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On the Shuffling Algorithm for Domino Tilings*

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Abstract

We study the dynamics of a certain discrete model of interacting interlaced particles that comes from the so called shuffling algorithm for sampling a random tiling of an Aztec diamond. It turns out that the transition probabilities have a particularly convenient determinantal form. An analogous formula in a continuous setting has recently been obtained by Jon Warren studying certain model of interlacing Brownian motions which can be used to construct Dyson's non-intersecting Brownian motion.

We conjecture that Warren's model can be recovered as a scaling limit of our discrete model and prove some partial results in this direction. As an application to one of these results we use it to rederive the known result that random tilings of an Aztec diamond, suitably rescaled near a turning point, converge to the GUE minor process.

Key words: random tilings; Brownian motion; random matrices.

AMS 2000 Subject Classification: Primary 60C05; 60G50.

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1 Introduction

There has been a lot of work in recent years connecting tilings of various planar regions with random matrices. One particular model that has been intensely studied is domino tilings of a so called *Aztec diamond*. One way of analysing that model, [Joh05; Joh01; JN06], is to define a particle process corresponding to the tilings so that the uniform measure on all tilings induces some measure on this particle process.

In this article we study the so called *shuffling algorithm*, described in [EKLP92; Pro03], which in various variants can be used either to count or to enumerate all tilings of the Aztec diamond or to sample a random such tiling.

The sampling of a random tiling by this method is an iterative process. Starting with a tiling of an order $n-1$ Aztec diamond, a certain procedure is performed, producing a random tiling of order n. This procedure is usually described in terms of the dominoes which should be moved and created according to a certain procedure. We will instead look at this algorithm as a certain dynamics on the particle process mentioned above.

The detailed dynamics of the particle process will be presented in section 2 and how it is obtained from the traditional formulation of the shuffling algorithm is presented in section 3. For now, consider a process $X(t) = (X^1(t), \dots, X^m(t))$ for $t = 0, 1, 2, \dots$, where $X^n(t) = (X_1^n(t), \dots, X_n^n(t)) \in \mathbb{Z}^n$. The quantity $X_i^n(t)$ represents the position of the i:th particle on line n after $t-n$ steps of the shuffling algorithm have been performed. At each time a certain interlacing condition (3) is maintained. (The reason for the $t-n$ is technical convenience.)

Denote by $X^N(t) = (X^{N,1}(t), \dots, X^{N,m}(t))$, for $t \in [0,\infty)$, a version of $X(t)$ rescaled according to

$$X_i^{N,n}(t) = \frac{X_i^n(Nt) - \frac{1}{2}Nt}{\frac{1}{2}\sqrt{N}}, \quad t \in \frac{1}{N}\mathbb{N}_0, \tag{1}$$

and extended by linear interpolation to non-integer values of Nt. We will prove the following in section 7.

Theorem 1.1. *For fixed n, as $N \to \infty$, the process $(X^{N,n}(t))_{t\in[0,\infty)}$ converges weakly to a Dyson Brownian motion with all n particles started at the origin.*

The full process $(X(t))_{t\in\mathbb{N}_0}$ has remarkable similarities to, and is we believe a discretization of, a process studied recently by Warren, [War07]. It consists of many interlaced Dyson Brownian motions and is here briefly described in section 4. We will denote that process $(\mathbf{X}(t))_{t\in[0,\infty)}$. We show the following in section 7 along with the stronger statement Theorem 7.4.

Theorem 1.2. *Let $t \geq 0$ be fixed. Then $X^N(t)$ converges in distribution to $\mathbf{X}(t)$ as $N \to \infty$.*

The key to our asymptotic analysis of the shuffling algorithm is that the transition probabilities of $(X^n(t), X^{n+1}(t))_{t\in\mathbb{N}_0}$ can be written in a convenient determinantal form, see proposition 4.2. These formulas mirror in a beautiful way formulas obtained by Warren.

As an application of our results we will use it to rederive an asymptotic result about random tilings near the point where the arctic circle touches the edge of the diamond. This result was first stated in [Joh05] and proved in [JN06].

Recall that the *Gaussian Unitary Ensemble*, or GUE for short, is a probability measure on $m \times m$ Hermitian matrices with density $Z_m^{-1} e^{-\operatorname{Tr} H^2/2}$ where Z_m is a normalisation constant. Let $H = (h_{rs})_{1 \le r,s \le m}$ be a GUE matrix and denote its principal minors by $H_n = (h_{rs})_{1 \le r,s \le n}$. Let $\lambda^n = (\lambda_1^n, \ldots, \lambda_n^n)$ be the vector of eigenvalues of H_n ordered so that $\lambda_i^n \le \lambda_{i+1}^n$ for $i = 1, \ldots, n-1$. Then $\Lambda = (\lambda^1, \ldots, \lambda^m) \in \mathbb{R}^{m(m+1)/2}$ is in [JN06] called the GUE minor process.

Corollary 1.3 (Theorem 1.5 in [JN06]). *$X^N(1) \to \Lambda$ in distribution as $N \to \infty$.*

Proof. Warren in [War07] shows that Λ has the same distribution as $\mathbf{X}(1)$. □

To put this in perspective, let us note that a similar result for lozenge tilings is known from Okounkov and Reshetikhin [OR06]. They discuss the fact that, for quite general regions, that close to a so called turning point the GUE minor process can be obtained in a limit. A turning point is, just as in our situation, where the disordered region is tangent to the domain boundary.

After this work appeared as a preprint, Borodin & Ferrari published [BF08] treating a class of models where the present particle model appears as a special case. They however approach the problem from a different angle and study different scaling limits. The transition probabilities they give are on a different form from the ones we give in section 2. It is not obvious how to algebraically relate these two expressions for the same thing, and it would be interesting to understand this.

Borodin & Gorin in [BG08] do a similar analysis to the one in this article in the case of tilings of a hexagon with rhombuses. Their construction also fits into the general framework of [BF08].

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2 The Aztec Diamond Particle Process

We will here content ourselves with stating the rules of the particle dynamics that we will study. The reader will in section 3 find a summary of the traditional formulation of the shuffling algorithm and how it relates to the formulas below.

Consider the process $(X(t)) = (X^1(t), \ldots, X^m(t))$ for $t = 0, 1, 2, \ldots$, where $X^n(t) = (X_1^n(t), \ldots, X_n^n(t)) \in \mathbb{Z}^n$. It satisfies the initial condition

$$X^n(0) = \bar{x}^n \tag{2}$$

for $n = 0, \ldots, m$ where $\bar{x}_i^n = i$ for $1 \le i \le n$. At each time $t \in \mathbb{N}_0$ the process fulfils the interlacing condition

$$X_i^n(t) \le X_i^{n-1}(t) < X_{i+1}^n(t), \quad \text{for } 1 \le i < n, \tag{3}$$

and evolves in time according to

$$
\begin{aligned}
X_1^1(t) &= X_1^1(t-1) + \beta_1^1(t), \\
X_1^n(t) &= X_1^n(t-1) + \beta_1^n(t) \\
&\qquad - \mathbf{1}\{X_1^n(t-1) + \beta_1^n(t) = X_1^{n-1}(t) + 1\}, \quad \text{for } n \ge 2, \\
X_n^n(t) &= X_n^n(t-1) + \beta_n^n(t) \\
&\qquad + \mathbf{1}\{X_n^n(t-1) + \beta_n^n(t) = X_{n-1}^{n-1}(t)\}, \qquad \text{for } n \ge 2, \\
X_i^j(t) &= X_i^j(t-1) + \beta_i^j(t) \\
&\qquad - \mathbf{1}\{X_i^j(t-1) + \beta_i^j(t) = X_i^{j-1}(t) + 1\} \\
&\qquad + \mathbf{1}\{X_i^j(t-1) + \beta_i^j(t) = X_{i-1}^{j-1}(t)\}, \qquad \text{for } n \ge 3 \text{ and } 1 < i < n,
\end{aligned}
\tag{4}
$$

and $t \in \mathbb{N}_0$. All the $\beta_i^n(t)$ for $1 \le i \le n$ and $t \in \mathbb{N}$ are i.i.d. unbiased coin tosses, satisfying $\mathbb{P}[\beta_1^1(1) = 0] = \mathbb{P}[\beta_1^1(1) = 1] = \frac{1}{2}$.

One way to think about this is that at each time t, this is a set of particles on m lines. The n:th line has n particles on it at positions $X_1^n < \cdots < X_n^n$. At each time step each of these particles either stays or jumps one unit step forward independent of all others except that the particles on line n can push or block the particles on line $n+1$ to enforce the interlacing condition (3). The lines are updated in sequence starting with line 1 and ending in line m. On each line the order of update of the particles is irrelevant.

As mentioned we can write down transition probabilities for this process on a particularly convenient determinantal form. Define the delta function $\delta_i : \mathbb{Z} \to \mathbb{R}$ such that $\delta_i(x) = 1$ if $i = x$ and $\delta_i(x) = 0$ otherwise. Let us first introduce some notation.

$$
\begin{aligned}
(\phi * \psi)(x) &= \sum_{s+t=x} \phi(s)\psi(t) && \text{(Convolution product)} \\
\phi^{(0)} &= \delta_0 \\
\phi^{(n)} &= \phi^{(n-1)} * \phi && \text{for } n = 1, 2, \ldots \\
\Delta\phi &= (\delta_0 - \delta_1) * \phi && \text{(Backward difference)} \\
\Delta^{-1}\phi(x) &= \sum_{y=-\infty}^{x} \phi(y) = (\delta_0 + \delta_1 + \ldots) * \phi
\end{aligned}
$$

These convolutions have the following probabilistic meaning. Let X_1, X_2, $\ldots$, be a sequence of i.i.d. random variables with probability distribution ϕ. Then $S_n = X_1 + \cdots + X_n$ has probability distribution $\phi^{(n)}$. Backward difference operator Δ and the summation operator Δ^{-1} have no such simple probabilistic meaning. In the rest of this article $\phi = \frac{1}{2}(\delta_0 + \delta_1)$ which is the case of a Bernoulli random walk.

Let $\mathcal{W}^{n+1,n} = \{(x, y) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^n : x_1 \le y_1 < x_2 \le \cdots \le y_n < x_{n+1}\}$. For $(x, y), (x', y') \in \mathcal{W}^{n+1,n}$ and $t = \mathbb{N}_0$, define

$$
q_t^n((x, y), (x', y')) = \det \begin{bmatrix} A_t(x, x') & B_t(x, y') \\ C_t(y, x') & D_t(y, y') \end{bmatrix} \tag{5}
$$

where

- $A_t(x,x')$ is an $(n+1)\times(n+1)$-matrix where element (i,j) is $\phi^{(t)}(x'_i - x_j)$,
- $B_t(x,y')$ is an $(n+1)\times(n)$-matrix where element (i,j) is $\Delta^{-1}\phi^{(t)}(y'_i - x_j) - \mathbf{1}\{j \geq i\}$,
- $C_t(y,x')$ is an $n\times(n+1)$-matrix where element (i,j) is $\Delta\phi^{(t)}(x'_i - y_j)$ and
- $D_t(y,y')$ is an $n\times n$-matrix where element (i,j) is $\phi^{(t)}(y'_i - y_j)$.

Note that the expression $\Delta^{-1}\phi^{(t)}$ is taken to mean $\Delta^{-1}(\phi^{(t)})$, not $(\Delta^{-1}\phi)^{(t)}$. As a side note, and this will be useful in later sections, convolution is a commutative operation. So for example $\Delta^{-1}(\phi^{(t)}) = (\Delta^{-1}\phi) * \phi^{(t-1)}$ for $t \in \mathbb{N}$.

Let $\mathcal{W}^n = \{x \in \mathbb{Z}^n : x_1 < x_2 < \cdots < x_n\}$ and for $x \in \mathcal{W}^n$ let the Vandermonde determinant be

$$h_n(x) = \prod_{1\leq i<j\leq n} (x_j - x_i). \tag{6}$$

Theorem 2.1. *The transition probabilities of $(X^n(t), X^{n+1}(t))_{t\in\mathbb{N}_0}$ from the process $(\mathcal{X}(t)_{t\in\mathbb{N}_0}$ above are*

$$q_t^{n,+}((x,y),(x',y')) := \frac{h_n(y')}{h_n(y)} q_t^n((x,y),(x',y')), \tag{7}$$

that is

$$\mathbb{P}[(X^{n+1}(s+t), X^n(s+t)) = (x',y') | (X^{n+1}(s), X^n(s)) = (x,y)] \\ = q_t^{n,+}((x,y),(x',y')). \tag{8}$$

A proof is given in section 6. It is a very straightforward computation to integrate out the x component in expression (7). We find that the transition probabilities of $(X^n)_{t\in\mathbb{N}_0}$ from the process $(\mathcal{X})_{t\in\mathbb{N}_0}$ are

$$p_t^{n,+}(y,y') := \frac{h_n(y')}{h_n(y)} p_t^n(y,y') \tag{9}$$

where $p_t^n(y,y') := D_t(y,y')$ given above. We recognise this transition probability as a Karlin-MacGregor type determinant with a Doob h-conditioning. This leads to the following important observation.

Corollary 2.2. *The component $(X^n(t))_{t\in\mathbb{N}_0}$ of $(\mathcal{X}(t))_{t\in\mathbb{N}_0}$ is the positions of n walkers started at $\bar{x}^n$, taking steps with distribution ϕ and conditioned never to intersect.*

This fits nicely with theorem 1.1. The process $(X^n(t))_{t\in\mathbb{N}_0}$ is a discrete Dyson Brownian motion of n particles and its limit under suitable rescaling is Brownian motions conditioned never to intersect, which is exactly what $(X^n(t))_{t\geq 0}$ from Warren's process $(\mathbf{X}(t))_{t\geq 0}$ is.

3 The Shuffling algorithm

We will now relate some well known facts about sampling random tilings of an Aztec diamond before showing how to get the particle dynamics in section 2.

The Aztec diamond of order n, denoted A_n, is an area in the plane that is the union of those lattice squares $[a, a+1] \times [b, b+1] \subset \mathbb{R}^2$ for $a,\ b \in \mathbb{Z}$ that are entirely contained in the square $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq n+1\}$. A_n can be tiled in $2^{n(n+1)/2}$ ways by dominoes that are of size 2×1. We will be interested picking a random tiling. By random tiling in this article we will always mean that all possible tilings are given the same probability.

A key ingredient of almost all results concerning tilings of the Aztec diamond is the realization that one can distinguish four kinds of dominoes present in a typical tiling. The obvious distinction to the casual observer is the difference between horizontal and vertical dominoes. These can be subdivided further. Colour the underlying lattice squares black and white according to a checkerboard fashion in such a way that the left square on the top line is black. Let a horizontal domino be of type N or north if its leftmost square is black, and of type S or south otherwise. Likewise let a vertical domino be of type W or west if its topmost square is black and type E or east otherwise. In figures 1 and 2 the S and E type dominoes have been shaded for convenience.

One way of sampling from this measure is the so called shuffling algorithm, first described in [EKLP92], and very nicely explained and generalised in [Pro03]. It is an iterative procedure that produces a random tiling of A_{n+1} given a random tiling of A_n and some number of coin-tosses. One starts with the empty tiling on A_0 and one repeats this process until one has a tiling of the desired size. It is a theorem that this procedure gives all tilings with equal probability, provided that the coin-tosses made along the way were fair.

The algorithm works in three stages. Start with a tiling of A_n.

Destruction All 2×2 blocks consisting of an S-domino directly above an N-domino are removed. Likewise all 2×2 blocks of consisting of an E-domino directly to left of a W-domino are removed.

Shuffling All N, S, E and W-dominoes respectively move one unit length up, down, right and left respectively.

Creation The result is a tiling of a subset of A_{n+1}. The empty parts can be covered in a unique way by 2×2 squares. Toss a coin to fill these with two horizontal or two vertical dominoes with equal probability.

Figure 1 illustrates the process. In the leftmost column there are tilings of successively larger diamonds. From column one to column two, the destruction step is carried out. From there to the third column, shuffling is performed. These figures contain several dots which will concern us later in this exposition. The creation step of the algorithm applied to a diamond in the last column gives (with positive probability) the diamond in the first column on the next row.

To study more detailed properties of random tilings it is useful to introduce a coordinate system suited to the setting and a particle process such that the possible tilings correspond to particle configurations.

In the left picture in figure 2, the S and E type dominoes are shaded and a coordinate system is imposed on the tiling. For each tile there is exactly one of the x lines and exactly one of the y lines

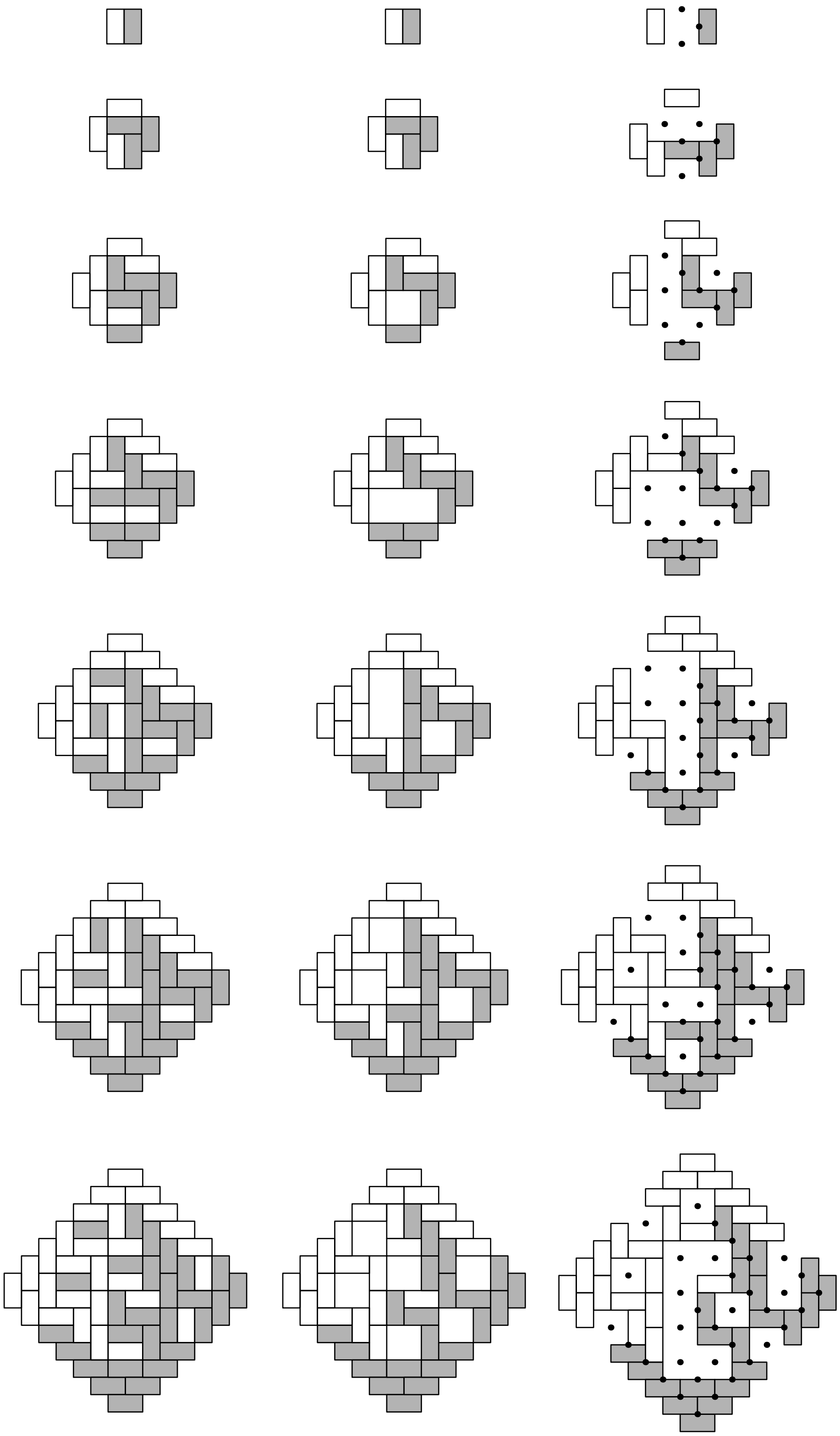

Figure 1: The shuffling procedure. S- and E-type dominoes are shaded.

that passes through its interior. Indeed we can uniquely specify the location of a tile by giving its coordinates (x, y) and type (N, S, E or W). One can see that along the line $y = n$ there are exactly n shaded tiles, for $y = 1, \ldots, 8$ where 8 is the order of the diamond. The generalisation of that statement is true for tilings of A_n for any n. We shall call the occurrence of a shaded tile a particle. The right picture in figure 2 is the same tiling but with dots marking the particles.

Just to fix some notation, let x_i^j be the x-coordinate of the i:th particle along the line $y = j$. It is clear from the definitions that these satisfy an interlacing condition,

$$x_i^j \le x_i^{j-1} \le x_{i+1}^j. \tag{10}$$

We will now see how the shuffling algorithm described above acts on these particles.

It turns out that the positions of the particles is uniquely determined before the creation stage of the last iteration of the shuffling algorithm, and we have marked these with dots in the last column in figure 1. As can be seen in that figure, running the shuffling algorithm to produce tilings of successively larger Aztec diamonds imposes certain dynamics on these particles. That is the central object of study in this article.

Let us first consider the trajectory of x_1^1. As can easily be seen in figure 1, on the $y = 1$ line there are always a number of W-dominoes, then the particle, then a number of N-dominoes. Depending on whether the creation stage of the algorithm fills the empty space in between these with a pair of horizontal or vertical dominoes, either the particle stays or its x-coordinate will increase by one in the next step. Thus the first particle performs the simple random walk

$$x_1^1(t) = x_1^1(t-1) + \gamma_1^1(t). \tag{11}$$

were $\gamma_j^i(t)$ are independent coin tosses, i.e. $P[\gamma_i^j(t) = 1] = P[\gamma_i^j(t) = 0] = \frac{1}{2}$, for $t, j = 1, \ldots$ and $0 \le i \le j$.

Consider now the particles on row $y = 2$. For x_1^2, while $x_1^2(t) < x_1^1(t)$ it performs a random walk independently of x_1^1, at each time either staying or adding one with equal probability. However, when there is equality, $x_1^2(t) = x_1^1(t)$, then the particle must be represented by a vertical (S) tile. Thus it does not contribute to growth of the west polar region, thus the particle will remain fixed. In order to represent this as a formula, we subtract one if the particle attempts to jump past x_1^1.

$$x_1^2(t) = x_1^2(t-1) + \gamma_1^2(t) - \mathbf{1}\{x_1^2(t-1) + \gamma_1^2(t) = x_1^1(t-1) + 1\} \tag{12}$$

Symmetry completes our analysis of this row with the relation

$$x_2^2(t) = x_2^2(t-1) + \gamma_2^2(t) + \mathbf{1}\{x_2^2(t-1) + \gamma_2^2(t) = x_1^1(t-1)\}. \tag{13}$$

For the third row, our previous analysis applies to the first and last particle.

$$x_1^3(t) = x_1^3(t-1) + \gamma_1^3(t) - \mathbf{1}\{x_1^3(t-1) + \gamma_1^3(t) = x_1^2(t-1) + 1\} \tag{14}$$

$$x_3^3(t) = x_3^3(t-1) + \gamma_3^3(t) + \mathbf{1}\{x_3^3(t-1) + \gamma_3^3(t) = x_2^2(t-1)\} \tag{15}$$

On $y = 3$ between x_1^2 and x_2^2 there must be first a sequence of zero or more E dominoes, then x_2^3, then a sequence of zero or more N dominoes. While x_2^3 is in the interior of this area it performs the customary random walk. It must interact with x_1^2 and x_2^2 in the same way as we have seen other particles interacting above.

So

$$x_2^3(t) = x_2^3(t-1) + \gamma_2^3(t) - \mathbf{1}\{x_2^3(t-1) + \gamma_2^3(t) = x_2^2(t-1) + 1\}$$
$$+ \mathbf{1}\{x_2^3(t-1) + \gamma_2^3(t) = x_1^2(t-1)\}. \tag{16}$$

The same pattern repeats itself evermore.

$$x_1^j(t) = x_1^j(t-1) + \gamma_1^j(t) - \mathbf{1}\{x_1^j(t-1) + \gamma_1^j(t) = x_1^{j-1}(t-1) + 1\} \tag{17}$$
$$x_j^j(t) = x_j^j(t-1) + \gamma_j^j(t) + \mathbf{1}\{x_j^j(t-1) + \gamma_j^j(t) = x_{j-1}^{j-1}(t-1)\} \tag{18}$$
$$x_i^j(t) = x_i^j(t-1) + \gamma_i^j(t) - \mathbf{1}\{x_i^j(t-1) + \gamma_i^j(t) = x_j^{j-1}(t-1) + 1\} \tag{19}$$
$$+ \mathbf{1}\{x_i^j(t-1) + \gamma_i^j(t) = x_{j-1}^{j-1}(t-1)\}. \tag{20}$$

with initial conditions $x_i^j(j) = i$ for $j = 2, \dots$ and $1 \le i \le j$.

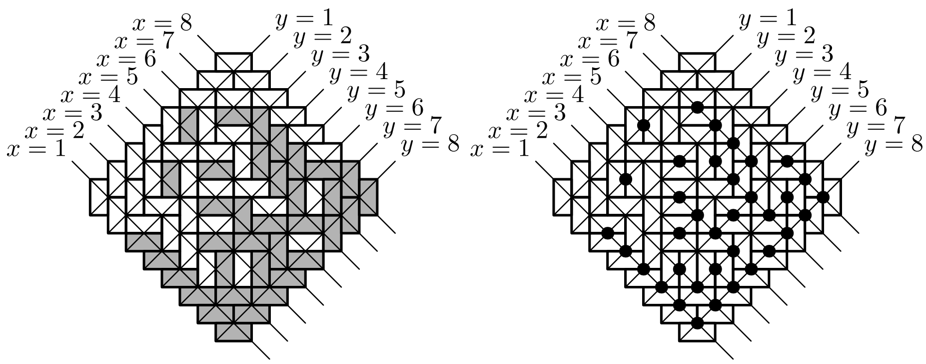

Figure 2: Same diamond

In order to analyse this process it is suitable to perform a change of variables,

$$X_i^j(t) = x_i^j(t-j), \tag{21}$$

which gives the equations given in section 2.

4 Interlacing Brownian motions

We will now digress a bit and summarise Warren's work in [War07], so as to fix notation and to emphasise the similarities between his continuous process and our discrete process. The reader is referred to that reference for more details of the construction. Consider an $\mathbb{R}^{n+1} \times \mathbb{R}^n$-valued stochastic process $(Q(t))_{t\ge 0} = (X(t), Y(t))_{t\ge 0}$ satisfying an interlacing condition

$$X_1(t) \le Y_1(t) \le X_2(t) \le \cdots \le Y_n(t) \le X_{n+1}(t), \tag{22}$$

and equations

$$Y_i(t) = y_i + \beta_i(t \wedge \tau), \qquad \text{for } i = 1, \dots, n, \tag{23}$$
$$X_i(t) = y_i + \gamma_i(t \wedge \tau) + L_i^-(t \wedge \tau) - L_i^+(t \wedge \tau) \qquad \text{for } i = 1, \dots, n+1, \tag{24}$$

where

β_i for $i = 1, \ldots, n$ and γ_i for $i = 1, \ldots, n+1$ are independent Brownian motions,

$\tau = \inf\{t \geq 0 : Y_i(t) = Y_{i+1}(t)$ for some $i\}$,

$L_1^- \equiv L_{n+1}^+ \equiv 0$ and

$$L_i^+(t) = \int_0^t \mathbf{1}(X_i(s) = Y_i(s))\, dL_i^+(s) \qquad L_i^-(t) = \int_0^t \mathbf{1}(X_i(s) = Y_{i-1}(s))\, dL_i^-(s) \tag{25}$$

are twice the semimartingale local times at zero of $X_i - Y_i$ and $X_i - Y_{i-1}$ respectively.

This process can be constructed from the Brownian motions β_i and γ_i by using Skorokhod's construction to push X_i up from Y_{i-1} and down from Y_i. The process is killed when τ is reached, i.e. when two of the Y_i meet.

Warren then goes on to show that the transition densities of this process have a determinantal form similar to what we have seen in the previous section. Let $\varphi_t(x) = (2\pi t)^{-1/2} e^{-x^2/2t}$ and $\Phi_t(x) = \int_{-\infty}^x \varphi_t(y)\, dy$. Let $W^{n,n+1} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^{n+1} : x_1 < y_1 < x_2 < \cdots < y_n < x_{n+1}\}$.

Define $q_t^n((x,y),(x',y'))$ for $(x,y),\ (x',y') \in W^{n,n+1}$ and $t > 0$ to be the determinant of the matrix

$$\begin{bmatrix} A_t(x,x') & B_t(x,y') \\ C_t(y,x') & D_t(y,y') \end{bmatrix} \tag{26}$$

where

$A_t(x,x')$ is an $(n+1) \times (n+1)$-matrix where element (i,j) is $\varphi_t(x'_i - x_j)$,

$B_t(x,y')$ is an $(n+1) \times (n)$-matrix where element (i,j) is $\Phi_t(y'_i - x_j) - \mathbf{1}(j \geq i)$,

$C_t(y,x')$ is an $n \times (n+1)$-matrix where element (i,j) is $\varphi'_t(x'_i - y_j)$ and

$D_t(y,y')$ is an $n \times n$-matrix where element (i,j) is $\varphi_t(y'_i - y_j)$.

Proposition 4.1 (Prop 2 in [War07]). *The process (X,Y) killed at time τ has transition densities q_t^n, that is*

$$q_t^n((x,y),(x,y))\, dx' dy' = \mathbb{P}^{x,y}[X(t) \in dx', Y(t) \in dx'; t < \tau] \tag{27}$$

Warren goes on to condition the Y_i not to intersect via so called the Doob h-transform. The transition densities for the transformed process are given in terms of the those for the killed process by

$$q_t^{n,+}((x,y),(x',y')) = \frac{h_n(y')}{h_n(y)} q_t^n((x,y),(x',y')). \tag{28}$$

He also shows that you can start all the X_i and Y_i of the transformed process at the origin by giving a so called entrance law,

$$\nu_t^n(x,y) := \frac{n!}{Z_{n+1}} t^{-(n+1)^2/2} \exp\left\{ -\sum_i x_i^2/(2t) \right\} h_{n+1}(x) h_n(y), \tag{29}$$

that is, showing (lemma 4 of [War07]) that this expression satisfies

$$\nu_{t+s}^n(x',y') = \int_{W^{n,n+1}} \nu_s^n(x,y) q_t^{n,+}((x,y),(x',y'))\, dx\, dy. \tag{30}$$

It is possible to integrate out the X components in that transition density and entrance law. The result is transition density

$$p_t^{n,+}(y,y') := \frac{h(y')}{h(y)} \det D_t(y,y') \tag{31}$$

and entrance law

$$\mu_t^n(y) := \frac{1}{Z_n} t^{-n^2/2} \exp\left\{-\sum_i y_i^2/(2t)\right\} (h_n(y))^2. \tag{32}$$

Now comes the interesting part. Let $\mathbf{K}$ be the cone of points $x = (x^1, \ldots, x^m)$ where $x^n = (x_1^n, \ldots, x_n^n) \in \mathbb{R}^n$ subject to the interlacing condition

$$x_i^n \leq x_i^{n-1} \leq x_{i+1}^n \tag{33}$$

for $n = 1, \ldots, m$ and $i = 1, \ldots, n$.

Warren defines a process $\mathbf{X}(t)$ taking values in $\mathbf{K}$ such that

$$X_i^n(t) = x_i^n + \gamma_i^n(t) + L_i^{n,-}(t) - L_i^{n,+}(t) \tag{34}$$

where the $(\gamma_i^n)_{i,n}$ are independent Brownian motions and $L_i^{n,+}$ and $L_i^{n,+}$ are continuous, increasing processes growing only when $X_i^n(t) = X_i^{n-1}(t)$ and $X_i^n(t) = X_{i-1}^n(t)$ respectively and the special cases $L_k^{n,+}$ and $L_1^{n,-}$ are identically zero for all n.

Think of this as essentially $m(m+1)/2$ particles performing independent Brownian motions except that the n particles in X^n can push or block the particles in X^{n+1} to enforce the interlacing condition that the whole process should stay in $\mathbf{K}$.

This full process process can be constructed inductively as follows.

1. The process $(X^n(t))_{t\geq 0}$ has transition densities $p_t^{n,+}$ and entrance law μ_t^n.
2. The process $(X^n(t), X^{n+1}(t))_{t\geq 0}$ has transition densities $q_t^{n,+}$ and entrance law ν_t^n.
3. For $n = 2, \ldots, m-1$ the process $(X^{n+1}(t))_{t\geq 0}$ is conditionally independent of $(X^1(t), \ldots, X^{n-1}(t))_{t\geq 0}$ given $(X^n(t))_{t\geq 0}$.
4. This implies (by some explicit calculations) that $(X^{n+1}(t))_{t\geq 0}$ has transition densities $p_t^{n+1,+}$ and entrance law μ_t^{n+1}.

This argument shows that the following.

Proposition 4.2 (Warren)**.** *There exists such a process $\mathbf{X}(t)$ started at the origin and it satisfies that for $n = 1, \ldots, m-1$, the process (X^n, X^{n+1}) has entrance law ν_t^n and transition probabilities $q^{n,+}$.*

It is this process $(\mathbf{X}(t))_{t\geq 0}$ that is the continuous analog of our discrete process $(\mathcal{X}(t))_{t\in\mathbb{N}_0}$.

5 Transition probabilities on two lines

In order to analyse the dynamics described in section 3 we follow Warren's example and first consider just two lines at a time. What we do in this section is very similar to section 2 of [War07].

Consider the $\mathcal{W}^{n+1,n}$-valued process process $(Q^n(t)) = (X(t), Y(t))$ for $t \in \mathbb{N}_0$ with components $X(t) = (X_1(t), \ldots, X_{n+1}(t))$ and $Y(t) = (Y_1(t), \ldots, Y_n(t))$, satisfying the equations

$$\begin{aligned} Y_i(t+1) &= Y_i(t) + \beta_i(t) \\ X_1(t+1) &= X_1(t) + \alpha_1(t) - \mathbf{1}\{X_1(t) + \alpha_1(t) = Y_1(t+1) + 1\} \\ X_i(t+1) &= X_i(t) + \alpha_i(t) + \mathbf{1}\{X_i(t) + \alpha_i(t) = Y_{i-1}(t+1)\} \\ &\quad - \mathbf{1}\{X_i(t) + \alpha_i(t) = Y_i(t+1) + 1\} \\ X_{n+1}(t+1) &= X_i(t) + \alpha_{n+1}(t) + \mathbf{1}\{X_{n+1}(t) + \alpha_{n+1}(t) = Y_n(t+1)\} \end{aligned} \tag{35}$$

where $\alpha_i(t)$ and $\beta_i(t)$ are i.i.d. coin tosses, s.t. $\mathbb{P}[\alpha_i(t) = 0] = \mathbb{P}[\alpha_i(t) = 1] = \frac{1}{2}$. They evolve until the stopping time

$$\tau = \min\left\{t : Y_i(t) = Y_{i+1}(t) \text{ for some } i \in \{1, \ldots, n-1\}\right\}. \tag{36}$$

At the time τ the process is killed and remains constant for all time after that. These are very simple dynamics, each Y_i either stays or increases by one independently of all others. The X_i do the same but are sometimes pushed up or blocked by Y_{i-1} or Y_i, respectively, so as to stay in the cone $\mathcal{W}^{n,n+1}$. This is the discrete analog of the process $(Q(t))_{t>0}$ defined in section 4 of this paper.

Define the forward difference operator and its inverse as

$$\bar{\Delta}\phi = (-\delta_0 + \delta_{-1}) * \phi \tag{37}$$

$$\bar{\Delta}^{-1}\phi(x) = \sum_{y=-\infty}^{x-1} \phi(y). \tag{38}$$

Lemma 5.1. *For any* $f : \mathcal{W}^{n+1,n} \to \mathbb{R}$,

$$\sum_{(x',y') \in \mathcal{W}^{n+1,n}} q_0^n((x,y),(x',y')) f(x',y') = f(x,y). \tag{39}$$

Proof. Let $m = 2n+1$ and $z_1 = x_1$, $z_2 = y_1$, $\ldots$, $z_{m-1} = y_n$, $z_m = x_{n+1}$. Equation (42) in [War07] states that

$$\det \begin{Bmatrix} \mathbf{1}\{z_i \le z'_j\} & i \ge j \\ -\mathbf{1}\{z_i \le z'_j\} & i < j \end{Bmatrix} = \mathbf{1}\{z_1 \le z'_1, z_2 \le z'_2, \ldots, z_m \le z'_m\} \tag{40}$$

for $z, z' \in \mathcal{W}^n$. Applying the operator $\Delta_{z'_1}(-\bar{\Delta}_{z_2})\Delta_{z'_3} \ldots (-\bar{\Delta}_{z_{m-1}})\Delta_{z'_m}$ to both sides of that equality turns the left hand side into $q_0^n((x,y),(x',y'))$ and the right hand side into $\mathbf{1}\{z_1 = z'_1, z_2 = z'_2, \ldots, z_m = z'_m\}$. □

Proposition 5.2. q_t*, for* $t = 0, 1, \ldots$*, are the transition probabilities for the process* (X, Y)*, i.e. for* (x,y)*,* $(x',y') \in \mathcal{W}^{n+1,n}$*,*

$$q_t^n((x,y),(x',y')) = \mathbb{P}^{(x,y)}[X(t) = x', Y(t) = y'; t < \tau] \tag{41}$$

Proof. Take some test function $f : \mathcal{W}^{n+1,n} \to \mathbb{R}$. Let

$$F(t,(x,y)) := \sum_{(x',y')\in\mathcal{W}^{n+1,n}} q_t^n((x,y),(x',y'))f(x',y') \tag{42}$$

and

$$G(t,(x,y)) := \mathbb{E}^{(x,y)}[f(X_t,Y_t); t < \tau] \tag{43}$$

We want of course to prove that F and G are equal and we will do this by showing that they satisfy the same recursion equation with the same boundary values. By lemma 5.1 we already know that

$$F(0,\cdot) \equiv G(0,\cdot) \equiv f(\cdot). \tag{44}$$

The master equation satisfied by G is

$$G(t+1,(x,y)) = \frac{1}{2^{2n+1}} \sum_{a_i,b_i\in\{0,1\}} G(t, x_1+a_1, y_1+b_1, x_2+a_2, \dots, y_n+b_n, x_{n+1}+a_{n+1}). \tag{45}$$

This formula simply encodes the dynamics that each particle either stays or jumps forward one step. This needs to be supplemented with some boundary conditions that have to do with the interactions between particles.

When two of the y_i-particles coincide, this corresponds to the event $t = \tau$, which does not contribute to the expectation in (43). Thus

$$G(t,\dots,y_{i-1}=z, x_i, y_i=z,\dots) := 0. \tag{46}$$

Also, the particle x_i cannot jump past y_i,

$$G(t,\dots,x_i=z+1, y_i=z,\dots) := G(t,\dots,x_i=z, y_i=z,\dots) \tag{47}$$

and x_{i+1} must not drop below y_i+1,

$$G(t,\dots,y_i=z, x_{i+1}=z,\dots) := G(t,\dots,y_i=z, x_{i+1}=z+1,\dots). \tag{48}$$

$G(t+1,\cdot)$ is uniquely determined from $G(t,\cdot)$ using the recursion equation and boundary values above. It follows that G is uniquely defined by the recursion equation (45) and the boundary conditions (44,46,47,48).

Observe that all functions $g : \mathbb{Z} \to \mathbb{R}$ satisfy

$$\frac{1}{2}(g(x)+g(x+1)) = g(x) + \frac{1}{2}\bar{\Delta} g(x). \tag{49}$$

Using this identity many times on (45) shows that

$$G(t+1,(x,y)) = (1+\frac{1}{2}\bar{\Delta}_{x_1})(1+\frac{1}{2}\bar{\Delta}_{y_1})(1+\frac{1}{2}\bar{\Delta}_{x_2})\cdots(1+\frac{1}{2}\bar{\Delta}_{y_n})(1+\frac{1}{2}\bar{\Delta}_{x_{n+1}})G(t,x_1,y_1,\dots,y_n,x_{n+1}) \tag{50}$$

which can be rewritten as

$$\bar{\Delta}_t G(t,(x,y)) = \left(\prod_{i=1}^{n+1} (1 + \frac{1}{2}\bar{\Delta}_{x_i}) \prod_{i=1}^{n} (1 + \frac{1}{2}\bar{\Delta}_{y_i}) - 1 \right) G(t,(x,y)). \tag{51}$$

The boundary conditions, equations (46,47,48), can in this notation be rewritten as

$$G(t,(x,y)) = 0 \qquad \text{when } y_i = y_{i+1}, \tag{52}$$

$$\bar{\Delta}_{x_i} G(t,(x,y)) = 0 \qquad \text{when } x_i = y_i \text{ and} \tag{53}$$

$$\bar{\Delta}_{x_{i+1}} G(t,(x,y)) = 0 \qquad \text{when } x_{i+1} = y_i. \tag{54}$$

Now consider F. The observation (49) gives that $\phi * \psi = (1 + \frac{1}{2}\Delta)\psi$. In particular, $\phi^{(n+1)}(y-x) = (1 + \frac{1}{2}\bar{\Delta}_x)\phi^{(n)}(y-x)$.

$$F(t+1,(x,y)) =$$

$$\begin{bmatrix} \phi^{(t+1)}(x_1'-x_1) & \Delta^{-1}\phi^{(t+1)}(y_1'-x_1)-1 & \phi^{(t+1)}(x_2'-x_1) & \dots \\ \Delta\phi^{(t+1)}(x_1'-y_1) & \phi^{(t+1)}(y_1'-y_1) & \Delta\phi^{(t+1)}(x_2'-y_1) & \dots \\ \phi^{(t+1)}(x_1'-x_2) & \Delta^{-1}\phi^{(t+1)}(y_1'-x_2) & \phi^{(t+1)}(x_2'-x_2) & \dots \\ \vdots & & & \end{bmatrix} =$$

$$(1 + \frac{1}{2}\bar{\Delta}_{x_1}) \begin{bmatrix} \phi^{(t)}(x_1'-x_1) & \Delta^{-1}\phi^{(t)}(y_1'-x_1)-1 & \phi^{(t)}(x_2'-x_1) & \dots \\ \Delta\phi^{(t+1)}(x_1'-y_1) & \phi^{(t+1)}(y_1'-y_1) & \Delta\phi^{(t+1)}(x_2'-y_1) & \dots \\ \phi^{(t+1)}(x_1'-x_2) & \Delta^{-1}\phi^{(t+1)}(y_1'-x_2) & \phi^{(t+1)}(x_2'-x_2) & \dots \\ \vdots & & & \end{bmatrix} =$$

$$= \dots = \prod_{i=1}^{n+1} (1 + \frac{1}{2}\bar{\Delta}_{x_i}) \prod_{i=1}^{n} (1 + \frac{1}{2}\bar{\Delta}_{y_i}) F(t,(x,y))$$

which shows that F satisfies the same recursion (51) as G.

Now let us consider the boundary values. The probability $q_t^n((x,y),(x',y'))$ is zero when $y_i = y_{i+1}$ because two of its rows are then equal. When $y_i = x_i$ for some i then $\bar{\Delta}_{x_i} q_t^n((x,y),(x',y')) = 0$ because two rows will be equal when you take the difference operator into the determinant. The same argument shows that $\bar{\Delta}_{x_{i+1}} q_t^n((x,y),(x',y')) = 0$ when $y_i = x_{i+1}$. Applying this knowledge to the sum F, shows that

$$F(t,(x,y)) = 0 \qquad \text{when } y_i = y_{i+1} \tag{55}$$

$$\bar{\Delta}_{x_i} F(t,(x,y)) = 0 \qquad \text{when } x_i = y_i \tag{56}$$

$$\bar{\Delta}_{x_{i+1}} F(t,(x,y)) = 0 \qquad \text{when } x_{i+1} = y_i \tag{57}$$

Since F and G satisfy the same recursion equation with the same boundary values, they must be equal. □

Again, following the example of Warren, we observe that it is possible to condition the processes never to leave $\mathcal{W}^{n,n+1}$ via a so called Doob h-transform. See for example [KOR02] for details about

h-transforms for discrete processes. Recall the definition of Vandermonde determinant in (6). The h-transform of the process above has transition probabilities

$$q_t^{n,+}((x,y),(x',y')) = \frac{h_n(y')}{h_n(y)} q_t^n((x,y),(x',y')). \tag{58}$$

For the sake of notation, call the transformed process $(Q^{n,+}(t))$.

The idea now is to stitch together the process $\mathcal{X}$ from processes $Q^{k,+}$ for $k = 1, \ldots, n-1$, just like Warren does in the continuous case. For this we need to establish some auxiliary results about Q^n and $Q^{n,+}$.

It is that it is possible to integrate out the x variables from $q_t^{n,+}$.

$$p_t^{n,+}(y,y') := \int_{x_1' \le y_1' < \cdots \le y_n' < x_{n+1}'} q_t^{n,+}((x,y),(x',y'))\,dx' = \frac{h_n(y')}{h_n(y)} \det[\phi^{(t)}(y_j' - y_i)]_{1 \le i,j \le n} \tag{59}$$

where dx' is counting measure on $\mathcal{W}^{n+1}$. The reader might recognise $p_t^{n,+}$, as a h-transformed version of the transition probabilities from the Lindström-Gessel-Viennot theorem. Thus this can be seen as the transition probabilities for a process on $\mathcal{W}^n$, where all n particles perform independent random walks but are conditioned never to intersect, i.e. never to leave $\mathcal{W}^n$. We state this as a proposition.

Fix $n > 0$ and let $\bar{x} = (1, \ldots, n+1) \in \mathbb{Z}^{n+1}$ and $\bar{y} = (1, \ldots, n) \in \mathbb{Z}^n$.

Proposition 5.3. *Consider the process $(Q^{n,+}(t)) = (X(t), Y(t))$ for $t \in \mathbb{N}_0$ started in $(X(0), Y(0)) = (\bar{x}, \bar{y})$. The process $(Y(t))_{t \in \mathbb{N}_0}$ is governed by $p^{n,+}$.*

Now for a technical lemma. For $x \in \mathcal{W}^{n+1}$ let $\mathcal{W}^n(x) = \{y \in \mathbb{Z}^n : x_1 \le y_1 < \cdots \le y_n < x_{n+1}\}$ and for $y \in \mathcal{W}^n(x)$ let

$$\lambda^n(x,y) = n! \frac{h_n(y)}{h_{n+1}(x)}. \tag{60}$$

It is not a difficult calculation to show that $\lambda^n(x,\cdot)$ is a probability measure on $\mathcal{W}^n(x)$. To show this rewrite $h_n(y)$ as a Vandermonde matrix and perform the summation over all y.

Lemma 5.4.

$$\int_{W^n(x)} \lambda^n(x,y) q_t^{n,+}((x,y),(x',y'))\,dy = p_t^{n+1,+}(x,x') \lambda^n(x',y') \tag{61}$$

where dy is counting measure on $W^n(x)$, that is integer vectors y that intertwine with x.

Proof. An elementary calculation given the explicit formula for $q_t^{n,+}$. □

Theorem 5.5. *Consider the process $(Q^{n,+}(t)) = (X(t), Y(t))$ for $t \in \mathbb{N}_0$ started in $(X(0), Y(0)) = (\bar{x}, \bar{y})$. The process $(X(t))_{t \in \mathbb{N}_0}$ is governed by $p^{n+1,+}$.*

Our proof of this is almost identical to the proof of proposition 5 in [War07] and is omitted.

6 Transition probabilities for the Aztec Diamond Process

Let us now return to the process $(X(t))_{t\in\mathbb{N}_0}$ that came from the shuffling algorithm. Observe in the recursions (4), the formulas that define $(X^{n+1})_{t\in\mathbb{N}_0}$ contain $(X^n)_{t\in\mathbb{N}_0}$ but not $(X^k)_{t\in\mathbb{N}_0}$ for $k<n$. Thus X^{n+1} is conditionally independent of $(X^1,\dots,X^{n-1})$ given (X^n). Also, the dependence of (X^{k+1}) on (X^k) is the same as the dependence of X on Y in Q^k and in $Q^{k,+}$, see (35). This, together with theorem 5.5 lends itself to an inductive procedure for constructing the process X.

1. The process $(X^k(t), t=0,1,\dots)$ is started at $X^k(0)=\bar{x}^k$ and has transition probabilities governed by $p^{k,+}$ for $k=1, 2, \dots, k$,
2. By proposition 5.3, X^k can be considered as the Y component of the process $Q^{k,+}$. By the observation above, the pair of processes $(X^k(t), X^{k+1}(t))$ has the same distribution as $Q^{k,+}$ started at $(\bar{x}^k, \bar{x}^{k+1})$ and are thus governed by transition probabilities $q^{k,+}$,
3. The process X^{k+1} is conditionally independent of $(X^1,\dots,X^{k-1})$ given X^k.
4. By theorem 5.5, the process X^{k+1} is governed by transition probabilities $p^{k+1,+}$ and started at $X^{k+1}(0)=\bar{x}^{k+1}$.

This proves theorem 2.1.

7 Asymptotics

Let us rescale time by $t=N\,t$ and space by $x_i=\frac{1}{2}N\,t+\frac{1}{2}\sqrt{N}x_i$ for $i=0,\dots,n$ in the above process. First let us show how to recover the entrance law for Warrens process. Recall that the Aztec diamond process the discrete process starts at $X^n(0)=\bar{x}^n$ and $X^{n+1}(0)=\bar{x}^{n+1}$.

Lemma 7.1. *Fix t and n.*

$$\left(\frac{\sqrt{N}}{2}\right)^n p_t^{n,+}(\bar{x}^n, x)\to\mu_t^n(x) \tag{62}$$

as $N\to\infty$, uniformly for x in compact sets. μ_t^n is defined in (32).

Proof. The expression can be written explicitly as

$$2^{-2n-1}\sqrt{N}^{2n+1}p_t^{n,+}((\bar{x}^{n+1},\bar{x}^n),(x,y))=2^{-n}\sqrt{N}^n\frac{h_n(x)}{h_n(\bar{x}^n)}\det\left[2^{-t}\binom{t}{x_i-j}\right]_{i,j=1}^n \tag{63}$$

which can be evaluated by a formula of Krattenthaler (Theorem 26 of [Kra99]). Applying Stirling's approximation to the result shows our lemma. □

Lemma 7.2. *Fix t and n.*

$$\left(\frac{\sqrt{N}}{2}\right)^{2n+1} q_t^{n,+}((\bar{x}^n,\bar{x}^{n+1}),(x,y))\to\nu_t^n(x,y) \tag{64}$$

as $N\to\infty$ for x and y in compact sets where $\nu_t^n(x,y)$ is given by (29).

Proof. We can write

$$2^{-2n-1}\sqrt{N}^{2n+1} q_t^{n,+}((\bar{x}^{n+1}, \bar{x}^n), (x, y)) =$$
$$2^{-2n+1}\sqrt{N}^{2n+1} \int_{\bar{x}^n \in \mathcal{W}(\bar{x}^{n+1})} \lambda^n(\bar{x}, \bar{y}) q_t^{n,+}((\bar{x}, \bar{y}), (x, y))\, d\bar{x}^n \quad (65)$$

where $d\bar{x}^n$ is counting measure on the space $\mathcal{W}(\bar{x}^{n+1})$ which has only one element. Then we apply lemma 5.4.

$$= 2^{-2n+1}\sqrt{N}^{2n+1} p_t^{n+1,+}(\bar{x}^{n+1}, x)\lambda^n(x, y) \quad (66)$$

Applying Lemma 7.1 proves this lemma. □

Likewise, Warren's expression for q_t^n and p_t^n can be recovered as a scaling limit from our expression for q_t^n and p_t^n. By Stirling's approximation,

$$\frac{1}{2}\sqrt{N}\phi^{(t)}(x) \to \varphi_t(x), \qquad \Delta^{-1}\phi^{(t)}(x) \to \int_{-\infty}^{x} \varphi_t(y)\, dy \quad (67)$$

and

$$\frac{1}{4}N\Delta\phi^{(t)}(x) \to \frac{d}{dx}\varphi_t(x) \quad (68)$$

uniformly on compact sets as $N \to \infty$ where

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}. \quad (69)$$

Lemma 7.3. *Let $x_i = \frac{1}{2}Ns + \frac{1}{2}\sqrt{N}x_i$, $x_i' = \frac{1}{2}N(s+t) + \frac{1}{2}\sqrt{N}x_i'$, and the same relations for y and y'.*

$$(\frac{\sqrt{N}}{2})^{2n+1} q_{Nt}^n((x, y), (x', y')) \to q_t^n((x, y), (x', y')) \quad (70)$$

and

$$(\frac{\sqrt{N}}{2})^{n} p_{Nt}^n((x, y), (x', y')) \to p_t^n((x, y), (x', y')) \quad (71)$$

uniformly on compact sets as $N \to \infty$.

Proof. Just insert the limit relations given above for ϕ and φ in the explicit expression for q_t^n and p_t^n. □

We now prove the first theorem from section 1.

Proof of Theorem 1.1. Given Corollary 2.2 it is clear that these non-intersecting random walks fit in as a special case of a model studied in [EK08]. Lemma 4.7 in that article gives the desired conclusion.

For completeness, we here provide our proof of convergence of finite dimensional distributions. Pick times $t_1 < \cdots < t_m \in \mathbb{R}$ and compact sets $A_1, \ldots, A_m \in W^n$. We need to show the following.

$$\lim_{N\to\infty} \int_{A_1} dx_1 \cdots \int_{A_m} dx_m p_{t_1}^{n,+}(\bar{x}, x_1) p_{t_2-t_1}^{n,+}(x_1, x_2) \cdots p_{t_m-t_{m-1}}^{n,+}(x_{m-1}, x_m) = \int_{A_1} dx_1 \cdots \int_{A_m} dx_m p_{t_1}^{n,+}(\bar{x}, x_1) p_{t_2-t_1}^{n,+}(x_1, x_2) \cdots p_{t_m-t_{m-1}}^{n,+}(x_{m-1}, x_m) \quad (72)$$

where dx is point measure on $\mathcal{W}^n$. This is a Riemann sum converging to an integral. By the uniform convergence of p_t^n in Lemmas 7.1 and 7.3 we can interchange the order of integration and taking the limit.

For tightness we shall not here repeat the argument in [EK08]*Lemma 4.7. Suffice it to say that they show that there exists an $\alpha > 2$ such that

$$\mathbb{E}[|X^{N,n}(t_2) - X^{N,n}(t_1)|^\alpha] \le C|t_2 - t_1|^{\alpha/2} \quad (73)$$

which according to for example Billingsley [Bil99, Theorem 12.3], is a sufficient condition for tightness. □

Finally, one of the main results of this article is the following.

Theorem 7.4. *The process $(X^{N,n}(t), X^{N,n+1}(t))_{t\ge0}$ defined in (1) and extended by linear interpolation to non-integer values of of Nt, converges weakly to the process $(X^n(t), X^{n+1}(t))_{t\ge0}$ from Warren's process $(\mathbf{X}(t))_{t\ge0}$ as $N \to \infty$.*

Proof. For times $t_1, \ldots, t_m$, and compact sets $A_1, \ldots, A_m \in W^{n,n+1}$, we need to study

$$\lim_{N\to\infty} \int_{A_1} dx_1 \cdots \int_{A_m} dx_m \times q_{t_1}^{n,+}((\bar{x}^{n+1}, \bar{x}^n), (x_1, y_1)) q_{t_2-t_1}^{n,+}((x_1, y_1), (x_2, y_2)) \cdots q_{t_m-t_{m-1}}^{n,+}((x_{m-1}, y_{m-1}), (x_m, x_m)) = \quad (74)$$

where of course $dx_i dy_i$ is point measure on $\mathcal{W}^{n,n+1}$. Again this is a Riemann-sum converging to an integral. By the uniform convergence of q_t^n in lemmas 7.2 and 7.3, we can interchange the order of integration and taking the limit. This gives

$$= \int_{A_1} dx_1 dy_1 \cdots \int_{A_k} dx_k dy_k \times \times \nu_{t_1}^n(x_1, y_1) q_{t_2-t_1}^n((x_1, y_1), (x_2, y_2)) \cdots q_{t_k-t_{k-1}}^n((x_{k-1}, y_{k-1}), (x_k, y_k)) \quad (75)$$

which proves convergence of finite dimensional distributions.

For tightness observe that

$$\mathbb{E}[|(X^{N,n}(t_2) - X^{N,n}(t_1), X^{N,n+1}(t_2) - X^{N,n+1}(t_1))|^\alpha] \le 2^{\alpha/2-1}(\mathbb{E}[|(X^{N,n}(t_2) - X^{N,n}(t_1)|^\alpha] + \mathbb{E}[|(X^{N,n}(t_2) - X^{N,n}(t_1)|^\alpha]) \le 2^{\alpha/2} C|t_2 - t_1|^{\alpha/2} \quad (76)$$

where the first inequality is due to convexity and the second one is from (73). □

We feel that given this theorem together with the fact that X^{n+1} is conditionally independent of $X^1, \ldots, X^{n-1}$ given X^n, lends a lot of credibility to the following conjecture.

Conjecture. *The process* $(X^N(t)_{t\geq 0}$ *defined in (1) converges weakly to Warren's process* $(\mathbf{X}(t))_{t\geq 0}$ *as* $N \to \infty$.

The difficulty in proving this is that we cannot write the transition probabilities for the whole process on a convenient form. Some trick is needed here.

We can say something about the limit at a fixed time.

Proof of Theorem 1.2. Let $\mathcal{K}^m$ be the cone of points $x = (x^1, \ldots, x^m)$ with $x^n = (x_1^n, \ldots, x_n^n) \in \mathbb{Z}^n$ such that

$$x_i^{n+1} \leq x_i^n < x_{i+1}^{n+1} \quad \text{for } i = 1, \ldots, n. \tag{77}$$

For each $x^m \in \mathcal{W}^m$ we will denote by $\mathcal{K}(x^m)$ the set of all $(x^1, \ldots, x^{m-1})$ such that $(x^1, \ldots, x^{m-1}, x^m) \in \mathcal{K}^m$. The number of points in $\mathcal{K}(x^m)$ is

$$\operatorname{card}(\mathcal{K}(x^m)) = \frac{h_n(x^m)}{\prod_{n<m} n!}. \tag{78}$$

It follows from the characterisation in section 6 that at a fixed time the distribution of $X^{n-1}(t)$ given $X^n(t)$ is $\lambda^{n-1}(X^n(t), \cdot)$. Together with the conditional independence noted in that section this implies that the distribution of $(X^1(t), \ldots, X^{n-1}(t))$ given $X^n(t)$ is uniform in $\mathcal{K}(X^n(t))$. So the probability distribution of $X(t)$ for some fixed $t \in \mathbb{N}_0$ is

$$m_t^m(x) = p_t^m(\bar{x}^m, x^m) \frac{\chi(x^1, x^2) \ldots \chi(x^{m-1}, x^m)}{\operatorname{card}(\mathcal{K}(x^m))} \tag{79}$$

where $\chi(x^n, x^{n+1})$ is one if $x_i^{n+1} \leq x_i^n < x_{i+1}^{n+1}$ for all $i = 1, \ldots, n$ and zero otherwise.

For $x^m \in W^m$, define $\mathbf{K}(x^m)$ as the set of $(x^1, \ldots, x^{m-1})$ where $x^n \in \mathbb{R}^n$ satisfying $x_i^{n+1} \leq x_i^n \leq x_{i+1}^{n+1}$. The $n(n-1)/2$-dimensional volume of $\mathbf{K}(x^m)$ is

$$\operatorname{vol}(\mathbf{K}(x^m)) = \frac{h_n(x^m)}{\prod_{n<m} n!}. \tag{80}$$

Inserting the rescaling (1) in the expression in (79) and letting $N \to \infty$ using Lemma 7.1 gives

$$\mu_t^m(x) \frac{\chi(x^1, x^2) \ldots \chi(x^{m-1}, x^m)}{\operatorname{vol}(\mathbf{K}(x^m))} \tag{81}$$

which by [War07, equation (31)] is exactly the probability density for $\mathbf{X}(t)$. □

8 Closing Remarks

Looking at the expression of transition probabilities $q_t^{n,+}$ it is natural to ask the question, what happens if we plug in a different ϕ than $\frac{1}{2}(\delta_0 + \delta_1)$ into that determinantal formula? It turns out that for many other ϕ this gives a valid transition probability, although we do not fully understand why the Doob h-conditioning still works in that case. It would be interesting to see sufficient and necessary conditions on ϕ for this construction to work.

We should mention an article by Dieker and Warren, [DW08]. They study only the top and bottom particles separately from our model, i.e. in our language $(X_1^1(t), X_2^2(t), \ldots, X_n^n(t))$ and $(X_1^1(t), X_1^2(t), \ldots, X_1^n(t))$ from $X(t)$. They consider both geometric jumps ($\phi = (1-q)(\delta_0 + q\delta_1 + q^2\delta^2 + \ldots)$, for $0 < q < 1$) and Bernoulli jumps ($\phi = p\delta_0 + q\delta_1$ where $p + q = 1$). They write down transition probabilities but do not do the rescaling to obtain a process in continuous time and space.

Another reference worthy of attention is [Joh07], by Johansson. He considers only geometric jumps and studies only the top particles $(X_1^1(t), X_2^2(t), \ldots, X_n^n(t))$ from our model, with a slight change of variables that is of no real importance. He not only writes down transition probabilities, but also recovers the top particles Warren's process $\mathbf{X}$ as the limit of his process properly rescaled.

All the results proved in this article can be generalised to $\phi = p\delta_0 + q\delta_1$ where $p + q = 1$. It is also not very difficult given the calculations in section 5 to write down transition probabilities for the top particles and to rescale that to obtain the top particles in Warren's continuous process, analogous to Johansson [Joh07] but with Bernoulli jumps. We have not included that calculation here since it does not add much to our knowledge of these processes, but it is a fact that adds to the plausibility of the above conjecture.

References

[BF08] Alexei Borodin and Patrik L. Ferrari. Anisotropic growth of random surfaces in 2+1 dimensions. *Arxiv*, 0804.3035, 2008. MR2438811

[BG08] Alexei Borodin and Vadim Gorin. Shuffling algorithm for boxed plane partitions. *Arxiv*, 0804.3071, 2008. MR2493180

[Bil99] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication. MR1700749

[DW08] A. B. Dieker and J. Warren. Determinantal transition kernels for some interacting particles on the line. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44(6):1162–1172, 2008. MR2469339

[EK08] Peter Eichelsbacher and Wolfgang König. Ordered random walks. *Electron. J. Probab.*, 13:no. 46, 1307–1336, 2008. MR2430709

[EKLP92] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp. Alternating-sign matrices and domino tilings. II. *J. Algebraic Combin.*, 1(3):219–234, 1992. MR1194076

[JN06] Kurt Johansson and Eric Nordenstam. Eigenvalues of GUE minors. *Electron. J. Probab.*, 11:no. 50, 1342–1371 (electronic), 2006. MR2268547

[Joh01] Kurt Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math. (2)*, 153(1):259–296, 2001. MR1826414

[Joh05] Kurt Johansson. The arctic circle boundary and the Airy process. *Ann. Probab.*, 33(1):1–30, 2005. MR2118857

[Joh07] Kurt Johansson. A multi-dimenstional Markov chain and the Meixner ensemble. *Arxiv*, 0707.0098v1, 2007. MR2288065

[KOR02] Wolfgang König, Neil O'Connell, and Sébastien Roch. Non-colliding random walks, tandem queues, and discrete orthogonal polynomial ensembles. *Electron. J. Probab.*, 7:no. 5, 24 pp. (electronic), 2002. MR1887625

[Kra99] C. Krattenthaler. Advanced determinant calculus. *Sém. Lothar. Combin.*, 42:Art. B42q, 67 pp. (electronic), 1999. The Andrews Festschrift (Maratea, 1998). MR1701596

[OR06] Andrei Okounkov and Nicolai Reshetikhin. The birth of a random matrix. *Mosc. Math. J.*, 6(3):553–566, 2006. MR2274865

[Pro03] James Propp. Generalized domino-shuffling. *Theoret. Comput. Sci.*, 303(2-3):267–301, 2003. Tilings of the plane. MR1990768

[War07] Jon Warren. Dyson's Brownian motions, intertwining and interlacing. *Electron. J. Probab.*, 12:no. 19, 573–590 (electronic), 2007. MR2299928