DOI: 10.1214/EJP.v11-308

Electronic Journal of Probability

Vol. 11 (2006), Paper no. 6, pages 146–161.

Journal URL
http://www.math.washington.edu/∼ejpecp/

Two-sided estimates on the density of the Feynman-Kac semigroups of stable-like processes

Renming Song *
Department of Mathematics, University of Illinois
Urbana, IL 61801, USA
Email: rsong@math.uiuc.edu
http://www.math.uiuc.edu/∼rsong

Abstract

Suppose that $\alpha \in (0, 2)$ and that X is an α-stable-like process on $\mathbf{R}^d$. Let μ be a signed measure on $\mathbf{R}^d$ belonging to the class $\mathbf{K}_{d,\alpha}$ and A^μ_t be the continuous additive functional of X associated with μ. In this paper we show that the Feynman-Kac semigroup $\{T^\mu_t : t \geq 0\}$ defined by

$$T^\mu_t f(x) = \mathbf{E}_x \left(e^{-A^\mu_t} f(X_t) \right)$$

has a density q^μ and that there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1 e^{-c_2 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha} \leq q^\mu(t, x, y) \leq c_3 e^{c_4 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha}$$

for all $(t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$. We also provide similar estimates for the densities of two other kinds of Feynman-Kac semigroups of X.

Key words: Stable processes, stable-like processes, Kato class, Feynman-Kac semigroups, continuous additive functionals, continuous additive functionals of zero energy, purely discontinuous additive functionals.

AMS 2000 Subject Classification: Primary: 60J45, 60J35; Secondary: 60J55, 60J57, 60J75, 60G51, 60G52.

Submitted to EJP on November 9, 2004. Final version accepted on February 21, 2006.

*The research of this author is supported in part by a joint US-Croatia grant INT 0302167.

1 Introduction

Suppose that $X = (X_t, \mathbf{P}_x)$ is a Brownian motion on $\mathbf{R}^d$ and that V is a function on $\mathbf{R}^d$ belonging to the Kato class of X, i. e., a function satisfying the condition

$$\lim_{t\downarrow 0} \sup_{x\in\mathbf{R}^d} \mathbf{E}_x \int_0^t |V|(X_s)ds = 0.$$

It is well known (see [11] for instance) that the Feynman-Kac semigroup $\{T_t^V : t \geq 0\}$ with potential V

$$T_t^V f(x) = \mathbf{E}_x \left(\exp(-\int_0^t V(X_s)ds) f(X_t) \right),$$

has a transition density $q^V(t,x,y)$ with respect to the Lebesgue measure and that q^V has both an upper and a lower Gaussian estimates, that is there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1 e^{-c_2 t} t^{-\frac{d}{2}} \exp(-\frac{3|x-y|^2}{4t}) \leq q^V(t,x,y) \leq c_3 e^{c_4 t} t^{-\frac{d}{2}} \exp(-\frac{|x-y|^2}{4t}) \tag{1.1}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$. This result can be easily generalized (see [2] for instance) to the case when V is replaced by a signed measure satisfying

$$\lim_{t\downarrow 0} \sup_{x\in\mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} t^{-\frac{d}{2}} \exp(-\frac{|x-y|^2}{2t}) |\mu|(dy)ds = 0$$

and $\int_0^t V(X_s)ds$ is replaced by the continuous additive functional A_t^μ of X associated with μ.

Now suppose that $\alpha \in (0,2)$ and that $X = (X_t, \mathbf{P}_x)$ is a symmetric α-stable process on $\mathbf{R}^d$. The question that we are going to address in this paper is the following: can one establish two-sided estimates for the density of the Feynman-Kac semigroup of the symmetric α-stable process X? As far as we know, this question has not been addressed in the literature. The proof of (1.1) in [11] and [2] can not be adapted to the case of discontinuous stable processes. It seems that, to answer the question above, one has to use some new ideas. In this paper, we are going to tackle the question above by adapting an idea used in [13] and [14] to establish heat kernel estimates for diffusions to the present case. Actually, instead of dealing with symmetric stable processes, we are going to deal with the more general stable-like processes introduced in [4].

The content of this paper is organized as follows. In section 2, we first recall the definition of the Kato class with respect to symmetric α-stable processes and some basic facts about stable-like processes, and then we present some preliminary results on Feynman-Kac semigroups. In section 3 we establish two-sided estimates on the density of Feynman-Kac semigroups with potentials given by measures belonging to the Kato class. In the last section we deal with two other kinds of Feynman-Kac semigroups of stable-like processes. The first kind consists of Feynman-Kac semigroups given by purely discontinuous additive functionals, and the second kind consists of Feynman-Kac semigroups given by continuous additive functionals of zero energy.

In this paper we will use the following convention on the labeling of constants. The values of the constants $M_1, M_2, \cdots$ will remain the same throughout this paper, while the values of the constants $C_1, C_2, \cdots$ might change from one appearance to the next. The labeling of the constants $C_1, C_2, \cdots$ starts anew in the statement of each result.

2 Kato Class and Basic Properties of Feynman-Kac Semigroups

In this paper we will always assume that $\alpha \in (0, 2)$. We will use $X^0 = \{X_t^0, \mathbf{P}_x^0\}$ to denote a symmetric α-stable process in $\mathbf{R}^d$ whose transition density $p^0(t, x, y) = p^0(t, x - y)$ with respect to the Lebesgue measure satisfies

$$\int_{\mathbf{R}^d} e^{ix\cdot\xi} p^0(t, x) dx = e^{-t|\xi|^\alpha}, \quad t > 0.$$

It is known (see [3]) that there exist positive constants $M_1 < M_2$ such that

$$M_1 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x - y|}\right)^{d+\alpha} \le p^0(t, x, y) \le M_2 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x - y|}\right)^{d+\alpha} \tag{2.1}$$

for all $(t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$. For any $\lambda > 0$, we define

$$G_\lambda^0(x, y) = G_\lambda^0(x - y) = \int_0^\infty e^{-\lambda t} p^0(t, x - y) dt$$

for all $x, y \in \mathbf{R}^d$. When $\alpha < d$, the process X^0 is transient and its potential density $G^0(x, y) = G^0(x - y)$ is given by

$$G^0(x - y) = \int_0^\infty p^0(t, x - y) dt = 2^{-\alpha} \pi^{-\frac{d}{2}} \Gamma\left(\frac{d - \alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)^{-1} |x - y|^{\alpha - d}.$$

The Dirichlet form $(\mathcal{E}^0, \mathcal{F})$ of X^0 is given by

$$\mathcal{E}^0(u, v) = \frac{1}{2} \mathcal{A}(d, -\alpha) \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dxdy$$

$$\mathcal{F} = \left\{ u \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dxdy < \infty \right\},$$

where

$$\mathcal{A}(d, -\alpha) = \frac{|\alpha| \, \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha} \, \pi^{d/2} \Gamma(1 - \frac{\alpha}{2})}.$$

For any function V on $\mathbf{R}^d$ and $t > 0$, we define

$$N_V(t) = \sup_{x \in \mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} p^0(s, x, y) |V(y)| dyds.$$

By a signed measure we mean in this paper the difference of two nonnegative measures at most one of which can have infinite total mass. For any signed measure on $\mathbf{R}^d$, we use μ^+ and μ^- to denote its positive and negative parts, and $|\mu| = \mu^+ + \mu^-$ its total variation. For any $t > 0$, we define

$$N_\mu(t) = \sup_{x\in\mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} p^0(s, x, y)|\mu|(dy)ds.$$

Definition 2.1 *We say that a function V on R^d belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{t\downarrow 0} N_V(t) = 0$. We say that a signed Radon measure μ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{t\downarrow 0} N_\mu(t) = 0$.*

Rigorously speaking a function V in $\mathbf{K}_{d,\alpha}$ may not give rise to a signed measure μ in $\mathbf{K}_{d,\alpha}$ since it may not give rise to a signed measure at all. However, for the sake of simplicity we use the convention that whenever we write that a signed measure μ belongs to $\mathbf{K}_{d,\alpha}$ we are implicitly assuming that we are covering the case of all the functions in $\mathbf{K}_{d,\alpha}$ as well.

The following result is well known, see [1] and [12] for instance.

Proposition 2.1 *Suppose that μ is a signed measure on $\mathbf{R}^d$. Then $\mu \in \mathbf{K}_{d,\alpha}$ if and only if*

$$\lim_{\lambda\to\infty} \sup_{x\in\mathbf{R}^d} \int_{\mathbf{R}^d} G^0_\lambda(x, y)|\mu|(dy) = 0.$$

When $\alpha < d$, $\mu \in \mathbf{K}_{d,\alpha}$ is also equivalent to the condition

$$\lim_{r\to 0} \sup_{x\in\mathbf{R}^d} \int_{|x-y|<r} \frac{|\mu|(dy)}{|x-y|^{d-\alpha}} = 0.$$

We assume from now on that m is a measure on R^d given by $m(dx) = M(x)dx$, where M is a function satisfying

$$M_3 \le M(x) \le M_4, \quad x \in \mathbf{R}^d \tag{2.2}$$

for some positive constants $M_3 < M_4$. We will fix a symmetric function $c(x, y)$ on $\mathbf{R}^d \times \mathbf{R}^d$ which is bounded between two fixed positive constants. If for any $u, v \in \mathcal{F}$ we define

$$\mathcal{E}(u, v) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \frac{c(x, y)(u(x) - u(y))(v(x) - v(y))}{|x-y|^{d+\alpha}} m(dx)m(dy),$$

then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbf{R}^d, m)$. It is shown in [4] that, associated with this Dirichlet form, there is an m-symmetric Hunt process $X = \{X_t, \mathbf{P}_x\}$ on $\mathbf{R}^d$ which can start from any point $x \in \mathbf{R}^d$. We will use $\{\mathcal{M}_t; t \ge 0\}$ to denote the natural filtration of X. The process X is called an α-stable-like process in [4]. It is also shown in [4] that the process X admits a transition density $p(t, x, y)$ with respect to m and that p is jointly continuous on $(0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$ and satisfies the condition

$$M_5 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \le p(t, x, y) \le M_6 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \tag{2.3}$$

for all $(t, x, y) \in (0, 1] \times \mathbf{R}^d \times \mathbf{R}^d$. By using the scaling property (see the proof of Proposition 4.1 in [4]), one can show that (2.3) is valid for all $(t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$. We may and do assume that $M_5 < 1 < M_6$.

From the display above and estimates (2.1) one can easily see that a signed measure μ is in $\mathbf{K}_{d,\alpha}$ if and only if

$$\lim_{t \downarrow 0} \sup_{x \in \mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} p(s, x, y)|\mu|(dy)ds = 0.$$

By repeating the argument in the proof of Theorem 2.1 in [2], we can show the following

Lemma 2.2 *Let μ be a signed Radon measure on $\mathbf{R}^d$. Then $\mu \in \mathbf{K}_{d,\alpha}$ if and only the following conditions are satisfied:*

1. *$|\mu|$ is a smooth measure in the sense of [9],*
2. *the continuous additive functional A_t associated with μ can be defined without exceptional set,*
3. *$\lim_{t \downarrow 0} \sup_{x \in \mathbf{R}^d} \mathbf{E}_x |A|_t = 0$, where $|A|_t = A^+_t + A^-_t$, A^+_t and A^-_t being the positive continuous additive functionals of X associated with μ^+ and μ^- respectively.*

Furthermore, if $\mu \in \mathbf{K}_{d,\alpha}$ and A_t is the continuous additive functional of X associated with μ, then

$$\mathbf{E}_x A_t = \int_0^t \int_{\mathbf{R}^d} p(s, x, y)\mu(dy)ds, \quad \forall (t, x) \in (0, \infty) \times \mathbf{R}^d.$$

Proof. We omit the details. □

In the sequel, whenever we have a signed measure $\mu \in \mathbf{K}_{d,\alpha}$, we will use A^μ_t to denote the continuous additive functional of X associated with μ. Using Khas'minskii's lemma (see Lemma 2.6 of [2]), we can easily show the following

Lemma 2.3 *Suppose that $\mu \in \mathbf{K}_{d,\alpha}$ and A^μ_t is the continuous additive functional of X associated with μ. There exist positive constants C_1 and C_2, depending on μ only via the rate at which $N_\mu(t)$ goes to zero, such that*

$$\sup_{x \in \mathbf{R}^d} \mathbf{E}_x \left(e^{A^{|\mu|}_t} \right) \le C_1 e^{C_2 t}, \quad t > 0.$$

The meaning of the phrase "depending on μ only via the rate at which $N_\mu(t)$ goes to zero" will become clear in the proof of Theorem 3.3. It roughly means that if $w(t)$ is a increasing function on $(0, \infty)$ with $\lim_{t \to 0} w(t) = 0$, then there exist positive constants C_1 and C_2 such that for any signed measure μ with

$$N_\mu(t) \le w(t), \quad t > 0,$$

we have

$$\sup_{x\in\mathbf{R}^d} \mathbf{E}_x\left(e^{A_t^{|\mu|}}\right) \le C_1 e^{C_2 t}, \quad t > 0.$$

For any $\mu \in \mathbf{K}_{d,\alpha}$, we define the Feynman-Kac semigroup $\{T_t^\mu : t \ge 0\}$ with potential μ by

$$T_t^\mu f(x) = \mathbf{E}_x\left(e^{-A_t^\mu} f(X_t)\right).$$

When μ is given by $\mu(dx) = U(x)dx$ for some function U, we will sometimes write T_t^μ as T_t^U. The following result is well known, see [12] and [5].

Theorem 2.4 *Suppose that $\mu \in \mathbf{K}_{d,\alpha}$, then*

1. *For any $p \in [1,\infty)$, $\{T_t^\mu : t \ge 0\}$ is a strongly continuous semigroup in $L^p(\mathbf{R}^d, m)$;*
2. *For each $t > 0$, T_t^μ maps $L^\infty(\mathbf{R}^d, m)$ into bounded continuous functions on $\mathbf{R}^d$;*
3. *For any $p \in [1,\infty)$ and $t > 0$, T_t^μ maps $L^p(\mathbf{R}^d, m)$, $p \in [0,\infty)$, into bounded continuous functions on $\mathbf{R}^d$ which converges to zero at infinity, and there exist positive constants C_1 and C_2, depending on μ only via the rate at which $N_\mu(t)$ goes to zero, such that*

$$\|T_t^\mu\|_{p,p} \le \|T_t^\mu\|_{\infty,\infty} \le C_1 e^{C_2 t}, \quad t > 0,$$

where, for any $p, q \in [1,\infty]$, $\|T_t^\mu\|_{p,q}$ stands for the norm of T_t^μ as an operator from $L^p(\mathbf{R}^d, m)$ into $L^q(\mathbf{R}^d, m)$.

Using an argument similar to that of the proof of Theorem 3.1 in [2], we can show the following

Theorem 2.5 *For any $\mu \in \mathbf{K}_{d,\alpha}$, there exists a function $q^\mu(t,x,y)$ such that*

1. *q^μ is jointly continuous on $(0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$;*
2. *there exist positive constants C_1 and C_2 depending on μ only via the rate at which $N_\mu(t)$ goes to zero such that*

$$0 < q^\mu(t,x,y) \le C_1 e^{C_2 t} t^{-\frac{d}{\alpha}}, \quad \forall (t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d;$$

3. *$T_t^\mu f(x) = \int_{\mathbf{R}^d} q^\mu(t,x,y) f(y) m(dy)$ for all $(t,x) \in (0,\infty) \times \mathbf{R}^d$ and all bounded function f on $\mathbf{R}^d$;*
4. *$\int_{\mathbf{R}^d} q^\mu(t,x,z) q^\mu(s,z,y) m(dz) = q^\mu(t+s,x,y)$ for all $t,s > 0$ and $(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$;*
5. *$q^\mu(t,x,y)$ is symmetric in x and y;*

6. if f is a bounded function continuous at $x \in \mathbf{R}^d$, then

$$\lim_{t\downarrow 0} \int_{\mathbf{R}^d} q^\mu(t,x,y) f(y) m(dy) = f(x).$$

Proof. We omit the details. $\square$

Corollary 2.6 *For any $\mu \in \mathbf{K}_{d,\alpha}$, the function q^μ in the theorem above satisfies the equation*

$$q^\mu(t,x,y) = p(t,x,y) - \int_0^t \int_{\mathbf{R}^d} p(s,x,z) q^\mu(t-s,z,y)\mu(dz)ds, \tag{2.4}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

Proof. Since for any $t > 0$

$$e^{-A_t^\mu} = 1 - \int_0^t e^{-(A_t^\mu - A_s^\mu)} dA_s^\mu,$$

we have

$$\mathbf{E}_x \left(e^{-A_t^\mu} f(X_t) \right) = \mathbf{E}_x f(X_t) - \mathbf{E}_x \left(f(X_t) \int_0^t e^{-(A_t^\mu - A_s^\mu)} dA_s^\mu \right)$$

for all $(t,x) \in (0,\infty) \times \mathbf{R}^d$ and all bounded functions f on $\mathbf{R}^d$. Now the conclusion of the corollary follows easily from the Markov property, Fubini's theorem and the two theorems above. $\square$

When the measure μ is given by $\mu(dx) = U(x)dx$ form some function U, we will sometimes write q^μ as q^U.

3 Two-sided Estimates for Densities of Local Feynman-Kac transforms

In this section we shall establish two-sided estimates for the densities of Feynman-Kac semigroups with potentials belonging to $\mathbf{K}_{d,\alpha}$. The following elementary lemma will play an important role.

Lemma 3.1 *Suppose that a, b, c, d are positive numbers. If $a < d$ and $c < b$, then we have*

$$(1 \wedge \frac{a}{b})(1 \wedge \frac{d}{c}) \le (1 \wedge \frac{a}{c})(1 \wedge \frac{d}{b}). \tag{3.1}$$

Proof. We prove this lemma by looking at all the different cases.

In the first case we assume that $a \ge b$. In this case we have $c < b \le a < d$, so the left and right hand sides of (3.1) are both equal to 1. Thus (3.1) is valid in this case.

In the second case we assume that $c \le a \le b$. We further divide this case into two subcases. In the first subcase we assume that $c \le a \le b \le d$. In this subcase the left hand side of (3.1) is equal to $\frac{a}{b}$ and the right hand side is equal to 1. In the second subcase we assume that $c \le a < d \le b$. In this subcase the left hand side of (3.1) is equal to $\frac{a}{b}$ and the right hand side is equal to $\frac{d}{b}$. Thus (3.1) is valid in this case.

In the third case we assume that $a \le c$. We further divide this case into three subcases. In the first subcase we assume that $a \le c < b \le d$. In this subcase the left hand side of (3.1) is equal to $\frac{a}{b}$ and the right hand side is equal to $\frac{a}{c}$. In the second subcase we assume that $a \le c \le d \le b$. In this subcase the left hand side of (3.1) is equal to $\frac{a}{b}$ and the right hand side is equal to $\frac{ad}{bc} \ge \frac{a}{b}$. In the third subcase we assume that $a < d \le c < b$. In this subcase the left and right hand sides of (3.1) are both equal to $\frac{ad}{bc}$. Thus (3.1) is also valid in this case. □

The following lemma is similar to Lemma 3.1 of [14] and is crucial in establishing the main estimates of this paper.

Lemma 3.2 *There exists a positive constant C depending only on d and α such that for any measure ν on $\mathbf{R}^d$ and $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$,*

$$\begin{aligned}
&\int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{\alpha}} \left(1 \wedge \frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha} (t-s)^{-\frac{d}{\alpha}} \left(1 \wedge \frac{(t-s)^{1/\alpha}}{|z-y|}\right)^{d+\alpha} \nu(dz)ds \\
&\le CM_1^{-1} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} N_\nu(\frac{t}{2}).
\end{aligned}$$

Proof. Put

$$J(t,x,y) = \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{\alpha}} \left(1 \wedge \frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha} (t-s)^{-\frac{d}{\alpha}} \left(1 \wedge \frac{(t-s)^{1/\alpha}}{|z-y|}\right)^{d+\alpha} \nu(dz)ds.$$

We can rewrite J as

$$J(t,x,y) = \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right) \int_{\mathbf{R}^d} \cdots \nu(dz)ds := J_1(t,x,y) + J_2(t,x,y).$$

We estimate $J_1(t,x,y)$ by estimating the integrand separately on the region

$$R_1 := \{(s,z) : s \in (0, \frac{t}{2}), |z-y| \ge \frac{1}{2}|x-y|\}$$

and the region

$$R_2 := \{(s,z) : s \in (0, \frac{t}{2}), |z-y| < \frac{1}{2}|x-y|\}.$$

On R_1 we have

$$
\begin{aligned}
& s^{-\frac{d}{\alpha}}\left(1\wedge\frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha}(t-s)^{-\frac{d}{\alpha}}\left(1\wedge\frac{(t-s)^{1/\alpha}}{|z-y|}\right)^{d+\alpha} \\
& \le 2^{\frac{d}{\alpha}}s^{-\frac{d}{\alpha}}\left(1\wedge\frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{2t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \\
& \le 2^{\frac{d}{\alpha}+d+\alpha}s^{-\frac{d}{\alpha}}\left(1\wedge\frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.
\end{aligned}
$$

On R_2 we have $|x-z|\ge|x-y|-|y-z|\ge\frac{1}{2}|x-y|$. So by applying Lemma 3.1 with $a=s^{1/\alpha}$, $b=|x-z|$, $c=|z-y|$ and $d=(t-s)^{1/\alpha}$ we get that

$$
\begin{aligned}
& s^{-\frac{d}{\alpha}}\left(1\wedge\frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha}(t-s)^{-\frac{d}{\alpha}}\left(1\wedge\frac{(t-s)^{1/\alpha}}{|z-y|}\right)^{d+\alpha} \\
& \le s^{-\frac{d}{\alpha}}\left(1\wedge\frac{s^{1/\alpha}}{|z-y|}\right)^{d+\alpha}(t-s)^{-\frac{d}{\alpha}}\left(1\wedge\frac{(t-s)^{1/\alpha}}{|x-z|}\right)^{d+\alpha} \\
& \le 2^{\frac{d}{\alpha}}s^{-\frac{d}{\alpha}}\left(1\wedge\frac{s^{1/\alpha}}{|y-z|}\right)^{d+\alpha}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{2t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \\
& \le 2^{\frac{d}{\alpha}+d+\alpha}s^{-\frac{d}{\alpha}}\left(1\wedge\frac{s^{1/\alpha}}{|y-z|}\right)^{d+\alpha}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
& J_1(t,x,y) \\
& \le 2^{\frac{d}{\alpha}+d+\alpha}t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}\int_0^{\frac{t}{2}}\int_{\mathbf{R}^d}s^{-\frac{d}{\alpha}}\left((1\wedge\frac{s^{1/\alpha}}{|x-z|})+(1\wedge\frac{s^{1/\alpha}}{|y-z|})\right)^{d+\alpha}\nu(dz)ds \\
& \le M_1^{-1}2^{\frac{d}{\alpha}+d+\alpha+1}N_\nu(\frac{t}{2})t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.
\end{aligned}
$$

By a similar argument we get

$$
J_2(t,x,y)\le M_1^{-1}2^{\frac{d}{\alpha}+d+\alpha+1}N_\nu(\frac{t}{2})t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.
$$

Consequently we have

$$
J(t,x,y)\le M_1^{-1}2^{\frac{d}{\alpha}+d+\alpha+2}N_\nu(\frac{t}{2})t^{-\frac{d}{\alpha}}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \tag{3.2}
$$

for all $(t,x,y)\in(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$. □

Theorem 3.3 *For any $\mu \in \mathbf{K}_{d,\alpha}$, there exists a positive constant T, depending on μ only via the rate at which $N_\mu(t)$ goes to zero, such that*

$$C_1 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \le q^\mu(t,x,y) \le C_2 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}$$

for some constants C_1 and C_2 depend only on M_5 and for all $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$.

Proof. For $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$, we define $I_n(t,x,y)$ recursively for $n \ge 0$ by

$$\begin{aligned} &I_0(t,x,y) = p(t,x,y), \\ &I_{n+1} = \int_0^t \int_{\mathbf{R}^d} p(s,x,z) I_n(t-s,z,y)\mu(dz)ds. \end{aligned}$$

We claim that there exists a positive constant T, depending on μ only via the rate at which $N_\mu(t)$ goes to zero, such that for all $n \ge 1$ and $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$

$$I_n(t,x,y) \le (\frac{M_5}{2})^n t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}. \tag{3.3}$$

We will prove this claim by induction. In fact, for $n=1$, we have

$$\begin{aligned} &|I_1(t,x,y)| = |\int_0^t \int_{\mathbf{R}^d} p(s,x,z)p(t-s,z,y)\mu(dz)ds| \\ &\le M_6^2 \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{\alpha}} \left(1 \wedge \frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha} (t-s)^{-\frac{d}{\alpha}} \left(1 \wedge \frac{(t-s)^{1/\alpha}}{|z-y|}\right)^{d+\alpha} |\mu|(dz)ds. \end{aligned}$$

Applying Lemma 3.2 we get that there exists a constant $c_1 > 0$ depending only on d and α such that

$$|I_1(t,x,y)| \le c_1 M_1^{-1} M_6^2 N_\mu(\frac{t}{2}) t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.$$

Take $T > 0$ small enough so that

$$c_1 M_1^{-1} M_6^2 N_\mu(\frac{t}{2}) \le \frac{M_5}{2}, \quad t \le T.$$

Obviously, this T depends on μ only via the rate at which $N_\mu(t)$ goes to zero and

$$|I_1(t,x,y)| \le \frac{M_5}{2} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}$$

for all $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$. Thus the claim above is valid for $n=1$. Now suppose that the claim is valid for n. Then we have

$$\begin{aligned} &|I_{n+1}(t,x,y)| = |\int_0^t \int_{\mathbf{R}^d} p(s,x,z) I_n(t-s,z,y)\mu(dz)ds| \\ &\le M_6 (\frac{M_5}{2})^n \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{\alpha}} \left(1 \wedge \frac{s^{1/\alpha}}{|x-z|}\right)^{d+\alpha} (t-s)^{-\frac{d}{\alpha}} \left(1 \wedge \frac{(t-s)^{1/\alpha}}{|z-y|}\right)^{d+\alpha} |\mu|(dz)ds. \end{aligned}$$

Applying Lemma 3.2 again we get that

$$\begin{aligned}|I_{n+1}(t,x,y)| &\leq c_1 M_1^{-1} M_6 (\frac{M_5}{2})^n N_\mu(\frac{t}{2}) t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \\ &\leq (\frac{M_5}{2})^{n+1} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}\end{aligned}$$

for all $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$. Therefore the claim above is valid.

It follows from the claim above that, for $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$, the series $\sum_{n=0}^{\infty} |I_n(t,x,y)|$ is uniformly absolutely convergent and

$$\sum_{n=0}^{\infty} |I_n(t,x,y)| \leq \sum_{n=0}^{\infty} (\frac{M_5}{2})^n t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} := c_2 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.$$

Using Corollary 2.6 and Lemma 3.2 we see that

$$q^\mu(t,x,y) = \sum_{n=0}^{\infty} (-1)^n I_n(t,x,y) \leq c_2 t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}$$

for all $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$.

Using the claim above again we get that, for $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$,

$$\sum_{n=1}^{\infty} |I_n(t,x,y)| \leq \frac{M_5}{2-M_5} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}.$$

Therefore we have

$$q^\mu(t,x,y) \geq \frac{1-M_5}{2-M_5} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}$$

for all $(t,x,y) \in (0,T] \times \mathbf{R}^d \times \mathbf{R}^d$. □

As a consequence of the theorem above and the semigroup property (Theorem 2.5.4), we immediately get the following

Theorem 3.4 *For any $\mu \in \mathbf{K}_{d,\alpha}$, there exist positive constant C_1, C_2, C_3, C_4, depending on μ only via the rate at which $N_\mu(t)$ goes to zero, such that*

$$C_1 e^{-C_2 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \leq q^\mu(t,x,y) \leq C_3 e^{C_4 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

4 Feynman-Kac Semigroups Given by Discontinuous Additive Functionals and Continuous Additive Functionals of Zero Energy

We first deal with a class of Feynman-Kac semigroups given by a purely discontinuous additive functional. To do this, we need to recall a definition and introduce some notations.

Definition 4.1 *Suppose that F is a function on $\mathbf{R}^d \times \mathbf{R}^d$. We say that F belongs to the class $\mathbf{J}_{d,\alpha}$ if F is bounded, vanishing on the diagonal, and the function*

$$x \mapsto \int_{\mathbf{R}^d} \frac{|F(x,y)|}{|x-y|^{d+\alpha}} dy$$

belongs to $\mathbf{K}_{d,\alpha}$.

It is easy to see from the definition above that if $F \in \mathbf{J}_{d,\alpha}$, then e^{-F} is also in $\mathbf{J}_{d,\alpha}$.

The process X has a Lévy system (N, H) given by $H_t = t$ and

$$N(x, dy) = 2c(x,y)|x-y|^{-(d+\alpha)} m(dy),$$

that is, for any nonnegative function f on $\mathbf{R}^d \times \mathbf{R}^d$ vanishing on the diagonal

$$\mathbf{E}_x \left(\sum_{s \le t} f(X_{s-}, X_s) \right) = \mathbf{E}_x \int_0^t \int_{\mathbf{R}^d} \frac{2c(X_s, y) f(X_s, y)}{|X_s - y|^{d+\alpha}} m(dy) ds$$

for every $x \in \mathbf{R}^d$ and $t > 0$.

For any F belonging to $\mathbf{J}_{d,\alpha}$, we put

$$B_t^F = \sum_{s \le t} F(X_{s-}, X_s), \quad t \ge 0.$$

We can define the following so-called non-local Feynman-Kac semigroup

$$S_t^F f(x) = \mathbf{E}_x \left(e^{-B_t^F} f(X_t) \right).$$

This semigroup has been studied in [12] and [6].

Theorem 4.1 *Suppose that $F \in \mathbf{J}_{d,\alpha}$ is a symmetric function. The semigroup $\{S_t^F, t \ge 0\}$ admits a density $k^F(t,x,y)$ with respect to m and that k^F is jointly continuous on $(0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$. Furthermore, there exist positive constants C_1, C_2, C_3 and C_4 such that*

$$C_1 e^{-C_2 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha} \le k^F(t,x,y) \le C_3 e^{C_4 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

Proof. Put $G = e^{-F} - 1$ and

$$V(x) = \int_{\mathbf{R}^d} \frac{2c(x,y)G(x,y)}{|x-y|^{d+\alpha}} m(dy).$$

Using the definition of Lévy systems we can see that $B_t^G - A_t^V$ is a $\mathbf{P}_x$-martingale for every $x \in \mathbf{R}^d$. It follows from the Doleans-Dade formula that

$$M_t = e^{-B_t^F - A_t^V} \tag{4.1}$$

is a local martingale under $\mathbf{P}_x$ for every $x \in \mathbf{R}^d$. M_t is a clearly a multiplicative functional, so M_t is supermartingale multiplicative functional of X. Therefore by Theorem 62.19 of [10], M_t defines a family of probability measures $\{\tilde{\mathbf{P}}_x, x \in \mathbf{R}^d\}$ by $d\tilde{\mathbf{P}}_x = M_t d\mathbf{P}_x$ on $\mathcal{M}_t$. We will use $\tilde{X} = (X_t, \tilde{\mathbf{P}}_x)$ denote this new process. It follows from [6] that $\tilde{X}$ is a symmetric Hunt process on $\mathbf{R}^d$ whose Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is given by $\tilde{\mathcal{F}} = \mathcal{F}$ and

$$\tilde{\mathcal{E}}(u,u) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \frac{e^{-F(x,y)}c(x,y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}} m(dx)m(dy), \quad u \in \mathcal{F}.$$

Thus $\tilde{X}$ is an α-stable-like process in the sense of [4]. It follows from (4.1) that for any nonnegative function f on $\mathbf{R}^d$, any $t > 0$ and any $x \in R^d$ we have

$$\mathbf{E}_x\left(e^{-B_t^F} f(X_t)\right) = \tilde{\mathbf{E}}_x\left(e^{A_t^V} f(X_t)\right).$$

Therefore S_t^F can be regarded as a Feynman-Kac semigroup of $\tilde{X}$ with a potential $-V$. Now our assertion follows immediately from Theorem 3.4. □

Now we deal with Feynman-Kac semigroups given by continuous additive functionals of zero energy. To do this, we need to recall some facts from the theory of Dirichlet forms.

We denote by $\mathcal{F}_e$ the family of functions u on $\mathbf{R}^d$ that is finite almost everywhere and there is an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ almost everywhere on $\mathbf{R}^d$. $(\mathcal{E}, \mathcal{F}_e)$ is called the extended Dirichlet space of $(\mathcal{E}, \mathcal{F})$. It is well known that any $u \in \mathcal{F}_e$ has a quasi-continuous version $\tilde{u}$. In this paper, whenever we talk about a function $u \in \mathcal{F}_e$, we implicitly assume that we are dealing with its quasi-continuous version. It is known (see [9]) that, for any $u \in \mathcal{F}_e$, $u(X_t)$ has the following Fukushima's decomposition

$$u(X_t) = u(X_0) + M_t^u + N_t^u, \quad t \geq 0.$$

Here M_t^u is a martingale additive functional of X and N_t^u is a continuous additive functional of X with zero quadratic variation. Note that in general, N_t^u is not a process of finite variation. The martingale part M_t^u is given by

$$\begin{aligned} M_t^u \;&=\; \lim_{n\to\infty}\Bigg\{\sum_{0<s\leq t}(u(X_s)-u(X_{s-}))1_{\{|u(X_s)-u(X_{s-})|>1/n\}} \\ &\quad - \int_0^t \left(\int_{\{y\in\mathbf{R}^d:|u(y)-u(X_s)|>1/n\}} \frac{2c(X_s,y)(u(y)-u(X_s))}{|X_s-y|^{d+\alpha}} m(dy)\right) ds\Bigg\}. \end{aligned}$$

Let $\mu_{<u>}$ be the Revuz measure associated with the sharp bracket positive continuous additive functional $< M^u >$. Then

$$\mu_{<u>}(dx) = \int_{\mathbf{R}^d} \frac{2c(x,y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}} m(dy)m(dx).$$

It follows from [8] that when $u \in \mathcal{F}_e$ satisfies the condition $\mu_{<u>} \in \mathbf{K}_{d,\alpha}$, the additive functionals M_t^u and N_t^u can be taken as additive functionals in the strict sense.

For any quasi-continuous function $u \in \mathcal{F}_e$ with $\mu_{<u>} \in \mathbf{K}_{d,\alpha}$, we will consider the following Feynman-Kac semigroup $\{R_t^u : t \geq 0\}$:

$$R_t^u f(x) = \mathbf{E}_x \left(e^{N_t^u} f(X_t) \right), \quad t \geq 0.$$

This semigroup has been studied in [7].

Theorem 4.2 *Suppose that u is bounded quasi-continuous function belonging to $\mathcal{F}_e$ and that $\mu_{<u>} \in \mathbf{K}_{d,\alpha}$. The semigroup $\{R_t^u, t \geq 0\}$ admits a density $r^u(t,x,y)$ with respect to m and r^u is jointly continuous on $(0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$. Furthermore, there exist positive constants C_1, C_2, C_3 and C_4 such that*

$$C_1 e^{-C_2 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha} \leq r^u(t,x,y) \leq C_3 e^{C_4 t} t^{-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha}$$

for all $(t,x,y) \in (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$.

Proof. Put $\rho(x) = e^{-u(x)}$ and $\rho(\partial) = 1$. It is easy to check that $\rho - 1 \in \mathcal{F}_e$. Thus if we define $M^\rho := M^{\rho-1}$ and $N^\rho := N^{\rho-1}$, then we have the Fukushima's decomposition for $\rho(X_t)$:

$$\rho(X_t) = \rho(X_0) + M_t^\rho + N_t^\rho.$$

Define a square integrable martingale M by

$$M_t = \int_0^t \frac{1}{\rho(X_{s-})} dM_s^\rho.$$

Let L_t^ρ be the solution of the following SDE:

$$L_t^\rho = 1 + \int_0^t L_{s-}^\rho dM_s.$$

It follows from the Doleans-Dade formula that

$$\begin{aligned}
L_t^\rho &= \exp(M_t) \prod_{0<s\leq t} (1 + M_s - M_{s-}) e^{-(M_s - M_{s-})} \\
&= \exp(M_t) \prod_{0<s\leq t} \frac{\rho(X_s)}{\rho(X_{s-})} \exp\left(1 - \frac{\rho(X_s)}{\rho(X_{s-})} \right) \\
&= \exp\left(M_t^{-u} + \int_0^t \int_{\mathbf{R}^d} \frac{2c(X_s,y)(u(X_s) - u(y) + 1 - e^{u(X_s)-u(y)})}{|X_s - y|^{d+\alpha}}) m(dy)ds \right), \quad (4.2)
\end{aligned}$$

where the last equality is shown on page 487 of [7]. L^ρ_t is a nonnegative local martingale and therefore a supermartingale multiplicative functional of X. Therefore by Theorem 62.19 of [10] L^ρ_t defines a family of probability measures $\{\tilde{\mathbf{P}}_x, x \in E\}$ by $d\tilde{\mathbf{P}}_x = L^\rho_t d\mathbf{P}_x$ on $\mathcal{M}_t$. We will use $\tilde{X} = (X_t, \tilde{\mathbf{P}}_x)$ denote this new process. Put $\nu(dx) = \rho^2(x)m(dx)$. It follows from [7] that $\tilde{X}$ is a ν-symmetric Hunt process on R^d whose Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\mathbf{R}^d, \nu)$ is given by $\tilde{\mathcal{F}} = \mathcal{F}$ and

$$\tilde{\mathcal{E}}(u,u) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \frac{\rho(x)\rho(y)c(x,y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}} m(dx)m(dy), \quad u \in \mathcal{F}.$$

Thus $\tilde{X}$ is an α-stable-like process in the sense of [4]. Using the boundedness of u and the assumption $\mu^{<u>} \in \mathbf{K}_{d,\alpha}$ we can easily check that the function

$$V(x) = \int_{\mathbf{R}^d} \frac{2c(x,y)(u(x)-u(y)+1-e^{u(x)-u(y)})}{|x-y|^{d+\alpha}} m(dy)$$

belongs to $\mathbf{K}_{d,\alpha}$. It follows from (4.2) that for any nonnegative function f on $\mathbf{R}^d$, any $t > 0$ and any $x \in \mathbf{R}^d$ we have

$$\begin{aligned} \mathbf{E}_x\left(e^{N^u_t} f(X_t)\right) &= e^{-u(x)}\mathbf{E}_x\left(L^\rho_t \exp(-\int_0^t V(X_s)ds)(fe^u)(X_t)\right) \\ &= e^{-u(x)}\tilde{\mathbf{E}}_x\left(\exp(-\int_0^t V(X_s)ds)(fe^u)(X_t)\right). \end{aligned}$$

Now applying Theorem 3.4 to the process $\tilde{X}$ and the potential V, we see that there is a function $r(t,x,y)$ defined on $(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$ such that

$$\begin{aligned} \mathbf{E}_x\left(e^{N^u_t} f(X_t)\right) &= e^{-u(x)}\int_{\mathbf{R}^d} r(t,x,y)(fe^u)(y)\nu(dy) \\ &= e^{-u(x)}\int_{\mathbf{R}^d} r(t,x,y)e^{-u(y)}f(y)m(dy) \end{aligned} \tag{4.3}$$

for all $(t,x) \in (0,\infty)\times\mathbf{R}^d$ and all nonnegative function f on $\mathbf{R}^d$ and that there exist positive constants c_1, c_2, c_3, c_4 such that

$$c_1 e^{-c_2 t} t^{-\frac{d}{\alpha}}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \le r(t,x,y) \le c_3 e^{c_4 t} t^{-\frac{d}{\alpha}}\left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}$$

for all $(t,x,y) \in (0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$. It follows from (4.3) that R^u_t admits a density with respect to m given by

$$r^u(t,x,y) = e^{-u(x)}r(t,x,y)e^{-u(y)}$$

for all $(t,x,y) \in (0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$. The last assertion of the theorem follows easily from the boundedness of u. □

Remark 4.3 *Of course, one can combine Theorems 3.4, 4.1 and 4.2 into one theorem about the density of Feynman-Kac semigroup given by additive functionals involving all three components: a continuous part with finite variation, a continuous part of zero energy and a purely discontinuous part. We leave this to the reader.*

Acknowledgment: I thank Panki Kim, Murali Rao and Zoran Vondracek for their helpful comments on the first version of this paper.

References

[1] S. Albeverio and Z. M. Ma, Perturbation of Dirichlet forms–lower semiboundedness, closability and form cores, *J. Funct. Anal.*, **99**(1991), 332–356.

[2] Ph. Blanchard and Z. M. Ma, Semigroup of Schrödinger operators with potentials given by Radon measures, In *Stochastic processes, physics and geometry*, 160–195, World Sci. Publishing, Teaneck, NJ, 1990.

[3] R. M. Blumenthal and R. K. Getoor, Some theorems on stable processes, *Trans. Amer. Math. Soc.*, **95**(1960), 263–273.

[4] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d-sets, *Stochastic Process. Appl.*, **108**(2003), 27–62.

[5] Z.-Q. Chen and R. Song, Intrinsic ultracontractivity and conditional gauge for symmetric stable processes, *J. Funct. Anal.*, **150** (1997), 204–239.

[6] Z.-Q. Chen and R. Song, Conditional gauge theorem for non-local Feynman-Kac transforms, *Probab. Th. rel. Fields*,**125**(2003), 45–72.

[7] Z.-Q. Chen and T. Zhang, Girsanov and Feynman-Kac type transformations for symmetric Markov processes, *Ann. Inst. H. Poincaré Probab. Statist.*, **38**(2002), 475–505.

[8] M. Fukushima, On a decomposition of additive functionals in the strict sense for a symmetric Markov process. In *Dirichlet forms and stochastic processes (Beijing, 1993)*, 155–169, de Gruyter, Berlin, 1995.

[9] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Walter De Gruyter, Berlin, 1994.

[10] M. Sharpe, *General Theory of Markov Processes*, Academic Press, Boston, 1988.

[11] B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc. (N.S.)*, **7**(1982), 447–526.

[12] R. Song, Feynman-Kac semigroup with discontinuous additive functionals, *J. Theoret. Probab.*, **8** (1995), 727–762.

[13] Qi S. Zhang, A Harnack inequality for the equation $\nabla(a\nabla u) + b\nabla u = 0$, when $|b| \in K_{n+1}$, *Manuscripta Math.*, **89** (1996), 61–77.

[14] Qi S. Zhang, Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u)+B\nabla u-u_t = 0$, *Manuscripta Math.*, **93** (1997), 381–390.