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PITMAN'S $2M - X$ THEOREM FOR SKIP-FREE RANDOM WALKS WITH MARKOVIAN INCREMENTS

B.M. HAMBLY
Mathematical Institute, University of Oxford, 24-29 St Giles, Oxford OX1 3LB, UK
email: `hambly@maths.ox.ac.uk`

J.B. MARTIN
Cambridge University
email: `jbm11@cus.cam.ac.uk`

N. O'CONNELL
BRIMS, HP Labs, Bristol BS34 8QZ, UK
email: `noc@hplb.hpl.hp.com`

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Abstract
Let $(\xi_k,\ k \geq 0)$ be a Markov chain on $\{-1,+1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1|\ \xi_k = 1) = a$ and $P(\xi_{k+1} = -1|\ \xi_k = -1) = b < a$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0\leq k\leq n} X_k$. We prove that the process $2M - X$ has the same law as that of X conditioned to stay non-negative.

Pitman's representation theorem [19] states that, if $(X_t,\ t \geq 0)$ is a standard Brownian motion and $M_t = \max_{s\leq t} X_s$, then $2M-X$ has the same law as the 3-dimensional Bessel process. This was extended in [20] to the case of non-zero drift, where it is shown that, if X_t is a standard Brownian motion with drift, then $2M - X$ is a certain diffusion process. This diffusion has the significant property that it can be interpreted as the law of X conditioned to stay positive (in an appropriate sense). Pitman's theorem has the following discrete analogue [19, 18]: if X is a simple random walk with non-negative drift (in continuous or discrete time) then $2M - X$ has the same law as X conditioned to stay non-negative (for the symmetric random walk this conditioning is in the sense of Doob).

Here we present a version of Pitman's theorem for a random walk with Markovian increments. Let $(\xi_k,\ k \geq 0)$ be a Markov chain on $\{-1,+1\}$ with $\xi_0 = 1$ and transition probabilities $P(\xi_{k+1} = 1|\ \xi_k = 1) = a$ and $P(\xi_{k+1} = -1|\ \xi_k = -1) = b$. We will assume that $1 > a > b > 0$. Set $X_0 = 0$, $X_n = \xi_1 + \cdots + \xi_n$ and $M_n = \max_{0\leq k\leq n} X_k$.

Theorem 1 *The process $2M - X$ has the same law as that of X conditioned to stay non-negative.*

Note that, if $b = 1 - a$, then X is a simple random walk with drift and we recover the original statement of Pitman's theorem in discrete time.
To prove Theorem 1, we first consider a two-sided stationary version of ξ, which we denote by $(\eta_k,\ k \in \mathbb{Z})$, and define a stationary process $\{Q_n, n \in \mathbb{Z}\}$ by

$$Q_n = \max_{m \le n} \left(- \sum_{j=m}^{n} \eta_j \right)^+ .$$

Note that Q satisfies the Lindley recursion $Q_{n+1} = (Q_n - \eta_{n+1})^+$, and we have the following queueing interpretation. The number of customers in the queue at time n is Q_n; if $\eta_{n+1} = -1$ a new customer arrives at the queue and $Q_{n+1} = Q_n + 1$; if $\eta_{n+1} = 1$ and $Q_n > 0$, a customer departs from the queue and $Q_{n+1} = Q_n - 1$; otherwise $Q_{n+1} = Q_n$.
Note that the process η can be recovered from Q, as follows:

$$\eta_n = \begin{cases} -1 & \text{if } Q_n > Q_{n-1} \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

For $n \in \mathbb{Z}$, set $\bar{Q}_n = Q_{-n}$.

Theorem 2 *The processes Q and $\bar{Q}$ have the same law.*

Proof: We first note that it suffices to consider a single excursion of the process Q from zero. This follows from the fact that, at the beginning and end of a single excursion, the values of η are determined, and so these act as regeneration points for the process. To see that the law of a single excursion is reversible, note that the probability of a particular excursion path depends only on the numbers of transitions (in the underlying Markov chain η) of each type which occur within that excursion path, and these numbers are invariant under time-reversal.
□

Thus, if we define, for $n \in \mathbb{Z}$,

$$\hat{\eta}_n = \begin{cases} -1 & \text{if } Q_n > Q_{n+1} \\ 1 & \text{otherwise,} \end{cases} \tag{2}$$

we have the following corollary of Theorem 2.

Corollary 3 *The process $\hat{\eta}$ has the same law as η.*

Proof of Theorem 1: Note that we can write $\hat{\eta}_n = \eta_{n+1} + 2(Q_{n+1} - Q_n)$. Summing this, we obtain, for $n \ge 1$,

$$\sum_{j=0}^{n-1} \hat{\eta}_j = \tilde{X}_n + 2(Q_n - Q_0). \tag{3}$$

where $\tilde{X}_n = \sum_{j=1}^{n} \eta_j$. If we adopt the convention that empty sums are zero, and set $\tilde{X}_0 = 0$, then this formula remains valid for $n = 0$. It follows that, on $\{Q_0 = 0\}$,

$$\sum_{j=0}^{n-1} \hat{\eta}_j = 2\tilde{M}_n - \tilde{X}_n, \tag{4}$$

where $\tilde{M}_n = \max_{0 \le m \le n} \tilde{X}_m$.
Note also that, from the definitions, for $m \in \mathbb{Z}$,

$$Q_m = (Q_{m+1} - \hat{\eta}_m)^+ = \max_{n \ge m} \left(- \sum_{j=m}^{n} \hat{\eta}_j \right)^+ . \tag{5}$$

The law of X conditioned to stay non-negative is the same as the law of $\tilde{X}$ conditioned to stay non-negative, since the events $X_1 \ge 0$ and $\tilde{X}_1 \ge 0$ respectively require that $\xi_1 = 1$ and $\eta_1 = 1$, and so the difference in law between ξ and η becomes irrelevant. By Corollary 3, the law of $\tilde{X}$ conditioned to stay non-negative is the same as the law of the process

$$\left(\sum_{j=0}^{n-1} \hat{\eta}_j, \ n \ge 0 \right)$$

given that

$$Q_0 = \max_{n \ge 0} \left(- \sum_{j=0}^{n-1} \hat{\eta}_j \right) = 0.$$

By (4) this is the same as the law of $2\tilde{M} - \tilde{X}$ given that $Q_0 = 0$ or, equivalently, that $\eta_0 = 1$; but this is the same as the law of $2M - X$, so we are done. □
In the queueing interpretation, $\hat{\eta} = -1$ whenever there is a departure from the queue and $\hat{\eta} = 1$ otherwise. Thus, Corollary 3 states that the process of departures from the queue has the same law as the process of arrivals to the queue; it can therefore be regarded as an extension of the celebrated theorem in queueing theory, due to Burke [5], which states that the output of a stable $M/M/1$ queue in equilibrium has the same law as the input (both are Poisson processes; by considering the embedded chain in the M/M/1 queue, Burke's theorem is equivalent to the statement of Corollary 3 with $b = 1 - a$). Our proof of Theorem 2 is inspired by the kind of reversibility arguments used often in queueing theory. For general discussions on the role of reversibility in queueing theory, see [4, 13, 22]; the idea of using reversibility to prove Burke's theorem is originally due to Reich [21].
To describe the finite dimensional distributions of the process $2M - X$ appearing in Theorem 1, one can consider the Markov chain (X, ξ) conditioned on X staying non-negative; this is a h-transform of (X, ξ) with

$$h(k, -1) = 1 - \left(\frac{b}{a} \right)^{k+1}$$

and

$$h(k, 1) = 1 - \left(\frac{b}{a} \right)^{k} \left(\frac{1-a}{1-b} \right).$$

It is well-known (see, for example, [11]) that the particular case of Theorem 1 with $b = 1 - a$ is more or less equivalent to a collection of random walk analogues of Williams' path-decomposition and time-reversal results relating Brownian motion and the three-dimensional Bessel process. The same is true for general a and b. For example, X conditioned to stay non-negative has a shift-homogeneous regenerative property at last exit times, like the three-dimensional Bessel process. Moreover, if we set $\hat{X}_n = \sum_{j=0}^{n-1} \hat{\eta}$, then, by Corollary 3, $(\hat{X}_n, \ n \ge 1)$ has the same law as $(\tilde{X}_n, \ n \ge 1)$, and this can be interpreted as the analogue of Williams'

path-decomposition for Brownian motion with drift. The analogue of Williams' time-reversal theorem for the three-dimensional Bessel process can also be verified. In this case we have, setting $R = 2M - X$, $L_k = \max\{n : \ R_n = k\}$ and $T_k = \min\{n : \ X_n = k\}$, that $\{k - X_{T_{k+1}-1-n},\ 0 \le n \le T_{k+1} - 1\}$ and $\{R_n,\ 0 \le n \le L_k\}$ have the same law.
Finally, we remark that the following analogue of Theorem 1 holds in continuous time: let $(\xi_t,\ t \ge 0)$ be a continuous-time Markov chain on $\{-1, +1\}$ with $\xi_0 = 1$, and set $X_t = \int_0^t \xi_s ds$, $M_t = \max_{0 \le s \le t} X_s$. We assume that the transition rates of the chain are such that the event that X remains non-negative forever has positive probability. Then $2M - X$ has the same law as that of X conditioned to stay non-negative. The proof is identical to that of Theorem 1; in particular, the following analogues of Theorem 2 and Corollary 3 also hold: if we let $(\eta_t,\ t \in \mathbb{R})$ be a stationary version of ξ and, for $t \in \mathbb{R}$, set

$$Q_t = \max_{s \le t} \left(- \int_s^t \eta_s ds \right),$$

then $\bar{Q}$ (defined as $\bar{Q}_t = Q_{-t}$) has the same law as Q, and $\hat{\eta}$, defined by

$$\hat{\eta}_t = \begin{cases} -1 & \text{if } \eta_t = 1 \text{ and } Q_t > 0 \\ 1 & \text{otherwise,} \end{cases} \tag{6}$$

has the same law as η. The process X in this setting is sometimes called the *telegrapher's random process*, because it is connected with the telegrapher equation. It was introduced by Kac [12], where it is also shown to be related to the Dirac equation. There is a considerable literature on this process and its connections with relativistic quantum mechanics (see, for example, [6, 7] and references therein).
For other variants and multidimensional extensions of Pitman's theorem see [1, 2, 9, 10, 15, 8, 16, 17, 18] and references therein. In [16] a version of Pitman's theorem for geometric functionals of Brownian motion is presented. In [17] connections with Burke's theorem are discussed. In [18], a representation for non-colliding Brownian motions is given (the case of two motions is equivalent to Pitman's theorem); this extends a partial representation (for the rightmost motion at a single epoch) given in [1, 8]. The corresponding result for continuous-time random walks is also presented in [18]. The corresponding discrete-time random walk result is presented in [15], and this extends a partial representation given in [10]. See also [9] for a related but not yet well understood representation; this is also discussed in [15]. (See also [14].) In [2] an extension of Pitman's theorem is given for spectrally positive Lévy processes. A partial extension of Pitman's theorem for Brownian motion in a wedge of angle $\pi/3$ is presented in [3].

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