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Asymptotic stability of neutral stochastic functional integro-differential equations*

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Abstract

This paper is concerned with the existence and asymptotic stability in the p-th moment of mild solutions of nonlinear impulsive stochastic delay neutral partial functional integro-differential equations. We suppose that the linear part possesses a resolvent operator in the sense given in [8], and the nonlinear terms are assumed to be Lipschitz continuous. A fixed point approach is used to achieve the required result. An example is provided to illustrate the theory developed in this work. .

Keywords: Resolvent operators;C_0-semigroup; impulsive stochastic neutral partial functional integro-differential equations;Wiener process; mild solution;asymptotic stability..
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1 Introduction

Neutral stochastic differential equation occurs in many areas of science and engineering having received much attention over the past decades. Partial integro-differential equations have wide applications in the field of mechanics, electricity, etc (see [8] for more details). For abstract models of partial integro-differential equations with resolvent operators, see for instance [3, 5, 8]. The deterministic model often fluctuates due to noise, and for this reason, it is sensible to use stochastic model problems instead of deterministic ones. For more specific details the reader is referred to [1, 4, 6, 12].

In recent years, impulsive differential equations have been used to model interesting problems from applications, see [19]. Considerable work in the field of fixed impulsive type equations may be found in [9, 16] and the references therein. The study of impulsive stochastic differential equations (ISDEs) is a new area of research and few publications on that subject can be found in the literature. For example, non-autonomous and random dynamical systems perturbed by impulses are investigated in [20]. Jun Yang *et al.* [22] studied the stability analysis of ISDEs with delays, Zhiguo Yang *et al.* [23] analyzed the exponential p-th stability of ISDEs with delays, while in [17, 18], R. Sakthivel and J. Luo investigated the existence and asymptotic stability in the p-th moment of mild solutions to ISDEs, with and without infinite delays, through the fixed point theory. Motivated by the works [14, 15, 17, 18] we study in this paper the existence and asymptotic

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stability in the p-th moment of mild solutions of nonlinear neutral impulsive stochastic integro-differential equations (ISNIDEs) with delays under Lipschitz conditions.

Here we shall apply the Banach fixed point principle to investigate the existence and asymptotic stability of mild solutions of this class of equations.

The rest of the paper is organized as follows. In Section 2, we summarize several important working tools on the Wiener process and deterministic integro-differential equations that will be used to develop our results. Section 3 is devoted to the existence and asymptotic stability of mild solutions. Finally, in Section 4, we provide an example to illustrate our main approach.

2 Preliminaries

2.1 Wiener process

Throughout this work, $\mathbb{H}$ and $\mathbb{K}$ are two real separable Hilbert spaces; we denote by $\langle\cdot,\cdot\rangle_{\mathbb{H}}$,$\langle\cdot,\cdot\rangle_{\mathbb{K}}$ their inner products and by $|\cdot|_{\mathbb{H}}, |\cdot|_{\mathbb{K}}$ their associated norms, respectively. $\mathcal{L}(\mathbb{H},\mathbb{K})$ denotes the space of all bounded linear operators from $\mathbb{H}$ into $\mathbb{K}$, equipped with the usual operator norm $\|\cdot\|$. In the sequel, we use the same symbol $\|\cdot\|$ to denote the norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Moreover, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right-continuous while $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets).

Let $\{w(t) : t \geq 0\}$ denote a $\mathbb{K}$-valued Wiener process defined on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$, with covariance operator Q, that is, $E\langle w(t),x\rangle_{\mathbb{K}}\langle w(s),y\rangle_{\mathbb{K}} = (t\wedge s)\langle Qx,y\rangle_{\mathbb{K}}$, for all $x,y\in\mathbb{K}$, where Q is a positive, self-adjoint, trace class operator on $\mathbb{K}$. In particular, we denote by $w(t)$ a $\mathbb{K}$-valued Q-Wiener process with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. To define the stochastic integrals with respect to the Q-Wiener process $w(t)$, we introduce the subspace $\mathbb{K}_0 = Q^{1/2}\mathbb{K}$ of $\mathbb{K}$ endowed with the inner product $\langle u,v\rangle_{\mathbb{K}_0} = \left\langle Q^{-1/2}u, Q^{-1/2}v\right\rangle_{\mathbb{K}}$ as a Hilbert space. We assume that there exist a complete orthonormal system $\{e_i\}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i$, $i = 1,2,\ldots$, and a sequence $\{\beta_i(t)\}_{i>1}$ of independent standard Brownian motions such that

$$w(t) = \sum_{i=1}^{\infty}\sqrt{\lambda_i}\beta_i(t)e_i \ \text{ for } \ t\geq 0$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0\leq s\leq t\}$. Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0,\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $\mathbb{K}_0$ to $\mathbb{H}$, which turns out to be a separable Hilbert space equipped with the norm $\|v\|_{\mathcal{L}_2^0} = tr((vQ^{1/2})(vQ^{1/2})^*)$ for any $v\in\mathcal{L}_2^0$. For any bounded operator $v\in\mathcal{L}_2^0$, its norm is reduced to $\|v\|^2_{\mathcal{L}_2^0} = tr(vQv^*)$.

2.2 Partial integro–differential equations in Banach spaces

In the present section, we recall some definitions, notations and properties needed in the sequel. Let Z_1 and Z_2 denote two Banach spaces. We denote by $\mathcal{L}(Z_1,Z_2)$ the Banach space of bounded linear operators from Z_1 into Z_2 endowed with the operator norm and we abbreviate this notation to $\mathcal{L}(Z_1)$ when $Z_1 = Z_2$.

In what follows, $\mathbb{H}$ will denote a Banach space, A and $B(t)$ are closed linear operators on $\mathbb{H}$. Y represents the Banach space $D(A)$, the domain of operator A, equipped with the graph norm

$$|y|_Y := |Ay| + |y| \quad \text{for } \ y\in Y.$$

The notation $C([0,+\infty);Y)$ stands for the space of all continuous functions from $[0,+\infty)$ into Y. We then consider the following Cauchy problem

$$\begin{cases} v'(t) &=& Av(t) + \int_0^t B(t-s)v(s)ds \ \text{ for } \ t \geq 0, \\ v(0) &=& v_0 \in \mathbb{H}. \end{cases} \tag{2.1}$$

Definition 2.1. *([8]) A resolvent operator for Eq.* (2.1) *is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{H})$ for $t \geq 0$, satisfying the following properties :*

(i) $R(0) = I$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For each $x \in \mathbb{H}$, $R(t)x$ is strongly continuous for $t \geq 0$.

(iii) For $x \in Y,\ R(\cdot)x \in C^1([0,+\infty);\mathbb{H}) \cap C([0,+\infty);Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds \quad \text{ for } \ t \geq 0. \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [8, 13]. The resolvent operator plays an important role to study the existence of solutions and to establish a variation of constants formula for nonlinear systems. For this reason, we need to know when the linear system (2.1) possesses a resolvent operator. Theorem 2.2 below provides a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H1) A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathbb{H}$.

(H2) For all $t \geq 0$, $B(t)$ is a continuous linear operator from $(Y, |\cdot|_Y)$ into $(\mathbb{H}, |\cdot|_{\mathbb{H}})$. Moreover, there exists an integrable function $c : [0,+\infty) \to \mathbb{R}^+$ such that for any $y \in Y$, $y \mapsto B(t)y$ belongs to $W^{1,1}([0,+\infty),\mathbb{H})$ and

$$\left| \frac{d}{dt} B(t)y \right|_{\mathbb{H}} \leq c(t)|y|_Y \ \text{ for } \ y \in Y \ \text{ and } \ t \geq 0.$$

Theorem 2.2. *([8]) Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Then Eq.* (2.1) *admits a resolvent operator $(R(t))_{t\geq 0}$.*

Theorem 2.3. *([11]) Assume that hypotheses* **(H1)** *and* **(H2)** *hold. Let $T(t)$ be a compact operator for $t > 0$. Then, the corresponding resolvent operator $R(t)$ of Eq.* (2.1) *is continuous for $t > 0$ in the operator norm, namely, for all $t_0 > 0$, it holds that $\lim_{h\to 0} \|R(t_0 + h) - R(t_0)\| = 0$.*

In the sequel, we recall some results on the existence of solutions for the following integro–differential equation

$$\begin{cases} v'(t) &=& Av(t) + \int_0^t B(t-s)v(s)ds + q(t) \ \text{ for } \ t \geq 0, \\ v(0) &=& v_0 \in \mathbb{H}, \end{cases} \tag{2.2}$$

where $q : [0,+\infty[\to \mathbb{H}$ is a continuous function.

Definition 2.4. *([8]) A continuous function $v : [0,+\infty) \to \mathbb{H}$ is said to be a strict solution of Eq.* (2.2) *if* *(i)* $v \in C^1([0,+\infty);\mathbb{H}) \cap C([0,+\infty);Y)$,

(ii) v *satisfies Eq.* (2.2) *for* $t \geq 0$.

Remark 2.5. From this definition we deduce that $v(t) \in D(A)$, and the function $B(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0,+\infty)$.

Theorem 2.6. *([8]) Assume that* ***(H1)-(H2)*** *hold. If v is a strict solution of Eq.* (2.2)*, then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \text{for} \quad t \geq 0. \tag{2.3}$$

Accordingly, we can establish the following definition.

Definition 2.7. *([8]) A function $v : [0,+\infty) \to \mathbb{H}$ is called a mild solution of* (2.2)*, for $v_0 \in \mathbb{H}$, if v satisfies the variation of constants formula* (2.3)*.*

The next theorem provides sufficient conditions ensuring the regularity of solutions of Eq. (2.2).

Theorem 2.8. *([8]) Let $q \in C^1([0,+\infty);\mathbb{H})$ and let v be defined by* (2.3)*. If $v_0 \in D(A)$, then v is a strict solution of Eq.* (2.2)*.*

In this paper we will examine the impulsive stochastic semilinear neutral differential equations with delays of the form

$$\begin{cases} d\left[u(t) + H(t,u(t-\tau(t)))\right] = A\left[u(t) + H(t,u(t-\tau(t)))\right]dt \\ \qquad + \left[\int_0^t B(t-s)[u(s) + H(s,u(s-\tau(s)))]ds + F(t,u(t-\tau(t)))\right]dt, \\ \qquad + G(t,u(t-\tau(t)))dw(t) \quad\quad t \geq 0,\ t \neq t_k, \\ \Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k)) \quad\quad t = t_k,\ \ k = 1,2,..., \\ u_0(\cdot) = \phi \in D_{\mathcal{F}_0}([-r,0];\mathbb{H}), \ \ r > 0, \end{cases} \tag{2.4}$$

Here $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is a closed linear operator, for all $t \geq 0$, $B(t)$ is a closed linear operator with domain $D(B(t)) \supset D(A)$.
Let $D_{\mathcal{F}_0} := D_{\mathcal{F}_0}([-r,0];\mathbb{H})$ be the space of all almost surely bounded $\mathcal{F}_0$-measurable function ϕ from $[-r,0] \times \Omega$ into $\mathbb{H}$ that are almost surely continuous everywhere except for a finite number of points s at which the left and right limits, $\phi(s-)$ and $\phi(s+)$, exist, with $\phi(s-) = \phi(s)$, and which is equipped with the supremum norm $\|\phi\|_0 = essup_{\omega\in\Omega}\sup_{t\in[-r,0]}|\phi(t)(\omega)|_{\mathbb{H}}$. Moreover, $u(t_k^+)$ and $u(t_k^-)$ denote the right-hand and left-hand limits of $u(t)$ at $t = t_k$, respectively and the fixed moments of time t_k satisfy $0 < t_1 < ... < t_k < .... < ....$ and $\lim_{k\to\infty} t_k = \infty$; $\Delta(u(t_k)) = u(t_k^+) - u(t_k^-)$ denotes the jump in the state u at time t_k with $I_k(\cdot) : \mathbb{H} \to \mathbb{H}(k = 1,2,\cdots)$ determining the size of the jumps; the mappings $F : \mathbb{R}_+ \times \mathbb{H} \to \mathbb{H}$, $G : \mathbb{R}_+ \times \mathbb{H} \to \mathcal{L}_2^0(\mathbb{K},\mathbb{H})$ and $H : \mathbb{R}_+ \times \mathbb{H} \to \mathbb{H}$ are all Borel measurable, and $\tau : [0,+\infty) \to [0,r]$ is continuous.
Let us recall the definition of mild solution for the stochastic system (2.4) and recall the definitions of p-th moment stability and asymptotically stability in the p-th moment.

Definition 2.9. *An $\mathbb{H}$-valued stochastic process $\{u(t),\ t \geq 0\}$ is called a mild solution to Eq. (2.4) if*

(a) $u(t)$ is adapted to $\mathcal{F}_t$ and $\int_0^T |u(t)|_{\mathbb{H}}^p\,dt < +\infty$ *almost surely;*

(ii) $u(t)$ has càdlàg paths on $[0,+\infty[$ almost surely, and for $t\in[0,+\infty[$, $u(t)$ satisfies the following integral equation

$$\begin{aligned} u(t)+H(t,u(t-\tau(t))) &= R(t)\left[\phi(0)+H(0,\phi)\right]+\int_0^t R(t-s)F(s,u(s-\tau(s)))ds \\ &\quad +\int_0^t R(t-s)G(s,u(s-\tau(s)))dw(s) \qquad (2.5) \\ &\quad +\sum_{0<t_k<t} R(t-t_k)I_k(u(t_k)), \ \text{ for } \ t\geq 0, \end{aligned}$$

and the initial value condition $u_0(\cdot)=\phi\in D_{\mathcal{F}_0}$, a.s.

Definition 2.10. *Let $p\geq 2$ be an integer and assume that $u\equiv 0$ is solution to Eq. (2.4). It is said that the zero solution to Eq. (2.4) is stable in p-th moment if for arbitrary given $\epsilon>0$ there exists a $\delta>0$ such that $\|\phi\|_0<\delta$ guarantees that*

$$\mathbb{E}\left(|u(t)|_{\mathbb{H}}^p\right)<\epsilon, \ \text{for all} \ \ t\geq 0,$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and u is the mild solution of Eq. (2.4) corresponding to the initial value $\phi\in D_{\mathcal{F}_0}$.

Definition 2.11. *Let $p\geq 2$ be an integer and assume that $u\equiv 0$ is solution to Eq. (2.4). It is said that the zero solution to Eq. (2.4) is asymptotically stable in p-th moment if it is stable in p-th moment and for any $\phi\in D_{\mathcal{F}_0}$,*

$$\lim_{t\to\infty}\mathbb{E}\left(|u(t)|_{\mathbb{H}}^p\right)=0.$$

where u is the mild solution of Eq. (2.4) corresponding to the initial value $\phi\in D_{\mathcal{F}_0}$.

In order to obtain our main result, we need to impose the following assumptions:

(H3) The resolvent operator given by Theorem 2.2 satisfies the following condition:

$$\|R(t)\|\leq Me^{-at} \ \text{ for } \ t\geq 0, \ \text{ where } \ M\geq 1 \ \text{ and } \ a>0.$$

(H4) The functions F,G and H satisfy Lipschitz conditions, and there exists a constant $K>0$ such that for every $t\geq 0$ and $\eta,\zeta\in\mathbb{H}$

$$|F(t,\zeta)-F(t,\eta)|_{\mathbb{H}}\leq K\,|\zeta-\eta|_{\mathbb{H}}\,,$$

$$\|G(t,\zeta)-G(t,\zeta)\|_{\mathcal{L}_2^0}\leq K\,|\zeta-\zeta|_{\mathbb{H}}\,,$$

$$|H(t,\zeta)-H(t,\eta)|_{\mathbb{H}}\leq K\,|\zeta-\eta|_{\mathbb{H}}\,.$$

(H5) $I_k\in C(\mathbb{H},\mathbb{H})$ and there exists a constant q_k such that

$$|I_k(\zeta)-I_k(\eta)|_{\mathbb{H}}\leq q_k\,|\zeta-\eta|_{\mathbb{H}} \ \text{ for each } \ \eta,\zeta\in\mathbb{H} \ \ (k=1,2,...,)$$

Now, let us state the following well-known lemma (Da Prato and Zabczyk, 1992), which will be used in the proofs of our main results.

Lemma 2.12. *([4]) For any $l\geq 1$, and for arbitrary $\mathcal{L}_2^0$-valued predictable process ϕ*

$$\sup_{0\leq s\leq t} E\left|\int_0^s \phi(l)dw(l)\right|_{\mathbb{H}}^{2l}\leq C_r\left(\int_0^t E\,\|\phi(s)\|_{L_2^0}^{2l}\,d(s)\right)^l \qquad (2.6)$$

where $C_l=(l(2l-1))^l$

3 Existence and Asymptotic Stability for Eq. (2.4)

In this section, using definitions and lemmas stated in Section 2, we will prove the existence of mild solution for (2.4), and will also consider the asymptotic stability in p-th moment of mild solutions. We shall assume that for any
$t \geq 0, \;\; F(t,0) \equiv 0, \; G(t,0) \equiv 0, \;\; H(t,0) \equiv 0,$ and $I_k(0) \equiv 0 \;\; (k = 1,2,\cdots,)$. In this case, when $\phi \equiv 0$, it is easy to see that Eq. (2.4) has a trivial solution and we can prove the following result.

Theorem 3.1. *Let $p \geq 2$. In addition to hypotheses* **(H1)-(H5)***, assume that the following conditions are also satisfied:*

(i) there exists a constant $\tilde{q}$ such that $q_k \leq \tilde{q}(t_k - t_{k-1}), k = 1,2,\cdots,$

(ii) $4^{p-1}\Big(K^p + M^pK^pa^{-p} + M^pK^pC_p(2a)^{-p/2} + M^p\tilde{q}^pa^{-p}\Big) < 1$, *where* $C_p = \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$.

Then, the zero solution of Eq. (2.4) is asymptotically stable in the p-th moment.

Proof of Theorem 3.1. Given $\phi \in D_{\mathcal{F}_0}$, denote by $\mathcal{S}$ the space of all $\mathcal{F}_t$-adapted processes $\Psi(t,\omega) : [-r,\infty[\times\Omega \to \mathbb{H}$ which are almost surely continuous in $t \neq t_k$ $(k = 1,2,\cdots)$ for fixed $\omega \in \Omega$, $\lim_{t\to t_k^-}\Psi(t)$ and $\lim_{t\to t_k^+}\Psi(t)$ exist, and $\lim_{t\to t_k^-}\Psi(t) = \Psi(t_k)$.
Moreover, $\Psi(s,\omega) = \phi(s)$ for $s \in [-r,0]$ and $\mathbb{E}\,|\Psi(t,\omega)|^p_{\mathbb{H}} \to 0$ as $t \to \infty$. If we define

$$|\Psi(\cdot,\cdot)|_{\mathcal{S}} := \sup_{s\geq 0}\mathbb{E}\,|\Psi(s,\omega)|^p_{\mathbb{H}}\,, \tag{3.1}$$

then $\mathcal{S}$ becomes a complete metric space with respect to the distance induced by (3.1). We will use now the contraction mapping principle for a suitable mapping defined on the space $\mathcal{S}$. Indeed, let us define operator $\pi \;:\; \mathcal{S} \to \mathcal{S}$ by $(\pi u)(t) = \phi(t)$ for $t \in [-r,0]$, and, for all $t \geq 0$,

$$\begin{aligned}
(\pi u)(t) \;\;&=\;\; R(t)[\phi(0) + H(0,\phi)] - H(t,u(t-\tau(t))) + \int_0^t R(t-s)F(s,u(s-\tau(s)))ds \\
&\quad + \int_0^t R(t-s)G(s,u(s-\tau(s)))dw(s) + \sum_{0<t_k<t} R(t-t_k)I_k(u(t_k)) \\
&= \sum_{i=1}^{5}\Delta_i(t).
\end{aligned} \tag{3.2}$$

Now, we will split the proof into three steps.
Firstly, we prove that π is continuous in the p-th moment on $[0,\infty[$. Let $u \in \mathcal{S}$, $t_1 \geq 0$ and $|\gamma|$ be sufficiently small, then

$$\mathbb{E}\,|(\pi u)(t_1+\gamma) - (\pi u)(t_1)|^p_{\mathbb{H}} \leq 5^{p-1}\sum_{i=1}^{5}\mathbb{E}\,|\Delta_i(t_1+\gamma) - \Delta_i(t_1)|^p_{\mathbb{H}}\,.$$

We can easily see that $\mathbb{E}\,|\Delta_i(t_1+\gamma) - \Delta_i(t_1)|^p_{\mathbb{H}} \to 0, \;\; i = 1,2,3$ as $\gamma \to 0$. For the case $i = 5$, taking into account assumption (i), we have

$$\begin{aligned}\mathbb{E}|\Delta_5(t_1+\gamma)-\Delta_5(t_1)| &\leq \mathbb{E}\left(\sum_{0<t_k<t} q_k\|R(t_1+\gamma-t_k)-R(t_1-t_k)\|\left|u(t_k^-)\right|_{\mathbb{H}}\right)^p \\ &\leq \mathbb{E}\left(\sum_{0<t_k<t} \tilde{q}\|R(t_1+\gamma-t_k)-R(t_1-t_k)\|\left|u(t_k^-)(t_k-t_{k-1})\right|_{\mathbb{H}}\right)^p \\ &\leq \mathbb{E}\left(\int_0^t \tilde{q}\|R(t_1+\gamma-s)-R(t_1-s)\|\left|u(s)\right|_{\mathbb{H}} ds\right)^p \\ &\leq \tilde{q}^p t^{p-1}\left|u\right|_S \int_0^t \|R(t_1+\gamma-s)-R(t_1-s)\|^p ds. \qquad (3.3)\end{aligned}$$

Using the norm continuity of $R(t)$ for $t>0$ and applying Lebesgue's dominated convergence theorem, it follows that $\mathbb{E}|\Delta_5(t_1+\gamma)-\Delta_5(t_1)| \to 0$ as $\gamma \to 0$. Moreover, by using Hölder's inequality and Lemma 2.12, we obtain

$$\begin{aligned}\mathbb{E}\left|\Delta_4(t_1+\gamma)-\Delta_4(t_1)\right|_{\mathbb{H}}^p &\leq 2^{p-1}c_p\Big[\int_0^{t_1}(\mathbb{E}\left|(R(t_1+\gamma-s)-R(t_1-s))G(t,u(s-\tau(s)))\right|_{\mathbb{H}}^p)^{\frac{2}{p}}ds\Big]^{\frac{p}{2}} \\ &\quad +2^{p-1}c_p\Big[\int_{t_1}^{t_1+\gamma}(\mathbb{E}\left|(R(t_1+\gamma-s)G(t,u(s-\tau(s)))\right|_{\mathbb{H}}^p)^{\frac{2}{p}}ds\Big]^{\frac{p}{2}} \qquad (3.4) \\ &\quad \to 0 \qquad (3.5)\end{aligned}$$

as $\gamma \to 0$, where $C_p = \left(\frac{(p(p-1)}{2}\right)^{\frac{p}{2}}$. Thus, π is indeed continuous in the p-th mean on $[0,\infty[$. Next, we show that $\pi(\mathcal{S}) \subset \mathcal{S}$. It follows from (3.2) that

$$\begin{aligned}\mathbb{E}\left|(\pi u)(t)\right|_{\mathbb{H}}^p &\leq 5^{p-1}\mathbb{E}\left|R(t)[\phi(0)+H(0,\phi)]\right|_{\mathbb{H}}^p + 5^{p-1}\mathbb{E}\left|H(t,u(t-\tau(t)))\right|_{\mathbb{H}}^p \\ &\quad +5^{p-1}E\left|\int_0^t R(t-s)F(s,u(s-\tau(s)))ds\right|_{\mathbb{H}}^p \\ &\quad +5^{p-1}E\left|\int_0^t R(t-s)G(s,u(s-\tau(s)))dw(s)\right|_{\mathbb{H}}^p \qquad (3.6) \\ &\quad +5^{p-1}\sum_{0<t_k<t}\mathbb{E}\left|R(t-s)I_k(u(t_k))\right|_{\mathbb{H}}^p \\ &=: 5^{p-1}(J_1+J_2+J_3+J_4+J_5).\end{aligned}$$

Now, we estimate the terms on the right-hand side of (3.6). From assumption **(H3)** we obtain

$$J_1 \leq M^p e^{-apt}2^{p-1}\Big(1+K^p\Big)\mathbb{E}\left\|\phi\right\|_0^p \to 0 \text{ as } t\to\infty. \qquad (3.7)$$

By assumption **(H4)** it follows that

$$J_2 \leq K^p\mathbb{E}\left|u(t-\tau(t))\right|_{\mathbb{H}}^p \to 0 \text{ as } t\to\infty.$$

Now from assumptions $(\mathbf{H3}), (\mathbf{H5})$ and Hölder's inequality we have

$$\begin{aligned} J_3 &\leq \mathbb{E}\left|\int_0^t Me^{-a(t-s)}F(s,u(s-\tau(s)))ds\right|_{\mathbb{H}}^p \\ &\leq M^pK^p\Big(\int_0^t e^{-a(t-s)}ds\Big)^{p-1}\int_0^t e^{-a(t-s)}\mathbb{E}\left|u(s-\tau(s))\right|_{\mathbb{H}}^p ds \\ &\leq M^pa^{1-p}K^p\int_0^t e^{-a(t-s)}E\left|u(s-\tau(s))\right|_{\mathbb{H}} ds. \end{aligned} \tag{3.8}$$

For any $u \in \mathcal{S}$, and any $\epsilon > 0$, there exists a $t_1 > 0$, such that $\mathbb{E}\left|u(t-\tau(t))\right|_{\mathbb{H}}^p < \epsilon$, for $t \geq t_1$. Thus from (3.8), we obtain

$$J_3 \leq M^pa^{1-p}K^pe^{-at}\int_0^{t_1} e^{as}\mathbb{E}\left|u(s-\tau(s))\right|_{\mathbb{H}} ds + M^pK^pa^{-p}\epsilon. \tag{3.9}$$

As $e^{-at} \to 0$ as $t \to \infty$, by (ii), there exists a $t_2 \geq t_1$ such that, for $t \geq t_2$, we have

$$M^pa^{1-p}K^Pe^{-at}\int_0^t e^{as}\mathbb{E}\left|u(s-\tau(s))\right|_{\mathbb{H}} ds \leq \epsilon - M^pK^pa^{-p}\epsilon. \tag{3.10}$$

From (3.9) and (3.10), we obtain for any $t \geq t_2$

$$J_3 < \epsilon.$$

In other words,

$$J_3 \to 0 \text{ as } t \to \infty. \tag{3.11}$$

Now, for any $u \in \mathcal{S}$, $t \in [0, \infty[$, we obtain

$$J_4 \leq C_pM^pK^p\Big(\int_0^t e^{-2a(t-s)}\Big(\mathbb{E}\left|u(s-\tau(s))\right|_{\mathbb{H}}^p\Big)^{\frac{2}{p}}ds\Big)^{\frac{p}{2}}. \tag{3.12}$$

Further, similar to the proof of (3.11), from (3.12), we have

$$J_4 \to 0 \text{ as } t \to \infty. \tag{3.13}$$

Now, we estimate the impulsive term, from the condition (i), we obtain

$$\begin{aligned} J_5 &\leq \mathbb{E}\left(\sum_{0<t_k<t} Me^{-a(t-t_k)}q_k\left|u(t_k^-)\right|_{\mathbb{H}}\right)^p \\ &\leq \mathbb{E}\left(\sum_{0<t_k<t} Me^{-a(t-t_k)}\tilde{q}\left|u(t_k^-)(t_k-t_{k-1})\right|_{\mathbb{H}}\right)^p \\ &\leq \mathbb{E}\left(\int_0^t Me^{-a(t-s)}\tilde{q}\left|u(s)\right|_{\mathbb{H}} ds\right)^p \\ &\leq M^p\tilde{q}^p\left(\int_0^t e^{-a(t-s)}ds\right)^{p-1}\int_0^t e^{-a(t-s)}\mathbb{E}\left|u(s)\right|_{\mathbb{H}}^p ds. \end{aligned} \tag{3.14}$$

From (3.14) we obtain $J_5 \to 0$ as $t \to \infty$. Using (3.7), (3.14), (3.11) and (3.13) in (3.6), we obtain $\mathbb{E}\left|(\pi u)(t)\right|^p \to 0$ as $t \to \infty$. Thus, we conclude that $\pi(\mathcal{S}) \subset \mathcal{S}$.

Next, we prove that π is a contraction mapping. To see this, let $u, v \in \mathcal{S}$. Then, for

$t \in [0, T]$ we obtain

$$\begin{aligned} &\sup_{t\in[0,T]} \mathbb{E}\,|(\pi u)(t) - (\pi v)(t)|_{\mathbb{H}}^p \qquad (3.15)\\ &\quad \le 4^{p-1} \sup_{t\in[0,T]} \mathbb{E}\,|H(t, u(t-\tau(t))) - H(t, v(t-\tau(t)))|_{\mathbb{H}}^p \\ &\quad +4^{p-1} \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t R(t-s)\Big(F(s, u(s-\tau(s))) - F(s, v(s-\tau(s)))\Big)ds\right|_{\mathbb{H}}^p \\ &\quad +4^{p-1} \sup_{t\in[0,T]} \mathbb{E}\left|\int_0^t R(t-s)\Big(G(s, u(s-\tau(s))) - G(s, v(s-\tau(s)))\Big)dw(s)\right|_{\mathbb{H}}^p \\ &\quad +4^{p-1} \sup_{t\in[0,T]} \mathbb{E}\left|\sum_{0<t_k<t} R(t-t_k)\left(I_k(u(t_k)) - I_k(v(t_k))\right)\right|_{\mathbb{H}}^p \\ &\quad \le 4^{p-1}\Big(K^p + M^pK^pa^{-p} + M^pK^pC_p(2a)^{-p/2} + M^p\tilde{q}^pa^{-p}\Big) \sup_{t\ge 0} \mathbb{E}\,|u(t) - v(t)|_{\mathbb{H}}^p, \end{aligned}$$

and as this holds for arbitrary $T > 0$ and the right-hand side of (3.15) is independent of T, thanks to condition (ii), it follows that π is a contraction. Thus, applying the Banach fixed point principle, it follows that there exists a unique $u \in \mathcal{S}$ which is a mild solution of Eq. (2.4).

To obtain the asymptotic stability, we have to prove that the zero solution to (2.4) is stable in p-th moment.
Let $\varepsilon > 0$ be given and choose $\hat{\delta} > 0$ such that

$$5^{p-1}\left\{M^p2^{p-1}(1+K^p) + \frac{1}{4^{p-1}}\right\}\hat{\delta} < \varepsilon.$$

If $u(t) = u(t, 0, \phi)$ is a mild solution of (2.4) with $\|\phi\|_0^p < \hat{\delta}$, then $\pi(u)(t) = u(t)$ defined in (2.5). We claim that $\mathbb{E}\,|u(t)|_{\mathbb{H}}^p < \varepsilon$ for all $t \ge 0$. Indeed, noticing that $\mathbb{E}\,|u(t)|_{\mathbb{H}}^p < \varepsilon$ on $t \in [-r, 0]$, if there exists $t^* > 0$ such that $\mathbb{E}\,|u(t^*)|_{\mathbb{H}}^p = \varepsilon$ and $\mathbb{E}\,|u(t)|_{\mathbb{H}}^p < \varepsilon$ for $-r \le s < t^*$, then it easily follows from (3.6), taking into account assumption (ii), that

$$\begin{aligned} \mathbb{E}\,|u(t^*)|_{\mathbb{H}}^p &\le 5^{p-1}\left\{M^p2^{p-1}(1+K^p) + K^p + MK^pa^{-p} + M^pK^pC_p(2a)^{-p/2} + M^p\tilde{q}^pa^{-p}\right\}\hat{\delta} \\ &\le 5^{p-1}\left\{M^p2^{p-1}(1+K^p) + \frac{1}{4^{p-1}}\right\}\hat{\delta} \\ &< \varepsilon, \end{aligned}$$

which contradicts the definition of t^*. This shows that the mild solution of (2.4) is asymptotically stable in the p-th moment if assumption in Theorem 3.1 holds. This completes the proof. □

Remark 3.2. In particular, if $m = 0$, then system (2.4) reduces to

$$\begin{cases} d[u(t) + H(t, u(t-\tau(t)))] = A[u(t) + H(t, u(t-\tau(t)))]dt \\ \qquad + \left[\int_0^t B(t-s)\Big(u(s) + H(s, u(s-\tau(s)))\Big)ds + F(t, u(t-\tau(t)))\right]dt \\ \qquad +G(t, u(t-\tau(t)))dw(t) \;\text{ for }\; t \ge 0 \\ u_0(\cdot) = \phi \in C^b_{\mathcal{F}_0}([-r, 0]; \mathbb{H}), \quad r > 0. \end{cases} \qquad (3.16)$$

By applying Theorem 3.1 under the hypotheses **(H1)-(H4)**, the existence and asymptotic stability of system (3.16) are guaranted.

In particular, when $p = 2$, from Theorem 3.1 we obtain

Corollary 3.3. *Suppose that assumptions* **(H1)-(H5)** *hold. Then, the trivial solution to Eq. (2.4) is mean square asymptotically stable if*

$$4\Big(K^2 + M^2K^2a^{-2} + M^2K^2(2a)^{-1}\Big) < 1. \tag{3.17}$$

4 Application

Impulsive dynamical systems exhibit the various evolutionary processes, including those in engineering, biology and population dynamics, undergo abrupt changes in their state at certain moments between intervals of continuous evolution, since many evolution process, optimal control models in economics, stimulated neutral networks, frequency–modulated systems and some motions of missiles or aircrafts are characterized by that impulsive dynamical behavior.
To illustrate our abstract results we consider the following model

$$\begin{cases} \dfrac{\partial}{\partial t}\left[x(t,\xi) + h(t, x(t-\delta(t),\xi))\right] = \dfrac{\partial^2}{\partial \xi^2}\left[x(t,\xi) + h(t, x(t-\delta(t),\xi))\right] \\ \qquad\qquad + \displaystyle\int_0^t b(t-s)\frac{\partial^2}{\partial \xi^2}\left[x(s,\xi) + h(s, x(s-\delta(s),\xi))\right] ds \\ \qquad\qquad + f(t, x(t-\delta(t),\xi))dt + g(t, x(t-\delta(t),\xi))dw(t) \;\text{ for }\; t \neq t_k \;\text{ and }\; t \geq 0 \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)) \;\text{ for }\; t = t_k, \;\; k = 1, 2, \cdots, \\ x(t,0) + h(t, x(t-\delta(t),0)) = 0 \quad \text{for} \quad t \geq 0, \\ x(t,\pi) + h(t, x(t-\delta(t),\pi)) = 0 \quad \text{for} \;\; t \geq 0, \\ x(\theta,\xi) = x_0(\theta,\xi) \quad \text{for} \quad \theta \in [-r, 0] \quad \text{and} \quad \xi \in [0,\pi], \end{cases} \tag{4.1}$$

where $h, f, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $\delta : [0, +\infty) \to [0, r]$, and $|x_0|_0 < \infty$.
Let $\mathbb{H} = L^2([0,\pi])$ and $e_n := \sqrt{\frac{2}{\pi}}\sin(nx)$, $\;(n = 1, 2, 3, \cdots)$ denote the complete orthonormal basis in $\mathbb{H}$. Let $w(t) := \sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n$ $\;(\lambda_n > 0)$, where $\beta_n(t)$ are one dimensional standard Brownian motions mutually independent on a usual complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.
Define $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $A = \frac{\partial^2}{\partial z^2}$, with domain $D(A) = H^2([0,\pi]) \cap H_0^1([0,\pi])$.
A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{H}$, which is given by $T(t)\phi = \sum_{n=1}^{\infty} e^{-n^2t} < \phi, e_n > e_n$, $\;\phi \in D(A)$.
Let $B : D(A) \subset \mathbb{H} \to \mathbb{H}$ be the operator defined by

$$B(t)(z) = b(t)Az \;\text{ for }\; t \geq 0 \;\text{ and }\; z \in D(A).$$

We suppose that

(A1) For $t \geq 0$, $= f(t,0) = g(t,0)$ and $I_k(0) = 0$ for $k = 1, 2, \cdots,$.

(A2) There exist positive constants $l_1,\ l_2,\ l_{\mathbb{H}},\ l_k,\ k = 1, 2, \cdots$, such that

$$|f(t,\zeta_1) - f(t,\zeta_2)| \leq K|\zeta_1 - \zeta_2|,$$

$$|g(t,\zeta_1)-g(t,\zeta_2)|\leq K|\zeta_1-\zeta_2|,$$

$$|h(t,\zeta_1)-h(t,\zeta_2)|\leq K|\zeta_1-\zeta_2|,$$

for $t\geq 0$ and $\zeta_1,\zeta_2\in\mathbb{R}$,

and

$$|I_k(\zeta_1)-I_k(\zeta_2)|\leq h_k|\zeta_1-\zeta_2| \text{ for } k=1,2,\cdots, \text{ and } \zeta_1,\zeta_2\in\mathbb{R}.$$

Let $D=D([-r,0],\mathbb{H})$, for $t\geq 0$ and $\phi\in\mathbb{H}$, define the operators $H,F,G:\mathbb{R}^+\times D\to\mathbb{H}$ for $\xi\in[0,\pi]$ by

$$F(t,\phi)(\xi)=f(t,\phi(-\tau_1)(\xi)),$$
$$G(t,\phi)(\xi)=g(t,\phi(-\tau_1)(\xi)),$$
$$H(t,\phi)(\xi)=h(t,\phi(-\tau_1)(\xi)).$$

If we put

$$\begin{cases} u(t)(\xi) &=& x(t,\xi) \text{ for } t\geq 0 \text{ and } \xi\in[0,\pi] \\ \varphi(\theta)(\xi) &=& x_0(\theta,\xi) \text{ for } \theta\in[-r,0] \text{ and } \xi\in[0,\pi], \end{cases}$$

then Eq. (4.1) takes the following abstract form

$$\begin{cases} d\left[u(t)+H(t,u(t-\tau(t)))\right]=A\left[u(t)+H(t,u(t-\tau_1))\right]dt \\ \qquad +\left[\int_0^t B(t-s)[u(s)+H(s,u(s-\tau_1))]ds+F(t,u(t-\tau_1))\right]dt \\ \qquad +G(t,u(t-\tau_1))dw(t) \text{ for } t\geq 0,\ t\neq t_k \\ \Delta u(t_k)=x(t_k^+)-x(t_k^-)=I_k(u(t_k)) \text{ for } t=t_k,\ \ k=1,2,..., \\ u_0(\cdot)=\phi\in D_{\mathcal{F}_0}([-r,0];\mathbb{H}),\ \ r>0, \end{cases}$$

Moreover, if b is bounded and C^1 function such that b' is bounded and uniformly continuous, then **(H1)** and **(H2)** are satisfied and hence, by Theorem 2.2, Eq. (4.1) has a resolvent operator $(R(t))_{t\geq 0}$ on $\mathbb{H}$. As a consequence of the continuity of f and g and assumption (A1) it follows that F and G are continuous. By assumption (A2), we have

$$|F(t,\phi_1)-F(t,\phi_2)|\leq K\left|\phi_1-\phi_2\right|,$$

$$|G(t,\phi_1)-G(t,\phi_2)|\leq K\left|\phi_1-\phi_2\right|,$$

$$|H(t,\phi_1)-H(t,\phi_2)|\leq K\left|\phi_1-\phi_2\right|,$$

and

$$|I_k(\zeta)-I_k(\eta)|\leq q_k\left|\zeta-\eta\right| \text{ for } k=1,2,...,.$$

Moreover, if we suppose that

$$\|R(t)\|\leq Ne^{-at} \text{ for } t\geq 0, \text{ where } N\geq 1 \text{ and } a>0,$$

then all the assumptions of Theorem 3.1 are fulfilled. Therefore, Eq (4.1) has a unique mild solution which is asymptotically stable in the p-th moment provided

$$4^{p-1}\Big(3^pK^p + M^p3^pK^pa^{-p} + M^p3^pK^pc_p(2a)^{-p/2} + M^p\tilde{q}^pa^{-p}\Big) < 1,$$

where $C_p = \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}$.

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