of the present paper is fivefold. We want to prove

1. existence of weak solutions up to the critical case $q=\frac{p+1}{2}$ independently of the dimension of the spatial domain $\mathbb{R}^d$,
2. global weak solutions exist for data in the **local** Sobolev space $W^{1,2}_{loc}(\mathbb{R}^d)\oplus L^2_{loc}(\mathbb{R}^d)$ and have trajectories weakly continuous in this local space,
3. solutions in the local Sobolev space satisfy a local energy inequality (Theorem 5.2),
4. include dependence on first derivatives in the non-linearities in the equation (1.1)
5. study systems of stochastic wave equations, i.e. when f and g are $\mathbb{R}^n$-valued.

Let us briefly comment on the issues.

Ad (1): As mentioned above, existence of weak solutions in the critical case $q=\frac{p+1}{2}$ is known to hold only in particular cases depending on the dimension d. We will prove that no additional assumption is, in fact, necessary.

Ad (2): To our knowledge, stochastic equations with polynomially growing non-linearities have not been studied in local spaces yet despite it is well known that solutions of wave equations propagate at finite speed and the commonly used restriction to global spaces is therefore unimportant. *Nota bene*, existence of solutions in global spaces follows trivially from existence of solutions in local spaces by the energy estimate in Theorem 5.2, as demonstrated e.g. in Example 5.5.

As a consequence of the "local" approach to the wave equation (1.1), the second order differential operator $\mathscr{A}$ in (1.1) need not be *uniformly* elliptic (as is usually assumed) and mere ellipticity of $\mathscr{A}$ is sufficient (see (2.1)). In particular, $\mathscr{A}$ may even decay or explode at infinity, cf. Example 5.4 and 5.5.

The localization of the wave problem is interesting by itself, though it is not very difficult to establish. The main importance of the local approach to the wave equation dwells in our primary interest to prove the subtle existence result in the critical case. We remark at this point that attempts to prove existence of solutions of (1.1) in the critical case while studying the wave equation in global spaces failed (see [29]).

Ad (3): Energy inequalities are a sort of a twin result to any existence theorem in the theory of wave equations as the solutions of the wave equation are, in fact, stationary points of certain Lagrangians and the energy functionals represent their conservation laws (see e.g. Chapter 2 in [42]). On the other hand, energy inequalities also describe basic behaviour of the solution such as the finite speed of propagation mentioned above, long time behaviour of the paths or the conditional dependence on the initial condition (see Theorem 5.2).

Ad (4) & (5): We are not aware of existence results for stochastic wave equations with non-linearities depending on first derivatives of the solution (the velocity and the spatial gradient). This issue is closely related to the fact that we aim at studying systems of stochastic wave equations (1.1). Such generality is not very substantial for the present paper, however, the corresponding results are essential in the newly started research in the field of stochastic wave equations in Riemannian manifolds with possible applications in physical theories and models such as harmonic gauges in general relativity, non-linear σ-models in particle systems, electro-vacuum Einstein equations or Yang-Mills field theory. These models require the target space of the solutions to be a Riemannian manifold (see [18], [42] for deterministic systems and [3], [4] for stochastic ones). For instance, if the unit sphere $\mathbb{S}^{n-1}$ is the considered Riemannian manifold, the stochastic geometric wave equation has the form

$$u_{tt} = \Delta u + (|\nabla_x u|^2 - |u_t|^2)u + g(u, u_t, \nabla_x u)\,\dot{W}, \qquad |u|_{\mathbb{R}^n} = 1$$

where $g(p,\cdot,\cdot) \in T_p\mathbb{S}^{n-1}$, $p \in \mathbb{S}^{n-1}$, see e.g. [4]. We do not cover these particular equations here but the present paper is partly intended as a preparation for further applications and as a citation/reference paper for a companion paper on stochastic wave equations in compact Riemannian homogeneous space by Z. Brzeźniak and the author.

Finally, we remark that our proof of the main theorem is based on a new general method of constructing weak solutions of SPDEs, that does not rely on any kind of martingale representation theorem and that might be of interest itself. First applications were done already in [4] and, in the finite-dimensional case, also in [20].

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2 Notation and Conventions

We consider complete filtrations in this paper. We say that a filtration $(\mathscr{F}_t)$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is complete provided that $\mathscr{F}_0$ contains all $\mathbb{P}$-negligible sets of $\mathscr{F}$. We denote by

- $\mathbb{R}_+$ the set of all nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, \infty)$,
- $\mathscr{B}(X)$ the Borel σ-algebra on a topological space X,
- B_R the open ball in $\mathbb{R}^d$ with center at the origin and of radius R,
- $L^p = L^p(\mathbb{R}^d, \mathbb{R}^n)$, $W^{k,p} = W^{k,p}(\mathbb{R}^d, \mathbb{R}^n)$,
- $L^p_{\mathrm{loc}} = L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ and $W^{k,p}_{\mathrm{loc}} = W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^n)$ equipped with the metrics

$$(u,v) \mapsto \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{1, \|u-v\|_{L^p(B_j)}\}, \qquad (u,v) \mapsto \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{1, \|u-v\|_{W^{k,p}(B_j)}\},$$

- $\mathscr{H}_R^k = W^{k+1,2}(B_R) \oplus W^{k,2}(B_R)$, $\mathscr{H}_R := \mathscr{H}_R^0$,
- $\mathscr{H}^k = W^{k+1,2} \oplus W^{k,2}$, $\mathscr{H} := \mathscr{H}^0$,
- $\mathscr{H}_{\mathrm{loc}} = W^{1,2}_{\mathrm{loc}} \oplus L^2_{\mathrm{loc}}$,
- $\mathscr{D} = \mathscr{D}(\mathbb{R}^d, \mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued compactly supported C^∞-functions,
- $\mathscr{S}$ is the Schwartz spaces of complex rapidly decreasing C^∞-functions on $\mathbb{R}^d$ ([40]),
- $\mathscr{S}'$ is the space of tempered distributions on $\mathscr{S}$, i.e. the real dual space to $\mathscr{S}$,
- $\xi \mapsto \widehat{\xi}$ the Fourier transformation on $\mathscr{S}'$,
- C_b^k the space of k-times continuously differentiable functions on $\mathbb{R}^d$ with bounded derivatives up to order k equipped with the supremum norm of all derivatives up to order k,
- $C^\gamma([a,b],X)$ the Banach space of X-valued γ-Hölder continuous functions on $[a,b]$ with the norm

$$\|h\| = \sup\{\|h(t)\|_X : t \in [a,b]\} + \sup\left\{\frac{\|h(t)-h(s)\|_X}{(t-s)^\gamma} : a \le s < t \le b\right\}$$

  where X is a Banach space,
- $\mathscr{A}$ a second order elliptic operator

$$\mathscr{A} = \sum_{k=1}^d \sum_{l=1}^d \frac{\partial}{\partial x_k}\left(\mathbf{a}_{kl}(x)\frac{\partial}{\partial x_l}\right)$$

  where $\mathbf{a}(x)$ is a symmetric, strictly positive, $(d \times d)$-real-matrix for every $x \in \mathbb{R}^d$ and $\mathbf{a}$ is a continuous, $W^{1,\infty}_{loc}(\mathbb{R}^d)$-valued function such that

$$\inf\left\{t^{-2}\sup\left\{\left|\mathbf{a}(x)y\right|_{\mathbb{R}^d} : |x|_{\mathbb{R}^d} < t,\, |y|_{\mathbb{R}^d} = 1\right\} : t > 0\right\} = 0, \tag{2.1}$$

  see also Remark 2.1.
- π_R various restriction maps to the ball B_R, for example

$$\pi_R : L^2_{\mathrm{loc}} \ni v \mapsto v|_{B_R} \in L^2(B_R) \quad \text{or} \quad \pi_R : \mathscr{H}_{\mathrm{loc}} \ni z \mapsto z|_{B_R} \in \mathscr{H}_R$$

- $\mathscr{L}(X,Y)$ the space of continuous linear operators from a topological vector space X to a topological vector space Y and we equip it with the strong σ-algebra, i.e. the σ-algebra generated by the family of maps $\mathscr{L}(X,Y) \ni B \mapsto Bx \in Y$, $x \in X$. If X and Y are Banach spaces then $\mathscr{L}(X,Y)$ is equipped with the usual operator norm,
- $\mathscr{L}_2(X,Y)$ the space of Hilbert-Schmidt operators from a Hilbert space X to a Hilbert space Y and is equipped with the strong σ-algebra, i.e. the σ-algebra generated by the family of maps $\mathscr{L}_2(X,Y) \ni B \mapsto Bx \in Y$, $x \in X$,
- $C(\mathbb{R}_+, Z)$ the space of continuous functions from $\mathbb{R}_+$ to a metric space (Z,ρ) and we equip it with the metric

$$(a,b) \mapsto \sum_{j=1}^\infty \frac{1}{2^j}\min\{1, \sup_{t\in[0,j]} \rho(a(t),b(t))\}.$$

  If, in addition, Z is a vector space, $C_0(\mathbb{R}_+;Z) = \{h \in C(\mathbb{R}_+;Z) : h(0) = 0\}$,

- $C_w(\mathbb{R}_+;X)$ the space of weakly continuous functions from $\mathbb{R}_+$ to a locally convex space X and we equip it with the locally convex topology generated by the a family $\|\cdot\|_{m,\varphi}$ of pseudonorms defined by

$$\|a\|_{m,\varphi} = \sup_{t\in[0,m]} |\varphi(a(t))|, \qquad m\in\mathbb{N}, \quad \varphi\in X^*,$$

- ζ a symmetric C^∞-density on $\mathbb{R}^d$ supported in the unit ball and we define

$$\zeta_m(x) = m^d\zeta(mx), \qquad x\in\mathbb{R}^d,$$

- $\mathscr{X} = C_w(\mathbb{R}_+, W^{1,2}_{loc}) \times C_w(\mathbb{R}_+, L^2_{loc})$,
- capital bold scripts the conic energy functions, i.e. if a measurable function $F:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}_+$, $x\in\mathbb{R}^d$, $\lambda>0$ and $T>0$ are given then

$$\mathbf{F}_{\lambda,x,T}(t,u,v) = \int_{B(x,T-\lambda t)} \left\{ \frac{1}{2}\sum_{k=1}^d\sum_{l=1}^d \mathbf{a}_{kl}\left\langle u_{x_k},u_{x_l}\right\rangle_{\mathbb{R}^n} + \frac{1}{2}|v|^2_{\mathbb{R}^n} + F(y,u)\right\} dy \tag{2.2}$$

is defined for $t\in[0,\frac{T}{\lambda}]$ and $(u,v)\in\mathscr{H}_{loc}$.

Remark 2.1. The condition (2.1) is equivalent with the following: given $R>0$, there exists $T>0$ such that the cylinder $\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d : t\in[0,R],\ |x|_{\mathbb{R}^d}\le R\}$ is contained in a centered backward cone $\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d : |x|+t\lambda_T\le T\}$ where

$$\lambda_T = \sup_{w\in B(0,T)} \|\mathbf{a}(w)\|^{\frac{1}{2}}. \tag{2.3}$$

3 Spatially homogeneous Wiener process

Given a stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\ge0},\mathbb{P})$, an $\mathscr{S}'$-valued process $W=\left(W_t\right)_{t\ge0}$ is called a spatially homogeneous Wiener process with a spectral measure μ that we assume to be positive, symmetric and to satisfy $\mu(\mathbb{R}^d)<\infty$ throughout the paper, provided that

- $W\varphi := \left(W_t\varphi\right)_{t\ge0}$ is a real $(\mathscr{F}_t)$-Wiener process, for every $\varphi\in\mathscr{S}$,
- $W_t(a\varphi+\psi) = aW_t(\varphi)+W_t(\psi)$ almost surely for all $a\in\mathbb{R}$, $t\in\mathbb{R}_+$ and $\varphi,\psi\in\mathscr{S}$,
- $\mathbb{E}\{W_t\varphi_1W_t\varphi_2\} = t\langle\widehat{\varphi}_1,\widehat{\varphi}_2\rangle_{L^2(\mu)}$ for all $t\ge0$ and $\varphi_1,\varphi_2\in\mathscr{S}$.

Remark 3.1. "Spatial homogeneity" refers to the fact that the process W can be represented as a centered $(\mathscr{F}_t)$-adapted Gaussian random field $(\mathscr{W}(t,x): t\ge0, x\in\mathbb{R}^d)$ so that

$$\mathbb{P}\left[W_t\varphi = \int_{\mathbb{R}^d}\varphi(x)\mathscr{W}(t,x)\,dx\right] = 1, \qquad \varphi\in\mathscr{S}$$

and

$$\mathbb{E}\left[\mathscr{W}(t,x)\mathscr{W}(s,y)\right] = \min\{t,s\}\Gamma(x-y), \qquad t,s\ge0, \quad x,y\in\mathbb{R}^d$$

where $\Gamma=(2\pi)^{-\frac{d}{2}}\widehat{\mu}$ is a bounded continuous function. The reader is referred to [5], [12], [36] and [37] for further details and examples of spatially homogeneous Wiener processes.

Let us denote by $H_\mu \subseteq \mathscr{S}'$ the reproducing kernel Hilbert space of the $\mathscr{S}'$-valued random vector $W(1)$, see e.g. [11]. Then W is an H_μ-cylindrical Wiener process. Moreover, see [5] and [37], if we denote by $L^2_{(s)}(\mathbb{R}^d,\mu)$ the subspace of $L^2(\mathbb{R}^d,\mu;\mathbb{C})$ consisting of all ψ such that $\psi=\psi_{(s)}$, where $\psi_{(s)}(\cdot)=\overline{\psi(-\cdot)}$, then we have the following result:

Proposition 3.2.

$$\begin{aligned} H_\mu &= \{\widehat{\psi\mu}:\ \psi\in L^2_{(s)}(\mathbb{R}^d,\mu)\},\\ \langle\widehat{\psi\mu},\widehat{\varphi\mu}\rangle_{H_\mu} &= \int_{\mathbb{R}^d}\psi(x)\overline{\varphi(x)}\,d\mu(x),\quad \psi,\varphi\in L^2_{(s)}(\mathbb{R}^d,\mu). \end{aligned}$$

The following lemma states that, under some assumptions, H_μ is a function space and that multiplication operators are Hilbert-Schmidt from H_μ to L^2 (see [30] for a proof). In that case, we can calculate the Hilbert-Schmidt norm explicitly.

Lemma 3.3. *Assume that $\mu(\mathbb{R}^d)<\infty$. Then the reproducing kernel Hilbert space H_μ is continuously embedded in $C_b(\mathbb{R}^d)$, the multiplication operator $m_g=\{H_\mu\ni\xi\mapsto g\cdot\xi\in L^2(D)\}$ is Hilbert-Schmidt and there exists a constant* **c** *such that*

$$\|m_g\|_{\mathscr{L}_2(H_\mu,L^2(D))}=\mathbf{c}\|g\|_{L^2(D)} \tag{3.1}$$

whenever $D\subseteq\mathbb{R}^d$ is Borel and $g\in L^2(D)$.

Remark 3.4. A stochastic integral with respect to a spatially homogeneous Wiener process is understood in the classical way, see e.g. [11], [36] or [37].

4 Solution

Definition 4.1. *Let $f^1,\dots,f^d:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ be Borel matrix-valued functions, let $f^{d+1},g^{d+1}:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}^n$ be Borel functions, μ a given finite spectral measure on $\mathbb{R}^d$, $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P})$ a completely filtered probability space with a spatially homogeneous $(\mathscr{F}_t)$-Wiener process W with spectral measure μ. An $(\mathscr{F}_t)$-adapted process $z=(u,v)$ with weakly continuous paths in $\mathscr{H}_{loc}$ is a solution of (1.1) where, for $(x,y,z)=(x,y,z_0,\dots,z_d)\in\mathbb{R}^d\times\mathbb{R}^n\times\prod_{i=0}^d\mathbb{R}^n$,*

$$f(x,y,z)=\sum_{i=0}^d f^i(x,y)z_i+f^{d+1}(x,y),\quad g(x,y,z)=\sum_{i=0}^d g^i(x,y)z_i+g^{d+1}(x,y) \tag{4.1}$$

provided that

$$\mathbb{P}\left[\int_0^T\left\{\|f(\cdot,u(s),v(s),\nabla_x u(s))\|_{L^1(B_T)}+\|g(\cdot,u(s),v(s),\nabla_x u(s))\|^2_{L^2(B_T)}\right\}ds\right]=1 \tag{4.2}$$

holds for every $T > 0$ and

$$\begin{aligned}
\langle u(t),\varphi\rangle &= \langle u(0),\varphi\rangle + \int_0^t \langle v(s),\varphi\rangle\,ds \\
\langle v(t),\varphi\rangle &= \langle v(0),\varphi\rangle + \int_0^t \langle u(s),\mathscr{A}\varphi\rangle\,ds + \int_0^t \left\langle f(\cdot,u(s),v(s),\nabla_x u(s)),\varphi\right\rangle\,ds \\
&+ \int_0^t \langle g(\cdot,u(s),v(s),\nabla u(s))\,dW_s,\varphi\rangle
\end{aligned} \tag{4.3}$$

holds for every $t \geq 0$ a.s. whenever $\varphi \in \mathscr{D}$.

Remark 4.2. The assumption (4.2) guarantees existence of integrals in (4.3). Let us verify the convergence of the stochastic integral in (4.3). For, let us denote $\rho = g(\cdot,u,v,\nabla u)$, let $\xi_j = \widehat{\psi_j\mu}$ be an ONB in H_μ (i.e. by Proposition 3.2, (ψ_j) is an ONB in $L^2_{(s)}(\mathbb{R}^d,\mu)$) and let $T > 0$ be such that the support of φ is contained in B_T. Then

$$\begin{aligned}
\|\langle\rho(s)\cdot,\varphi\rangle\|^2_{\mathscr{L}_2(H_\mu,\mathbb{R})} &= \sum_j \left|\langle\rho(s)\xi_j,\varphi\rangle\right|^2 \\
&= (2\pi)^{-d}\sum_j \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\langle x,y\rangle}\langle\rho(s,x),\varphi(x)\rangle_{\mathbb{R}^n}\psi_j(x)\,dx\,\mu(dy)\right|^2 \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} e^{-i\langle x,y\rangle}\langle\rho(s,x),\varphi(x)\rangle_{\mathbb{R}^n}\,dx\right|^2 \mu(dy) \\
&\leq c_\circ \left\|\langle\rho(s,x),\varphi\rangle_{\mathbb{R}^n}\right\|^2_{L^1} \leq c_\circ \left\|\varphi\right\|^2_{L^2(B_T)} \left\|\rho(s,x)\right\|^2_{L^2(B_T)}
\end{aligned}$$

where $c_\circ = (2\pi)^{-d}\mu(\mathbb{R}^d)$ and, by (4.2), $\int_0^t \|\xi \mapsto \langle\rho(s)\xi,\varphi\rangle\|^2_{\mathscr{L}_2(H_\mu,\mathbb{R})}\,ds < \infty$ for all $t > 0$ a.s. Hence, the stochastic integral in (4.3) is well defined e.g. by [11].

5 The main result

Assume that

i) $f^i, g^i : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $i \in \{0,\dots,d\}$,

ii) $f^{d+1}, g^{d+1} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$,

iii) $F : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$

are measurable functions, $\kappa \in \mathbb{R}_+$, $(\alpha_{r,R})_{r,R>0}$ are real numbers such that $\lim_{R\to\infty}\alpha_{r,R} = 0$ for every $r > 0$, and, for every $y \in \mathbb{R}^n$, $r > 0$ and $R > 0$,

iv) $F(w,\cdot) \in C^1(\mathbb{R}^n)$,

v) $f^i(w,\cdot), g^i(w,\cdot) \in C(\mathbb{R}^n)$, $i \in \{0,\dots,d+1\}$

vi) $|f^0(w,y)|^2 + |g^0(w,y)|^2 \le \kappa$,

vii)

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\left|\mathbf{a}^{-\frac{1}{2}}(w)\begin{pmatrix} f^1_{jk}(w,y) \\ \vdots \\ f^d_{jk}(w,y)\end{pmatrix}\right|^2_{\mathbb{R}^d} + \left|\mathbf{a}^{-\frac{1}{2}}(w)\begin{pmatrix} g^1_{jk}(w,y) \\ \vdots \\ g^d_{jk}(w,y)\end{pmatrix}\right|^2_{\mathbb{R}^d}\right] \le \kappa,$$

viii) $\left|g^{d+1}(w,y)\right|^2 + \left|\nabla_y F(w,y) + f^{d+1}(w,y)\right|^2 \le \kappa F(w,y)$,

ix) $|f^{d+1}(w,y)| \le \kappa F(w,y)$,

x) $\mathbf{1}_{[|w|\le r]\cap[|y|\ge R]}|f^{d+1}(w,y)| \le \alpha_{r,R} F(w,y)$,

xi) $\|F_{\max}(\cdot,r)\|_{L^1(B_r)} < \infty$ where

$$F_{\max}(x,r) = \sup_{|y|\le r} F(x,y) \tag{5.1}$$

hold for almost every $w \in \mathbb{R}^d$.

Theorem 5.1 (Existence). *Let μ be a finite spectral measure and let Θ be a Borel probability measure on $\mathscr{H}_{loc}$ such that*

$$\Theta\{(u,v) :\in \mathscr{H}_{loc} : \|F_{\max}(\cdot, |u(\cdot)|+1)\|_{L^1(B_r)} < \infty\} = 1, \qquad r > 0. \tag{5.2}$$

Then there exists a completely filtered stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ with a spatially homogeneous $(\mathscr{F}_t)$-Wiener process W with spectral measure μ and an $(\mathscr{F}_t)$-adapted process $z = (u,v)$ with weakly continuous paths in $\mathscr{H}_{loc}$ which is a solution of the equation (1.1) in the sense of Definition 4.1 and $\mathbb{P}[z(0) \in A] = \Theta(A)$ for every $A \in \mathscr{B}(\mathscr{H}_{loc})$.

Theorem 5.2 (Energy estimate). *Let μ, Θ, κ and $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}, W, z)$ be the same as in Theorem 5.1, let (5.2) hold and let $\tilde{\kappa} \in \mathbb{R}_+$. Let $G : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function such that*

- *$G(w,\cdot) \in C^1(\mathbb{R}^n)$,*
- *$|g^{d+1}(w,y)|^2 + |\nabla_y G(w,y) + f^{d+1}(w,y)|^2 \le \tilde{\kappa} G(w,y)$, $y \in \mathbb{R}^n$*

hold for almost every $w \in \mathbb{R}^d$ and let

$$\Theta\{(u,v) :\in \mathscr{H}_{loc} : \|G_{\max}(\cdot, |u(\cdot)|+1)\|_{L^1(B_r)} < \infty\} = 1, \qquad r > 0$$

where

$$G_{\max}(w,r) = \sup_{|y|\le r} G(w,y), \qquad r > 0.$$

Then there exists a constant $\rho \in \mathbb{R}_+$ depending only on the numbers $\mu(\mathbb{R}^d)$ and $\max\{\kappa, \tilde{\kappa}\}$ such that, given $x \in \mathbb{R}^d$, $T > 0$,

$$\lambda \ge \sup_{w \in B(x,T)} \|\mathbf{a}(w)\|^{\frac{1}{2}}, \tag{5.3}$$

a non-decreasing function $L \in C(\mathbb{R}_+) \cap C^2(0,\infty)$ such that

$$rL'(r) + r^2 \max\{L''(r), 0\} \le \tilde{\kappa} L(r), \qquad r > 0, \tag{5.4}$$

the estimate

$$\mathbb{E}\left\{\mathbf{1}_A(z(0))\sup_{r\in[0,t]} L(\mathbf{G}(r,z(r)))\right\} \le 4e^{\rho t}\mathbb{E}\left\{\mathbf{1}_A(z(0))L(\mathbf{G}(0,z(0)))\right\} \tag{5.5}$$

holds with the convention $0\cdot\infty=0$ *for every* $A\in\mathscr{B}(\mathscr{H}_{loc})$ *and* $t\in[0,T/\lambda]$ *where* $\mathbf{G}=\mathbf{G}_{\lambda,x,T}$ *is the conic energy function for* G *defined as in (2.2).*

Remark 5.3. Let us observe that Theorem 5.2 is a sort of extension of Theorem 5.1 that claims that the particular solution constructed in Theorem 5.1 satisfies the infinite number of qualitative properties (i.e. given whichever entries G, L etc.) in Theorem 5.2. We however cannot exclude, at this moment, the possibility of existence of a weak solution of (1.1) for which (5.5) is not satisfied.

5.1 Examples

Example 5.4 (Local space). Let $\alpha\in(-\infty,2)$, let $a_i,b_i:\mathbb{R}^{d+1}\to\mathbb{R}$, $i\in\{0,\dots,d\}$ be bounded measurable functions continuous in the last variable, let $A:\mathbb{R}^d\to\mathbb{R}_+$, $B:\mathbb{R}^{d+1}\to\mathbb{R}$ be measurable functions such that B is continuous in the last variable, let $B^2\le\rho A$ for some $\rho\ge 0$, let A be locally integrable, let

$$0<p,\qquad 0\le q\le\frac{p+1}{2},\qquad \beta_i\le\frac{\alpha}{2},\qquad \gamma_i\le\frac{\alpha}{2},\qquad i\in\{1,\dots,d\},$$

let Θ be a Borel probability measure on $\mathscr{H}_{loc}$ such that

$$\int_{B_r} A(x)|u(x)|^{p+1}\,dx<\infty,\qquad r>0$$

holds for Θ-almost every $(u,v)\in\mathscr{H}_{loc}$ and let μ be a finite spectral measure. Then the equation

$$\begin{aligned} u_{tt} &= (1+|x|)^{\alpha}\Delta u+\left[a_0(x,u)u_t+\sum_{i=1}^{d}(1+|x|)^{\beta_i}a_i(x,u)\frac{\partial u}{\partial x_i}-A(x)u|u|^{p-1}\right]\\ &+ \left[b_0(x,u)u_t+\sum_{i=1}^{d}(1+|x|)^{\gamma_i}b_i(x,u)\frac{\partial u}{\partial x_i}+B(x,u)|u|^{q}\right]\dot W \end{aligned}\tag{5.6}$$

has a weak solution $z=(u,v)$ with weakly $\mathscr{H}_{loc}$-continuous paths, with the initial distribution Θ and

$$\sup_{t\in[0,r]}\int_{B_r} A(x)|u(t,x,\omega)|^{p+1}\,dx<\infty,\qquad r>0 \tag{5.7}$$

holds a.s. where W is a spatially homogeneous Wiener process with spectral measure μ. It is straightforward to verify that the hypotheses i) - xi) at the beginning of Section 5 hold if we put $F(x,y)=A(x)\left(\frac{|y|^{p+1}}{p+1}+1\right)$ and $\alpha_{r,R}=\sup_{t\ge R}t^p(\frac{t^{p+1}}{p+1}+1)^{-1}$.

Example 5.5 (Global space). The equation

$$\begin{aligned} u_{tt} &= \Delta u+a_0(x,u)u_t+\sum_{i=1}^{d}a_i(x,u)u_{x_i}-u|u|^{p-1}\\ &+ \left[b_0(x,u)u_t+\sum_{i=1}^{d}b_i(x,u)u_{x_i}+b_{d+1}(x,u)|u|^{q}\right]\dot W \end{aligned}\tag{5.8}$$

is a particular case of (5.6) with $\alpha = \beta_i = \gamma_i = 0$, $A = 1$ and $B = b_{d+1}$, so we know, by Example 5.4, which we develop here further, that if $a_i, b_i : \mathbb{R}^{d+1} \to \mathbb{R}$, $i \in \{0, \dots, d+1\}$ are bounded measurable functions continuous in the last variable,

$$1 \leq q \leq \frac{p+1}{2},$$

Θ is a Borel probability measure on $\mathscr{H}$ such that $u \in L^{p+1}$ holds for Θ-almost every $(u,v) \in \mathscr{H}$ and μ is a finite spectral measure, that the equation (5.8) has a weak solution $z = (u,v)$ with weakly $\mathscr{H}_{loc}$-continuous paths, with the initial distribution Θ and W is a spatially homogeneous Wiener process with spectral measure μ. Notwithstanding, the estimate (5.7) can be further strengthened to

$$\sup_{t \in [0,r]} \left(\|z(t)\|_{\mathscr{H}} + \|u(t)\|_{L^{p+1}} \right) < \infty, \qquad r > 0, \quad \text{a.s.} \tag{5.9}$$

by applying Theorem 5.2 on the function $G(x,y) = |y|^{p+1}/(p+1) + |y|^2/2$, $L(x) = x$ and $\lambda = 1$ (which satisfy the assumptions of Section 5). In particular, paths of the solution z are not only weakly $\mathscr{H}_{loc}$-continuous, but weakly $\mathscr{H}$-continuous a.s.

Proof. The assumptions of Theorem 5.2 are satisfied by a direct verification so if we define

$$H_R = \{(u,v) \in \mathscr{H} : \|(u,v)\|_{\mathscr{H}} + \|u\|_{L^{p+1}} \leq R\}, \qquad R > 0$$

then, for a fixed $\rho > 0$ independent of z, R and T,

$$\begin{aligned} \mathbb{E}\, \mathbf{1}_{H_R}(z(0)) \sup_{r \leq t} \left[\frac{\|z(r)\|^2_{\mathscr{H}_{T-t}}}{2} + \frac{\|u(r)\|^{p+1}_{L^{p+1}(B_{T-t})}}{p+1} \right] \quad &\leq \quad 4e^{\rho t} \int_{H_R} \left[\frac{\|(u,v)\|^2_{\mathscr{H}_T}}{2} + \frac{\|u\|^{p+1}_{L^{p+1}(B_T)}}{p+1} \right] d\Theta \\ &\leq \quad 4e^{\rho t} \left(\frac{R^2}{2} + \frac{R^{p+1}}{p+1} \right) =: C_{t,R} \end{aligned}$$

holds for every $T > 0$ and $t \in [0,T)$ by Theorem 5.2. Letting $T \to \infty$, we obtain

$$\mathbb{E}\, \mathbf{1}_{H_R}(z(0)) \sup_{r \leq t} \left[\frac{\|z(r)\|^2_{\mathscr{H}}}{2} + \frac{\|u(r)\|^{p+1}_{L^{p+1}}}{p+1} \right] \leq C_{t,R}$$

by the Fatou lemma. Thus

$$\mathbb{P} \left[\mathbf{1}_{H_R}(z(0)) \sup_{r \leq t} \left[\frac{\|z(r)\|^2_{\mathscr{H}}}{2} + \frac{\|u(r)\|^{p+1}_{L^{p+1}}}{p+1} \right] < \infty, \quad R > 0 \right] = 1$$

whence we get (5.9). □

6 Guideline through the paper

The present work generalizes the state of art in the five directions mentioned in the Introduction. Let us, though, illustrate the progress with respect to the most related paper [29] on the example of the equation

$$u_{tt} = \Delta u + f(u) + g(u)\, dW \tag{6.1}$$

in the global space $W^{1,2}(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$. Here, the nonnegative function F is just the potential of $-f$, i.e. $f + \nabla F = 0$ (provided it exists) and its purpose is to control the growth of the norm of the solutions in the energy space - see the apriori estimates (8.4) and (10.4) which generalize the conservation of energy law in the theory of deterministic wave equations (see e.g. [38]).

If the equation (6.1) is scalar, existence of a weak solution of (6.1) was proved, independently of the dimension d, in [29] provided that, roughly speaking, f and g are continuous, the primitive function F to $-f$ is positive and

$$\lim_{|t|\to\infty} \frac{|f(t)| + |g(t)|^2}{F(t)} = 0, \tag{6.2}$$

i.e. only the case of subcritically growing polynomial nonlinearities $q < (p+1)/2$ was covered for the equation (1.2) (cf. the hypothesis (1.3)). The condition (6.2) was induced in [29] by the method of proof in global spaces and it was not surprising because it just accompanied the Strauss hypothesis $\lim_{|t|\to\infty} |f(t)|/F(t) = 0$ on the drift in the deterministic equation (see [44]) by the expected hypothesis $\lim_{|t|\to\infty} |g(t)|^2/F(t) = 0$ on the diffusion. However, the subtle arguments in Section 11 based on the local nature of the equation show that mere eventual boundedness of $|g|^2/F$ is sufficient for the proofs to go through and, consequently, the critical case $q = (p+1)/2$ for the equation (1.2) is covered.

The proofs of both Theorem 5.1 and Theorem 5.2 are based on *a refined stochastic compactness method* (adapted from [17]) which consists in the following: a sequence of solutions of suitably constructed approximating equations is shown to be tight in the path space of weakly continuous vector-valued functions $\mathscr{Z} = C_w(\mathbb{R}_+, W^{1,2}_{loc}) \times C_w(\mathbb{R}_+, L^2_{loc})$. This space is not metrizable hence the Jakubowski-Skorokhod theorem [21] is applied (instead of the Skorokhod representation theorem) to model the Prokhorov weak convergence of laws as an almost sure convergence of processes on a fixed probability space to a limit process which is, eventually, proved to be the desired weak solution. Apart from the Jakubowski-Skorokhod theorem, another novelty of the method relies in the fact that the identification of the limit process with the solution is not done via a martingale representation theorem (which is not available in our setting anyway) but by a few tricks with quadratic variations (see Sections 9 and 10).

Let us briefly comment on the structure of the proofs: The stochastic Cauchy problem (1.1) is first reduced (by a localization in Section 7) to the global Hilbert space $W^{1,2}(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ where an apriori estimate (8.4) independent of the localization is proved. The stochastic compactness method is then applied in two steps:

First, existence of a weak solution with $W^{1,2}(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$-continuous paths is established in Section 9 for sub-linearly growing Lipschitz functions $f = (f^i)$, $g = (g^i)$ using the apriori estimate (8.4), where the nonlinearities are simply mollified by smooth densities. This step is not trivial since the Nemytski operators associated to f and g are not "locally Lipschitz" on the state space $W^{1,2}(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ - yet, the stochastic compactness method is employed in its standard form (see e.g. [2], [17] or [29]).

Subsequently, a refined apriori estimate (10.4) adapted to finely approximated nonlinearities is established in Section 10 and the full strength of the stochastic compactness method based on the Jakubowski-Skorokhod theorem is carried out in Section 11 which is also the core of the paper. The technicalities are caused mainly by the local space setting of the problem. Finally, Theorem 5.2 is proved collaterally in Section 11.6.

7 Localization of the operator $\mathscr{A}$

Our differential operator $\mathscr{A}$ must be localized first in order general theorems on generating C_0-semigroups can be applied. For, let us define the C^1-function $h : \mathbb{R} \to [0,1]$ by

$$h(t) = 1, \quad t \in (-\infty, 1], \qquad h(t) = (2t-1)(t-2)^2, \quad t \in [1,2], \qquad h(t) = 0, \quad t \in [2, \infty),$$

$$\phi^m(x) := h(|x|/m)x, \qquad x \in \mathbb{R}^d, \quad m \in \mathbb{N}. \tag{7.1}$$

We may verify quite easily that ϕ^m is diffeomorphic on $\{x : |x| < \frac{11+\sqrt{57}}{16}\}$ and also on $\{x : \frac{11+\sqrt{57}}{16} < |x| < 2\}$ hence the set $B_{2m} \cap [\phi^m \in C]$ is negligible for the Lebesgue measure whenever so is $C \in \mathscr{B}(\mathbb{R}^d)$, hence $\mathbf{a} \circ \phi^m$ is uniformly continuous on $\mathbb{R}^d$, belongs to $W^{1,\infty}(\mathbb{R}^d)$ where

$$\frac{\partial(\mathbf{a} \circ \phi^m)}{\partial x_j} = \mathbf{1}_{B_{2m}} \sum_{l=1}^{d} \frac{\partial \mathbf{a}}{\partial x_l}(\phi^m) \frac{\partial \phi_l^m}{\partial x_j}, \qquad j \in \{1, \dots, d\}, \quad m \in \mathbb{N}$$

and if we define

$$\mathscr{A}^m = \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_i} \left((\mathbf{a}_{ij} \circ \phi^m) \frac{\partial}{\partial x_j} \right), \qquad m \in \mathbb{N} \tag{7.2}$$

then we have the following result which concerns a realization of the differential operator $\mathscr{A}^m$ in L^2 on the Sobolev space $W^{2,2}$, states its funcional-analytic properties and allows to introduce a matrix infinitesimal generator of a wave C_0-group.

Proposition 7.1. *For $m \in \mathbb{N}$, it holds that*

- *the operator $\mathscr{A}^m$ with* Dom $\mathscr{A}^m = W^{2,2}$ *is uniformly elliptic, selfadjoint and negative on L^2,*
- Dom $(I - \mathscr{A}^m)^{\frac{1}{2}} = W^{1,2}$ *with equivalence of the graph norm and the $W^{1,2}$-norm,*
- *the graph norm on* Dom $\mathscr{A}^m$ *is equivalent with the $W^{2,2}$-norm,*
- *the operator*

  $$\mathscr{G}^m = \begin{pmatrix} 0 & I \\ \mathscr{A}^m & 0 \end{pmatrix}, \qquad \operatorname{Dom} \mathscr{G}^m = W^{2,2} \oplus W^{1,2}$$

  generates a C_0-group (S_t^m) on the space $W^{1,2} \oplus L^2$,
- *the graph norm on* Dom $\mathscr{G}^m$ *is equivalent with the $W^{2,2} \oplus W^{1,2}$-norm.*

Proof. The operator $\mathscr{A}^m$ is selfadjoint e.g. by a consequence of Theorem 4 in Section 1.6 in [24], the operator

$$\begin{pmatrix} 0 & I \\ \mathscr{A}^m - I & 0 \end{pmatrix}$$

is skew-adjoint in Dom$(I - \mathscr{A}^m)^{\frac{1}{2}} \oplus L^2$ hence generates a unitary C_0-group by the Stone theorem (see e.g. Theorem 10.8 in Chapter 1.10 in [34]). The operator

$$\begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$$

is bounded on $\mathrm{Dom}(I-\mathscr{A}^m)^{\frac{1}{2}} \oplus L^2$ so $\mathscr{G}^m$ generates a C_0-group by Theorem 1.1 in Chapter 3.1 in [34].

The equivalence of the graph norm on Dom $\mathscr{A}^m$ with the $W^{2,2}$-norm follows from Theorem 5 in Section 1.6 in [24]. □

We close this section by introducing the *localized conic energy function* relative to F and to the operator $\mathscr{A}^m$ analogously to (2.2). Toward this end, given $x \in \mathbb{R}^d$, $T > 0$, $\lambda > 0$ and a measurable function $F : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$, we define

$$\mathbf{F}_{m,\lambda,x,T}(t,u,v) = \int_{B(x,T-\lambda t)} \left\{ \frac{1}{2} \sum_{k=1}^{d} \sum_{l=1}^{d} (\mathbf{a}_{kl} \circ \phi^m) \left\langle u_{x_k}, u_{x_l} \right\rangle_{\mathbb{R}^n} + \frac{1}{2} |v|^2_{\mathbb{R}^n} + F(y,u) \right\} dy \tag{7.3}$$

for $t \in [0, \frac{T}{\lambda}]$ and $(u,v) \in \mathscr{H}_{loc}$.

Remark 7.2. Observe that the integrand in (7.3) coincides with the integrand of (2.2) on the centered ball B_m, hence the localized conic energy function $\mathbf{F}_{m,\lambda,x,T}$ relative to F and to the operator $\mathscr{A}^m$ is really a “localization” of the conic energy function $\mathbf{F}_{\lambda,x,T}$ relative to F and to the operator $\mathscr{A}$.

8 A local energy inequality

In this technical section we shall establish a backward cone energy estimate that, on one hand, makes it possible to find uniform bounds for a suitable approximating sequences of processes that will later on yield a solution by invoking a compactness argument, and on the other hand, imply finite propagation property of solutions of (1.1).

Proposition 8.1. *Let $m \in \mathbb{N}$, $T > 0$, let U be a separable Hilbert space and W a U-cylindrical Wiener process. Let α and β be progressively measurable processes with values in L^2 and $\mathscr{L}_2(U, L^2)$ respectively such that*

$$\mathbb{P}\left[\int_0^T \left\{ \|\alpha(s)\|_{L^2} + \|\beta(s)\|^2_{\mathscr{L}_2(U,L^2)} \right\} ds < \infty \right] = 1. \tag{8.1}$$

Assume that $z = (u, v)$ is an adapted process with continuous paths in $\mathscr{H}$ such that

$$\begin{aligned} \langle u(t), \varphi \rangle_{L^2} &= \langle u(0), \varphi \rangle_{L^2} + \int_0^t \langle v(s), \varphi \rangle_{L^2} \, ds \\ \langle v(t), \varphi \rangle_{L^2} &= \langle v(0), \varphi \rangle_{L^2} + \int_0^t \langle u(s), \mathscr{A}^m \varphi \rangle_{L^2} \, ds + \int_0^t \langle \alpha(s), \varphi \rangle_{L^2} \, ds + \int_0^t \langle \beta(s) \, dW_s, \varphi \rangle_{L^2} \end{aligned} \tag{8.2}$$

holds a.s. for every $t \geq 0$ and every $\varphi \in \mathscr{D}$. Assume that $F : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ is such that

(a) *$F(w, \cdot) \in C^1(\mathbb{R}^n)$ for every $w \in \mathbb{R}^d$,*

(b) *$F(\cdot, y)$ measurable for every $y \in \mathbb{R}^n$*

(c) *and*

$$\sup\left\{\frac{F(w,y)}{1+|y|^2}+\frac{\left|\nabla_y F(w,y)\right|}{1+|y|}:|w|\le r,y\in\mathbb{R}^n\right\}<\infty,\qquad r>0.$$

Fix $x\in\mathbb{R}^d$ and λ so that

$$\lambda\ge\sup_{w\in B(x,T)}\|\mathbf{a}(\phi^m(w))\|^{\frac{1}{2}}$$

where ϕ^m was defined in (7.1), and consider the conic energy function $\mathbf{F}=\mathbf{F}_{m,\lambda,x,T}$ for F (see (7.3)). Assume also that a nondecreasing function $L:\mathbb{R}_+\to\mathbb{R}$ is of C^2-class on $[0,\infty)$ and put

$$\begin{aligned}V(t,z) &= \frac{1}{2}L''(\mathbf{F}(t,z))\sum_l\langle v,\beta(t)e_l\rangle^2_{L^2(B(x,T-\lambda t))}+\frac{1}{2}L'(\mathbf{F}(t,z))\|\beta(t)\|^2_{\mathscr{L}_2(U,L^2(B(x,T-\lambda t)))}\\ &+ L'(\mathbf{F}(t,z))\langle v,\nabla_y F(\cdot,u)+\alpha(t)\rangle_{L^2(B(x,T-\lambda t))}\end{aligned}\tag{8.3}$$

for $t\in[0,\frac{T}{\lambda}]$ and $z=(u,v)\in\mathscr{H}$ where (e_l) is any ONB in U. Then

$$\begin{aligned}L(\mathbf{F}(t,z(t))) &\le L(\mathbf{F}(r,z(r)))+\int_r^t V(s,z(s))\,ds\\ &+\int_r^t L'(\mathbf{F}(s,z(s)))\langle v(s),\beta(s)\,dW_s\rangle_{L^2(B(x,T-\lambda s))}\end{aligned}\tag{8.4}$$

is satisfied for every $0\le r<t\le\frac{T}{\lambda}$ almost surely.

Proof. By density, (8.2) holds for every $\varphi\in W^{2,2}$ and if we test with $\varphi=\varepsilon^2(\varepsilon-\mathscr{A}^m)^{-2}\psi$ for $\varepsilon>0$ and $\psi\in L^2$ then

$$\begin{aligned}u_\varepsilon(t) &= u_\varepsilon(0)+\int_0^t v_\varepsilon(s)\,ds\\ v_\varepsilon(t) &= v_\varepsilon(0)+\int_0^t\left[\mathscr{A}^m u_\varepsilon(s)+\alpha_\varepsilon(s)\right]\,ds+\int_0^t\beta_\varepsilon(s)\,dW_s\end{aligned}$$

holds for every $t\ge 0$ a.s. where

$$\begin{array}{llllll}u_\varepsilon= & \varepsilon^2(\varepsilon-\mathscr{A}^m)^{-2}u, & v_\varepsilon= & \varepsilon^2(\varepsilon-\mathscr{A}^m)^{-2}v, & z_\varepsilon= & (u_\varepsilon,v_\varepsilon)\\ \alpha_\varepsilon= & \varepsilon^2(\varepsilon-\mathscr{A}^m)^{-2}\alpha, & \beta_\varepsilon\xi= & \varepsilon^2(\varepsilon-\mathscr{A}^m)^{-2}[\beta\xi], & \xi\in & U\end{array}$$

and the integrals converge in Dom $\mathscr{A}^m=W^{2,2}$ whose norms are equivalent by Proposition 7.1. If $F^j:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}_+$ satisfies

i) $F^j(w,y)=0$ for every $w\in\mathbb{R}^d$ and $|y|>j$,

ii) $F^j(w,\cdot)\in C^\infty(\mathbb{R}^n)$ for every $w\in\mathbb{R}^d$,

iii) $\sup\left\{|D_y^\gamma F^j(w,y)|:|z|\le r,y\in\mathbb{R}^n\right\}<\infty$ for every multiindex γ and $r>0$

then

$$\mathbf{F}^j = \mathbf{F}^j_{m,\lambda,x,T} \in C^{1,2}\left([0,T/\lambda] \times W^{2,2} \oplus W^{2,2}\right)$$

as

$$\|\varphi\|_{L^2(\partial B(y,r))} \le 2\|\varphi\|_{L^2}\|\nabla\varphi\|_{L^2}, \qquad \varphi \in W^{1,2}$$

holds for every $y \in \mathbb{R}^d$ and $r > 0$. We may thus apply the Ito formula (see [11]) on $L(\mathbf{F}^j(z_\varepsilon))$ to obtain

$$L(\mathbf{F}^j(t, z_\varepsilon(t))) - L(\mathbf{F}^j(r, z_\varepsilon(r))) =$$

$$\begin{aligned}
- \quad & \lambda \int_r^t L'(\mathbf{F}^j(s, z_\varepsilon(s))) \|F^j(\cdot, u_\varepsilon(s))\|_{L^1(\partial B(x,T-\lambda s))}\, ds \\
- \quad & \frac{\lambda}{2} \int_r^t L'(\mathbf{F}^j(s, z_\varepsilon(s))) \left\| \sum_{i=1}^d \sum_{k=1}^d (\mathbf{a}_{ik} \circ \phi^m) \left\langle \frac{\partial u_\varepsilon(s)}{\partial x_i}, \frac{\partial u_\varepsilon(s)}{\partial x_k} \right\rangle_{\mathbb{R}^n} \right\|_{L^1(\partial B(x,T-\lambda s))} ds \\
- \quad & \frac{\lambda}{2} \int_r^t L'(\mathbf{F}^j(s, z_\varepsilon(s))) \left\| v_\varepsilon(s) \right\|^2_{L^2(\partial B(x,T-\lambda s))} ds \\
+ \quad & \int_r^t L'(\mathbf{F}^j(s, z_\varepsilon(s))) \left\{ \int_{B(x,T-\lambda s)} \sum_{i=1}^d \sum_{k=1}^d (\mathbf{a}_{ik} \circ \phi^m) \left\langle \frac{\partial u_\varepsilon(s)}{\partial x_i}, \frac{\partial v_\varepsilon(s)}{\partial x_k} \right\rangle_{\mathbb{R}^n} \right\} ds \\
+ \quad & \int_r^t L'(\mathbf{F}^j(s, z_\varepsilon(s))) \left\{ \left\langle v_\varepsilon(s), \mathscr{A}^m u_\varepsilon(s) + \alpha_\varepsilon(s) + \nabla_y F^j(\cdot, u_\varepsilon(s)) \right\rangle_{L^2(B(x,T-\lambda s))} \right\} ds \\
+ \quad & \int_r^t L'(\mathbf{F}^j(s, z_\varepsilon(s))) \left\{ \langle v_\varepsilon(s), \beta_\varepsilon(s)\, dW_s \rangle_{L^2(B(x,T-\lambda s))} \right\} ds \\
+ \quad & \frac{1}{2} \int_r^t L'(\mathbf{F}^j(s, z_\varepsilon(s))) \|\beta_\varepsilon(s)\|^2_{\mathscr{L}_2(U, L^2(B(x,T-\lambda s)))}\, ds \\
+ \quad & \frac{1}{2} \sum_l \int_r^t L''(\mathbf{F}^j(s, z_\varepsilon(s))) \left\langle v_\varepsilon(s), \beta_\varepsilon(s) e_l \right\rangle^2_{L^2(B(x,T-\lambda s))}\, ds
\end{aligned} \tag{8.5}$$

for every $0 \le r < t \le T$ a.s. We may, in fact, find the functions F^j satisfying i)-iii) even so that

iv) $F^j(w, \cdot) \to F(w, \cdot)$ uniformly on compacts in $\mathbb{R}^n$, for every $w \in \mathbb{R}^d$,

v) $\nabla_y F^j(w, \cdot) \to \nabla_y F(w, \cdot)$ uniformly on compacts in $\mathbb{R}^n$, for every $w \in \mathbb{R}^d$,

vi) and

$$\sup \left\{ \frac{F_j(w,y)}{1+|y|^2} + \frac{\left|\nabla_y F_j(w,y)\right|}{1+|y|} : |w| \le r, y \in \mathbb{R}^n, j \in \mathbb{N} \right\} < \infty, \qquad r > 0.$$

Thus

$$\begin{array}{ll}
\lim_{j\to\infty} \mathbf{F}^j(t, z_\varepsilon(t,\omega)) = \mathbf{F}(t, z_\varepsilon(t,\omega)), & t \in [0, T/\lambda], \\
\sup \left\{ \mathbf{F}^j(t, z_\varepsilon(t,\omega)) : t \in [0,T/\lambda], j \in \mathbb{N} \right\} < \infty, & \\
\lim_{j\to\infty} \|\nabla_y F^j(\cdot, u_\varepsilon(t,\omega)) - \nabla_y F(\cdot, u_\varepsilon(t,\omega))\|_{L^2(B(x,T-\lambda t))} = 0, & t \in [0, T/\lambda], \\
\sup \left\{ \|\nabla_y F^j(\cdot, u_\varepsilon(t,\omega))\|_{L^2(B(x,T-\lambda t))} : t \in [0,T/\lambda], j \in \mathbb{N} \right\} < \infty &
\end{array}$$

for every $\omega\in\Omega$. Moreover, by the Gauss theorem,

$$\left|\int_{B(x,T-\lambda s)}\left[\sum_{i=1}^d\sum_{k=1}^d(\mathbf{a}_{ik}\circ\phi^m)\left\langle\frac{\partial u_\varepsilon(s)}{\partial x_i},\frac{\partial v_\varepsilon(s)}{\partial x_k}\right\rangle_{\mathbb{R}^n}+\langle v_\varepsilon(s),\mathscr{A}^m u_\varepsilon(s)\rangle_{\mathbb{R}^n}\right]\,ds\right|=$$

$$\begin{aligned}
&= \left|\sum_{i=1}^d\sum_{k=1}^d\int_{\partial B(x,T-\lambda s)}(\mathbf{a}_{ik}\circ\phi^m)\left\langle v_\varepsilon,\frac{\partial u_\varepsilon}{\partial x_i}\right\rangle_{\mathbb{R}^n}\frac{y_k-x_k}{T-\lambda s}\,dy\right|\\
&\le \int_{\partial B(x,T-\lambda s)}|v_\varepsilon|\left[\sum_{l=1}^n\|(\mathbf{a}\circ\phi^m)\nabla u_\varepsilon^l\|^2\right]^{\frac12}dy\\
&\le \lambda\int_{\partial B(x,T-\lambda s)}|v_\varepsilon|\left[\sum_{l=1}^n\|(\mathbf{a}^{\frac12}\circ\phi^m)\nabla u_\varepsilon^l\|^2\right]^{\frac12}dy\\
&= \lambda\int_{\partial B(x,T-\lambda s)}|v_\varepsilon|\left|\sum_{i=1}^d\sum_{k=1}^d(\mathbf{a}_{ik}\circ\phi^m)\left\langle\frac{\partial u_\varepsilon}{\partial x_i},\frac{\partial u_\varepsilon}{\partial x_k}\right\rangle_{\mathbb{R}^n}\right|^{\frac12}dy\\
&\le \frac{\lambda}{2}\|v_\varepsilon\|^2_{L^2(\partial B(x,T-\lambda s))}+\frac{\lambda}{2}\left\|\sum_{i=1}^d\sum_{k=1}^d(\mathbf{a}_{ik}\circ\phi^m)\left\langle\frac{\partial u_\varepsilon}{\partial x_i},\frac{\partial u_\varepsilon}{\partial x_k}\right\rangle_{\mathbb{R}^n}\right\|_{L^1(\partial B(x,T-\lambda s))}
\end{aligned}\tag{8.6}$$

so, after applying (8.6) and letting $j\to\infty$ in (8.5),

$$\begin{aligned}
&L(\mathbf{F}(t,z_\varepsilon(t)))-L(\mathbf{F}(r,z_\varepsilon(r)))\le\\
&+ \int_r^t L'(\mathbf{F}(s,z_\varepsilon(s)))\left\{\left\langle v_\varepsilon(s),\alpha_\varepsilon(s)+\nabla_y F(\cdot,u_\varepsilon(s))\right\rangle_{L^2(B(x,T-\lambda s))}\right\}ds\\
&+ \int_r^t L'(\mathbf{F}(s,z_\varepsilon(s)))\left\{\langle v_\varepsilon(s),\beta_\varepsilon(s)\,dW_s\rangle_{L^2(B(x,T-\lambda s))}\right\}ds\\
&+ \frac12\int_r^t L'(\mathbf{F}(s,z_\varepsilon(s)))\|\beta_\varepsilon(s)\|^2_{\mathscr{L}_2(U,L^2(B(x,T-\lambda s)))}ds\\
&+ \frac12\sum_l\int_r^t L''(\mathbf{F}(s,z_\varepsilon(s)))\langle v_\varepsilon(s),\beta_\varepsilon(s)e_l\rangle^2_{L^2(B(x,T-\lambda s))}ds
\end{aligned}\tag{8.7}$$

for every $0\le r<t\le T$ a.s. by the Lebesgue dominated convergence theorem and the convergence theorem for stochastic integrals (see e.g. Proposition 4.1 in [33]).

Now, since

$$\sup_{\varepsilon>1}\|\varepsilon^2(\varepsilon-\mathscr{A}^m)^{-2}\|_{\mathscr{L}(L^2)}<\infty\quad\text{and}\quad\lim_{\varepsilon\to\infty}\|\varepsilon^2(\varepsilon-\mathscr{A}^m)^{-2}\varphi-\varphi\|_{L^2}=0,\quad\varphi\in L^2,$$

there is

$$\lim_{\varepsilon\to\infty}\left[\sup_{t\in[0,T/\lambda]}\|z_\varepsilon(t,\omega)-z(t,\omega)\|_{W^{1,2}\oplus L^2}+\sup_{t\in[0,T/\lambda]}|\mathbf{F}(t,z_\varepsilon(t,\omega))-\mathbf{F}(t,z(t,\omega))|\right]=0$$

$$\sup\{\|z_\varepsilon(t,\omega)\|_{W^{1,2}\oplus L^2}+\mathbf{F}(t,z_\varepsilon(t,\omega)):\varepsilon>0,\,t\in[0,T/\lambda]\}<\infty$$

for every $\omega\in\Omega$ so we get the result from (8.7) by the Lebesgue dominated convergence theorem and a convergence result for stochastic integrals (e.g. Proposition 4.1 in [33]). □

9 Linear growth + Global space case

We first prove existence of weak solutions for a localized equation with regular nonlinearities. The proof is based on a compactness method: local energy estimates yield tightness of an approximating sequence of solutions. This sequence converges, on another probability space, to a limit due to the Jakubowski-Skorokhod theorem and finally, it is shown, that this limit is the desired weak solution of the localized equation (9.1).

Lemma 9.1. *Let μ be a finite spectral measure on $\mathbb{R}^d$, let $m \in \mathbb{N}$, let ν be a Borel probability measure supported in a ball in $\mathscr{H}$, let $f^i, g^i : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ for $i \in \{0, \dots, d\}$, $f^{d+1}, g^{d+1} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be measurable functions such that*

$$\sup \left\{ |f^i(w, y)| + |g^i(w, y)| : |w| \le r,\, y \in \mathbb{R}^n, i \in \{0, \dots, d\} \right\} < \infty$$

$$\sup \left\{ \frac{|f^{d+1}(w, y)| + |g^{d+1}(w, y)|}{1 + |y|} : |w| \le r,\, y \in \mathbb{R}^n \right\} < \infty,$$

$$\sup \left\{ \frac{|f^i(w, y_1) - f^i(w, y_2)|}{|y_1 - y_2|} : |w| \le r,\, y_1 \ne y_2 \in \mathbb{R}^n, i \in \{0, \dots, d+1\} \right\} < \infty$$

$$\sup \left\{ \frac{|g^i(w, y_1) - g^i(w, y_2)|}{|y_1 - y_2|} : |w| \le r,\, y_1 \ne y_2 \in \mathbb{R}^n, i \in \{0, \dots, d+1\} \right\} < \infty$$

hold for every $r > 0$. Then there exists a completely filtered stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ with a spatially homogeneous $(\mathscr{F}_t)$-Wiener process W with the spectral measure μ and an $(\mathscr{F}_t)$-adapted process z with continuous paths in $\mathscr{H}$ which is a solution of the equation

$$u_{tt} = \mathscr{A}^m u + \mathbf{1}_{B_m} f(\cdot, u, u_t, \nabla u) + \mathbf{1}_{B_m} g(\cdot, u, u_t, \nabla u)\,\dot{W} \tag{9.1}$$

in the sense of Section 4 with the notation (4.1), (4.1), ν is the law of $z(0)$ and B_m is the open centered ball in $\mathbb{R}^d$ with radius m.

The proof of Lemma 9.1 will be carried out in a sequence of lemmas. For, let us introduce the mappings $f_k : \mathscr{H} \to \mathscr{H}$ and $g_k : \mathscr{H} \to \mathscr{L}_2(H_\mu, \mathscr{H})$ defined by

$$\begin{aligned} f_k(u, v) &= \mathbf{1}_{B_m} \begin{pmatrix} 0 \\ f^0(\cdot, u)(\zeta_k * v) + \sum_{i=1}^d f^i(\cdot, u)(\zeta_k * u_{x_i}) + f^{d+1}(\cdot, u) \end{pmatrix}, \\ g_k(u, v) &= \mathbf{1}_{B_m} \begin{pmatrix} 0 \\ g^0(\cdot, u)(\zeta_k * v) + \sum_{i=1}^d g^i(\cdot, u)(\zeta_k * u_{x_i}) + g^{d+1}(\cdot, u) \end{pmatrix} \xi, \quad \xi \in H_\mu. \end{aligned}$$

Lemma 9.2. *For every $k \in \mathbb{N}$, there exists a completely filtered stochastic basis $(\Omega^k, \mathscr{F}^k, \mathbb{P}^k)$ with a spatially homogeneous $(\mathscr{F}^k_t)$-Wiener process W^k with spectral measure μ and an $(\mathscr{F}^k_t)$-adapted process $z^k = (u^k, v^k)$ with $\mathscr{H}$-continuous paths such that ν is the law of $z^k(0)$ under $\mathbb{P}^k$ and*

$$z^k(t) = S^m_t z^k(0) + \int_0^t S^m_{t-s} f_k(z^k(s))\, ds + \int_0^t S^m_{t-s} g_k(z^k(s))\, dW^k_s, \qquad t \ge 0.$$

Moreover, for every $p \in [2, \infty)$, there exists a constant $K^{(1)}_{p,l} = K^{(1)}_{l,m,g,f,p,\nu,\mathbf{c},\mathbf{a}}$ such that

$$\mathbb{E}^k \sup_{s\in[0,l]} \|z^k(s)\|^{2p}_{\mathscr{H}} \le K^{(1)}_{p,l}, \qquad k, l \in \mathbb{N} \tag{9.2}$$

and, if $q \in (1,\infty)$ and $\gamma > 0$ are such that $\gamma + \frac{1}{q} < \frac{1}{2}$ then there exists a constant $K^{(2)}_{q,l} = K^{(2)}_{l,m,\gamma,q,f,g,a^{(2,m)},\mathbf{c},\nu}$ such that

$$\mathbb{E}^k \|v^k(t)\|^{2q}_{C^\gamma([0,l];W^{-1,2})} \le K^{(2)}_{q,l}, \qquad k,l \in \mathbb{N}. \tag{9.3}$$

Proof. The mappings $f_k : \mathscr{H} \to \mathscr{H}$ and $g_k : \mathscr{H} \to \mathscr{L}_2(H_\mu, \mathscr{H})$ are Lipschitz on bounded sets and have at most linear growth hence there exists a completely filtered stochastic basis $(\Omega^k, \mathscr{F}^k, \mathbb{P}^k)$ with a spatially homogeneous $(\mathscr{F}^k_t)$-Wiener process W^k with spectral measure μ and an $(\mathscr{F}^k_t)$-adapted process z^k with $\mathscr{H}$-continuous paths such that ν is the law of $z^k(0)$ under $\mathbb{P}^k$ and

$$z^k(t) = S^m_t z^k(0) + \int_0^t S^m_{t-s} f_k(z^k(s))\,ds + \int_0^t S^m_{t-s} g_k(z^k(s))\,dW^k_s, \qquad t \ge 0$$

by e.g. [11] extended in the sense of Theorem 12.1 in Chapter V.2.12 in [39] whose generalization to SPDE is possible and can be proved in the same way as in [39]) since

$$\langle \mathscr{G}^m z, z \rangle_{\mathrm{Dom}\,(I-\mathscr{A}^m)^{\frac{1}{2}} \oplus L^2} \le \frac{1}{2} \|z\|^2_{\mathrm{Dom}\,(I-\mathscr{A}^m)^{\frac{1}{2}} \oplus L^2}, \qquad z \in \mathrm{Dom}\,\mathscr{G}^m$$

hence the square norm of the local solution $\left\|z^k\right\|^2_{\mathrm{Dom}\,(I-\mathscr{A}^m)^{\frac{1}{2}} \oplus L^2}$ cannot explode in finite time and so z^k is a global solution in the sense of (4.3) by the Chojnowska-Michalik theorem (see [9] or Theorem 12 in [33]).

By Proposition 8.1 applied on $T > 0$, $x = 0$, $l(r) = \log(1 + r^p)$ for $p \in [2,\infty)$, $F(y) = |y|^2/2$, $\lambda_0 = \sup_{w \in \mathbb{R}^d} \|\mathbf{a}(\phi^m(w))\|^{\frac{1}{2}}$, with the notation $\mathbf{F}_T = \mathbf{F}_{m,\lambda_0,0,T}$ and

$$\mathbf{F}_\infty(u,v) = \frac{1}{2} \int_{\mathbb{R}^d} \left\{ \sum_{i=1}^d \sum_{l=1}^d (\mathbf{a}_{il} \circ \phi^m) \left\langle u_{x_i}, u_{x_l} \right\rangle_{\mathbb{R}^n} + |v|^2_{\mathbb{R}^n} + |u|^2_{\mathbb{R}^n} \right\} dy,$$

there is

$$l(\mathbf{F}_T(t, z^k(t))) \le l(\mathbf{F}_T(0, z^k(0))) + M_{T,k}(t) - \frac{1}{2} \langle M_{T,k} \rangle (t)$$

$$\begin{aligned}
&+ \quad \frac{\mathbf{c}^2}{2} \int_0^t \frac{p(p-1)\mathbf{F}_T^{p-2}(s,z^k(s))}{1 + \mathbf{F}_T^p(s,z^k(s))} \|v^k(s)\|^2_{L^2(B_{T-\lambda_0 s})} \|g(\cdot, u^k(s), v^k(s), \nabla u^k(s))\|^2_{L^2(B_m \cap B_{T-\lambda_0 s})}\,ds \\
&+ \quad \frac{\mathbf{c}^2}{2} \int_0^t l'(\mathbf{F}_T(s,z^k(s))) \|g(\cdot, u^k(s), v^k(s), \nabla u^k(s))\|^2_{L^2(B_m \cap B_{T-\lambda_0 s})}\,ds \\
&+ \quad \int_0^t l'(\mathbf{F}_T(s,z^k(s))) \|v^k(s)\|_{L^2(B_{T-\lambda_0 s})} \|u^k(s)\|_{L^2(B_{T-\lambda_0 s})}\,ds \\
&+ \quad \int_0^t l'(\mathbf{F}_T(s,z^k(s))) \|v^k(s)\|_{L^2(B_{T-\lambda_0 s})} \|f(\cdot, u^k(s), v^k(s), \nabla u^k(s))\|_{L^2(B_m \cap B_{T-\lambda_0 s})}\,ds \\
&\le \quad l(\mathbf{F}_\infty(z^k(0))) + M_{T,k}(t) - \frac{1}{2} \langle M_{T,k} \rangle (t) \\
&+ \quad K \int_0^t \frac{1 + \mathbf{F}_\infty^2(z^k(s))}{1 + \mathbf{F}_T^2(s,z^k(s))}\,ds + K \int_0^t \frac{1 + \mathbf{F}_\infty(z^k(s))}{1 + \mathbf{F}_T(s,z^k(s))}\,ds
\end{aligned}$$

for $t \in [0, T/\lambda_0]$ by Lemma 3.3 where K depends only on p, $\mathbf{c}$, m, $\mathbf{a}$, g, f and

$$M_{T,k}(t) = p \int_0^t \frac{\mathbf{F}_T^{p-1}(s, z^k(s))}{1 + \mathbf{F}_T^p(s, z^k(s))} \left\langle v^k(s), g(\cdot, u^k(s), v^k(s), \nabla u^k(s))\, dW_s^k \right\rangle_{L^2(B_m \cap B_{T-\lambda_0 s})}$$

for $t \in [0, T/\lambda_0]$. Thus, letting $T \to \infty$, we obtain

$$l(\mathbf{F}_\infty(z^k(t))) \le l(\mathbf{F}_\infty(z^k(0))) + M_k(t) - \frac{1}{2}\langle M_k\rangle(t) + 2Kt$$

for $t \in \mathbb{R}_+$ by the Lebesgue dominated convergence theorem and a convergence result for stochastic integrals (e.g. Proposition 4.1. in [33]) where

$$\begin{aligned} M_k(t) &= p \int_0^t \frac{\mathbf{F}_\infty^{p-1}(z^k(s))}{1 + \mathbf{F}_\infty^p(z^k(s))} \left\langle v^k(s), g(\cdot, u^k(s), v^k(s), \nabla u^k(s))\, dW_s^k \right\rangle_{L^2(B_m)}, \quad t \in \mathbb{R}_+, \\ \langle M_k\rangle(t) &= p^2 \sum_l \int_0^t \frac{\mathbf{F}_\infty^{2(p-1)}(z^k(s))}{[1 + \mathbf{F}_\infty^p(z^k(s))]^2} \left\langle v^k(s), g(\cdot, u^k(s), v^k(s), \nabla u^k(s)) e_l \right\rangle_{L^2(B_m)}^2 ds \\ &\le p^2 \mathbf{c}^2 \int_0^t \frac{\mathbf{F}_\infty^{2(p-1)}(z^k(s))}{[1 + \mathbf{F}_\infty^p(z^k(s))]^2} \left\| v^k(s) \right\|_{L^2(B_m)}^2 \left\| g(\cdot, u^k(s), v^k(s), \nabla u^k(s)) \right\|_{L^2(B_m)}^2 ds \\ &\le K_0 \int_0^t \frac{\mathbf{F}_\infty^{2(p-1)}(z^k(s))}{[1 + \mathbf{F}_\infty^p(z^k(s))]^2} (1 + \mathbf{F}_\infty(z^k(s)))^2\, ds \le K_3 t, \qquad t \in \mathbb{R}_+. \end{aligned}$$

Hence, applying the exponential on both sides, we get

$$\sup_{s \in [0,t]} \mathbf{F}_\infty^p(z^k(t)) \le e^{2Kt}[1 + \mathbf{F}_\infty^p(z^k(0))] \sup_{s \in [0,t]} e^{M_k(t) - \frac{1}{2}\langle M_k\rangle(t)}, \qquad t \in \mathbb{R}_+$$

so

$$\begin{aligned} \mathbb{E}^k \sup_{s \in [0,t]} \mathbf{F}_\infty^p(z^k(t)) &\le e^{2Kt} \left\{ \mathbb{E}^k\, [1 + \mathbf{F}_\infty^p(z^k(0))]^2 \right\}^{\frac{1}{2}} \left\{ \mathbb{E}^k \sup_{s \in [0,t]} e^{2M_k(s) - \langle M_k\rangle(s)} \right\}^{\frac{1}{2}} \\ &\le 2K_1 e^{2Kt} \left\{ \mathbb{E}^k\, e^{2M_k(t) - \langle M_k\rangle(t)} \right\}^{\frac{1}{2}} \le 2K_1 e^{(2K + \frac{K_3}{2})t} \left\{ \mathbb{E}^k\, e^{2M_k(t) - 2\langle M_k\rangle(t)} \right\}^{\frac{1}{2}} \\ &\le 2K_1 e^{(2K + \frac{K_3}{2})t} \end{aligned}$$

by the Doob maximal inequality for martingales and the Novikov criterion, where

$$K_1^2 = \mathbb{E}^k\, [1 + \mathbf{F}_\infty^p(z^k(0))]^2 = \int_{\mathscr{H}} [1 + \mathbf{F}_\infty^p(z)]^2\, d\nu < \infty.$$

Since $\mathbf{F}_\infty^{\frac{1}{2}}$ is an equivalent norm on $\mathscr{H}$, we have proved (9.2).

Next, by the Chojnowska-Michalik theorem (see [9] or Theorem 12 in [33])

$$\mathbb{P}^k \left[\int_0^t u^k(s)\, ds \in \operatorname{Dom} \mathscr{A}^m \right] = 1, \qquad t \in \mathbb{R}_+$$

and

$$
\begin{aligned}
v^k(t) &= v^k(0)+\mathscr{A}^m\int_0^t u^k(s)\,ds+\int_0^t f_k^{(2)}(z^k(s))\,ds+\int_0^t g_k^{(2)}(z^k(s))\,dW_s^k \qquad (9.4)\\
v^k(t) &= v^k(0)+\mathscr{A}^m I_1(t)+I_2(t)+I_3(t)
\end{aligned}
$$

almost surely for every $t\in\mathbb{R}_+$, where $f_k^{(2)}$ and $g_k^{(2)}$ are the second components of f_k and g_k, respectively, and the integrals converge in L^2. We get that

$$
\begin{aligned}
\mathbb{E}^k\,\|v^k\|^{2q}_{C^\gamma([0,l];W^{-1,2})} &\le 4^{2q-1}\left[\mathbb{E}^k\,\|v^k(0)\|^{2q}_{L^2}+\mathbb{E}^k\,\|\mathscr{A}^m I_1\|^{2q}_{C^\gamma([0,l];W^{-1,2})}\right]\\
&+ 4^{2q-1}\left[\mathbb{E}^k\,\|I_2\|^{2q}_{C^\gamma([0,l];W^{-1,2})}+\mathbb{E}^k\,\|I_3\|^{2q}_{C^\gamma([0,l];W^{-1,2})}\right]\\
&\le 4^{2q-1}\left[\mathbb{E}^k\,\|v^k(0)\|^{2q}_{L^2}+c_{a^{(2,m)}}\mathbb{E}^k\,\|I_1\|^{2q}_{C^\gamma([0,l];W^{1,2})}\right]\\
&+ 4^{2q-1}\left[\mathbb{E}^k\,\|I_2\|^{2q}_{C^\gamma([0,l];L^2)}+\mathbb{E}^k\,\|I_3\|^{2q}_{C^\gamma([0,l];L^2)}\right]\\
&\le c_{\mathbf{a},\gamma,q,l,f,g,m}\left(1+\mathbb{E}^k\sup_{s\in[0,t]}\|z^k(s)\|^{2q}_{\mathscr{H}}\right)\le K_l^{(2)}
\end{aligned}
$$

by the inequality (18) in the proof of Lemma 4 in [29] and (9.2). □

Lemma 9.3. *The sequence of processes (z^k) constructed in Lemma 9.2 is tight in the space $\mathscr{Z}=C_w(\mathbb{R}_+;W^{1,2}_{loc})\times C_w(\mathbb{R}_+;L^2_{loc})$.*

Proof. It holds, by the Chojnowska-Michalik theorem (see [9] or Theorem 12 in [33]), that

$$
u^k(t)=u^k(0)+\int_0^t v^k(s)\,ds,\qquad t\in\mathbb{R}_+ \tag{9.5}
$$

almost surely where the integral converges in L^2. So, if $\gamma\in(0,1)$ then

$$
\|u^k\|_{C^\gamma([0,l];L^2)}\le(1+l)\sup_{s\in[0,l]}\|z^k(s)\|_{\mathscr{H}}.
$$

Hence, if we fix $\varepsilon>0$, $q\in(1,\infty)$ and $\gamma>0$ such that $\gamma+\frac{1}{q}<\frac{1}{2}$ and we assume

$$
a_l>(2+l)\left[\frac{4^l}{\varepsilon}(K^{(1)}_{q,l}+K^{(2)}_{q,l})\right]^{\frac{1}{2q}},
$$

there is

$$
\mathbb{P}^k\left[\sup_{s\in[0,l]}\|u^k(s)\|_{W^{1,2}(B_l)}+\|u^k(s)\|_{C^\gamma([0,l];L^2(B_l))}>a_l\right]\le
$$

$$
\le\mathbb{P}^k\left[\sup_{s\in[0,l]}\|z^k(s)\|_{\mathscr{H}}>\frac{a_l}{2+l}\right]\le\left(\frac{2+l}{a_l}\right)^{2q}\mathbb{E}^k\sup_{s\in[0,l]}\|z^k(s)\|^{2q}_{\mathscr{H}}\le\frac{\varepsilon}{4^l}
$$

and

$$
\mathbb{P}^k\left[\sup_{s\in[0,l]}\|v^k(s)\|_{L^2(B_l)}+\|v^k(s)\|_{C^\gamma([0,l];\mathbb{W}^{-1,2}_l)}>a_l\right]\le
$$

$$
\begin{aligned}
&\leq \mathbb{P}^k\left[\sup_{s\in[0,l]}\|z^k(s)\|_{\mathscr{H}} > \frac{a_l}{2}\right] + \mathbb{P}^k\left[\|v^k(s)\|_{C^\gamma([0,l];\mathbb{W}_l^{-1,2})} > \frac{a_l}{2}\right] \\
&\leq \left(\frac{2}{a_l}\right)^{2q}\left[\mathbb{E}^k\sup_{s\in[0,l]}\|z^k(s)\|^{2q}_{\mathscr{H}} + \mathbb{E}^k\|v^k(s)\|^{2q}_{C^\gamma([0,l];W^{-1,2})}\right] \leq \frac{\varepsilon}{4^l},
\end{aligned}
$$

the sets

$$
\begin{aligned}
C_1 &= \left\{h \in C_w(\mathbb{R}_+; W^{1,2}_{loc}) : \|h\|_{L^\infty((0,l);W^{1,2}(B_l))} + \|h\|_{C^\gamma([0,l];L^2(B_l))} \leq a_l\right\} \\
C_2 &= \left\{h \in C_w(\mathbb{R}_+; L^2_{loc}) : \|h\|_{L^\infty((0,l);L^2(B_l))} + \|h\|_{C^\gamma([0,l];\mathbb{W}_l^{-1,2})} \leq a_l\right\}
\end{aligned}
$$

are such that $C_1 \times C_2$ is compact in $\mathscr{Z}$ by Corollary B.2 and $\mathbb{P}^k\,[z^k \in C_1 \times C_2] \geq 1 - \varepsilon$. □

We may now proceed to the proof of Lemma 9.1. Fixing an ONB (e_l) in H_μ, by the Jakubowski-Skorokhod theorem A.1 applied to the $\mathscr{Z} \times C_0(\mathbb{R}_+;\mathbb{R}^{\dim H_\mu})$-valued sequence $(z^k,(W^k(e_l))_l)_k$, there exists

- a subsequence (k_j), a probability space $(\Omega,\mathscr{F},\mathbb{P})$ with
- $C(\mathbb{R}_+;\mathscr{H})$-valued random variables Z^j, $j \in \mathbb{N}$,
- a $\mathscr{Z}$-valued random variable Z,
- $C_0(\mathbb{R}_+;\mathbb{R}^{\dim H_\mu})$-valued random variables β^j, $j \in \mathbb{N}$ and β

such that

i) the law of $(z^{k_j},(W^{k_j}(e_l))_l)$ under $\mathbb{P}^{k_j}$ coincides with the law of (Z^j,β^j) under $\mathbb{P}$ on

$$\mathscr{B}(C(\mathbb{R}_+;\mathscr{H})) \otimes \mathscr{B}(C_0(\mathbb{R}_+;\mathbb{R}^{\dim H_\mu}))$$

ii) (Z^j,β^j) converges in $\mathscr{Z} \times C_0(\mathbb{R}_+;\mathbb{R}^{\dim H_\mu})$ to (Z,β) on Ω.

Remark 9.4. We point out for completeness that tightness of the sequence $(z^k,(W^k(e_l))_l)_k$ in $\mathscr{Z} \times C_0(\mathbb{R}_+;\mathbb{R}^{\dim H_\mu})$ follows from Lemma 9.3 and from the fact that all $(W^k(e_l))_l$ have the same Radon law on the Polish space $C_0(\mathbb{R}_+;\mathbb{R}^{\dim H_\mu})$ for every $k \in \mathbb{N}$. Consequently, the sequence $(W^k(e_l))_l$ is tight in $C_0(\mathbb{R}_+;\mathbb{R}^{\dim H_\mu})$.

Remark 9.5. It should be also noted that the random variables Z^j are $\mathscr{Z}$-valued by the Jakubowski-Skorokhod theorem. However, since z^{k_j} and Z^j have the same law on $\mathscr{Z}$ and z^{k_j} are $C(\mathbb{R}_+;\mathscr{H})$-valued, we conclude that Z^j may be assumed to be $C(\mathbb{R}_+;\mathscr{H})$-valued satisfying the property i) above without loss of generality as $C(\mathbb{R}_+;\mathscr{H})$ is a measurable subset of $\mathscr{Z}$ by Corollary A.2.

Lemma 9.6. *If $p \in [2,\infty)$ then*

$$
\begin{aligned}
\mathbb{E}\sup_{s\in[0,l]}\|Z^j(s)\|^{2p}_{\mathscr{H}} &\leq K^{(1)}_{p,l}, \qquad j,l \in \mathbb{N} &(9.6)\\
\mathbb{E}\sup_{s\in[0,l]}\|Z(s)\|^{2p}_{\mathscr{H}} &\leq K^{(1)}_{p,l}, \qquad j,l \in \mathbb{N} &(9.7)
\end{aligned}
$$

where $K^{(1)}_{p,l}$ is the same constant as in (9.2).

Proof. The mapping

$$C(\mathbb{R}_+;\mathscr{H}) \to \mathbb{R}_+ : z \mapsto \sup_{s\in[0,l]} \|z(s)\|_{\mathscr{H}}^{2p}$$

is continuous hence Borel measurable and so

$$\mathbb{E} \sup_{s\in[0,l]} \|Z^j(s)\|_{\mathscr{H}}^{2p} = \mathbb{E}^{k_j} \sup_{s\in[0,l]} \|z^{k_j}(s)\|_{\mathscr{H}}^{2p} \le K_{p,l}^{(1)}, \qquad j,l\in\mathbb{N}$$

by the property i) and

$$\mathbb{E} \sup_{s\in[0,l]} \|Z(s)\|_{\mathscr{H}}^{2p} \le \liminf_{j\to\infty} \mathbb{E} \sup_{s\in[0,l]} \|Z^j(s)\|_{\mathscr{H}}^{2p} \le K_{p,l}^{(1)}, \qquad j,l\in\mathbb{N}$$

by the Fatou lemma and the property ii). □

Corollary 9.7. *The process Z has weakly continuous paths in $\mathscr{H}$ a.s.*

Lemma 9.8. *It holds that*

$$U(t) = U(0) + \int_0^t V(s)\,ds, \qquad t\in\mathbb{R}_+$$

almost surely where the integral converges in L^2.

Proof. The mapping

$$C(\mathbb{R}_+;\mathscr{H}) \to \mathbb{R} : z \mapsto \langle u(t),\varphi\rangle - \langle u(0),\varphi\rangle - \int_0^t \langle v(s),\varphi\rangle\,ds$$

is continuous hence Borel measurable for every $\varphi\in\mathscr{D}$ and so

$$\langle U^j(t),\varphi\rangle = \langle U^j(0),\varphi\rangle + \int_0^t \langle V^j(s),\varphi\rangle\,ds, \qquad t\in\mathbb{R}_+$$

holds almost surely for every $j\in\mathbb{N}$ by the property i) and (9.5). Letting $j\to\infty$, we get

$$\langle U(t),\varphi\rangle = \langle U(0),\varphi\rangle + \int_0^t \langle V(s),\varphi\rangle\,ds, \qquad t\in\mathbb{R}_+$$

$\mathbb{P}$-a.s. by the property ii). The result now follows from (9.7) and density of $\mathscr{D}$ in L^2. □

If we define the complete filtration

$$\mathscr{F}_t = \sigma\left(\sigma\left(Z(s),\beta(s) : s\in[0,t]\right) \cup \{N\in\mathscr{F} : \mathbb{P}(N)=0\}\right), \qquad t\in\mathbb{R}_+$$

then the following results.

Lemma 9.9. *The processes $\beta_1,\beta_2,\beta_3,\ldots$ are independent standard $(\mathscr{F}_t)$-Wiener processes.*

Proof. Let us consider the sequence (φ_i) from Corollary C.1, let $0 \le s < t$, $J \in \mathbb{N}$, $0 \le s_1 \le \dots \le s_J \le s$, let $h_0 : (\mathbb{R}^2)^{J\times J} \times (\mathbb{R}^{\dim H_\mu})^J \to [0,1]$ and $h_1 : \mathbb{R}^{\dim H_\mu} \to [0,1]$ be continuous functions and define

$$\begin{aligned} X_j &= \left(\left(\langle z^{k_j}(s_{i_0}), \varphi_{i_1} \rangle_{L^2} \right)_{i_0,i_1 \le J}, (W^{k_j}_{s_1}(e_l))_l, \dots, (W^{k_j}_{s_J}(e_l))_l \right) \\ \mathscr{X}_j &= \left(\left(\langle Z^j(s_{i_0}), \varphi_{i_1} \rangle_{L^2} \right)_{i_0,i_1 \le J}, \beta^j(s_1), \dots, \beta^j(s_J) \right) \\ \mathscr{X} &= \left(\left(\langle Z(s_{i_0}), \varphi_{i_1} \rangle_{L^2} \right)_{i_0,i_1 \le J}, \beta(s_1), \dots, \beta(s_J) \right) \end{aligned}$$

for $j \in \mathbb{N}$. Then, for every $j \in \mathbb{N}$,

$$\int_{\Omega^{k_j}} h_0(X_j) h_1((W^{k_j}_t(e_l) - W^{k_j}_s(e_l))_l)\, d\mathbb{P}^{k_j} = \int_{\Omega^{k_j}} h_0(X_j)\, d\mathbb{P}^{k_j} \int_{\Omega^{k_j}} h_1((W^{k_j}_t(e_l) - W^{k_j}_s(e_l))_l)\, d\mathbb{P}^{k_j}$$

by $\mathbb{P}^{k_j}$-independence of $\sigma((W^{k_j}_t(\xi) - W^{k_j}_s(\xi))_{\xi \in H_\mu})$ and $\mathscr{F}^{k_j}_s$. So, by the property i),

$$\mathbb{E}\left\{ h_0(\mathscr{X}_j) h_1(\beta^j(t) - \beta^j(s)) \right\} = \mathbb{E}\, h_0(\mathscr{X}_j)\, \mathbb{E}\, h_1(\beta^j(t) - \beta^j(s)), \qquad j \in \mathbb{N}$$

whence

$$\mathbb{E}\left\{ h_0(\mathscr{X}) h_1(\beta(t) - \beta(s)) \right\} = \mathbb{E}\, h_0(\mathscr{X})\, \mathbb{E}\, h_1(\beta(t) - \beta(s))$$

by the property ii) and we conclude that

$$\mathbb{E}\left\{ \mathbf{1}_{F_s} h_1(\beta(t) - \beta(s)) \right\} = \mathbb{P}(F_s)\, \mathbb{E}\, h_1(\beta(t) - \beta(s))$$

holds for every $F_s \in \mathscr{F}_s$ whenever $s < t$, i.e. $\sigma(\beta(t) - \beta(s))$ is $\mathbb{P}$-independent from $\mathscr{F}_s$. Since $(\beta^j_1(t) - \beta^j_1(s), \dots, \beta^j_l(t) - \beta^j_l(s))$ and $(W^{k_j}_t(e_1) - W^{k_j}_s(e_1), \dots, W^{k_j}_t(e_l) - W^{k_j}_s(e_l))$ have the normal centered distribution with covariance $(t-s)I_l$ for every $j, l \in \mathbb{N}$ and $0 \le s < t$ by the property i) preceding Remark 9.4 and the fact that W^{k_j} are cylindrical Wiener processes on H_μ by Section 3, we conclude that $(\beta_1(t) - \beta_1(s), \dots, \beta_l(t) - \beta_l(s))$ has the normal centered distribution with covariance $(t-s)I_l$ as well, as $\beta^j \to \beta$ on Ω by the property ii) preceding Remark 9.4. The proof of Lemma 9.9 is thus complete. □

Corollary 9.10. *Let (e_l) be the previously fixed ONB in H_μ. Then the cylindrical process*

$$W_t(\xi) = \sum_l \beta_l(t) \langle \xi, e_l \rangle_{H_\mu}, \qquad \xi \in H_\mu, \quad t \ge 0$$

is a spatially homogeneous $(\mathscr{F}_t)$-Wiener process with spectral measure μ.

Lemma 9.11.

Proof. Fix $\varphi \in \mathscr{D}$ and define the continuous operators

$$\begin{aligned} d^k_t : C(\mathbb{R}_+; \mathscr{H}) \to \mathbb{R} : z &\mapsto \langle v(t), \varphi \rangle - \langle v(0), \varphi \rangle - \int_0^t \left[\langle u(r), \mathscr{A}^m \varphi \rangle + \langle f^{(2)}_k(z(r)), \varphi \rangle \right] dr \\ D^{k,l}_t : C(\mathbb{R}_+; \mathscr{H}) \to \mathbb{R} : z &\mapsto \int_0^t \langle g^{(2)}_k(z(r)) e_l, \varphi \rangle\, dr \\ D^k_t : C(\mathbb{R}_+; \mathscr{H}) \to \mathbb{R} : z &\mapsto \sum_l \int_0^t \langle g^{(2)}_k(z(r)) e_l, \varphi \rangle^2\, dr \end{aligned}$$

where $f_k^{(2)}$ and $g_k^{(2)}$ are the second components of f_k and g_k, respectively. Then, fixing $0 \le s < t$ and with the notation of the proof of Lemma 9.9,

$$\mathbb{E}\, h_0(\mathscr{X}_j)\left\{d_t^{k_j}(Z^j) - d_s^{k_j}(Z^j)\right\} = \mathbb{E}^{k_j}\, h_0(X_j)\left\{d_t^{k_j}(z^{k_j}) - d_s^{k_j}(z^{k_j})\right\} = 0 \tag{9.8}$$

$$\mathbb{E}\, h_0(\mathscr{X}_j)\left\{d_t^{k_j}(Z^j)\beta_l^j(t) - D_t^{k_j,l}(Z^j) - d_s^{k_j}(Z^j)\beta_l^j(s) + D_s^{k_j,l}(Z^j)\right\} = \tag{9.9}$$

$$= \mathbb{E}^{k_j}\, h_0(X_j)\left\{d_t^{k_j}(z^{k_j})W_t^{k_j}(e_l) - D_t^{k_j,l}(z^{k_j}) - d_s^{k_j}(z^{k_j})W_s^{k_j}(e_l) + D_s^{k_j,l}(z^{k_j})\right\} = 0$$

$$\mathbb{E}\, h_0(\mathscr{X}_j)\left\{(d_t^{k_j}(Z^j))^2 - D_t^{k_j}(Z^j) - (d_s^{k_j}(Z^j))^2 + D_s^{k_j}(Z^j)\right\} = \tag{9.10}$$

$$= \mathbb{E}^{k_j}\, h_0(X_j)\left\{(d_t^{k_j}(z^{k_j}))^2 - D_t^{k_j}(z^{k_j}) - (d_s^{k_j}(z^{k_j}))^2 + D_s^{k_j}(z^{k_j})\right\} = 0$$

by the property i) since, by (9.4),

$$d_t^{k_j}(z^{k_j}) = \int_0^t g_{k_j}^{(2)}(z^{k_j}(s))\, dW_s^{k_j}, \qquad t \in \mathbb{R}_+$$

is an $L^2(\Omega^{k_j})$-integrable martingale in L^2 by (9.2) and Lemma 3.3, and the integrals (expectations) in (9.8)-(9.10) converge by (9.2) and (9.6). Since

$$\sup_{j\in\mathbb{N}} \mathbb{E}\left[|d_r^{k_j}(Z^j)|^q + |D_r^{k_j,l}(Z^j)|^q + |D_r^{k_j}(Z^j)|^q\right] < \infty$$

for every $r \in \mathbb{R}_+$, $l \in \mathbb{N}$ and $q > 0$ by (9.6), we get

$$\begin{aligned}
&\mathbb{E}\, h_0(\mathscr{X})\left\{d_t - d_s\right\} = 0\\
&\mathbb{E}\, h_0(\mathscr{X})\left\{d_t\beta_l(t) - d_s\beta_l(s) - \int_s^t \left\langle g^{(2)}(Z(r))e_l, \varphi\right\rangle\, dr\right\} = 0\\
&\mathbb{E}\, h_0(\mathscr{X})\left\{(d_t)^2 - (d_s)^2 - \sum_l \int_s^t \left\langle g^{(2)}(Z(r))e_l, \varphi\right\rangle^2\, dr\right\} = 0
\end{aligned}$$

by the property ii) where

$$\begin{aligned}
d_t &= \langle V(t), \varphi\rangle - \langle V(0), \varphi\rangle - \int_0^t \left[\langle U(r), \mathscr{A}^m\varphi\rangle + \langle f^{(2)}(Z(r)), \varphi\rangle\right]\, dr\\
f^{(2)}(z) &= \mathbf{1}_{B_m} f(\cdot, u, v, \nabla u)\\
g^{(2)}(z) &= \mathbf{1}_{B_m} g(\cdot, u, v, \nabla u).
\end{aligned}$$

In particular, the processes

$$d, \quad d\cdot\beta_l - \int_0^\cdot \left\langle g^{(2)}(Z(r))e_l, \varphi\right\rangle\, dr, \quad d^2 - \sum_l \int_0^\cdot \left\langle g^{(2)}(Z(r))e_l, \varphi\right\rangle^2\, dr$$

are $(\mathscr{F}_t)$-martingales hence the quadratic variation

$$\left\langle d - \int_0^\cdot \left\langle g^{(2)}(Z(r))\, dW_r, \varphi\right\rangle\right\rangle = 0$$

and so

$$\langle V(t),\varphi\rangle=\langle V(0),\varphi\rangle+\int_0^t\left[\langle U(r),\mathscr{A}^m\varphi\rangle+\langle f^{(2)}(Z(r)),\varphi\rangle\right]dr+\int_0^t\left\langle g^{(2)}(Z(r))\,dW_r,\varphi\right\rangle.$$

Thus Z is a solution of (9.1). Moreover, by the Chojnowska-Michalik theorem (see [9] or Theorem 13 in [33]),

$$Z(t)=S^mZ(0)+\int_0^t S^m_{t-s}\begin{pmatrix}0\\ \mathbf{1}_{B_m}f(Z(s))\end{pmatrix}ds+\int_0^t S^m_{t-s}\begin{pmatrix}0\\ \mathbf{1}_{B_m}g(Z(s))\end{pmatrix}dW_s$$

holds a.s. for every $t\in\mathbb{R}_+$, hence paths of Z are $\mathscr{H}$-continuous almost surely. □

10 General growth + Local space case

In this section, we use the existence result for the localized equation (9.1) and mimic the compactness method of the previous section based on the local energy estimates, tightness of an approximating sequence of solutions, convergence to a limit on another probability space due to the Jakubowski-Skorokhod theorem and final identification of the limit with a solution. The construction-approximation procedure is, however, much more refined this time.

Lemma 10.1. *Let $\kappa\in\mathbb{R}_+$. Then there exists a constant $\rho\in\mathbb{R}_+$ depending only on κ and $\mathbf{c}$ (see Section 2) such that the following holds: If $f^i,g^i:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}^{n\times n}$ for $i\in\{0,\dots,d\}$ and $f^{d+1},g^{d+1}:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}^n$ are measurable functions satisfying the assumptions of Lemma 9.1, $m\in\mathbb{N}$, z is an $\mathscr{H}$-continuous solution of (9.1), $L:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous nondecreasing function in $C^2(0,\infty)$ satisfying (5.4) with κ, if $T>0$ and $x\in\mathbb{R}^d$ satisfy $B(x,T)\subseteq B_m$, if $F:\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}_+$ satisfies the assumptions (a)-(c) in Proposition 8.1 and, for every $y\in\mathbb{R}^n$, there is*

$$|f^0(w,y)|^2+|g^0(w,y)|^2\le\kappa \tag{10.1}$$

$$\sum_{j=1}^n\sum_{k=1}^n\left[\left|\mathbf{a}^{-\frac12}(w)\begin{pmatrix}f^1_{jk}(w,y)\\ \vdots\\ f^d_{jk}(w,y)\end{pmatrix}\right|^2_{\mathbb{R}^d}+\left|\mathbf{a}^{-\frac12}(w)\begin{pmatrix}g^1_{jk}(w,y)\\ \vdots\\ g^d_{jk}(w,y)\end{pmatrix}\right|^2_{\mathbb{R}^d}\right]\le\kappa \tag{10.2}$$

$$|g^{d+1}(w,y)|^2+|\nabla_yF(w,y)+f^{d+1}(w,y)|^2\le\kappa F(w,y) \tag{10.3}$$

for a.e. $w\in B(x,T)$, if λ satisfies (5.3) and $\mathbf{F}=\mathbf{F}_{\lambda,x,T}$ is the conic energy function for F defined as in (2.2) then

$$\mathbb{E}\left\{\mathbf{1}_{\Omega_0}\sup_{r\in[0,t]}L(\mathbf{F}(r,z(r)))\right\}\le 4e^{\rho t}\mathbb{E}\left\{\mathbf{1}_{\Omega_0}L(\mathbf{F}(0,z_0))\right\} \tag{10.4}$$

holds for every $t\in[0,T/\lambda]$ and every $\Omega_0\in\mathscr{F}_0$.

Proof. Define $l_\varepsilon(r)=\log(\varepsilon+L(\varepsilon+r))$ for $r\in\mathbb{R}_+$ and $\varepsilon>0$, put $e(t)=\mathbf{F}_{m,\lambda,x,T}(t,z(t))$ for $t\in[0,T/\lambda]$ and write shortly

$$f(z)=f(\cdot,u,v,\nabla u),\qquad g(z)=g(\cdot,u,v,\nabla u)\qquad\text{for}\qquad z=(u,v)\in\mathscr{H}_{loc}.$$

Then, by Proposition 8.1,

$$\begin{aligned}
l_\varepsilon(e(t)) &\le l_\varepsilon(e(0)) + M_\varepsilon(t) - \frac{1}{2}\langle M_\varepsilon\rangle(t) \\
&+ \frac{1}{2}\sum_l \int_0^t \frac{L''(\varepsilon+e(s))}{\varepsilon+L(\varepsilon+e(s))}\langle v(s), g(z(s))e_l\rangle^2_{L^2(B(x,T-\lambda s))}\,ds \\
&+ \frac{\mathbf{c}^2}{2}\sum_l \int_0^t \frac{L'(\varepsilon+e(s))}{\varepsilon+L(\varepsilon+e(s))}\|g(z(s))\|^2_{L^2(B(x,T-\lambda s))}\,ds \\
&+ \int_0^t \frac{L'(\varepsilon+e(s))}{\varepsilon+L(\varepsilon+e(s))}\langle v(s), \nabla_y F(\cdot,u(s)) + f(z(s))\rangle_{L^2(B(x,T-\lambda s))}\,ds \\
&\le l_\varepsilon(e(0)) + \rho(\kappa)t + M_\varepsilon(t) - \frac{1}{2}\langle M_\varepsilon\rangle(t), \qquad t\in[0,T/\lambda]
\end{aligned}$$

almost surely by (8.4) and Lemma 3.3 where $\rho(\kappa) = (12\mathbf{c}^2\kappa + 4\kappa + 1)\kappa$ and

$$M_\varepsilon(t) = \int_0^t \frac{L'(\varepsilon+e(s))}{\varepsilon+L(\varepsilon+e(s))}\langle v(s), g(z(s))\,dW_s\rangle_{L^2(B(x,T-\lambda s))}, \qquad t\in[0,T/\lambda]$$

as, for every $s\in[0,T/\lambda]$,

$$\begin{aligned}
\|g(z(s))\|^2_{L^2(B(x,T-\lambda s))} &\le 3\left\|g^0(\cdot,u(s))v(s)\right\|^2_{L^2(B(x,T-\lambda s))} + 3\left\|g^{d+1}(\cdot,u(s))\right\|^2_{L^2(B(x,T-\lambda s))} \\
&+ 3\left\|\sum_{i=1}^d g^i(\cdot,u(s))u_{x_i}(s)\right\|^2_{L^2(B(x,T-\lambda s))} \le 12\kappa e(s)
\end{aligned}$$

and, analogously,

$$\left\|\nabla_y F(\cdot,u(s)) + f(z(s))\right\|^2_{L^2(B(x,T-\lambda s))} \le 12\kappa e(s).$$

Hence, almost surely,

$$L(\varepsilon+e(t)) \le e^{\rho(\kappa)t}\left[\varepsilon + L(\varepsilon+e(0))\right] e^{M_\varepsilon(t) - \frac{1}{2}\langle M_\varepsilon\rangle(t)}, \qquad t\in[0,T/\lambda].$$

Since

$$\langle M_\varepsilon\rangle(t) = \sum_l \int_0^t \left[\frac{L'(\varepsilon+e(s))}{\varepsilon+L(\varepsilon+e(s))}\right]^2 \langle v(s), g(z(s))e_l\rangle^2_{L^2(B(x,T-\lambda s))}\,ds \le 24\kappa^3\mathbf{c}^2 t, \quad t\in[0,T/\lambda],$$

there is

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}\left\{\mathbf{1}_{\Omega_0\cap[e(0)\le\delta]}L(\varepsilon+e(s))\right\} &\le e^{\rho(\kappa)t}\,\mathbb{E}\sup_{s\in[0,t]} Y_{1,1}(s) \qquad (10.5)\\
&\le e^{\rho(\kappa)t}\,\mathbb{E}\sup_{s\in[0,t]}[Y_{\frac{1}{2},\frac{1}{2}}(s)]^2 \\
&\le 4e^{\rho(\kappa)t}\,\mathbb{E}\,[Y_{\frac{1}{2},\frac{1}{2}}(t)]^2 \\
&= 4e^{\rho(\kappa)t}\,\mathbb{E}\left\{Y_{1,1}(t)e^{\frac{1}{4}\langle M_\varepsilon\rangle(t)}\right\} \\
&\le 4e^{[6\kappa^3\mathbf{c}^2+\rho(\kappa)]t}\,\mathbb{E}\,Y_{1,1}(t) \\
&= 4e^{[6\kappa^3\mathbf{c}^2+\rho(\kappa)]t}\,\mathbb{E}\left\{\mathbf{1}_{\Omega_0\cap[e(0)\le\delta]}\left[\varepsilon + L(\varepsilon+e(0))\right]\right\}
\end{aligned}$$

by the Doob maximal inequality for submartingales where

$$Y_{\alpha,\beta}(t) = \mathbf{1}_{\Omega_0\cap[e(0)\le\delta]}\,[\varepsilon + L(\varepsilon + e(0))]^\alpha\, e^{\beta M_\varepsilon(t) - \frac{\beta^2}{2}\langle M_\varepsilon\rangle(t)}, \qquad t\in[0,T]$$

is a martingale for every $\alpha,\beta>0$ by the Novikov criterion. Now, we get the claim by letting $\varepsilon\downarrow 0$ (Fatou's lemma on the left hand side and Lebesgue's dominated convergence theorem on the right hand side) and $\delta\uparrow\infty$ (Levi's theorem on the left hand side) in (10.5). □

With the notation λ_T defined in (2.3), given $r>0$, let T_r be the smallest radius of the base of a backward cone

$$\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d : |x| + t\lambda_T \le T\}$$

that contains (houses) the cylinder $[0,r]\times B_r$ and, given $m\in\mathbb{N}$, let r_m be the radius of the largest cylinder $[0,r]\times B_r$ for which the radius of the housing backward cone

$$\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^d : |x| + t\lambda_{T_r} \le T_r\}$$

is not larger than m, i.e. $T_r\le m$. We can define these radii by

$$T_r = \inf\left\{T>0 : \frac{T}{1+\lambda_T}\ge r\right\}, \qquad r_m = \sup\{r>0 : T_r\le m\}. \tag{10.6}$$

Remark 10.2. Observe that $T_r<\infty$ for every $r>0$ and $r_m\in(0,m]$ satisfy $r_m\uparrow\infty$ by (2.1).

Given $r>0$, let use define extension operators

$$\begin{aligned} E_r\varphi(x) &= \varphi(x), && |x|<r \\ E_r\varphi(x) &= -\eta_r(x)\varphi(\mathscr{P}_r(x)), && |x|>r \end{aligned} \tag{10.7}$$

$$E_r^*\psi(x) = \psi(x) - \frac{r^{2d}}{|x|^{2d}}\eta_r(\mathscr{P}_r(x))\psi(\mathscr{P}_r(x)), \qquad 0<|x|<r$$

for $\varphi : B_r\to\mathbb{R}^n$ and $\psi:\mathbb{R}^d\to\mathbb{R}^n$ where $\eta_r(x)=\eta(x/r)$ and η is a smooth $[0,1]$-valued function such that $\eta(x)=1$ if $|x|\le 1$ and $\eta(x)=0$ if $|x|\ge 2$ and $\mathscr{P}_r(x) = r^2|x|^{-2}x$.

Lemma 10.3. *For every $p\in[1,\infty]$, the operator*

- *E_r maps $L^p(B_r)$ continuously to $L^p(\mathbb{R}^d)$,*
- *E_r^* maps $L^p(\mathbb{R}^d)$ continuously to $L^p(B_r)$,*
- *E_r^* maps $W^{1,2}(\mathbb{R}^d)$ continuously to $W_0^{1,2}(B_r)$,*

$$\|E_r\|_{\mathscr{L}(L^p(B_r),L^p)} + \|E_r^*\|_{\mathscr{L}(L^p,L^p(B_r))} \le c_{d,p}, \qquad \|E_r^*\|_{\mathscr{L}(W^{1,2},W_0^{1,2}(B_r))} \le c_d\left(1+\frac{1}{r}\right)$$

and

$$\int_{\mathbb{R}^d}\langle E_r\varphi,\psi\rangle_{\mathbb{R}^n}\,dx = \int_{B_r}\left\langle\varphi,E_r^*\psi\right\rangle_{\mathbb{R}^n}\,dx$$

hold for every $\varphi\in L^p(B_r)$, $\psi\in L^q(\mathbb{R}^d)$ whenever $p,q\in[1,\infty]$ are Hölder conjugate exponents.

Lemma 10.4. *Let $\kappa \in \mathbb{R}_+$, $R > 0$, $\delta > 0$, $d_* = \left[\frac{d}{2}\right] + 1$, $\gamma > 0$, $p \in (1,\infty)$ such that $\gamma + \frac{2}{p} < \frac{1}{2}$. Then there exists a constant $\tilde{\rho} \in \mathbb{R}_+$ depending only on R, d, p, κ, r_0, δ, γ and $\mathbf{c}$ (see Section 2 and (10.6)) such that the following holds. If $f^i, g^i : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $f^{d+1}, g^{d+1} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable functions satisfying the assumptions of Lemma 9.1, $m \in \mathbb{N}$, z is an $\mathscr{H}$-continuous solution of (9.1), $F : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ satisfies the assumptions (a)-(c) in Proposition 8.1 and, for every $y \in \mathbb{R}^n$, the inequalities (10.1)-(10.3) hold for a.e. $w \in B_{T_{R\wedge r_m}}$. If, for every $y \in \mathbb{R}^n$,*

$$|f^{d+1}(w,y)| \le \kappa F(w,y) \qquad \text{for a.e.} \quad w \in B_{r_m \wedge R} \tag{10.8}$$

and $\mathbf{F}_r = \mathbf{F}_{\lambda_{T_r},0,T_r}$ is the conic energy function defined as in (2.2) for the function F then

$$\mathbb{E}\left\{\mathbf{1}_{[\mathbf{F}_{r_m\wedge R}(0,z(0))\le\delta]} \|E_{r_m} v(\cdot \wedge r_m)\|^p_{C^\gamma([0,R],\mathbb{W}_R^{-d_*,2})}\right\} \le \tilde{\rho}$$

where the space $\mathbb{W}_R^{-d_*,2}$ is defined in Appendix B.

Proof. There is

$$\begin{array}{rll} \|E_{r_m} h\|_{\mathbb{W}_R^{-d_*,2}} & \le c_1 \|h\|_{L^2(B_{r_m\wedge R})}, & h \in L^2_{loc} \\ \|E_{r_m} h\|_{\mathbb{W}_R^{-d_*,2}} & \le c_2 \|h\|_{L^1(B_{r_m\wedge R})}, & h \in L^1_{loc} \\ \|E_{r_m} \mathscr{A}^m h\|^2_{\mathbb{W}_R^{-d_*,2}} & \le c_3^2 Q_{r_m\wedge R}(h,h), & h \in W^{2,2}_{loc} \end{array} \tag{10.9}$$

where

$$c_1 = 1, \quad c_2 = \| \subseteq \|_{\mathscr{L}(W^{d_*,2},L^\infty)}, \quad c_3 = \lambda_R$$

if $R \le r$,

$$c_1 = \|E_r\|_{\mathscr{L}(L^2(B_r),L^2)},\ c_2 = \| \subseteq \|_{\mathscr{L}(W^{d_*,2},L^\infty)} \|E_r^*\|_{\mathscr{L}(L^\infty,L^\infty(B_r))},\ c_3 = \lambda_r \|E_r^*\|_{\mathscr{L}(W^{1,2};W_0^{1,2}(B_r))}$$

if $R > r$ and

$$Q_\delta(h^1,h^2) = \sum_{i=1}^d \sum_{j=1}^d \int_{B_\delta} \mathbf{a}_{ij} \left\langle \frac{\partial h^1}{\partial x_i}, \frac{\partial h^2}{\partial x_j} \right\rangle_{\mathbb{R}^n} dw.$$

Observe that $\max\{c_1,c_2,c_3\}$ can be dominated by a constant c that depends only on d, R and r_0 by Lemma 10.3. Let us prove just the third inequality in case $R > r$ as the other cases are straightforward. For let $\varphi \in W_R^{d_*,2}$. Then

$$\begin{aligned} \langle E_r \mathscr{A}^m h, \varphi \rangle &= \int_{\mathbb{R}^d} \langle E_r \mathscr{A}^m h, \varphi \rangle_{\mathbb{R}^n} \, dw = \int_{B_r} \langle \mathscr{A}^m h, \psi \rangle_{\mathbb{R}^n} \, dw \\ &= \sum_{i=1}^d \sum_{j=1}^d \int_{B_r} \left[\frac{\partial}{\partial x_j} \left\{ \mathbf{a}_{ij} \left\langle \frac{\partial h}{\partial x_i}, \psi \right\rangle_{\mathbb{R}^n} \right\} - \mathbf{a}_{ij} \left\langle \frac{\partial h}{\partial x_i}, \frac{\partial \psi}{\partial x_i} \right\rangle_{\mathbb{R}^n} \right] dw \\ &= -\sum_{i=1}^d \sum_{j=1}^d \int_{B_r} \mathbf{a}_{ij} \left\langle \frac{\partial h}{\partial x_i}, \frac{\partial \psi}{\partial x_i} \right\rangle_{\mathbb{R}^n} dw \end{aligned}$$

where $\psi = E_r^* \varphi \in W_0^{1,2}(B_r)$ by Lemma 10.3. So

$$\begin{aligned}
|\langle E_r \mathscr{A}^m h, \varphi\rangle| &\le \lambda_r Q_r^{\frac{1}{2}}(h,h)\|\psi\|_{W_0^{1,2}(B_r)}\\
&\le \lambda_r Q_r^{\frac{1}{2}}(h,h)\|E_r^*\|_{\mathscr{L}(W^{1,2},W_0^{1,2}(B_r))}\|\varphi\|_{W^{1,2}}\\
&\le c_3\|\varphi\|_{W^{d_*,2}}.
\end{aligned}$$

If we define the processes

$$\begin{aligned}
I_1(t) &= v(0), & I_2(t) &= \int_0^t u(s)\,ds,\\
I_3(t) &= \int_0^t \mathbf{1}_{B_m} f(\cdot, z(s), \nabla u(s))\,ds, & I_4(t) &= \int_0^t \mathbf{1}_{B_m} g(\cdot, z(s), \nabla u(s))\,dW_s
\end{aligned}$$

where the integral I_2 converges in $W^{1,2}$ and the integrals I_3, I_4 converge in L^2 then

$$\mathbb{P}\left[I_2(t) \in W^{2,2}\right] = 1, \qquad \mathbb{P}\left[v(t) = I_1(t) + \mathscr{A}^m I_2(t) + I_3(t) + I_4(t)\right] = 1, \qquad t \in \mathbb{R}_+$$

by the Chojnowska-Michalik theorem (see [9] or Theorem 13 in [33]). Since $E_{r_m} \circ \mathscr{A}^m$ can be extended to a linear continuous operator from $W^{1,2}$ to $\mathbb{W}_R^{-d_*,2}$ by (10.9),

$$E_{r_m} v(t) = E_{r_m} I_1(t) + E_{r_m} \mathscr{A}^m I_2(t) + E_{r_m} I_3(t) + E_{r_m} I_4(t), \qquad t \in \mathbb{R}_+$$

in $\mathbb{W}_R^{-d_*,2}$ a.s. Since

$$\begin{aligned}
\|f(\cdot, z(t), \nabla u(t))\|_{L^1(B_{r_m \wedge R})} &\le c_d R^{\frac{d}{2}} \left\| f^0(\cdot, u(t)) v(t) + \sum_{i=1}^d f^i(\cdot, u(t)) u_{x_i}(t) \right\|_{L^2(B_{r_m \wedge R})}\\
&+ \|f^{d+1}(\cdot, u(t))\|_{L^1(B_{r_m \wedge R})}\\
&\le c_d R^{\frac{d}{2}} (1 + 2^{\frac{1}{2}}) \kappa^{\frac{1}{2}} \mathbf{F}_{r_m \wedge R}^{\frac{1}{2}}(t, z(t)) + \kappa \mathbf{F}_{r_m \wedge R}(t, z(t))\\
&\le c_{R,d,\kappa}\left[1 + \mathbf{F}_{r_m \wedge R}(t, z(t))\right]\\
\|g(\cdot, z(t), \nabla u(t))\|_{L^2(B_{r_m \wedge R})}^2 &\le 12 \kappa \mathbf{F}_{r_m \wedge R}(t, z(t))
\end{aligned}$$

holds for every $t\in[0,r_m\wedge R]$, we get, using (10.9),

$$\begin{aligned}
\|E_{r_m}I_1\|_{C^\gamma([0,r_m\wedge R],\mathbb{W}_R^{-d_*,2})} &\le c_1\|v(0)\|_{L^2(B_{r_m\wedge R})}\le c_1 2^{\frac12}\sup_{t\in[0,r_m\wedge R]}\mathbf{F}^{\frac12}_{r_m\wedge R}(t,z(t)),\\
\|E_{r_m}\mathscr{A}^m I_2\|_{C^\gamma([0,r_m\wedge R],\mathbb{W}_R^{-d_*,2})} &\le c_3(1+R^\gamma)\sup_{0\le s<t\le r_m\wedge R}\frac{Q^{\frac12}_{r_m\wedge R}\left(\int_s^t u(b)\,db,\int_s^t u(b)\,db\right)}{(t-s)^\gamma}\\
&\le c_3(1+R^\gamma)\sup_{0\le s<t\le r_m\wedge R}\frac{\int_s^t Q^{\frac12}_{r_m\wedge R}(u(b),u(b))\,db}{(t-s)^\gamma}\\
&\le c_3(1+R^\gamma)R^{1-\gamma}\sup_{t\in[0,r_m\wedge R_m]}\mathbf{F}^{\frac12}_{r_m\wedge R}(t,z(t))\\
\|E_{r_m}I_3\|_{C^\gamma([0,r_m\wedge R],\mathbb{W}_R^{-d_*,2})} &\le c_2(1+R^\gamma)R^{1-\gamma}\sup_{t\in[0,r]}\|f(\cdot,z(t),\nabla u(t))\|_{L^1(B_{r_m\wedge R})}\\
&\le c_{R,d,\kappa,\gamma}[1+\sup_{t\in[0,r_m\wedge R]}\mathbf{F}_{r_m\wedge R}(t,z(t))]\\
\mathbb{E}\left\{\mathbf{1}_{\Omega_0}\|E_{r_m}I_4\|^p_{C^\gamma([0,r_m\wedge R],\mathbb{W}_R^{-d_*,2})}\right\} &\le c_1\mathbb{E}\left\{\mathbf{1}_{\Omega_0}\|I_4\|^p_{C^\gamma([0,r_m\wedge R],L^2(B_{r_m\wedge R}))}\right\}\\
&\le c_{d,p,\gamma,R}\mathbb{E}\left\{\mathbf{1}_{\Omega_0}\int_0^{r_m\wedge R}\|g(\cdot,z,\nabla u)\|^p_{\mathscr{L}_2(H_\mu,L^2(B_{r_m\wedge R}))}\,ds\right\}\\
&\le \mathbf{c}^p c_{d,p,\gamma,R}\mathbb{E}\left\{\mathbf{1}_{\Omega_0}\int_0^{r_m\wedge R}\|g(\cdot,z,\nabla u)\|^p_{L^2(B_{r_m\wedge R})}\,ds\right\}\\
&\le (12\kappa)^{\frac p2}R\mathbf{c}^p c_{d,p,\gamma,R}\mathbb{E}\left\{\mathbf{1}_{\Omega_0}\sup_{t\in[0,r_m\wedge R]}\mathbf{F}^{\frac p2}_{r_m\wedge R}(t,z(t))\right\}
\end{aligned}$$

where $\Omega_0=[\mathbf{F}_{r_m\wedge R}(0,z(0))\le\delta]$, the estimate of I_4 follows from the Garsia-Rodemich-Rumsey lemma [16], the Burkholder inequality (see e.g. [33]) and Lemma 3.3. Altogether,

$$\mathbb{E}\left\{\mathbf{1}_{[\mathbf{F}_{r_m\wedge R}(0,z(0))\le\delta]}\|E_{r_m}v\|^p_{C^\gamma([0,r_m\wedge R],\mathbb{W}_R^{-d_*,2})}\right\}\le c_{d,R,r_0,\mathbf{c},\kappa,p,\gamma,\delta}$$

by the inequality (10.4). □

11 Compactness

The present section is the actual core of the paper. We list all preliminary results and assumptions prepared and discussed in previous parts of the paper (Section 11.1) and then we carry out, in a few steps, *the refined stochastic compactness method* as indicated in Section 6. That is to say, we prove tightness of a sequence of solutions of approximating equations (Section 11.2), then we verify the assumptions of the Jakubowski-Skorokhod theorem A.1 and prove that the limit process yielded by this theorem is the solution of (1.1) claimed in Theorem 5.1 (Sections 11.3 - 11.8), whereas the proof of Theorem 5.2 is given simultaneously in Section 11.6.

11.1 Assumptions

Let

i) μ be a finite spectral measure on $\mathbb{R}^d$,

ii) Θ be a Borel probability measure on $\mathscr{H}_{loc}$,

iii) for every $m \in \mathbb{N}$ and $i \in \{0, \ldots, d\}$,

$$f_m^i, g_m^i : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{n\times n} \quad \text{and} \quad f_m^{d+1}, g_m^{d+1} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$$

be measurable functions satisfying the assumptions of Lemma 9.1,

iv) for every $m \in \mathbb{N}$, $F_m : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ be measurable functions satisfying the assumptions (a)-(c) of Proposition 8.1,

v) $\kappa \in \mathbb{R}_+$ be such that (10.1)-(10.3) and (10.8) hold for $(f_m^i, g_m^i : i \in \{0, \ldots, d+1\})$, F_m and $y \in \mathbb{R}^n$ for a.e. $w \in B_m$, for every $m \in \mathbb{N}$,

vi) for $i \in \{0, \ldots, d\}$,

$$f^i, g^i : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{n\times n}, \quad f^{d+1}, g^{d+1} : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n, \quad F : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$$

be measurable functions such that, for a.e. $w \in \mathbb{R}^d$, there is

$$f_m^i(w, \cdot) \to f^i(w, \cdot), \qquad g_m^i(w, \cdot) \to g^i(w, \cdot), \qquad F_m(w, \cdot) \to F(w, \cdot)$$

uniformly on compact sets in $\mathbb{R}^n$ for every $i \in \{0, \ldots, d+1\}$,

vii) with the notation (4.1), (4.1), $z^m = (u^m, v^m)$ be $\mathscr{H}$-continuous $(\mathscr{F}_t^m)$-adapted solutions of

$$u_{tt} = \Delta u + \mathbf{1}_{B_m} f_m(\cdot, z, \nabla u) + \mathbf{1}_{B_m} g_m(\cdot, z, \nabla u)\, dW^m$$

on completely filtered stochastic bases $(\Omega^m, \mathscr{F}^m, (\mathscr{F}_t^m), \mathbb{P}^m)$ for some spatially homogeneous $(\mathscr{F}_t^m)$-Wiener processes W^m with spectral measure μ, such that $z^m(0)$ is supported on some ball in $\mathscr{H}$ (with a radius dependent on m) for every $m \in \mathbb{N}$ (see Lemma 9.1) and

$$\lim_{m\to\infty} \left\| \mathbb{P}^m \left[\pi_R(z^m(0)) \in \cdot \right] - \Theta \left[\pi_R \in \cdot \right] \right\| = 0 \tag{11.1}$$

holds for every $R > 0$ where the norm is taken in the total variation of measures on $\mathscr{B}(\mathscr{H}_R)$ and $\pi_R : \mathscr{H} \to \mathscr{H}_R$ is the restriction operator (see Section 2),

viii) it hold that

$$\Theta \{(u, v) \in \mathscr{H}_{loc} : \|F^*(\cdot, u)\|_{L^1(B_R)} < \infty\} = 1, \qquad R > 0 \tag{11.2}$$

where $F^* = \sup F_m$,

ix) given $r > 0$, $\mathscr{E}_r$ be an extension operator on $L^2(B_r)$ and on $W^{1,2}(B_r)$, i.e.

$$\mathscr{E}_r : L^2(B_r) \to L^2(\mathbb{R}^d), \qquad \mathscr{E}_r : W^{1,2}(B_r) \to W^{1,2}(\mathbb{R}^d) \tag{11.3}$$

are linear continuous operators such that $\mathscr{E}_r h = h$ a.e. on B_r,

x) us define the processes

$$Z^m(t) = (U^m(t), V^m(t)) := (\mathscr{E}_{r_m} u^m(t \wedge r_m), E_{r_m} v^m(t \wedge r_m)), \qquad t \in \mathbb{R}_+$$

where r_m and E_{r_m} were defined in (10.6) and (10.7),

xi) it hold that

$$\int_{B_R} \sup \{F^*(w,y) : |y| \le R\}\, dw < \infty, \qquad R > 0, \tag{11.4}$$

xii) given $r > 0$, there exist $\alpha_R = \alpha_{r,R}$ for $R > 0$ such that $\lim_{R\to\infty} \alpha_R = 0$ and

$$|f_m^{d+1}(w,y)| \le \alpha_R F_m(w,y), \qquad |y| \ge R \tag{11.5}$$

holds for every $m \in \mathbb{N}$ and almost every $w \in B_r$.

Remark 11.1. Observe that (11.1) implies $\mathbb{P}^m\left[Z^m(0) \in \cdot\right] \to \Theta$ weakly on $\mathscr{H}_{loc}$. Indeed, let $\mathscr{E}_r : \mathscr{H}_r \to \mathscr{H}$ be a continuous linear extension operator, i.e. $\mathscr{E}_r z = z$ a.e. on B_r, and let $\varphi : \mathscr{H}_{loc} \to [0,1]$ be a uniformly continuous function. Then, for every $\varepsilon > 0$, there is $r > 0$ such that

$$|\varphi(z) - \varphi(\mathscr{E}_r(\pi_r(z)))| \le \varepsilon, \qquad z \in \mathscr{H}_{loc}.$$

Thus

$$\lim_{m\to\infty} \int_{\mathscr{H}_{loc}} \varphi(Z^m(0))\, d\mathbb{P}^m = \int_{\mathscr{H}_{loc}} \varphi\, d\Theta$$

and the claim follows from Theorem 2.1 in [1].

Remark 11.2. Observe also that, given $R > 0$, the real valued sequence $\|F_m(\cdot, U^m(0))\|_{L^1(B_R)}$ is tight in $\mathbb{R}_+$. Indeed, there is

$$\begin{aligned}
\mathbb{P}^m\left[\|F_m(\cdot, U^m(0))\|_{L^1(B_R)} > \delta\right] &= \mathbb{P}^m\left[\|F_m(\cdot, \pi_R(u^m(0)))\|_{L^1(B_R)} > \delta\right] \\
&\le \|\mathbb{P}^m\left[\pi_R(z^m(0)) \in \cdot\right] - \Theta\left[\pi_R \in \cdot\right]\| \\
&+ \Theta\left\{(u,v) : \|F_m(\cdot,u)\|_{L^1(B_R)} > \delta\right\} \\
&\le \varepsilon_{R,m} + \Theta\left\{(u,v) : \|F^*(\cdot,u)\|_{L^1(B_R)} > \delta\right\}
\end{aligned}$$

for $m \in \mathbb{N}$ such that $r_m \ge R$ where $\varepsilon_{R,m} \to 0$ by (11.1). Tightness follows from (11.2).

11.2 Tightness

Lemma 11.3. *The sequence of processes (Z^m) is tight in $\mathscr{Z} = C_w(\mathbb{R}_+, W^{1,2}_{loc}) \times C_w(\mathbb{R}_+, L^2_{loc})$.*

Proof. Let $\varepsilon \in (0,1)$, let us define

$$\tilde{F}_m(w,y) = F_m(w,y) + |y|^2/2, \qquad \tilde{F}^*(w,y) = F^*(w,y) + |y|^2/2$$

and consider their conic energy functions

$$\tilde{\mathbf{F}}_{m,k} = (\tilde{\mathbf{F}}_m)_{\lambda_{T_{k\wedge r_m}},0,T_{k\wedge r_m}}, \qquad \tilde{\mathbf{F}}^*_k = (\tilde{\mathbf{F}}^*)_{0,0,T_k}$$

for $k \in \mathbb{N}$ defined as in (2.2) with the notation (2.3) and (10.6). Let also $p \in (1,\infty)$ and $\gamma \in (0,1)$ satisfy $\gamma + \frac{2}{p} < \frac{1}{2}$ and let $d_* = \left[\frac{d}{2}\right] + 1$. Since

$$
\begin{aligned}
|g_m^{d+1}(w,y)|^2 \quad &+ \quad |\nabla_y \tilde{F}_m(w,y) + g_m^{d+1}(w,y)|^2 \\
&= \quad |g_m^{d+1}(w,y)|^2 + |\nabla_y F_m(w,y) + y + g_m^{d+1}(w,y)|^2 \\
&\leq \quad 2|g_m^{d+1}(w,y)|^2 + 2|\nabla_y F_m(w,y) + g_m^{d+1}(w,y)|^2 + 2|y|^2 \\
&\leq \quad (2\kappa + 4)\tilde{F}_m(w,y)
\end{aligned}
$$

the assumptions (5.4)-(10.3), (10.8) are satisfied for κ, $\tilde{F}_m$ for every $y \in \mathbb{R}^n$ and a.e. $w \in B_m$ for the constant $\tilde{k}$ which depends on κ and p, and so Lemma 10.1 and Lemma 10.4 applied on $\tilde{F}_m$ and $L(x) = x^{\frac{p}{2}}$ yield, for every $\delta > 0$,

$$
\begin{aligned}
\int_{[\tilde{\mathbf{F}}_{m,k}(0,z^m(0)) \leq \delta]} \sup_{t \in [0,k\wedge r_m]} \tilde{\mathbf{F}}_{m,k}^{\frac{p}{2}}(t, z^m(t))\, d\mathbb{P}^m \quad &\leq \quad 4e^{\rho k} \int_{[\tilde{\mathbf{F}}_{m,k}(0,z^m(0)) \leq \delta]} \tilde{\mathbf{F}}_{m,k}^{\frac{p}{2}}(0, z^m(0))\, d\mathbb{P}^m \\
&\leq \quad 4e^{\rho k} \delta^{\frac{p}{2}}
\end{aligned}
$$

where $\rho = \rho_{\mathbf{c},\kappa,p}$ so

$$
\int_{[\tilde{\mathbf{F}}_{m,k}(0,z^m(0)) \leq \delta]} \left\{ \sup_{t \in [0,k]} \|U^m(t)\|^p_{W^{1,2}(B_k)} + \sup_{t \in [0,k]} \|V^m(t)\|^p_{L^2(B_k)} \right\} d\mathbb{P}^m \leq C_{k,\delta} \tag{11.6}
$$

as

$$
\sup_{t \in [0,k]} \|U^m(t)\|_{W^{1,2}(B_k)} \leq \max\{1, \tilde{\alpha}_k\} \sup_{t \in [0,k\wedge r_m]} \|u^m(t)\|_{W^{1,2}(B_{k\wedge r_m})},
$$

$$
\sup_{t \in [0,k]} \|V^m(t)\|_{L^2(B_k)} \leq \max\{1, \tilde{\alpha}_k\} \sup_{t \in [0,k\wedge r_m]} \|v^m(t)\|_{L^2(B_{k\wedge r_m})},
$$

$$
\|u^m(t)\|^2_{W^{1,2}(B_{k\wedge r_m})} + \|v^m(t)\|^2_{L^2(B_{k\wedge r_m})} \leq 2\max\{\alpha_k, 1\} \tilde{\mathbf{F}}_{m,k}(t, z^m(t)), \qquad t \in [0, k \wedge r_m]
$$

where

$$
\alpha_k = \sup_{w \in B_{T_k}} \|\mathbf{a}^{-1}(w)\|,
$$

$$
\tilde{\alpha}_k = \max\{\|E_{r_m}\|_{\mathscr{L}(L^2(B_{r_m}), L^2(\mathbb{R}^d))}, \|\mathscr{E}_{r_m}\|_{\mathscr{L}(W^{1,2}(B_{r_m}), W^{1,2}(\mathbb{R}^d))} : m \in \mathbb{N},\, r_m \leq k\},
$$

and

$$
\int_{[\tilde{\mathbf{F}}_{m,k}(0,z^m(0)) \leq \delta]} \left\{ \|U^m\|^p_{C^\gamma([0,k], L^2(B_k))} + \|V^m\|^p_{C^\gamma([0,k], \mathbb{W}_k^{-d_*,2})} \right\} d\mathbb{P}^m \leq C_{k,\delta} \tag{11.7}
$$

by Lemma 10.4 for some $C_{k,\delta} \in \mathbb{R}_+$ depending also on $\mathbf{c}$, $\mathbf{a}$, p, d, γ, $(r_j)_{j\in\mathbb{N}}$, $(E_{r_j})_{j\in\mathbb{N}}$, $(\mathscr{E}_{r_j})_{j\in\mathbb{N}}$ and κ since

$$
\|U^m\|_{C^\gamma([0,k]; L^2(B_k))} \leq \max\{1, \tilde{\beta}_k\} \left[\|u^m(0)\|_{L^2(B_{k\wedge r_m})} + 2k \sup_{t \in [0,k\wedge r_m]} \|v^m(t)\|_{L^2(B_{k\wedge r_m})} \right]
$$

where

$$
\tilde{\beta}_k = \max\{\|\mathscr{E}_{r_m}\|_{\mathscr{L}(L^2(B_{r_m}), L^2(\mathbb{R}^d))} : m \in \mathbb{N},\, r_m \leq k\}.
$$

Since

$$\begin{aligned}
\mathbb{P}^m\left[\tilde{\mathbf{F}}_{m,k}(0,z^m(0))>\delta\right] &= \mathbb{P}^m\left[\tilde{\mathbf{F}}_{m,k}(0,\pi_{T_k}(z^m(0)))>\delta\right]\le\Theta\left[\tilde{\mathbf{F}}_{m,k}(0,\pi_{T_k})>\delta\right]\\
&+ \sup_{A\mathscr{B}(\mathscr{H}_{T_k})}\left|\mathbb{P}^m\left[\pi_{T_k}(z^m(0))\in A\right]-\Theta\left[\pi_{T_k}\in A\right]\right|\\
&= \varepsilon_{m,k}+\Theta\left[\tilde{\mathbf{F}}_{m,k}(0,\cdot)>\delta\right]\le\varepsilon_{m,k}+\Theta\left[\tilde{\mathbf{F}}^*_k(0,\cdot)>\delta\right]
\end{aligned}$$

holds for every $m,k\in\mathbb{N}$ and δ where the norms are in the total variation of measures on $\mathscr{H}_{T_k}$, taking (11.1) and (11.2) into account, we can find $\delta_k>0$ and $a_k>0$ so that

$$\mathbb{P}^m\left[\tilde{\mathbf{F}}_{m,k}(0,z^m(0))>\delta_k\right]\le\frac{\varepsilon}{3\cdot 4^k},\qquad a_k\ge\left[\frac{6\cdot 4^k\cdot 2^p\cdot C_{k,\delta_k}}{\varepsilon}\right]^{\frac{1}{p}}$$

holds for every $m,k\in\mathbb{N}$. Then

$$\mathbb{P}^m\left[\sup_{t\in[0,k]}\|U^m\|_{W^{1,2}(B_k)}+\|U^m\|_{C^\gamma([0,k];L^2(B_k))}>a_k\right]$$

$$\begin{aligned}
&\le \mathbb{P}^m\left[\tilde{\mathbf{F}}_{m,k}(0,z^m(0))>\delta_k\right]+\mathbb{P}^m\left[\tilde{\mathbf{F}}_{m,k}(0,z^m(0))\le\delta_k,\ \sup_{t\in[0,k]}\|U^m\|_{W^{1,2}(B_k)}>\frac{a_k}{2}\right]\\
&+ \mathbb{P}^m\left[\tilde{\mathbf{F}}_{m,k}(0,z^m(0))\le\delta_k,\ \|U^m\|_{C^\gamma([0,k];L^2(B_k))}>\frac{a_k}{2}\right]\\
&\le \frac{\varepsilon}{3\cdot 4^k}+\frac{2^p}{a_k^p}\int_{[\tilde{\mathbf{F}}_{m,k}(0,z^m(0))\le\delta_k]}\left\{\sup_{t\in[0,k]}\|U^m\|^p_{W^{1,2}(B_k)}+\|U^m\|^p_{C^\gamma([0,k];L^2(B_k))}\right\}d\mathbb{P}^m\le\frac{\varepsilon}{4^k}
\end{aligned}$$

by (11.6) and (11.7), and analogously

$$\mathbb{P}^m\left[\sup_{t\in[0,k]}\|V^m\|_{L^2(B_k)}+\|V^m\|_{C^\gamma([0,k];\mathbb{W}_k^{-d_*,2})}>a_k\right]\le\frac{\varepsilon}{4^k}.$$

If

$$\begin{aligned}
K_1 &= \left\{h\in C_w(\mathbb{R}_+;W^{1,2}_{loc}):\|h\|_{L^\infty((0,k);W^{1,2}(B_k))}+\|h\|_{C^\gamma([0,k];L^2(B_k))}\le a_k,\ k\in\mathbb{N}\right\}\\
K_2 &= \left\{h\in C_w(\mathbb{R}_+;L^2_{loc}):\|h\|_{L^\infty((0,k);L^2(B_k))}+\|h\|_{C^\gamma([0,k];\mathbb{W}_k^{-d_*,2})}\le a_k,\ k\in\mathbb{N}\right\}
\end{aligned}$$

then $K_1\times K_2$ is compact in $\mathscr{X}$ by Corollary B.1 and

$$\mathbb{P}^m\left[Z^m\in K_1\times K_2\right]>1-\varepsilon,\qquad m\in\mathbb{N}.$$

□

11.3 Skorokhod representation

Since Z^m are tight in $\mathscr{Z}$ by Lemma 11.3,

$$(\|F_m(\cdot, U^m(0))\|_{L^1(T_k)})_{k\in\mathbb{N}}$$

are tight in $\mathbb{R}_+^{\mathbb{N}}$ (where T_k were defined in (10.6)) by Remark 11.2 and $\mathbb{P}^m\,[Z^m(0)\in\cdot]$ converge to Θ weakly on $\mathscr{H}_{loc}$ by Remark 11.1, i.e. $Z^m(0)$ are tight in $\mathscr{H}_{loc}$ by the Prokhorov theorem, fixing an ONB (e_l) in H_μ, we may apply Theorem A.1 on the sequence

$$(Z^m(0), Z^m, (W^m(e_l))_l, (\|F_m(\cdot, U^m(0))\|_{L^1(T_k)})_{k\in\mathbb{N}}) : \Omega^m \to \mathscr{H}_{loc} \times \mathscr{Z} \times C_0(\mathbb{R}_+; \mathbb{R}^{\dim H_\mu}) \times \mathbb{R}_+^{\mathbb{N}}$$

to claim that there exist

- a probability space $(\Omega, \mathscr{F}, \mathbb{P})$,
- a subsequence m_j,
- $C(\mathbb{R}_+; \mathscr{H})$-valued random variables $\mathbf{z}^j = (\mathbf{u}^j, \mathbf{v}^j)$ defined on Ω,
- $C_0(\mathbb{R}_+; \mathbb{R}^{\dim H_\mu})$-valued random variables $\beta^j = (\beta_l^j)$, $\beta = (\beta_l)$ defined on Ω,
- $\mathbb{R}_+^{\mathbb{N}}$-valued random variable $\nu = (\nu_k)_{k\in\mathbb{N}}$ defined on Ω,
- a $\mathscr{Z}$-valued random variable $\mathbf{z} = (\mathbf{u}, \mathbf{v})$ with σ-compact range defined on Ω

such that

(i) $(Z^{m_j}, (W^{m_j}(e_l))_l)$ has the same law under $\mathbb{P}^{m_j}$ as $(\mathbf{z}^j, \beta^j)$ under $\mathbb{P}$ on the space

$$\mathscr{B}(C(\mathbb{R}_+; \mathscr{H}) \times C_0(\mathbb{R}_+; \mathbb{R}^{\dim H_\mu}))$$

for every $j \in \mathbb{N}$,

(ii) $(\mathbf{z}^j, \beta^j)$ converges to $(\mathbf{z}, \beta)$ on Ω in the topology of $\mathscr{Z} \times C_0(\mathbb{R}_+; \mathbb{R}^{\dim H_\mu})$,

(iii) $\mathbf{z}^j(0)$ converges to $\mathbf{z}(0)$ on Ω in $\mathscr{H}_{loc}$,

(iv) $\|F_{m_j}(\cdot, \mathbf{u}^j(0))\|_{L^1(B_{T_k})}$ converges to ν_k for every $k \in \mathbb{N}$ on Ω.

Definition 11.4. *We also define, for completness,*

$$\tilde{\nu}_k = \nu_k + \frac{1}{2}\int_{B_{T_k}} \left[\sum_{i=1}^d \sum_{j=1}^d \mathbf{a}_{ij} \left\langle \frac{\partial \mathbf{u}(0)}{\partial x_i}, \frac{\partial \mathbf{u}(0)}{\partial x_j} \right\rangle_{\mathbb{R}^n} + |\mathbf{u}(0)|^2_{\mathbb{R}^n} + |\mathbf{v}(0)|^2_{\mathbb{R}^n}\right] dx \tag{11.8}$$

for $k \in \mathbb{N}$.

11.4 Property of β

Let us define

$$\mathscr{F}_t = \sigma\left(\sigma(v, \mathbf{z}(s), \beta(s) : s \le t) \cup \{N \in \mathscr{F} : \mathbb{P}(N) = 0\}\right), \qquad t \ge 0.$$

Apparently, the filtration $(\mathscr{F}_t)$ is complete. The proof of the following Lemma is analogous to the proof of Lemma 9.9.

Lemma 11.5. *The processes $\beta_1, \beta_2, \beta_3, \dots$ are independent standard $(\mathscr{F}_t)$-Wiener processes.*

Corollary 11.6. *The cylindrical process*

$$W_t(\xi) = \sum_l \beta_l(t)\langle \xi, e_l\rangle_{H_\mu}, \qquad \xi \in H_\mu, \quad t \ge 0$$

is a spatially homogeneous $(\mathscr{F}_t)$-Wiener process with spectral measure μ.

11.5 Property of u

Lemma 11.7. *There is*

$$\langle \mathbf{u}(t), \varphi\rangle_{L^2} = \langle \mathbf{u}(0), \varphi\rangle_{L^2} + \int_0^t \langle \mathbf{v}(s), \varphi\rangle_{L^2}\, ds, \qquad t \ge 0$$

almost surely for every $\varphi \in \mathscr{D}$.

Proof. If φ is supported in $B_{r_{m_j}}$ and $t \in [0, r_{m_j}]$ then

$$\begin{aligned} \left\langle U^{m_j}(t), \varphi\right\rangle_{L^2} - \left\langle U^{m_j}(0), \varphi\right\rangle_{L^2} &= \left\langle u^{m_j}(t), \varphi\right\rangle_{L^2} - \left\langle u^{m_j}(0), \varphi\right\rangle_{L^2} \\ &= \int_0^t \left\langle v^{m_j}(s), \varphi\right\rangle_{L^2}\, ds = \int_0^t \left\langle V^{m_j}(s), \varphi\right\rangle_{L^2}\, ds \end{aligned}$$

The rest of the proof is analogous with the proof of Lemma 9.8. □

11.6 Energy estimates

Lemma 11.8. *Let $T > 0$ and $x \in \mathbb{R}^d$, let $G^m : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ satisfy the assumptions (a)-(c) in Proposition 8.1, let $G : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function such that $G^m(w, \cdot)$ converges to $G(w, \cdot)$ uniformly on compact sets in $\mathbb{R}^n$ for a.e. $w \in \mathbb{R}^d$, let $\tilde{\kappa} \in \mathbb{R}_+$ be such that*

$$|g_m^{d+1}(w, y)|^2 + |\nabla_y G^m(w, y) + f_m^{d+1}(w, y)|^2 \le \tilde{\kappa} G^m(w, y), \qquad y \in \mathbb{R}^n, \quad m \ge |x| + T$$

holds for a.e. $w \in B(x, T)$, let λ satisfy (5.3), let $L : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function in $C^2(0, \infty)$ satisfying (5.4) with $\tilde{\kappa}$ and define $G^* = \sup_{m \in \mathbb{N}} G^m$. Assume that

$$\Theta\left\{(u, v) \in \mathscr{H}_{loc} : \|G^*(\cdot, u)\|_{L^1(B_R)} < \infty\right\} = 1, \qquad R > 0. \tag{11.9}$$

Then, for $A \in \mathscr{B}(\mathscr{H}_{loc})$ *and with the convention* $0 \cdot \infty = 0$*,*

$$\mathbb{E}\left\{\mathbf{1}_A(\mathbf{z}^j(0)) \sup_{r\in[0,t]} L(\mathbf{G}^{m_j}(r,\mathbf{z}^j(r)))\right\} \le 4e^{\rho t}\mathbb{E}\left\{\mathbf{1}_A(\mathbf{z}^j(0))L(\mathbf{G}^{m_j}(0,\mathbf{z}^j(0)))\right\} \tag{11.10}$$

holds for every $t \in [0, \min\{r_{m_j}, T/\lambda\}]$*,* $j \in \mathbb{N}$ *such that* $r_{m_j} \ge |x| + T$ *and*

$$\mathbb{E}\left\{\mathbf{1}_A(\mathbf{z}(0)) \sup_{r\in[0,t]} L(\mathbf{G}(r,\mathbf{z}(r)))\right\} \le 4e^{\rho t}\mathbb{E}\left\{\mathbf{1}_A(\mathbf{z}(0))L(\mathbf{G}(0,\mathbf{z}(0)))\right\} \tag{11.11}$$

holds for every $t \in [0, T/\lambda]$ *where* ρ *depends only on* $\mathbf{c}$ *and* $\max\{\kappa,\tilde{\kappa}\}$ *and* $\mathbf{G}^m = \mathbf{G}^m_{\lambda,x,T}$ *and* $\mathbf{G} = \mathbf{G}_{\lambda,x,T}$ *are the conic energy functions for* G^m *and* G *defined as in (2.2).*

Proof. The inequality (11.10) follows from (10.4) in Lemma 10.1 and (**i**) in Section 11.3 as

$$\begin{aligned}
\mathbb{E}\left\{\mathbf{1}_A(\mathbf{z}^j(0)) \sup_{r\in[0,t]} L(\mathbf{G}^{m_j}(r,\mathbf{z}^j(r)))\right\} &= \mathbb{E}^{m_j}\left\{\mathbf{1}_A(Z^{m_j}(0)) \sup_{r\in[0,t]} L(\mathbf{G}^{m_j}(r,Z^{m_j}(r)))\right\} \\
&= \mathbb{E}^{m_j}\left\{\mathbf{1}_A(Z^{m_j}(0)) \sup_{r\in[0,t]} L(\mathbf{G}^{m_j}(r,z^{m_j}(r)))\right\} \\
&\le 4e^{\rho t}\mathbb{E}^{m_j}\left\{\mathbf{1}_A(Z^{m_j}(0))L(\mathbf{G}^{m_j}(0,z^{m_j}(0)))\right\} \\
&= 4e^{\rho t}\mathbb{E}^{m_j}\left\{\mathbf{1}_A(Z^{m_j}(0))L(\mathbf{G}^{m_j}(0,Z^{m_j}(0)))\right\} \\
&= 4e^{\rho t}\mathbb{E}\left\{\mathbf{1}_A(\mathbf{z}^j(0))L(\mathbf{G}^{m_j}(0,\mathbf{z}^j(0)))\right\}.
\end{aligned}$$

Let ψ be a continuous density on $\mathbb{R}$ with support in $(1,2)$, let ϕ be the second antiderivative of $-\psi$ (i.e. a $C^2(\mathbb{R})$-function such that $\phi'' = -\psi$ on $\mathbb{R}$) and $\phi(0) = 0$, $\phi'(0) = 1$. Then $\phi(t) = t$ on $(-\infty, 1]$, $\phi'' \le 0 \le \phi' \le 1$ on $\mathbb{R}$ and ϕ is constant on $[2,\infty)$. If we define $\phi_k(t) = k\phi(t/k)$ for $t \in \mathbb{R}$, $k \in \mathbb{N}$ then

- $\phi_k(t) = t$ on $(-\infty, k]$,
- $\phi_k'' \le 0 \le \phi_k' \le 1$ on $\mathbb{R}$,
- ϕ_k is constant on $[2k,\infty)$,
- $t\phi_k'(t) \le \phi_k(t)$ for $t \in \mathbb{R}_+$ (which holds by monotonicity of $\phi_k(t) - t\phi_k'(t)$ on $\mathbb{R}_+$).

holds for every $k \in \mathbb{N}$. Consequently, $L_k = L \circ \phi_k \in C(\mathbb{R}_+) \cap C^2(0,\infty)$ is nondecreasing and satisfies (5.4) with the constant $\tilde{\kappa}$ for every $k \in \mathbb{N}$. Hence, if $h : \mathscr{H}_{loc} \to [0,1]$ is continuous, π_r and $\mathscr{E}_r$ are extension operators as in Remark 11.1 and $r_{m_j} \ge \max\{|x| + T, T/\lambda\}$ then

$$\begin{aligned}
&\mathbb{E}\left\{h(\mathscr{E}_R(\pi_R(\mathbf{z}^j(0)))) \sup_{r\in[0,t]} L_k(\mathbf{G}^{m_j}(r,\mathbf{z}^j(r)))\right\} \\
\le\ & 4e^{\rho t}\mathbb{E}\left\{h(\mathscr{E}_R(\pi_R(\mathbf{z}^j(0))))L_k(\mathbf{G}^{m_j}(0,\mathbf{z}^j(0)))\right\} \\
\le\ & 4e^{\rho t}L(\phi(2))\left\|\mathbb{P}^{m_j}\left[\pi_R(z^{m_j}(0)) \in \cdot\right] - \Theta\left[\pi_R \in \cdot\right]\right\|_{\text{total variation on } \mathscr{B}(\mathscr{H}_R)} \\
+\ & 4e^{\rho t}\int_{\mathscr{H}_{loc}} h(\mathscr{E}_R(\pi_R(z)))L_k(\mathbf{G}^{m_j}(0,z))\,d\Theta
\end{aligned} \tag{11.12}$$

holds for every $k \in \mathbb{N}$, $R \in [0, r_{m_j}]$ and $t \in [0, T/\lambda]$ by (11.10). Applying Fatou's lemma on the LHS of (11.12) and the Lebesgue dominated convergence theorem on the RHS of (11.12), we obtain, letting $j \to \infty$,

$$\mathbb{E}\left\{h(\mathscr{E}_R(\pi_R(\mathbf{z}(0)))) \sup_{r\in[0,t]} L_k(\mathbf{G}(r,\mathbf{z}(r)))\right\} \le 4e^{\rho t}\int_{\mathscr{H}_{loc}} h(\mathscr{E}_R(\pi_R(z)))L_k(\mathbf{G}(0,z))\,d\Theta$$

for every $t \in [0, T/\lambda]$ and $m \in \mathbb{N}$ by (11.1), (11.9) and (**ii**), (**iii**) in Section 11.3 as L_k is a bounded nondecreasing continuous and eventually constant function. Since $\mathscr{E}_R \circ \pi_R : \mathscr{H}_{loc} \to \mathscr{H}_{loc}$ converges uniformly to identity on $\mathscr{H}_{loc}$ as $R \to \infty$, we get

$$\mathbb{E}\left\{h(\mathbf{z}(0)) \sup_{r\in[0,t]} L_k(\mathbf{G}(r,\mathbf{z}(r)))\right\} \le 4e^{\rho t}\int_{\mathscr{H}_{loc}} h(z)L_k(\mathbf{G}(0,z))\,d\Theta \tag{11.13}$$

for every $t \in [0, T/\lambda]$ and $k \in \mathbb{N}$ by the Lebesgue dominated convergence theorem. Consequently, (11.13) holds also for $h = \mathbf{1}_K$ where K is closed in $\mathscr{H}_{loc}$, whence also for every F_σ-set and every Borel set $K \subseteq \mathscr{H}_{loc}$ by regularity of $\Theta = \mathbb{P}[\mathbf{z}(0) \in \cdot]$ (Remark 11.1). The claim now follows from Fatou's lemma when letting $k \to \infty$, applied on the LHS, since $L_k \le L$ for every $k \in \mathbb{N}$, applied on the RHS. □

11.7 Martingale property

Let us remind the reader that the integrals in the following Proposition converge by the assumption v) in Section 11.1 and by (11.11).

Proposition 11.9. *Let $\varphi \in \mathscr{D}$. Then*

$$\begin{aligned}\langle \mathbf{v}(t), \varphi\rangle &= \langle \mathbf{v}(0), \varphi\rangle + \int_0^t \langle \mathbf{u}(r), \mathscr{A}\varphi\rangle\,dr + \int_0^t \langle f(\cdot, \mathbf{u}(r), \mathbf{v}(r), \nabla\mathbf{u}(r)), \varphi\rangle\,dr \\ &+ \int_0^t \langle g(\cdot, \mathbf{u}(r), \mathbf{v}(r), \nabla\mathbf{u}(r))\,dW_r, \varphi\rangle\end{aligned}$$

holds a.s. for every $t \ge 0$ where W was defined in Corollary 11.6.

Proof. Let $k \in \mathbb{N}$, let $\varphi \in \mathscr{D}$ have support in B_k and, throughout this proof, consider only $j \in \mathbb{N}$ such that $r_{m_j} \ge T_k$, i.e. $j \ge j_0$ for some j_0 and it holds that

$$k \le T_k \le r_{m_j} \le T_{r_{m_j}} \le m_j, \qquad j \ge j_0.$$

Fixing $0 \le s < t \le k$, we consider the sequence (φ_i) from Corollary C.1. Let also $J \in \mathbb{N}$, $0 \le s_1 \le$

$\cdots \leq s_J \leq s$, let $\mathscr{H} : (\mathbb{R}^2)^{J\times J} \times (\mathbb{R}^{\dim H_\mu})^J \times \mathbb{R}_+^{\mathbb{N}} \to [0,1]$ be a continuous function and define

$$\begin{aligned}
X_j^1 &= \left(\left\langle \begin{pmatrix} \mathscr{E}_{r_{m_j}} u^{m_j}(s_{i_0} \wedge r_{m_j}) \\ E_{r_{m_j}} v^{m_j}(s_{i_0} \wedge r_{m_j}) \end{pmatrix}, \varphi_{i_1} \right\rangle_{L^2} \right)_{i_0,i_1 \leq J} \\
X_j^2 &= \left((W_{s_1}^{m_j}(e_l))_l, \ldots, (W_{s_J}^{m_j}(e_l))_l, \left(\left\| F_{m_j}(\cdot, \mathscr{E}_{r_{m_j}} u^{m_j}(0)) \right\|_{L^1(B_{T_\rho})} \right)_{\rho\in\mathbb{N}} \right) \\
X_j &= (X_j^1, X_j^2) \\
\mathscr{X}_j &= \left(\left(\langle \mathbf{z}^j(s_{i_0}), \varphi_{i_1} \rangle_{L^2} \right)_{i_0,i_1\leq J}, \beta^j(s_1), \ldots, \beta^j(s_J), \left(\left\| F_{m_j}(\cdot, \mathbf{u}^j(0)) \right\|_{L^1(B_{T_\rho})} \right)_{\rho\in\mathbb{N}} \right) \\
\mathscr{X} &= \left(\left(\langle \mathbf{z}(s_{i_0}), \varphi_{i_1} \rangle_{L^2} \right)_{i_0,i_1\leq J}, \beta(s_1), \ldots, \beta(s_J), \nu \right)
\end{aligned}$$

for $j \geq j_0$. If

$$h_\delta : \mathbb{R}_+ \to [0,1] \tag{11.14}$$

is any continuous function with support in $[0,\delta]$ such that $h_\delta = 1$ on $[0,\delta/2]$ then we also define continuous mappings

$$\begin{aligned}
d_q^j : C(\mathbb{R}_+;\mathscr{H}) &\to \mathbb{R} \\
(u,v) &\mapsto h_\delta(\tilde{\mathbf{F}}_{m_j}(0,u(0),v(0))) \left[\langle v(q), \varphi \rangle - \langle v(0), \varphi \rangle \right] \\
&- h_\delta(\tilde{\mathbf{F}}_{m_j}(0,u(0),v(0))) \int_0^q \langle u(r), \mathscr{A}\varphi \rangle \, dr \\
&- h_\delta(\tilde{\mathbf{F}}_{m_j}(0,u(0),v(0))) \int_0^q \langle f_{m_j}(\cdot, u(r), v(r), \nabla u(r)), \varphi \rangle \, dr \\
D_q^{j,l} : C(\mathbb{R}_+;\mathscr{H}) &\to \mathbb{R} \\
(u,v) &\mapsto h_\delta(\tilde{\mathbf{F}}_{m_j}(0,u(0),v(0))) \int_0^q \langle g_{m_j}(\cdot, u(r), v(r), \nabla u(r)) e_l, \varphi \rangle \, dr \\
D_q^j : C(\mathbb{R}_+;\mathscr{H}) &\to \mathbb{R} \\
(u,v) &\mapsto h_\delta^2(\tilde{\mathbf{F}}_{m_j}(0,u(0),v(0))) \sum_l \int_0^q \langle g_{m_j}(\cdot, u(r), v(r), \nabla u(r)) e_l, \varphi \rangle^2 \, dr
\end{aligned}$$

for $q \in [0,k]$, $j \geq j_0$ and l indexing the ONB (e_l) in H_μ that satisfy

$$|d_q^j(z)| + |D_q^{j,l}(z)| + |D_q^j(z)| \leq K \mathbf{1}_{[\tilde{\mathbf{F}}_{m_j}(0,z(0))\leq\delta]} [1 + \sup_{r\in[0,k]} \tilde{\mathbf{F}}_{m_j}(r, z(r))] \tag{11.15}$$

for $q \in [0,k]$, $z \in C(\mathbb{R}_+;\mathscr{H})$, $j \geq j_0$, l up to $\dim H_\mu$ and for some $K = K_{d,k,\kappa,\mathbf{a},\varphi,\mathbf{c}}$ as

$$\begin{aligned}
\|f_{m_j}(\cdot, u(r), v(r), \nabla u(r))\|_{L^1(B_k)} &\leq ((8\kappa)^{\frac{1}{2}} \operatorname{Leb}_d(B_k) + \kappa)[1 + \tilde{\mathbf{F}}_{m_j}(r, z(r))] \qquad (11.16) \\
\|g_{m_j}(\cdot, u(r), v(r), \nabla u(r))\|_{L^2(B_k)} &\leq (5\kappa)^{\frac{1}{2}} \tilde{\mathbf{F}}_{m_j}^{\frac{1}{2}}(r, z(r))
\end{aligned}$$

holds for every $r \in [0,k]$ where $\tilde{\mathbf{F}}_m = (\tilde{\mathbf{F}}_m)_{\lambda_{T_k},0,T_k}$ is the conic energy function for $\tilde{F}_m(w,y) = F_m(w,y) + |y|^2/2$ defined as in (2.2). Also, for every $p > 0$, there exist constants K_p depending also on d, k, κ, $\mathbf{a}$, φ and $\mathbf{c}$ such that

$$\mathbb{E} \sup_{q\in[0,k]} \left[|d_q^j(\mathbf{z}^j)|^p + |D_q^{j,l}(\mathbf{z}^j)|^p + |D_q^j(\mathbf{z}^j)|^p \right] \leq K_p < \infty \tag{11.17}$$

holds for every $j \geq j_0$ and every l by Lemma 11.8 and (11.15). Hence

$$\mathbb{E}\,\mathscr{H}(\mathscr{X}_j) \left\{ d_t^j(\mathbf{z}^j) - d_s^j(\mathbf{z}^j) \right\} = \mathbb{E}^{m_j}\,\mathscr{H}(X_j) \left\{ d_t^j(z^{m_j}) - d_s^j(z^{m_j}) \right\} = 0 \tag{11.18}$$

$$\mathbb{E}\,\mathscr{H}(\mathscr{X}_j) \left\{ d_t^j(\mathbf{z}^j)\beta_l^j(t) - D_t^{j,l}(\mathbf{z}^j) - d_s^j(\mathbf{z}^j)\beta_l^j(s) + D_s^{j,l}(\mathbf{z}^j) \right\} = \tag{11.19}$$

$$= \mathbb{E}^{m_j}\,\mathscr{H}(X_j) \left\{ d_t^j(z^{m_j}) W_t^{m_j}(e_l) - D_t^{j,l}(z^{m_j}) - d_s^j(z^{m_j}) W_s^{m_j}(e_l) + D_s^{j,l}(z^{m_j}) \right\} = 0$$

$$\mathbb{E}\,\mathscr{H}(\mathscr{X}_j) \left\{ (d_t^j(\mathbf{z}^j))^2 - D_t^j(\mathbf{z}^j) - (d_s^j(\mathbf{z}^j))^2 + D_s^j(\mathbf{z}^j) \right\} = \tag{11.20}$$

$$= \mathbb{E}^{m_j}\,\mathscr{H}(X_j) \left\{ (d_t^j(z^{m_j}))^2 - D_t^j(z^{m_j}) - (d_s^j(z^{m_j}))^2 + D_s^j(z^{m_j}) \right\} = 0$$

by the property (**i**) in Section 11.3 since, by vii) in Section 11.1,

$$d_q^j(z^{m_j}) = h_\delta(\tilde{\mathbf{F}}_{m_j}(0, z^{m_j}(0))) \int_0^q \left\langle g_{m_j}(\cdot, u^{m_j}(r), v^{m_j}(r), \nabla u^{m_j}(r))\, dW_r^{m_j}, \varphi \right\rangle_{L^2}$$

for every $q \in [0,k]$ which is an $L^2(\Omega^{m_j})$-integrable $(\mathscr{F}_t^{m_j})$-martingale by (11.17) and (**i**) in Section 11.3. Since v) in Section 11.1 was assumed, it holds that

$$\begin{aligned}
\lim_{j\to\infty} \int_{B_k} \left\langle f_{m_j}^0(\cdot, \mathbf{u}^j(r))\mathbf{v}^j(r), \varphi \right\rangle_{\mathbb{R}^n} dx &= \int_{B_k} \left\langle f^0(\cdot, \mathbf{u}(r))\mathbf{v}(r), \varphi \right\rangle_{\mathbb{R}^n} dx \qquad (11.21)\\
\lim_{j\to\infty} \int_{B_k} \left\langle g_{m_j}^0(\cdot, \mathbf{u}^j(r))\mathbf{v}^j(r)e_l, \varphi \right\rangle_{\mathbb{R}^n} dx &= \int_{B_k} \left\langle g^0(\cdot, \mathbf{u}(r))\mathbf{v}(r)e_l, \varphi \right\rangle_{\mathbb{R}^n} dx\\
\lim_{j\to\infty} \int_{B_k} \left\langle f_{m_j}^i(\cdot, \mathbf{u}^j(r))\frac{\partial \mathbf{u}^j(r)}{\partial x_i}, \varphi \right\rangle_{\mathbb{R}^n} dx &= \int_{B_k} \left\langle f^i(\cdot, \mathbf{u}(r))\frac{\partial \mathbf{u}(r)}{\partial x_i}, \varphi \right\rangle_{\mathbb{R}^n} dx\\
\lim_{j\to\infty} \int_{B_k} \left\langle g_{m_j}^i(\cdot, \mathbf{u}^j(r))\frac{\partial \mathbf{u}^j(r)}{\partial x_i}e_l, \varphi \right\rangle_{\mathbb{R}^n} dx &= \int_{B_k} \left\langle g^i(\cdot, \mathbf{u}(r))\frac{\partial \mathbf{u}(r)}{\partial x_i}e_l, \varphi \right\rangle_{\mathbb{R}^n} dx
\end{aligned}$$

for every $r \in [0,k]$, l and $i \in 1,\ldots,d$ on Ω by (**ii**) in Section 11.3. It remains to deal with convergence of the terms $i = d+1$. To this end, by the Lebesgue dominated convergence theorem, (**ii**) and (**iii**) in Section 11.3 and v) and xi) in Section 11.1, there is

$$\lim_{j\to\infty} \int_{B_k\cap[|\mathbf{u}^j(r,\omega)|_{\mathbb{R}^n}\leq R]} \left| h_\delta(\tilde{\mathbf{F}}_{m_j}(0, \mathbf{z}^j(0,\omega))) f_{m_j}^{d+1}(\cdot, \mathbf{u}^j(r,\omega)) - h_\delta(\tilde{v}_k(\omega)) f^{d+1}(\cdot, \mathbf{u}(r,\omega)) \right| dx = 0$$

for every $R > 0$ and every $(r,\omega) \in [0,k] \times \Omega$ such that

$$h_\delta(\tilde{v}_k(\omega)) \|f^{d+1}(\cdot, \mathbf{u}(r,\omega))\|_{L^1(B_k)} < \infty.$$

But

$$h_\delta(\tilde{v}_k(\omega)) \sup_{r\in[0,k]} \|f^{d+1}(\cdot,\mathbf{u}(r,\omega))\|_{L^1(B_k)} \leq \kappa \mathbf{1}_{[\tilde{v}_k\leq\delta]} \sup_{r\in[0,k]} \|F(\cdot,\mathbf{u}(r,\omega))\|_{L^1(B_k)} \leq \kappa \mathbf{1}_{[\tilde{\mathbf{F}}(0,\mathbf{z}(0))\leq\delta]} \sup_{r\in[0,k]} \|F(\cdot,\mathbf{u}(r,\omega))\|_{L^1(B_{T_k-r\lambda_{T_k}})} \tag{11.22}$$

by v) in Section 11.1 as $\tilde{\mathbf{F}}(0,\mathbf{z}(0)) \leq \tilde{v}_k$ on Ω, where $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_{\lambda_{T_k},0,T_k}$ is the conic energy function defined as in (2.2) for the function $\tilde{F}(w,y) = F(w,y) + |y|^2/2$, so the LHS of (11.22) is in $L^1(\Omega)$ as so is the RHS by (11.11). Consequently,

$$\lim_{j\to\infty} \mathbb{E}\int_0^k \int_{B_k\cap[|\mathbf{u}^j(r,\omega)|_{\mathbb{R}^n}\leq R]} \left|h_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0)))f^{d+1}_{m_j}(\cdot,\mathbf{u}^j(r)) - h_\delta(\tilde{v}_k)f^{d+1}(\cdot,\mathbf{u}(r))\right| dx\,dr = 0$$

holds for every $R > 0$ by the Lebesgue dominated convergence theorem. On the other hand

$$\begin{aligned} &\mathbb{E}\int_0^k \int_{B_k\cap[|\mathbf{u}^j(r,\omega)|_{\mathbb{R}^n}> R]} \left|h_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0)))f^{d+1}_{m_j}(\cdot,\mathbf{u}^j(r))\right| dx\,dr \leq \\ &\leq \alpha_{k,R}\mathbb{E}\int_0^k h_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0)))\|F_{m_j}(\cdot,\mathbf{u}^j(r))\|_{L^1(B_k)}\,dr \\ &\leq \alpha_{k,R}\mathbb{E}\int_0^k \mathbf{1}_{[\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0))\leq\delta]}\|F_{m_j}(\cdot,\mathbf{u}^j(r))\|_{L^1(B_{T_k-r\lambda_{T_k}})}\,dr \\ &\leq \alpha_{k,R}C_{\kappa,k,\delta,\mathbf{c}} \end{aligned}$$

holds for every $R > 0$ by (11.5) and (11.11) where $\lim_{R\to\infty}\alpha_{k,R} = 0$, and

$$\lim_{R\to\infty} \mathbb{E}\int_0^k \int_{B_k\cap[|\mathbf{u}^j(r,\omega)|_{\mathbb{R}^n}> R]} \left|h_\delta(\tilde{v}_k)f^{d+1}(\cdot,\mathbf{u}(r))\right| dx\,dr = 0$$

by the Lebesgue dominated convergence theorem based on (11.22) so

$$\lim_{j\to\infty} \mathbb{E}\int_0^k \left\|h_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0)))f^{d+1}_{m_j}(\cdot,\mathbf{u}^j(r)) - h_\delta(\tilde{v}_k)f^{d+1}(\cdot,\mathbf{u}(r))\right\|_{L^1(B_k)} dr = 0$$

and, altogether with (11.21), (11.16), (11.10), (11.11) and (11.17),

$$\lim_{j\to\infty} \mathbb{E}\left|d^j_q(\mathbf{z}^j) - d_q\right|^p = 0, \qquad q\in[0,k], \quad p > 0 \tag{11.23}$$

where

$$d_q = h_\delta(\tilde{v}_k)\left[\langle \mathbf{v}(q),\varphi\rangle - \langle \mathbf{v}(0),\varphi\rangle - \int_0^q \langle \mathbf{u}(r),\mathscr{A}\varphi\rangle\,dr - \int_0^q \langle f(\cdot,\mathbf{u}(r),\mathbf{v}(r),\nabla\mathbf{u}(r)),\varphi\rangle\,dr\right]$$

with the notation (4.1). Finally, fix l and define

$$\begin{aligned} \eta_j(r,\omega,x) &= h_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0,\omega)))\left\langle g^{d+1}_{m_j}(x,\mathbf{u}^j(r,\omega,x))e_l(x),\varphi(x)\right\rangle_{\mathbb{R}^n} \\ \eta(r,\omega,x) &= h_\delta(\tilde{v}_k(\omega))\left\langle g^{d+1}(x,\mathbf{u}(r,\omega,x))e_l(x),\varphi(x)\right\rangle_{\mathbb{R}^n}. \end{aligned}$$

Then $\eta_j \to \eta$ in the measure $\mathrm{Leb}_1 \otimes \mathbb{P} \otimes \mathrm{Leb}_d$ for variables $(r, \omega, x) \in [0, k] \times \Omega \times B_k$. Since, for any $p > 0$,

$$\mathbb{E}\int_0^k \|\eta_j(r,\omega,\cdot)\|^p_{L^2(B_k)}\,dr \le \kappa^{\frac{p}{2}}\|\varphi e_l\|^p_{L^\infty(B_k)}\mathbb{E}\int_0^k \mathbf{1}_{[\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}(0))\le\delta]}\tilde{\mathbf{F}}^{\frac{p}{2}}_{m_j}(r,\mathbf{z}(r))\,dr \le C_p$$

$$\mathbb{E}\int_0^k \|\eta(r,\omega,\cdot)\|^p_{L^2(B_k)}\,dr \le \kappa^{\frac{p}{2}}\|\varphi e_l\|^p_{L^\infty(B_k)}\mathbb{E}\int_0^k \mathbf{1}_{[\tilde{\mathbf{F}}(0,\mathbf{z}(0))\le\delta]}\tilde{\mathbf{F}}^{\frac{p}{2}}(r,\mathbf{z}(r))\,dr \le C_p$$

for some $C_p = C_{p,\delta,k,\kappa,\mathbf{c},\varphi,l}$ by (11.11),

$$\lim_{j\to\infty}\mathbb{E}\int_0^k\left(\int_{B_k}\mathbf{1}_{[|\eta_j-\eta|\le 1]}|\eta_j(r,\omega,x)-\eta(r,\omega,x)|\,dx\right)^2 dr = 0$$

by the Lebesgue dominated convergence theorem,

$$\mathbb{E}\int_0^k\left(\int_{B_k}\mathbf{1}_{[|\eta_j-\eta|>1]}|\eta_j(r,\omega,x)-\eta(r,\omega,x)|\,dx\right)^2 dr \le C\left[\mathbb{E}\int_0^k\int_{B_k}\mathbf{1}_{[|\eta_j-\eta|>1]}\,dx\,dr\right]^{\frac{1}{2}} \to 0$$

by a double application of the Cauchy-Schwarz inequality where $C = 4\,\mathrm{Leb}_d^{\frac{1}{2}}(B_k)C_4^{\frac{1}{2}}$ so, altogether with (11.21), (11.16), (11.10) and (11.11),

$$\lim_{j\to\infty}\mathbb{E}\int_0^k\left|h_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0)))\left\langle g_{m_j}(\cdot,\mathbf{z}^j,\nabla\mathbf{u}^j)e_l,\varphi\right\rangle - h_\delta(\tilde{v}_k)\left\langle g(\cdot,\mathbf{z},\nabla\mathbf{u})e_l,\varphi\right\rangle\right|^2 dr = 0 \qquad (11.24)$$

holds for every l. Whence

$$\lim_{j\to\infty}\mathbb{E}\left[|D^{j,l}_q(\mathbf{z}^j) - D^l_q|^p + |D^j_q(\mathbf{z}^j) - D_q|^p\right] = 0, \qquad q\in[0,k], \quad p>0 \qquad (11.25)$$

for every l by (11.17) where

$$D^l_q = h_\delta(\tilde{v}_k)\int_0^q \langle g(\cdot,\mathbf{u}(r),\mathbf{v}(r),\nabla\mathbf{u}(r))e_l,\varphi\rangle\,dr, \qquad q\in[0,k]$$

$$D_q = h^2_\delta(\tilde{v}_k)\sum_l\int_0^q \langle g(\cdot,\mathbf{u}(r),\mathbf{v}(r),\nabla\mathbf{u}(r))e_l,\varphi\rangle^2\,dr, \qquad q\in[0,k]$$

with the notation (4.1). This is indeed clear from (11.24) if $\dim H_\mu < \infty$. If $\dim H_\mu = \infty$ then

$$\begin{aligned}
&\mathbb{E}\sum_{l=l_0}^\infty\left[h^2_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0)))\int_0^k\left\langle g_{m_j}(\cdot,\mathbf{z}^j,\nabla\mathbf{u}^j)e_l,\varphi\right\rangle^2 dr + h^2_\delta(\tilde{v}_k)\int_0^k\langle g(\cdot,\mathbf{z},\nabla\mathbf{u})e_l,\varphi\rangle^2\,dr\right] \le \\
&\le \quad 5\kappa\varepsilon_{l_0}\mathbb{E}\left[h^2_\delta(\tilde{\mathbf{F}}_{m_j}(0,\mathbf{z}^j(0)))\int_0^k\tilde{\mathbf{F}}_{m_j}(r,\mathbf{z}^j(r))\,dr + h^2_\delta(\tilde{v}_k)\int_0^k\tilde{\mathbf{F}}(r,\mathbf{z}(r))\,dr\right] \\
&\le \quad \varepsilon_{l_0}C
\end{aligned}$$

by (11.16), (11.10) and (11.11) where $C = C_{\kappa,k,\mathbf{c}}$ and

$$\varepsilon_{l_0} = \sum_{l=l_0}^{\infty} \|\varphi e_l\|^2_{L^2(B_k)} \to 0$$

as $l_0 \to \infty$ by Lemma 3.3. The convergence results (11.23) and (11.25) together with (11.18)-(11.20) imply

$$\mathbb{E}\,\mathscr{H}(\mathscr{X})\left\{d_t - d_s\right\} = \mathbb{E}\,\mathscr{H}(\mathscr{X})\left\{d_t\beta_l(t) - D_t^l - d_s\beta_l(s) + D_s^l\right\} = 0$$
$$\mathbb{E}\,\mathscr{H}(\mathscr{X})\left\{d_t^2 - D_t - d_s^2 + D_s\right\} = 0$$

which means that $(d_q)_{q\in[0,k]}$ is an $L^2(\Omega)$-integrable $(\mathscr{F}_t)$-martingale whose quadratic variation and cross variation with β_l satisfy $\langle d\rangle_q = D_q$, $\langle d, \beta_l\rangle_q = D_q^l$ for $q \in [0,k]$ and $l \in \mathbb{N}$. Thus

$$\left\langle d - \int_0^{\cdot} h_\delta(\tilde{v}_k)\,\langle g(\cdot, \mathbf{z}(r), \nabla \mathbf{u}(r))\,dW_r, \varphi\rangle \right\rangle_k = 0$$

where W was defined in Corollary 11.6 whence the claim is proved after we let $\delta \to \infty$ as $h_\delta(\tilde{v}_k) \to 1$ on Ω. □

11.8 Approximation of nonlinearities

We use the C^1-functions

$$\phi^m : \mathbb{R}^n \to \mathbb{R}^n : y \mapsto h(|y|_{\mathbb{R}^n}/m)y$$

introduced analogously as in (7.1) where $h : \mathbb{R} \to [0,1]$ is the same as in (7.1) and smooth mollifiers $\zeta_m : \mathbb{R}^n \to \mathbb{R}_+ : \zeta_m(y) = m^n\zeta(my)$ supported in $B_{\frac{1}{m}}$ introduced analogously as in Section 2 that satisfy $\|\zeta_m\|_{L^1(\mathbb{R}^n)} = 1$ for every $m \in \mathbb{N}$. We first make a convention that functions f^i, g^i, F in Section 5 satisfy the assumptions therein for every $w \in \mathbb{R}^d$ and not for almost every $w \in \mathbb{R}^d$. This poses no loss of generality since redefinition of these functions by 0 on an Leb_d-exceptional set of $w \in \mathbb{R}^d$ does not modify the definition of a solution in Section 4. We then find numbers $\eta_m > 0$ such that the sets

$$O_m = \{x \in B_{2m} : \sup_{|y|\le 2m} F(x,y) \le \eta_m\}$$

satisfy

$$\mathrm{Leb}_d\,(B_{2m} \setminus O_m) \le \frac{1}{2^m}, \qquad m \in \mathbb{N},$$

we put

$$f_m^i(w,y) = \int_{\mathbb{R}^n} f^i(w,z)\zeta_m(y-z)\,dz, \qquad w \in \mathbb{R}^d, \quad y \in \mathbb{R}^n$$

$$g_m^i(w,y) = \int_{\mathbb{R}^n} g^i(w,z)\zeta_m(y-z)\,dz, \qquad w \in \mathbb{R}^d, \quad y \in \mathbb{R}^n$$

for $i \in \{0, \dots, d\}$,

$$f_m^{d+1}(w,y) = \mathbf{1}_{O_m}(w)\int_{\mathbb{R}^n} (\phi^m)'(z)f^{d+1}(w,\phi^m(z))\zeta_m(y-z)\,dz, \qquad w \in \mathbb{R}^d, \quad y \in \mathbb{R}^n$$

$$g_m^{d+1}(w,y) = \mathbf{1}_{O_m}(w)\int_{\mathbb{R}^n}(\phi^m)'(z)g^{d+1}(w,\phi^m(z))\zeta_m(y-z)\,dz, \qquad w\in\mathbb{R}^d, \quad y\in\mathbb{R}^n$$

and

$$F_m(w,y) = \mathbf{1}_{O_m}(w)\int_{\mathbb{R}^n}F(w,\phi^m(z))\zeta_m(y-z)\,dz, \qquad w\in\mathbb{R}^d, \quad y\in\mathbb{R}^n.$$

If we realize that $\phi^m[B_r]\subseteq B_r\cap B_{2m}$, $r>0$ and the matrix norm

$$\|(\phi^m)'(z)\| \le \min\{2, 12h^{\frac{1}{2}}(|z|/m)\}, \qquad z\in\mathbb{R}^n$$

holds for every $m\in\mathbb{N}$ then, concerning the assumptions in Section 11.1,

- iii), iv) are satisfied apparently,
- in v), the inequalities (10.1), (10.2) hold for κ, the inequality (10.3) holds for 4κ and the inequality (10.8) holds for 2κ,
- vi) holds as almost every $x\in\mathbb{R}^d$ belongs eventually to every O_m,
- the laws of $z^m(0)$ under $\mathbb{P}^m$ in (11.1) are constructed as follows: Let z_0 be an $\mathscr{H}_{loc}$-valued random variable on some probability space $(\Omega,\mathscr{F},\mathbb{P})$ with the law Θ, let $\xi_m\in\mathscr{D}(\mathbb{R}^d)$ satisfy $\xi_m=1$ on B_m and let $a_m>0$ satisfy $\mathbb{P}[\|\xi_m z^0\|_{\mathscr{H}}>a_m]\le 2^{-m}$ for every $m\in\mathbb{N}$. Denote by ι_m the law of $\mathbf{1}_{[\|\xi_m z^0\|_{\mathscr{H}}\le a_m]}\xi_m z^0$ and let $z^m(0)$ have the law ι_m. Then $z^m(0)$ is supported on some ball in $\mathscr{H}$, (11.1) holds and vii) is satisfied,
- xi) holds as
$$\sup_{|y|\le R}F^*(x,y)\le \sup_{|y|\le R+1}F(x,y)\in L^1(B_R),$$
- xii) is satisfied as $|f_m^{d+1}(w,y)|\le 2\kappa F_m(w,y)$ holds for every $w\in\mathbb{R}^d$, $y\in\mathbb{R}^n$ and $m\in\mathbb{N}$; so we may put $\tilde\alpha_{r,R}=2\kappa$ if $r>0$ and $R\in(0,2)$. If $r>0$, $R\ge 2$, $|w|\le r$, $|y|\ge R$ then $|f_m^{d+1}(w,y)|\le I_1+I_2$ where
$$\begin{aligned} I_1 &= 2\cdot\mathbf{1}_{O_m}(w)\int_{[|z|\ge R-1]\cap[|\phi^m(z)|>(R-1)^{\frac{1}{2}}]}|f^{d+1}(w,\phi^m(z))|\zeta_m(y-z)\,dz \\ &\le 2\cdot\alpha_{r,\sqrt{R-1}}\cdot F_m(w,y) \\ I_2 &= 12\cdot\mathbf{1}_{O_m}(w)\int_{[|z|\ge R-1]\cap[|\phi^m(z)|\le(R-1)^{\frac{1}{2}}]}h^{\frac{1}{2}}(|z|/m)|f^{d+1}(w,\phi^m(z))|\zeta_m(y-z)\,dz \\ &\le 12\cdot(R-1)^{-\frac{1}{4}}\cdot\kappa\cdot F_m(w,y) \end{aligned}$$
as $|z|\ge R-1$ and $|\phi^m(z)|\le(R-1)^{\frac{1}{2}}$ imply
$$h^{\frac{1}{2}}(|z|/m)\le\frac{(R-1)^{\frac{1}{4}}}{|z|^{\frac{1}{2}}}\le\frac{(R-1)^{\frac{1}{4}}}{(R-1)^{\frac{1}{2}}}$$
so we put
$$\tilde\alpha_{r,R}=\max\left\{2\alpha_{r,\sqrt{R-1}}, 12\kappa(R-1)^{-\frac{1}{4}}\right\}, \qquad R\ge 2.$$

A The Jakubowski-Skorokhod representation theorem

Theorem A.1. *Let X be a topological space such that there exists a sequence $\{f_m\}$ of continuous functions $f_m : X \to \mathbb{R}$ that separate points of X. Let us denote by $\mathscr{S}$ the σ-algebra generated by the maps $\{f_m\}$. Then*

(j1) every compact subset of X is metrizable,

(j2) every Borel subset of a σ-compact set in X belongs to $\mathscr{S}$,

(j3) every probability measure supported by a σ-compact set in X has a unique Radon extension to the Borel σ-algebra on X,

(j4) if (μ_m) is a tight sequence of probability measures on $(X, \mathscr{S})$, then there exists a subsequence (m_k), a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with X-valued Borel measurable random variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converge to ξ on Ω. Moreover, the law of ξ is a Radon measure.

Proof. See [21]. □

Corollary A.2. *Under the assumptions of Theorem A.1, iff Z is a Polish space and $b : Z \to X$ is a continuous injection, then $b[B]$ is a Borel set whenever B is Borel in Z.*

Proof. Since the map $F = (f_1, f_2, \dots) : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous injection, $F \circ b : Z \to \mathbb{R}^{\mathbb{N}}$ is also a continuous injection. Let us take a Borel set $B \subset Z$. Since both Z and $\mathbb{R}^{\mathbb{N}}$ are Polish spaces, we infer that $(F \circ b)[B]$ is a Borel set. Therefore $b[B] = F^{-1}[(F \circ b)[B]] \subset X$ is Borel set too. □

B The space $C_w(\mathbb{R}_+; W^{k,p}_{loc}(\mathbb{R}^d, \mathbb{R}^n))$, $k \geq 0$, $1 < p < \infty$

Let us introduce the spaces

$$\begin{array}{lll} W^{l,p}_m & = \{f \in W^{l,p}(\mathbb{R}^d, \mathbb{R}^n) : f = 0 \text{ on } \mathbb{R}^d \setminus B_m\}, & l \geq 0 \\ \mathbb{W}^{l,p}_m & = W^{l,p}(B_m) & l \geq 0 \\ \mathbb{W}^{-l,p}_m & = (W^{l,p'}_m)^*, & l > 0 \end{array}$$

where $\frac{1}{p'} + \frac{1}{p} = 1$.

Lemma B.1. *The maps J and L defined by*

$$J : (W^{k,p}_{loc}, w) \ni f \mapsto \left(f|_{B_m}\right)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} (W^{k,p}(B_m), w),$$

$$L : C_w(\mathbb{R}_+; W^{k,p}_{loc}) \ni h \mapsto \left((h|_{B_m})|_{[0,m]}\right)_{m=1}^{\infty} \in \prod_{m=1}^{\infty} C_w([0,m], W^{k,p}(B_m))$$

are both homeomorphisms onto closed sets.

Proof. The proof of Lemma B.1 is straightforward. □

Corollary B.2. *Let $a=(a_m)$ be a sequence of positive numbers and let $\gamma>0$, $1<r,p<\infty$, $-\infty<l\leq k$ satisfy*

$$\frac{1}{p}-\frac{k}{d}\leq\frac{1}{r}-\frac{l}{d}. \tag{B.1}$$

Then the set

$$K(a):=\{f\in C_w(\mathbb{R}_+;W^{k,p}_{loc}):\ \|f\|_{L^\infty([0,m],W^{k,p}(B_m))}+\|f\|_{C^\gamma([0,m],\mathbb{W}^{l,r}_m)}\leq a_m,\ m\in\mathbb{N}\}$$

is a metrizable compact set in $C_w(\mathbb{R}_+;W^{k,p}_{loc})$.

Proof. Let us define a set A_m, $m\in\mathbb{N}$, by

$$A_m=\{h\in C_w([0,m],W^{k,p}(B_m)):\|h\|_{L^\infty([0,m],W^{k,p}(B_m))}+\|h\|_{C^\gamma([0,m],\mathbb{W}^{l,r}_m)}\leq a_m\}.$$

Then $K(a)=L^{-1}\big(\prod_m A_m\big)$. It is enough to show that each A_m is a metrizable compact in $C_w([0,m],W^{k,p}(B_m))$. Indeed, if this is the case then $A:=\prod_m A_m$ is a metrizable compact and hence, since by Lemma B.1 $\mathrm{R}(L)$ (the range of L) is closed, $A\cap\mathrm{R}(L)$ is a metrizable compact. Therefore, as by Lemma B.1 $L^{-1}:\mathrm{R}(L)\to C_{\mathrm{w}}(\mathbb{R}_+;W^{k,p}_{\mathrm{loc}})$ is a continuous function, $K(a)=L^{-1}[A\cap\mathrm{R}(L)]$ is a metrizable compact. To this end let us fix $m\in\mathbb{N}$ and let $\{\varphi_j\}$ be a dense subset of $(W^{k,p}(B_m))^*$. Denote by τ the locally convex topology on $C_w([0,m],W^{k,p}(B_m))$ generated by the semi-norms $f\mapsto\sup_{t\leq m}|\varphi_j(f(t))|$. It is easy to see that τ coincides with the original topology of $C_w([0,m],W^{k,p}(B_m))$ on the set $\tilde{A}_m$ defined by

$$\tilde{A}_m=\{h\in C_w([0,m],W^{k,p}(B_m)):\|h\|_{L^\infty([0,m],W^{k,p}(B_m))}\leq a_m\}.$$

Hence the set A_m is metrizable. The compactness of A_m follows from the classical Arzela-Ascoli Theorem. Indeed, the balls in $(W^{k,p}(B_m),\mathrm{w})$ are compact metrizable spaces - towards this end, let (h_j) be an A_m-valued sequence. By the diagonal procedure we can find a subsequence h_{j_k} such that $h_{j_k}(t)$ is weakly convergent in $W^{k,p}(B_m)$ for every $t\in[0,m]\cap\mathbb{Q}$. Since in view of the assumption (B.1) by the celebrated Gagliardo-Nirenberg inequalities, see e.g. [15], $W^{k,p}(B_m)\subseteq\mathbb{W}^{l,r}_m$ continuously and h_j are bounded in $C^\gamma([0,m],\mathbb{W}^{l,r}_m)$, the sequence $\psi(h_{j_k}(t))$ is convergent for every $\psi\in(\mathbb{W}^{l,r}_m)^*$ and every $t\leq m$. And since h_j is uniformly bounded in $\mathbb{W}^{k,p}_m$ and $(\mathbb{W}^{l,r}_m)^*$ is dense in $(\mathbb{W}^{k,p}_m)^*$, the sequence $\varphi(h_{j_k}(t))$ is convergent for every $\varphi\in(\mathbb{W}^{k,p}_m)^*$, hence $h_{j_k}(t)$ is weakly convergent in $\mathbb{W}^{k,p}_m$ for every $t\leq m$. If we denote by h the pointwise limit of h_{j_k}, it is easy to show that $\varphi(h_{j_k})\to\varphi(h)$ uniformly on $[0,m]$ for every $\varphi\in(\mathbb{W}^{k,p}_m)^*$ and that $h\in A_m$. □

Proposition B.3. *The Skorokhod representation theorem A.1 holds for every tight sequence of probability measures defined on the σ-algebra generated by the following family of maps*

$$\{C_w(\mathbb{R}_+;W^{k,p}_{loc})\ni f\mapsto\langle\varphi,f(t)\rangle\in\mathbb{R}\}:\ \varphi\in\mathscr{D}(\mathbb{R}^d,\mathbb{R}^n),\ t\in[0,\infty).$$

Proof. By the Jakubowski-Skorokhod theorem [21], it is sufficient to verify that there exists a sequence $j_k:C_{\mathrm{w}}(\mathbb{R}_+;W^{k,p}_{\mathrm{loc}})\to\mathbb{R}$ of continuous functions that separate points of $C_{\mathrm{w}}(\mathbb{R}_+;W^{k,p}_{\mathrm{loc}})$. For, let φ_k be a countable sequence in $(W^{k,p}_{\mathrm{loc}})^*$ separating points of $W^{k,p}_{\mathrm{loc}}$. Then $j_{k,q}(f)=\varphi_k(f(q))$, $k\in\mathbb{N}$, $q\in\mathbb{Q}_+$ do the job. □

C A measurability lemma

Let X be a separable Fréchet space (with a countable system of pseudonorms $(\|\cdot\|_k)_{k\in\mathbb{N}}$, let X_k be separable Hilbert spaces and $i_k : X \to X_k$ linear mappings such that $\|i_k(x)\|_{X_k} = \|x\|_k$, $k \geq 1$. Let $\varphi_{k,j} \in X_k^*$, $j \in \mathbb{N}$ separate points of X_k. Then the mappings $(\varphi_{k,j} \circ i_k)_{k,j\in\mathbb{N}}$ generate the Borel σ-algebra on X.

Proof. Denote by σ_0 the σ-algebra generated by the mappings $(\varphi_{k,j} \circ i_k)_{k,j\in\mathbb{N}}$ and denote

$$V_k = \{\varphi \in X_k^* : \varphi \circ i_k \text{ is } \sigma_0\text{-measurable}\}.$$

Then V_k is a closed dense subspace in X_k^*, hence $V_k = X_k^*$. There exists $\psi_{k,j} \in X_k^*$ such that

$$\|z\|_{X_k} = \sup_{j\in\mathbb{N}} |\psi_{k,j}(z)|, \qquad x \in X_k,$$

and so the mapping

$$x \mapsto \rho(x,y) = \sum_{k=1}^{\infty} 2^{-k} \min\{1, \sup_j |\psi_{k,j} \circ i_k(x) - \psi_{k,j} \circ i_k(y)|\}$$

is σ_0-measurable for every $y \in X$. Consequently, the open balls in X are σ_0-measurable, and since every open set in X is a countable union of open balls in X, every open set in X is in σ_0. □

Corollary C.1. *There exists a countable system $\varphi_k \in \mathscr{D}(\mathbb{R}^d, \mathbb{R}^n)$ such that the mappings*

$$W^{m,2}_{loc} \ni h \mapsto \langle h, \varphi_k \rangle_{L^2} \in \mathbb{R}, \qquad k \in \mathbb{N}$$

generate the Borel σ-algebra on $W^{m,2}_{loc}$ whenever $m \geq 0$.

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