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Fractional smoothness of functionals of diffusion processes under a change of measure*

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Abstract

Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be the solution of the parabolic backward equation $\partial_t v + (1/2)\sum_{i,j}[\sigma\sigma^\top]_{i,j}\partial_{x_i}\partial_{x_j} v + \sum_i b_i \partial_{x_i} v + kv = 0$ with terminal condition g, where the coefficients are time- and state-dependent, and satisfy certain regularity assumptions. Let $X = (X_t)_{t\in[0,T]}$ be the associated $\mathbb{R}^d$-valued diffusion process on some appropriate $(\Omega, \mathcal{F}, \mathbb{Q})$. For $p \in [2,\infty)$ and a measure $d\mathbb{P} = \lambda_T d\mathbb{Q}$, where λ_T satisfies the Muckenhoupt condition A_p, we relate the behavior of

$$\|g(X_T) - \mathbb{E}_{\mathbb{P}}(g(X_T)|\mathcal{F}_t)\|_{L_p(\mathbb{P})}, \quad \|\nabla v(t,X_t)\|_{L_p(\mathbb{P})}, \quad \|D^2 v(t,X_t)\|_{L_p(\mathbb{P})}$$

to each other, where $D^2 v := (\partial_{x_i}\partial_{x_j} v)_{i,j}$ is the Hessian matrix.

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1 Introduction

We investigate the quantitative behavior of parabolic partial differential equations with respect to measures on the Wiener space generated by diffusions including a change of measure induced by a Muckenhoupt weight. This type of questions arises from the approximation theory of stochastic integrals and backward stochastic differential equations (BSDEs). The partial differential equation we consider is given by

$$\mathcal{L}v = 0 \quad \text{on} \quad [0,T) \times \mathbb{R}^d \qquad \text{and} \qquad v(T,\cdot) = g \quad \text{on} \quad \mathbb{R}^d \tag{1.1}$$

with

$$\mathcal{L} := \partial_t + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(t,x)\partial^2_{x_i,x_j} + \sum_{i=1}^d b_i(t,x)\partial_{x_i} + k(t,x), \tag{1.2}$$

where $A := (a_{i,j})_{i,j=1}^d = \sigma\sigma^\top$. It is well known [3] that under regularity conditions on σ, b and k there is a fundamental solution $\Gamma : \{0 \le t < \tau \le T\} \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ satisfying upper Gaussian bounds

$$|D^a_x D^b_t \Gamma(t,x;\tau,\xi)| \le c(\tau - t)^{-\frac{|a|+2b}{2}} \gamma^d_{\tau-t}\left((x-\xi)/c\right) \quad \text{with} \quad \gamma^d_s(x) := e^{-\frac{|x|^2}{2s}}/(\sqrt{2\pi s})^d$$

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for a and b up to a certain order. Under growth conditions on g these bounds transfer to estimates for the gradient and the Hessian of the solution to (1.1) obtained by

$$v(t,x) := \int_{\mathbb{R}^d} \Gamma(t,x;T,\xi) g(\xi) d\xi. \tag{1.3}$$

In our setting there will be a $\kappa_g \in [0,2)$ such that for $0 \le |a| + 2b \le 3$ the derivatives $D_x^a D_t^b v$ exist in any order, are continuous on $[0,T) \times \mathbb{R}^d$, and satisfy

$$|D_x^a D_t^b v(t,x)| \le c_{(1.4)} (T-t)^{-\frac{|a|+2b}{2}} \exp(c_{(1.4)} |x|^{\kappa_g}). \tag{1.4}$$

The point-wise estimates (1.4) serve often as a-priori estimates in stochastic analysis. However, they do not take into account regularities of g. Moreover, moment estimates of $D_x^a v(t,x)$ appear to be more natural in various situations. To explain this, let $p \in [2,\infty)$, $B = (B_t)_{t\in[0,T]}$ be a d-dimensional $(\mathcal{F}_t)_{t\in[0,T]}$-standard Brownian motion under a measure $\mathbb{Q}$, where the usual assumptions are satisfied, and consider the $\mathbb{R}^d$-valued diffusion

$$X_t = x_0 + \int_0^t \sigma(s,X_s) dB_s + \int_0^t b(s,X_s) ds,$$

with σ and b taken from (1.2). To consider L_p-time discretizations of the stochastic integrals

$$K_T^X g(X_T) = \mathbb{E}(K_T^X g(X_T)) + \int_0^T K_t^X \nabla v(t,X_t) \sigma(t,X_t) dB_t \quad \text{with} \quad K_t^X := e^{\int_0^t k(r,X_r)dr},$$

it turns out that the behavior of the L_p-norm of the Hessian $(\partial^2 v/\partial x_i \partial x_j)(t,X_t)$ determines this approximation; see [4, 6, 12] for $k=0$. A control of the blow-up of this L_p-norm as $t \to T$ enables the derivation of sharp convergence results. Similarly, the L_p-variation of the solution of a BSDE is triggered by the blow-up of the L_p-norm of the gradient of an associated semi-linear solution or an appropriate linear parabolic PDE, see [8, 5]. If one analyzes these examples, it turns out that one needs to relate to each other the quantitative behavior of

$$\|g(X_T) - \mathbb{E}(g(X_T)|\mathcal{F}_t)\|_{L_p(\mathbb{Q})}, \quad \|\nabla v(t,X_t)\|_{L_p(\mathbb{Q})}, \quad \text{and} \quad \|D^2 v(t,X_t)\|_{L_p(\mathbb{Q})}$$

with $D^2 = (\partial^2/\partial x_i \partial x_j)_{i,j=1}^d$. In this note we go even one step ahead, by establishing equivalence relations under an equivalent probability measure $\mathbb{P}$ that satisfies a Muckenhoupt condition. This gives considerably more insight into the quantitative behavior of the parabolic PDE and more flexibility in applications: among them, we mention the analysis of discrete-time hedging errors in mathematical finance [10, 9], where option prices are computed under the risk-neutral probability measure $\mathbb{Q}$ and hedging errors are analysed under the historical probability measure $\mathbb{P}$. An application to quadratic BSDEs is exposed in Remark 3.2(8).

Typically, setting $\mathbb{M} = \mathbb{P}$ or $\mathbb{Q}$, the terms $\|\nabla v(t,X_t)\|_{L_p(\mathbb{M})}$ and $\|D^2 v(t,X_t)\|_{L_p(\mathbb{M})}$ blow up as $t \uparrow T$ in case the terminal condition g is not sufficiently smooth. Firstly to measure the rates of these blows up and of the convergence to zero of $\|g(X_T) - \mathbb{E}_{\mathbb{M}}(g(X_T)|\mathcal{F}_t)\|_{L_p(\mathbb{M})}$, and secondly to establish relations between them in our main Theorem 3.1, we take advantage of the theory of real interpolation that provides for this purpose the functionals $\Phi_q(h) := \|h\|_{L_q\left([0,T),\frac{dt}{T-t}\right)}$ for a measurable function $h : [0,T) \to \mathbb{R}$ where $q \in [1,\infty]$.

We proceed as follows: Section 2 introduces the setting and needed tools, in Section 3 we formulate the main Theorem 3.1, and Section 4 contains the proof of Theorem 3.1.

2 Setting

Notation. Usually we denote by $|\cdot|$ the Euclidean norm of a vector. Given a matrix C considered as operator $C : \ell_2^n \to \ell_2^N$, the expression $|C|$ stands for the Hilbert-Schmidt norm and $C^\top$ for the transposed of C. The L_p-norm ($p \in [1,\infty]$) of a random vector $Z : \Omega \to \mathbb{R}^n$ or a random matrix $Z : \Omega \to \mathbb{R}^{n\times m}$ is denoted by $\|Z\|_p = \||Z|\|_{L_p}$. As usual, $D_x^a \varphi$ is the partial derivative of the order of a multi-index a (with length $|a| = \sum_i |a_i|$) with respect to x. The Hessian matrix of a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is abbreviated by $D^2\varphi$ and the gradient (as row vector) by $\nabla\varphi$. In particular, this means that D^2 and ∇ always refer to the state variable $x \in \mathbb{R}^d$. If we mention that a constant depends on b, σ or k, then we implicitly indicate a possible dependence on T and d as well. Finally, letting $h : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{n\times m}$ we use the notation $\|h\|_\infty := \sup_{t,x} |h(t,x)|$.

Parabolic PDE. Our assumptions on the Cauchy problem (1.1)-(1.2) are as follows:

(C1) The functions $\sigma_{i,j}, b_i, k$ are bounded and belong to $C_b^{0,2}([0,T] \times \mathbb{R}^d)$ and there is some $\gamma \in (0,1]$ such that the functions and their state-derivatives are γ-Hölder continuous with respect to the parabolic metric on each compactum of $[0,T] \times \mathbb{R}^d$. Moreover, σ is $1/2$-Hölder continuous in t, uniformly in x.

(C2) $\sigma(t,x)$ is an invertible $d \times d$-matrix with $\sup_{t,x} |\sigma^{-1}(t,x)| < +\infty$.

(C3) The terminal function $g : \mathbb{R}^d \to \mathbb{R}$ is measurable and exponentially bounded: for some $K_g \geq 0$ and $\kappa_g \in [0,2)$ we have $|g(x)| \leq K_g \exp(K_g |x|^{\kappa_g})$ for all $x \in \mathbb{R}^d$.

The condition (C2) implies that the operator $\mathcal{L}$ is uniformly parabolic. Under the above assumptions there exists a fundamental solution:

Proposition 2.1 ([3, Theorem 7, p. 260; Theorem 10, pp. 72-74]). *Under the assumptions* (C1) *and* (C2) *there exists a fundamental solution* $\Gamma(t,x;\tau,\xi) : \{0 \leq t < \tau \leq T\} \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ *for* $\mathcal{L}$ *and a constant* $c_{(2.1)} > 0$ *such that for* $0 \leq |a| + 2b \leq 3$ *the derivatives* $D_x^a D_t^b \Gamma$ *exist in any order, are continuous, and satisfy*

$$|D_x^a D_t^b \Gamma(t,x;\tau,\xi)| \leq c_{(2.1)} (\tau - t)^{-\frac{|a|+2b}{2}} \gamma_{\tau-t}^d \left(\frac{x-\xi}{c_{(2.1)}} \right) \quad \textit{with} \quad \gamma_s^d(x) = e^{-\frac{|x|^2}{2s}} / (\sqrt{2\pi s})^d. \tag{2.1}$$

For $0 \leq |a| + 2b \leq 3$ Proposition 2.1 implies that the derivatives $D_x^a D_t^b v$, with v defined in (1.3), exist in any order, are continuous on $[0,T) \times \mathbb{R}^d$ and satisfy

$$\mathcal{L}v = 0 \;\text{ on }\; [0,T) \times \mathbb{R}^d \qquad \text{and} \qquad |D_x^a D_t^b v(t,x)| \leq c(T-t)^{-\frac{|a|+2b}{2}} \exp(c|x|^{\kappa_g})$$

for $x \in \mathbb{R}^d$ and $t \in [0,T)$, where $c > 0$ depends at most on $(\kappa_g, K_g, c_{(2.1)}, T)$.

Stochastic differential equation. Let $(B_t)_{t\in[0,T]}$ be a d-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$, where $(\Omega, \mathcal{F}, \mathbb{Q})$ is complete, $(\mathcal{F}_t)_{t\in[0,T]}$ is right-continuous, $\mathcal{F} = \mathcal{F}_T$, $\mathcal{F}_0$ is generated by the null sets of $\mathcal{F}$ and where all local martingales are continuous. As we work on a closed time-interval we have to explain our understanding of a local martingale: we require that the localizing sequence of stopping times $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq T$ satisfies $\lim_n \mathbb{Q}(\tau_n = T) = 1$. So we think about the extension of the filtration by $\mathcal{F}_T$ to (T,∞) and that all local martingales $(N_t)_{t\in[0,T]}$ (in our setting) are extended by N_T to (T,∞). This yields the standard notion of a local martingale. We need this implicitly whenever we refer to results about the Muckenhoupt weights $A_\alpha(\mathbb{Q})$ from [15]. The process $X = (X_t)_{t\in[0,T]}$ is given as unique strong solution of

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dB_s + \int_0^t b(s, X_s) ds.$$

Introducing the standing notation

$$K_t^X = e^{\int_0^t k(r,X_r)dr} \quad \text{and} \quad M_t := K_t^X v(t,X_t),$$

Itô's formula implies, for $t \in [0,T)$, that

$$M_t = v(0,x_0) + \int_0^t K_s^X \nabla v(s,X_s)\sigma(s,X_s)dB_s. \tag{2.2}$$

Moreover,

$$\lim_{t\to T} M_t = M_T \quad \text{and} \quad \lim_{t\to T} v(t,X_t) = g(X_T) \tag{2.3}$$

almost surely and in any $L_r(\mathbb{Q})$ with $r \in [1,\infty)$. Using Proposition 2.1 for $k=0$ we also have $\mathbb{Q}(|X_t - x_0| > \lambda) \le c\exp\left(-\frac{\lambda^2}{c}\right)$ for all $\lambda \ge 0$ and $t \in [0,T]$, where $c = c(\sigma,b) > 0$ is independent of $x_0 \in \mathbb{R}^d$. It implies that $g(X_T) \in \bigcap_{r\in[1,\infty)} L_r(\mathbb{Q})$ so that Remark 2.6 below applies. We also use

Lemma 2.2 ([7], [8, Proof of Lemma 1.1], [5, Remark 3 in Appendix B]). *Assume* (C1) *and* (C2) *and let* $t \in (0,T]$, $h : \mathbb{R}^d \to \mathbb{R}$ *be a Borel function satisfying* (C3) *and* Γ_X *be the transition density of* X, *i.e.* Γ *from Proposition 2.1 for* $k=0$. *Define*

$$H(s,x) := \int_{\mathbb{R}^d} \Gamma_X(s,x;t,\xi)h(\xi)d\xi \quad \textit{for} \quad (s,x) \in [0,t) \times \mathbb{R}^d.$$

For $r \in [0,t)$ *and* $x \in \mathbb{R}^d$ *let* $(Z_u)_{u\in[r,t]}$ *be the diffusion based on* (σ,b) *starting in* x *defined on some* $(M,\mathcal{G},(\mathcal{G}_u)_{u\in[r,t]},\mu)$ *equipped with a standard* $(\mathcal{G}_u)_{u\in[r,t]}$*-Brownian motion, where* $(M,\mathcal{G},\mu)$ *is complete,* $(\mathcal{G}_u)_{u\in[r,t]}$ *is right-continuous and* $\mathcal{G}_r$ *is generated by the null sets of* $\mathcal{G}$. *Then, for* $q \in (1,\infty)$, $s \in [r,t)$, *and* $i = 1,2$ *one has a.s. that*

$$|\Delta_i H(s,Z_s)| \le \kappa_q (t-s)^{-\frac{i}{2}} [\mathbb{E}(|h(Z_t) - \mathbb{E}(h(Z_t)|\mathcal{G}_s)|^q|\mathcal{G}_s)]^{\frac{1}{q}},$$

where $\kappa_q > 0$ *depends at most on* (σ,b,q), $\Delta_1 := \nabla$, *and* $\Delta_2 := D^2$.

Muckenhoupt weights. The probabilistic Muckenhoupt weights provide a natural way to verify various martingale inequalities after a change of measure, see exemplary [14, 2, 15]. To use these weights we exploit an equivalent measure $\mathbb{P} \sim \mathbb{Q}$ in addition to the given measure $\mathbb{Q}$ and agree about the following standing assumption:

(P) There exists a martingale $Y = (Y_t)_{t\in[0,T]}$ with $Y_0 \equiv 0$ such that $\lambda_t := \mathcal{E}(Y)_t = e^{Y_t - \frac{1}{2}\langle Y\rangle_t}$ for $t \in [0,T]$ is a martingale and $d\mathbb{P} = \lambda_T d\mathbb{Q}$.

Definition 2.3. *Assume that condition* (P) *is satisfied.*

(i) *For* $\alpha \in (1,\infty)$ *we let* $\lambda \in A_\alpha(\mathbb{Q})$ *provided that there is a constant* $c > 0$ *such that for all stopping times* $\tau : \Omega \to [0,T]$ *one has that* $\mathbb{E}_{\mathbb{Q}}\left(|(\lambda_\tau/\lambda_T)|^{\frac{1}{\alpha-1}}|\mathcal{F}_\tau\right) \le c$ *a.s.*

(ii) *For* $\beta \in (1,\infty)$ *we let* $\lambda \in \mathcal{RH}_\beta(\mathbb{Q})$ *provided that there is a constant* $c > 0$ *such that for all stopping times* $\tau : \Omega \to [0,T]$ *one has that* $\mathbb{E}_{\mathbb{Q}}(|\lambda_T|^\beta|\mathcal{F}_\tau)^{\frac{1}{\beta}} \le c\lambda_\tau$ *a.s.*

The class $A_\alpha(\mathbb{Q})$ represents the probabilistic variant of the Muckenhoupt condition and $\mathcal{RH}$ stands for *reverse Hölder inequality*. Next we need

Definition 2.4. *A martingale* $Z = (Z_t)_{t\in[0,T]}$ *is called* BMO*-martingale if* $Z_0 \equiv 0$ *and there is a* $c > 0$ *with* $\mathbb{E}_{\mathbb{Q}}\left(|Z_T - Z_\tau|^2|\mathcal{F}_\tau\right) \le c^2$ *a.s. for all stopping times* $\tau : \Omega \to [0,T]$.

It is known [15, Theorem 2.3] that $(e^{Z_t - \frac{1}{2}\langle Z\rangle_t})_{t\in[0,T]}$ is a martingale for $Z \in \mathrm{BMO}$.

Proposition 2.5 ([15, Theorems 2.4 and 3.4])**.** *Under* (P) *the following is equivalent:*

$$Y \in \mathrm{BMO}, \qquad \mathcal{E}(Y) \in \bigcup_{\alpha\in(1,\infty)} A_\alpha(\mathbb{Q}), \qquad \textit{and} \qquad \mathcal{E}(Y) \in \bigcup_{\beta\in(1,\infty)} \mathcal{RH}_\beta(\mathbb{Q}).$$

Remark 2.6. *Under the assertions of Proposition 2.5 we have* $\lambda_T \in L_\beta(\mathbb{Q})$ *and* $1/\lambda_T \in L_{\alpha'}(\mathbb{P})$ *with* $1 = (1/\alpha) + (1/\alpha')$ *so that* $\bigcap_{r\in[1,\infty)} L_r(\mathbb{Q}) = \bigcap_{r\in[1,\infty)} L_r(\mathbb{P})$.

Proposition 2.7 ([15, Theorems 2.3 and 3.19])**.** *Let* Y *be a* BMO*-martingale so that* (P) *is satisfied. For all* $p \in (0,\infty)$ *there is a* $b_p(\mathbb{P}) > 0$ *such that for all* $\mathbb{Q}$*-martingales* N *with* $N_0 \equiv 0$ *and* $N_t^* := \sup_{s\in[0,t]} |N_s|$ *one has that*

$$(1/b_p(\mathbb{P}))\|N_T^*\|_{L_p(\mathbb{P})} \le \|\sqrt{\langle N\rangle_T}\|_{L_p(\mathbb{P})} \le b_p(\mathbb{P})\|N_T^*\|_{L_p(\mathbb{P})}.$$

Lastly, we will often use the notation $\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} U = \mathbb{E}_{\mathbb{Q}}(U|\mathcal{F}_t)$ and similarly $\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} U$.

3 The result

In the following $\theta \in (0,1]$ will be the main parameter of the fractional smoothness. As fine-tuning parameter we use $q \in [2,\infty]$ and define

$$\Phi_q(h) := \|h\|_{L_q\left([0,T),\frac{dt}{T-t}\right)}$$

for a measurable function $h : [0,T) \to \mathbb{R}$. The main result of the paper is:

Theorem 3.1. *Let* $p \in [2,\infty)$ *and* $\lambda \in A_p(\mathbb{Q})$*, and assume that* (C1), (C2) *and* (P) *are satisfied. Then, for* $\theta \in (0,1)$*,* $q \in [2,\infty]$*, a measurable function* $g : \mathbb{R}^d \to \mathbb{R}$ *satisfying* (C3) *and for* $d\mathbb{P} = \lambda_T d\mathbb{Q}$ *the following assertions are equivalent:*

(i$_\theta$) $\Phi_q\left((T-t)^{-\frac{\theta}{2}}\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})}\right) < +\infty.$

(ii$_\theta$) $\Phi_q\left((T-t)^{\frac{1-\theta}{2}}\|\nabla v(t,X_t)\|_{L_p(\mathbb{P})}\right) < +\infty.$

(iii$_\theta$) $\Phi_q\left((T-t)^{\frac{2-\theta}{2}}\|D^2 v(t,X_t)\|_{L_p(\mathbb{P})}\right) < +\infty.$

As explained in the introduction, the blow-up of $\|\nabla v(t,X_t)\|_{L_p(\mathbb{P})}$ and $\|D^2 v(t,X_t)\|_{L_p(\mathbb{P})}$ as $t \to T$ is used in [4, 6, 12] to study approximation properties of stochastic integrals and in [8, 5] to study the L_p-variation of the solutions of BSDEs. To illustrate Theorem 3.1 by two special cases, we again let $\Delta_1 = \nabla$ and $\Delta_2 = D^2$.

For $\underline{q = \infty}$ we obtain the equivalence of

(i) $\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})} \le c_1(T-t)^{\frac{\theta}{2}}$ for all $t \in [0,T)$, and

(ii) $\|\Delta_i v(t,X_t)\|_{L_p(\mathbb{P})} \le c_2(T-t)^{\frac{\theta-i}{2}}$ for all $t \in [0,T)$.

For $\underline{q = p}$ we use $\langle M\rangle_t = \int_0^t |K_s^X \nabla v(s,X_s)\sigma(s,X_s)|^2 ds$ to get an equivalence of moments of path-wise fractional integrals obtained by Riemann-Liouville operators:

$$\mathbb{E}_{\mathbb{P}} \int_0^T (T-t)^{-p\frac{\theta}{2}-1} |g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)|^p dt < \infty$$
$$\iff \quad \mathbb{E}_{\mathbb{P}} \int_0^T (T-t)^{p\frac{i-\theta}{2}-1} |\Delta_i v(t,X_t)|^p dt < \infty$$

$$\iff \quad \mathbb{E}_{\mathbb{P}} \int_0^T (T-t)^{\frac{p}{2}(1-\theta)-1} \left| \frac{d}{dt} \langle M \rangle_t \right|^{\frac{p}{2}} dt < \infty.$$

Note that for $p = 2/(1-\theta)$ the exponent of the weight in the last integral vanishes so that the quadratic intensity of M to the power $p/2$ is weighted uniformly on $[0,T)$.

Remark 3.2. (1) Often (i_θ) is reasonable easy to check in applications, so that one point of the paper is, that we derive the sharp controls (ii_θ)-(iii_θ) on the derivatives. Examples of functions g that satisfy (i_θ) are given in [4, 6, 11, 5]. For example, assume that $d = 1$ and $g : \mathbb{R} \to \mathbb{R}$ is a function of bounded variation (say $g(x) = \chi_{[K,\infty)}(x)$ for some $K \in \mathbb{R}$). Applying (4.1), we get $\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})} \le 2\|g(X_T) - g(X_t)\|_{L_p(\mathbb{P})}$ and [1, Theorem 2.4] yields upper bounds for the last expression.

(2) For $X = B$, $\mathbb{P} = \mathbb{Q}$, $T = 1$ and $k = 0$ the conditions of Theorem 3.1 are equivalent to g belonging to the Malliavin Besov space $B^\theta_{p,q}$ on $\mathbb{R}^d$ weighted by the standard Gaussian measure (see [12]). The case $p = 2$, $k = 0$, $b = 0$, and $q = \infty$ was considered in [4] for the one-dimensional case (in particular, the process X is a martingale).

(3) The case $\theta = 1$ and $q \in [2,\infty)$ yields to pathologies: Let $X = B$, $\mathbb{P} = \mathbb{Q}$, $T = 1$ and $k = 0$. Condition (i_1) implies (ii_1) by Lemma 4.2 below. Moreover, condition (ii_1) and the monotonicity of $\|\nabla v(t,B_t)\|_{L_p(\mathbb{P})}$ $((\nabla v(t,B_t))_{t\in[0,1)}$ is a martingale in this case) imply that $\nabla v(t,B_t) = 0$ a.s. so that $g(B_1)$ is almost surely constant.

(4) Instead of (i_θ) it is also natural to consider

$$(\mathrm{i}'_\theta) \;\; \Phi_q\big((T-t)^{-\frac{\theta}{2}} \|e^{\int_0^T k(r,X_r)dr} g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(e^{\int_0^T k(r,X_r)dr} g(X_T))\|_{L_p(\mathbb{P})}\big) < +\infty.$$

One can easily check that $(\mathrm{i}_\theta) \iff (\mathrm{i}'_\theta)$ for $\theta \in (0,1]$ and $q \in [1,\infty]$. Indeed, for any random variables U and V, bounded and in $L_p = L_p(\mathbb{P})$, respectively, observe that

$$\begin{aligned} & \|UV - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(UV)\|_{L_p} \\ \le\; & \big\|[U - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}U]V\big\|_{L_p} + \big\|\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(U)[V - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}V]\big\|_{L_p} + \big\|\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(U[\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(V) - V])\big\|_{L_p} \\ \le\; & \|[U - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}U]V\|_{L_p} + 2\|U\|_\infty \|V - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}V\|_{L_p}. \end{aligned}$$

For $U = e^{\int_0^T k(r,X_r)dr}$ and $V = g(X_T)$ we have $|U - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}U| \le 2\|k\|_\infty (T-t) e^{\|k\|_\infty T}$ and can therefore deduce that $(\mathrm{i}_\theta) \Longrightarrow (\mathrm{i}'_\theta)$. The converse is proved similarly.

(5) The case $\theta = 1$ and $q = \infty$: One has $(\mathrm{i}'_1) \iff (\mathrm{ii}_1) \Longrightarrow (\mathrm{iii}_1)$ which follows from (4.15), Lemmas 4.2 and 4.5 below, and $\Phi_\infty\Big((T-t)^{-\frac{1}{2}}\big(\int_t^T h(s)^2 ds\big)^{\frac{1}{2}}\Big) \le \Phi_\infty(h)$. The implication $(\mathrm{iii}_1) \Longrightarrow (\mathrm{ii}_1)$ is not true in general. Take $p = 2$, $q = \infty$, $X = B$, $\mathbb{P} = \mathbb{Q}$, $T = 1$, $k = 0$ and $d = 1$ and the counterexample $g(x) = \sqrt{x \vee 0}$ from [5].

(6) A change of drift of the diffusion X by a term $\int_0^t \beta_s ds$, where the process β is uniformly bounded, yields to the case that $d\mathbb{P}/d\mathbb{Q} \in A_\alpha(\mathbb{Q})$ for **all** $\alpha \in (1,\infty)$. Note that our main result Theorem 3.1 **only** requires $d\mathbb{P}/d\mathbb{Q} \in A_p(\mathbb{Q})$.

To explain this, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis satisfying the usual conditions with $\mathcal{F} = \mathcal{F}_T$. Assume that the filtration is the augmented natural filtration of a standard d-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$ starting in zero. It is known [17, Corollary 1 on p. 187] that on this stochastic basis all local martingales are continuous. Assume a progressively measurable d-dimensional process

$\beta = (\beta_t)_{t\in[0,T]}$ with $\sup_{t,\omega} |\beta_t(\omega)| < \infty$ and consider the unique strong solution of

$$X_t = x_0 + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds - \int_0^t \beta_s ds.$$

Letting $\gamma_s := \sigma^{-1}(s, X_s)\beta_s$, $B_t := W_t - \int_0^t \gamma_s ds$, $1/\lambda_t := e^{\int_0^t \gamma_s^\top dW_s - \frac{1}{2}\int_0^t |\gamma_s|^2 ds}$, and $d\mathbb{Q} := (1/\lambda_T) d\mathbb{P}$, Girsanov's Theorem gives that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$, $(B_t)_{t\in[0,T]}$ and $(X_t)_{t\in[0,T]}$ satisfy our assumptions. Moreover $\lambda \in A_\alpha(\mathbb{Q})$ for all $\alpha \in (1, \infty)$.

(7) In case the drift term in item (6) is Markovian, i.e. $\beta_t = \beta(t, X_t)$ for an appropriate $\beta : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, and if we let $y_t := v(t, X_t)$ and $z_t := \nabla v(t, X_t)\sigma(t, X_t)$, then

$$-dy_t = [k(t, X_t) y_t + z_t \sigma^{-1}(t, X_t)\beta(t, X_t)]dt - z_t dW_t \quad \text{with} \quad y_T = g(X_T).$$

Now we get analogues to $(i_\theta) \Leftrightarrow (ii_\theta)$ for $q = \infty$ because for $p \in [2, \infty)$, $\theta \in (0, 1]$, and a polynomially bounded g it is shown in [5] that under certain conditions

$$\Phi_\infty((T-t)^{\frac{1-\theta}{2}} \|z_t\|_{L_p(\mathbb{P})}) < +\infty \ \text{ iff } \ \Phi_\infty((T-t)^{-\frac{\theta}{2}} \|g(X_T) - \mathbb{E}^{\mathcal{F}_t}(g(X_T))\|_{L_p(\mathbb{P})}) < +\infty.$$

(8) We let $k \equiv 0$ and $g : \mathbb{R}^d \to \mathbb{R}$ be a bounded Borel function. By (2.2)-(2.3) one has

$$y_t^0 = g(X_T) - \int_t^T z_s^0 dB_s \quad \text{with} \quad y_t^0 := v(t, X_t) \text{ and } z_s^0 := \nabla v(s, X_s)\sigma(s, X_s)$$

for $t \in [0, T]$ and $s \in [0, T)$. Now we perturb this equation by a 1-variation term $\int_t^T f(s, X_s, y_s, z_s) ds$ and obtain a backward stochastic differential equation

$$y_t = g(X_T) + \int_t^T f(s, X_s, y_s, z_s) ds - \int_t^T z_s dB_s,$$

where the function f is called generator. As shown in [8, 5], a key tool to study variational properties of a BSDE (that are also the basis for discretization schemes) is the comparison of the exact solution to the solution for the zero-generator case, i.e. to study the difference $y_t - y_t^0$. The following example includes BSDEs of quadratic type. Our assumptions are:

(a) $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is continuous.

(b) There exists a progressively measurable scalar process $(\theta_s)_{s\in[0,T)}$ such that $\sup_{s,\omega} |\theta_s(\omega)| \le \eta_1 < \infty$ and $|f(s, X_s, y_s, z_s) - \theta_s |z_s|^2| \le \eta_2 < \infty$ on Ω for $s \in [0, T)$.

(c) $\mathbb{E}_\mathbb{Q}(\int_t^T |z_s|^2 ds | \mathcal{F}_t) \le c^2$ $\mathbb{Q}$-a.s. for all $t \in [0, T)$.

Using for example [13, Theorem 2.6], where one finds standard assumptions on f for the quadratic case, one can construct examples that satisfy our assumptions. The boundedness of g implies that $(z_s^0)_{s\in[0,T)}$ satisfies (possibly with another constant) the same property (c). Hence $Y := \int_0^\cdot \theta_s(z_s + z_s^0) dB_s$ is a BMO-martingale with respect to $\mathbb{Q}$. Letting $\lambda_t := \mathcal{E}(Y)_t$ and $d\mathbb{P} = \lambda_T d\mathbb{Q}$, we arrive in the setting of our paper as Proposition 2.5 implies that $\lambda \in A_\alpha(\mathbb{Q})$ and $\lambda \in \mathcal{RH}_\beta(\mathbb{Q})$ for some $\alpha, \beta \in (1, \infty)$. Letting $dW_s := dB_s - \theta_s(z_s + z_s^0) ds$, we obtain a $\mathbb{P}$-Brownian motion by Girsanov's Theorem. For $\Delta y_t := y_t - y_t^0$ and $\Delta z_t := z_t - z_t^0$ this yields

$$\Delta y_t = \int_t^T f(s, X_s, y_s, z_s) ds - \int_t^T \Delta z_s dB_s = \int_t^T \widetilde{f}(s, z_s^0) ds - \int_t^T \Delta z_s dW_s$$

with $\widetilde{f}(s,\omega,z_0) := f(s,X_s(\omega),y_s(\omega),z_s(\omega)) - \theta_s(\omega)(|z_s(\omega)|^2 - |z_s^0(\omega)|^2)$. Consequently,

$$|\Delta y_t| \le \mathbb{E}_{\mathbb{P}}\left(\int_t^T |\widetilde{f}(s,z_s^0)|ds|\mathcal{F}_t\right)$$

and, for $q \in [1,\infty)$, $\gamma := \left[\mathbb{E}_{\mathbb{P}}\lambda_T^{-\alpha'}\right]^{\frac{1}{\alpha' q}} < \infty$ ($\lambda \in A_\alpha(\mathbb{Q})$), $r := \alpha q$, and $p := 2r \in (2,\infty)$,

$$\begin{aligned}\|\Delta y_t\|_{L_q(\mathbb{Q})} \le \gamma\|\Delta y_t\|_{L_r(\mathbb{P})} &\le \eta_1\gamma\left\|\int_t^T |z_s^0|^2 ds\right\|_{L_r(\mathbb{P})} + \eta_2\gamma(T-t)\\ &\le \eta_1\gamma\int_t^T \|z_s^0\|^2_{L_p(\mathbb{P})}ds + \eta_2\gamma(T-t).\end{aligned}$$

Therefore, owing to Theorem 3.1 (two first items) the appropriate control of the above time-integral as $t \to T$ follows from the suitable time-integrability of $\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}g(X_T)\|_{L_p(\mathbb{P})}$, which can be directly checked according to the g considered.

4 Proof of Theorem 3.1

Given a probability space (M,Σ,μ) with a sub-σ algebra $\mathcal{G} \subseteq \Sigma$ and $Z \in L_p(M,\Sigma,\mu)$ with $p \in [1,\infty]$ we shall use the inequality:

$$\frac{1}{2}\|Z - \mathbb{E}(Z|\mathcal{G})\|_p \le \inf_{Z' \in L_p(M,\mathcal{G},\mu)} \|Z - Z'\|_p \le \|Z - \mathbb{E}(Z|\mathcal{G})\|_p. \tag{4.1}$$

Lemma 4.1. *For $1 < \alpha < \infty$, $\lambda \in A_\alpha(\mathbb{Q})$, $U \in L_\alpha(\Omega,\mathcal{F},\mathbb{P})$ and $c_{(4.2)} > 0$ such that $[\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}(|\frac{\lambda_t}{\lambda_T}|^{\frac{1}{\alpha-1}})]^{\frac{\alpha-1}{\alpha}} \le c_{(4.2)}$ a.s. we have that*

$$\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}|U| \le c_{(4.2)}\left[\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}|U|^\alpha\right]^{\frac{1}{\alpha}} \text{ a.s. and } \|\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}U\|_{L_\alpha(\mathbb{P})} \le c_{(4.2)}\|U\|_{L_\alpha(\mathbb{P})}. \tag{4.2}$$

Proof. Letting $1 = \frac{1}{\alpha} + \frac{1}{\alpha'}$ one has a.s. that

$$\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}|U| = \lambda_t\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(|U|/\lambda_T) \le \lambda_t[\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}|U|^\alpha]^{\frac{1}{\alpha}}[\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}\lambda_T^{-\alpha'}]^{\frac{1}{\alpha'}} \le c_{(4.2)}[\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}|U|^\alpha]^{\frac{1}{\alpha}}.$$

□

In the next step we will estimate $\nabla v(t,X_t)$ and $D^2v(t,X_t)$ from above by conditional moments of $M_T = K_T^X g(X_T)$ and $g(X_T)$ in Lemmas 4.2 and 4.5, and extend therefore Lemma 2.2 to the case $k \neq 0$ and allow at the same time a change of measure by Muckenhoupt weights.

Lemma 4.2. *For $p \in (1,\infty)$ and $d\mathbb{P} = \lambda_T d\mathbb{Q}$ with $\lambda \in A_p(\mathbb{Q})$ we have a.s. that*

$$|\nabla v(t,X_t)| \le c_{(4.3)}\Big[(T-t)^{-\frac{1}{2}}\Big(\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}M_T|^p\Big)^{\frac{1}{p}} + (T-t)\Big(\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}|M_T|^p\Big)^{\frac{1}{p}}\Big], \tag{4.3}$$

where $c_{(4.3)} > 0$ depends at most on $(\sigma,b,k,p,\mathbb{P})$.

Proof. I. First we follow a martingale approach (see, for example, [7]) and prove the statement for all $p \in (1,\infty)$ for the measure $\mathbb{Q}$.

(a) We define $(\nabla X_t)_{t\in[0,T]}$ to be the solution of the linear SDE (see [17, Chapter 5])

$$\nabla X_t = I_d + \sum_{j=1}^d \int_0^t \nabla\sigma_j(s,X_s)\nabla X_s dB_s^j + \int_0^t \nabla b(s,X_s)\nabla X_s ds$$

with $\sigma(.) = (\sigma_1(.), \dots, \sigma_d(.))$. This matrix-valued process is a.s. invertible with

$$[\nabla X_t]^{-1} = I_d - \sum_{j=1}^{d} \int_0^t [\nabla X_s]^{-1} \nabla \sigma_j(s, X_s) dB_s^j - \int_0^t [\nabla X_s]^{-1} (\nabla b(s, X_s) - \sum_{j=1}^{d} [\nabla \sigma_j(s, X_s)]^2) ds.$$

(b) Formally differentiating the martingale $(M_t)_{t\in[0,T]}$ with respect to the initial value $x_0 \in \mathbb{R}^d$ of $(X_t)_{t\in[0,T]}$, we obtain the process $(N_t)_{t\in[0,T)}$ with

$$N_t := K_t^X \nabla v(t, X_t) \nabla X_t + M_t \Big[\int_0^t \nabla k(s, X_s) \nabla X_s ds \Big]. \tag{4.4}$$

By [16, Section 3.1] and because of our quantitative bounds for the derivatives on v one can expect to obtain a martingale. Either one goes this way to check the fact that $(N_t)_{t\in[0,T)}$ is a $\mathbb{Q}$-martingale or, alternatively, one computes the Itô-process decomposition of N and uses the PDE to remove the bounded variation term.

(c) Exploiting the martingale property of N between t and some $S \in (t, T)$, we have

$$(S-t) N_t = \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \int_t^S N_r dr \tag{4.5}$$

$$= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \Big(\Big[\int_t^S K_r^X \nabla v(r, X_r) \sigma(r, X_r) dB_r \Big] \Big[\int_t^S (\sigma(r, X_r)^{-1} \nabla X_r)^\top dB_r \Big]^\top \Big) \tag{4.6}$$

$$+ (S-t) M_t \Big[\int_0^t \nabla k(s, X_s) \nabla X_s ds \Big] + \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \Big(M_S \int_t^S \Big[\int_t^r \nabla k(s, X_s) \nabla X_s ds \Big] dr \Big). \tag{4.7}$$

For the last equality, we have used the $\mathbb{Q}$-martingale property of $(M_t)_{t\in[0,T]}$ and the conditional Itô isometry. Inserting (4.4) into $(S-t)N_t$, the second term cancels with the first term from (4.7) and $(S-t) K_t^X \nabla v(t, X_t) \nabla X_t$ is left on the left-hand side. Interchanging the integrals over ds and dr in the second term of (4.7) and using the stochastic integral representation of $M_S - M_t$ in (4.6), we finally see that

$$\begin{aligned} (S-t) K_t^X \nabla v(t, X_t) \nabla X_t &= \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \Big([M_S - M_t] \Big[\int_t^S (\sigma(r, X_r)^{-1} \nabla X_r)^\top dB_r \Big]^\top \Big) \\ &\quad + \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} \Big(M_S \Big[\int_t^S (S-s) \nabla k(s, X_s) \nabla X_s ds \Big] \Big). \end{aligned}$$

Using that $M_S \to M_T$ in $L_2(\mathbb{Q})$ we derive the same equation with S replaced by T and multiplied with $[\nabla X_t]^{-1}$. Finally, observe that $\sup_{t\in[0,T)} \sup_{r\in[t,T]} \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} (|\nabla X_r [\nabla X_t]^{-1}|^q)$ is a bounded random variable for any $q \geq 1$; therefore, standard computations using the conditional Hölder inequality complete our assertion.

II. The statement for $\mathbb{P}$ will be deduced from the statement for $\mathbb{Q}$ proved for $q \in (1, p)$. By [15, Corollary 3.3] there is an $\alpha \in (1, p)$ such that also $\lambda \in A_\alpha(\mathbb{Q})$. Let $q := p/\alpha \in (1, p)$. For $\lambda \in A_\alpha(\mathbb{Q})$ we apply Lemma 4.1 with $U := |Z|^q$, where $Z \in \bigcap_{r\in[1,\infty)} L_r(\mathbb{Q})$ (cf. Remark 2.6), and get $\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |Z|^q \right)^{\frac{1}{q}} \leq c_{(4.2)}^{\frac{1}{q}} \left(\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} |Z|^p \right)^{\frac{1}{p}}$ and, by (4.1),

$$\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |Z - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} Z|^q \right)^{\frac{1}{q}} \leq 2 \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t} |Z - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} Z|^q \right)^{\frac{1}{q}} \leq 2 c_{(4.2)}^{\frac{1}{q}} \left(\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} |Z - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} Z|^p \right)^{\frac{1}{p}}.$$

□

For the following we let $m(t, x) := v(t, x) k(t, x)$.

Lemma 4.3. *For $0 \le r < t \le T$ and $1 < p_0 < p < \infty$ one has a.s. that*

$$\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|m(t,X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} m(t,X_t)|^{p_0}\right)^{\frac{1}{p_0}} \le c_{(4.8)}\left[\sqrt{t-r}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_t - M_r|^{p_0}\right)^{\frac{1}{p_0}}\right] \quad (4.8)$$

where $M^* := \sup_{s\in[0,T]}|M_s|$ and $c_{(4.8)} > 0$ depends at most on (p_0, p, σ, b, k).

Proof. Applying a telescoping sum argument and the conditional Hölder inequality to $m(s,X_s) = k(s,X_s)(K_s^X)^{-1}M_s$ we derive

$$\begin{aligned}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|m(t,X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} m(t,X_t)|^{p_0}\right)^{\frac{1}{p_0}} &\le 2\|k\|_\infty e^{T\|k\|_\infty}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_t - M_r|^{p_0}\right)^{\frac{1}{p_0}} \\ &\quad + 2\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|k(t,X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} k(t,X_t)|^{\beta}\right)^{\frac{1}{\beta}} e^{T\|k\|_\infty}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}} \\ &\quad + 2\|k\|_\infty\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|(K_t^X)^{-1} - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}(K_t^X)^{-1}|^{\beta}\right)^{\frac{1}{\beta}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}}\end{aligned}$$

for $\frac{1}{p_0} = \frac{1}{p} + \frac{1}{\beta}$. We conclude by

$$\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|k(t,X_t) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} k(t,X_t)|^{\beta}\right)^{\frac{1}{\beta}} \le 2\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|k(t,X_t) - k(t,X_r)|^{\beta}\right)^{\frac{1}{\beta}} \le c(k,b,\sigma,\beta)\sqrt{t-r}$$

and $\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|(K_t^X)^{-1} - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}(K_t^X)^{-1}|^{\beta}\right)^{\frac{1}{\beta}} \le 2\|k\|_\infty (t-r)e^{T\|k\|_\infty}$. □

Lemma 4.4. *For $0 \le r < t < T$ and $p \in (1,\infty)$ one has a.s. that*

$$\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_t - M_r|^p\right)^{\frac{1}{p}} \le c_{(4.9)}\left[\left(\frac{t-r}{T-t}\right)^{\frac{1}{2}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_T - M_r|^p\right)^{\frac{1}{p}} + (t-r)^{\frac{1}{2}}|M_r|\right] \quad (4.9)$$

where $c_{(4.9)} \ge 1$ depends at most on (p, σ, b, k).

Proof. Let $p_0 := \frac{1+p}{2}$, $\zeta_u := K_u^X \nabla v(u,X_u)\sigma(u,X_u)$ and $0 \le r \le u \le t$. Then Lemma 4.2 implies that

$$\begin{aligned}|\zeta_u|e^{-T\|k\|_\infty} &\le \|\sigma\|_\infty c_{(4.3),p_0}\left[(T-u)^{-\frac{1}{2}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u}|M_T - M_u|^{p_0}\right)^{\frac{1}{p_0}} + (T-u)\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u}|M_T|^{p_0}\right)^{\frac{1}{p_0}}\right] \\ &\le \|\sigma\|_\infty c_{(4.3),p_0}\left[(T-u)^{-\frac{1}{2}}2\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u}|M_T - M_r|^{p_0}\right)^{\frac{1}{p_0}}\right. \\ &\qquad \left. + (T-u)\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u}|M_T - M_r|^{p_0}\right)^{\frac{1}{p_0}} + (T-u)|M_r|\right] \\ &\le \|\sigma\|_\infty c_{(4.3),p_0}[2 + T^{\frac{3}{2}} + T]\left[(T-t)^{-\frac{1}{2}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u}|M_T - M_r|^{p_0}\right)^{\frac{1}{p_0}} + |M_r|\right].\end{aligned}$$

Letting $c := e^{T\|k\|_\infty}\|\sigma\|_\infty c_{(4.3),p_0}[2+T^{\frac{3}{2}}+T]$ we conclude the proof by using the Burkholder-Davis-Gundy and the Doob inequality in order to get

$$\begin{aligned}&\frac{1}{a_p}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_t - M_r|^p\right)^{\frac{1}{p}} \le \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\left(\int_r^t |\zeta_u|^2 du\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\ \le\ & c\left[(T-t)^{-\frac{1}{2}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\left(\int_r^t \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u}|M_T - M_r|^{p_0}\right)^{\frac{2}{p_0}} du\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} + \sqrt{t-r}|M_r|\right] \\ \le\ & c\left[\sqrt{\frac{t-r}{T-t}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\left(\sup_{u\in[r,t]}\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_u}|M_T - M_r|^{p_0}\right)^{\frac{p}{p_0}}\right)^{\frac{1}{p}} + \sqrt{t-r}|M_r|\right]\end{aligned}$$

$$
\begin{aligned}
&\leq c\Big[\Big(\frac{p/p_0}{(p/p_0)-1}\Big)^{\frac{1}{p_0}}\sqrt{\frac{t-r}{T-t}}\Big(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\Big(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}|M_T-M_r|^{p_0}\Big)^{\frac{p}{p_0}}\Big)^{\frac{1}{p}}+\sqrt{t-r}|M_r|\Big] \\
&\leq c\Big[\left(\frac{p}{p-p_0}\right)^{\frac{1}{p_0}}\sqrt{\frac{t-r}{T-t}}\Big(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_t}|M_T-M_r|^{p}\Big)^{\frac{1}{p}}+\sqrt{t-r}|M_r|\Big].
\end{aligned}
$$

□

Lemma 4.5. *For $p\in(1,\infty)$ and $d\mathbb{P}=\lambda_T d\mathbb{Q}$ with $\lambda\in A_p(\mathbb{Q})$ there is a constant $c_{(4.10)}>0$, depending at most on $(\sigma,b,k,p,\mathbb{P})$, such that one has a.s. that*

$$
|D^2v(r,X_r)|\leq c_{(4.10)}\left[\frac{\left(\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_r}\left|g(X_T)-\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_r}g(X_T)\right|^p\right)^{\frac{1}{p}}}{T-r}+\sqrt{T-r}\left(\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}}\right]. \tag{4.10}
$$

Proof. The statement for $\mathbb{P}$ can be deduced from the statement for $\mathbb{Q}$ for $q\in(1,p)$ as in Step II of the proof of Lemma 4.2. Now we show the estimate for the measure $\mathbb{Q}$. For $0\leq s\leq t\leq T$, a fixed $T_0\in(0,T)$ and $r\in[0,T_0]$ we let

$$
v^t(s,x):=\mathbb{E}_{\mathbb{Q}}\left(m(t,X_t)|X_s=x\right)\ \text{ and }\ v_h(r,x):=\mathbb{E}_{\mathbb{Q}}\left(v(T_0,X_{T_0})|X_r=x\right)
$$

where $m=vk$ (the superscript t stands for the time-horizon t and h for *homogenous*). Itô's formula applied to v gives for $r\in[0,T_0]$ that

$$
v(r,x)=\mathbb{E}_{\mathbb{Q}}\Big(v(T_0,X_{T_0})+\int_r^{T_0}(kv)(t,X_t)dt|X_r=x\Big)=v_h(r,x)+\int_r^{T_0}v^t(r,x)dt.
$$

Using Lemma 2.2 and the arguments from Remark 3.2(4) one can show for $0\leq r<t\leq T_0<T$ that

$$
|\nabla v^t(r,x)|\leq\gamma e^{\gamma|x|^{k_g}}\quad\text{and}\quad|D^2v^t(r,x)|\leq\frac{\gamma}{\sqrt{t-r}}e^{\gamma|x|^{k_g}}, \tag{4.11}
$$

where $\gamma>0$ depends at most on (σ,b,k,K_g,k_g,T_0). From this we deduce that

$$
D^2v(r,x)=D^2v_h(r,x)+\int_r^{T_0}D^2v^t(r,x)dt
$$

where (4.11) is used to interchange the integral and D^2. For $p_0:=\frac{1+p}{2}$, $0\leq r<t\leq T$ and $s\in[0,T_0)$ we again use Lemma 2.2 to get

$$
\begin{aligned}
|D^2v^t(r,X_r)| &\leq \frac{\kappa_{p_0}}{(t-r)}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\left|m(t,X_t)-\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}m(t,X_t)\right|^{p_0}\right)^{\frac{1}{p_0}}\quad\text{a.s.,}\\
|D^2v_h(s,X_s)| &\leq \frac{\kappa_p}{(T_0-s)}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_s}\left|v(T_0,X_{T_0})-\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_s}v(T_0,X_{T_0})\right|^{p}\right)^{\frac{1}{p}}\quad\text{a.s.}
\end{aligned}
$$

>From the first estimate we derive by Lemmas 4.3 and 4.4 (with p replaced by p_0) a.s. that

$$
\begin{aligned}
|D^2v^t(r,X_r)| &\leq \frac{\kappa_{p_0}}{(t-r)}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\left|m(t,X_t)-\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}m(t,X_t)\right|^{p_0}\right)^{\frac{1}{p_0}}\\
&\leq \frac{\kappa_{p_0}c_{(4.8)}}{(t-r)}\left[\sqrt{t-r}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}}+\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_t-M_r|^{p_0}\right)^{\frac{1}{p_0}}\right]\\
&\leq \kappa_{p_0}c_{(4.8)}[1+c_{(4.9)}]\frac{1}{\sqrt{t-r}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}}\\
&\quad+\kappa_{p_0}c_{(4.8)}c_{(4.9)}\frac{1}{\sqrt{T-t}\sqrt{t-r}}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_T-M_r|^{p_0}\right)^{\frac{1}{p_0}}
\end{aligned}
$$

and

$$\int_r^T |D^2 v^t(r,X_r)|dt \le c\left[\sqrt{T-r}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_T-M_r|^p\right)^{\frac{1}{p}}\right]$$

with $c := \kappa_{p_0} c_{(4.8)} \max\{2+2c_{(4.9)}, c_{(4.9)}\mathrm{Beta}(\frac{1}{2},\frac{1}{2})\}$. The second estimate yields by $T_0 \uparrow T$ and (2.3) that

$$|D^2 v_h(r,X_r)| \le \frac{\kappa_p}{(T-r)}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\left|g(X_T) - \mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} g(X_T)\right|^p\right)^{\frac{1}{p}}$$

and the upper bound is independent of T_0. Combining the estimates with

$$\begin{aligned}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M_T-M_r|^p\right)^{\frac{1}{p}} &\le 2e^{\|k\|_\infty T} \\ &\left[\|k\|_\infty (T-r) e^{\|k\|_\infty T}\left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}|M^*|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r}\left|g(X_T)-\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} g(X_T)\right|^p\right)^{\frac{1}{p}}\right]\end{aligned}$$

using the arguments from Remark 3.2(4) the proof is complete. □

Lemma 4.6. *For $p \in [2,\infty)$, $\lambda \in A_p(\mathbb{Q})$, $0 \le s < t < T$ and $l = 1,...,d$ we have that*

$$\begin{aligned}&\left\|K_t^X \partial_{x_l} v(t,X_t) - K_s^X \partial_{x_l} v(s,X_s)\right\|_{L_p(\mathbb{P})} \\ &\qquad \le c_{(4.12)}\left[\|M_T\|_{L_p(\mathbb{P})}\int_s^t \frac{dr}{\sqrt{T-r}} + \left(\int_s^t \|D^2 v(r,X_r)\|_{L_p(\mathbb{P})}^2 dr\right)^{\frac{1}{2}}\right]\end{aligned} \tag{4.12}$$

with $c_{(4.12)} > 0$ depending at most on $(\sigma, b, k, p, \mathbb{P})$.

Proof. Using the PDE for v to obtain that $w_l = \partial_{x_l} v$ solves

$$\mathcal{L} w_l = -\frac{1}{2}\sum_{i,j=1}^d (\partial_{x_l} a_{i,j})\, \partial^2_{x_i,x_j} v - \sum_{i=1}^d (\partial_{x_l} b_i)\, \partial_{x_i} v - (\partial_{x_l} k)\, v,$$

and exploiting Propositions 2.5 and 2.7 we get that

$$\begin{aligned}&\left\|K_t^X \partial_{x_l} v(t,X_t) - K_s^X \partial_{x_l} v(s,X_s)\right\|_{L_p(\mathbb{P})} \qquad (4.13)\\ \le\ & b_p(\mathbb{P})\left\|\left(\int_s^t |K_r^X (\nabla \partial_{x_l} v)(r,X_r)\sigma(r,X_r)|^2 dr\right)^{\frac{1}{2}}\right\|_{L_p(\mathbb{P})} \\ &+\frac{1}{2}\|\partial_{x_l} A\|_\infty \left\|\int_s^t |K_r^X D^2 v(r,X_r)| dr\right\|_{L_p(\mathbb{P})} \\ &+\|\partial_{x_l} b\|_\infty \left\|\int_s^t |K_r^X \nabla v(r,X_r)| dr\right\|_{L_p(\mathbb{P})} + \|\partial_{x_l} k\|_\infty \left\|\int_s^t |K_r^X v(r,X_r)| dr\right\|_{L_p(\mathbb{P})}.\end{aligned}$$

Lemma 4.1 yields $\sup_r \left\|K_r^X v(r,X_r)\right\|_{L_p(\mathbb{P})} = \sup_r \left\|\mathbb{E}_{\mathbb{Q}}^{\mathcal{F}_r} M_T\right\|_{L_p(\mathbb{P})} \le c_{(4.2)}\|M_T\|_{L_p(\mathbb{P})}$ and, by Lemma 4.2, $\|\nabla v(r,X_r)\|_{L_p(\mathbb{P})} \le c_{(4.3)}(T-r)^{-\frac{1}{2}}\left(2+T^{3/2}\right)\|M_T\|_{L_p(\mathbb{P})}$. Inserting these estimates into the above upper bound for (4.13) gives the result. □

Lemma 4.7 ([12, Proposition A.4])**.** *Let $0 < \theta < 1$, $2 \le q \le \infty$ and $d^k : [0,T) \to [0,\infty)$, $k = 0,1,2$, be measurable functions. Assume that there are $A \ge 0$ and $D \ge 1$ such that*

$$\frac{1}{D}(T-t)^{\frac{k}{2}} d^k(t) \le d^0(t) \le D\left(\int_t^T [d^1(s)]^2 ds\right)^{\frac{1}{2}} \quad \textit{and} \quad d^1(t) \le A + D\left(\int_0^t [d^2(u)]^2 du\right)^{\frac{1}{2}}$$

for $k = 1,2$ and $t \in [0,T)$. Then there is a constant $c_{(4.14)} > 0$, depending at most on (D,θ,q,T), such that, for $k,l \in \{0,1,2\}$,

$$A + \Phi_q\left((T-t)^{\frac{k-\theta}{2}} d^k(t)\right) \sim_{c_{(4.14)}} A + \Phi_q\left((T-t)^{\frac{l-\theta}{2}} d^l(t)\right). \tag{4.14}$$

Proof of Theorem 3.1: We let $d^0(t) := \sqrt{T-t} + \|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})}$,

$$d^1(t) := 1 + \|\nabla v(t,X_t)\|_{L_p(\mathbb{P})} \quad \text{and} \quad d^2(t) := 1 + \left\|D^2 v(t,X_t)\right\|_{L_p(\mathbb{P})}.$$

>From Lemma 4.2 we get that

$$\begin{aligned} d^1(t) &\leq 1 + c_{(4.3)}(T-t)^{-\frac{1}{2}}\|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} + c_{(4.3)}(T-t)\|M_T\|_{L_p(\mathbb{P})} \\ &\leq (T-t)^{-\frac{1}{2}}[1 + c_{(4.3)} + c_{(4.3)}T\|M_T\|_{L_p(\mathbb{P})}]d^0(t). \end{aligned}$$

>From Lemma 4.5 we get that

$$d^2(t) \leq 1 + c_{(4.10)}\left[\frac{\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})}}{T-t} + \sqrt{T-t}\|M^*\|_{L_p(\mathbb{P})}\right].$$

Using Remark 3.2(4) we have that

$$\|g(X_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} g(X_T)\|_{L_p(\mathbb{P})} \leq 2e^{\|k\|_\infty T}\left[\|k\|_\infty (T-t)\|M_T\|_{L_p(\mathbb{P})} + \|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})}\right].$$

Together with the previous estimate we obtain a $c = c(c_{(4.10)}, k, T, \|M^*\|_{L_p(\mathbb{P})}) > 0$ such that $d^2(t) \leq c(T-t)^{-1} d^0(t)$. >From

$$\|M_T - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t} M_T\|_{L_p(\mathbb{P})} \leq 2b_p(\mathbb{P}) e^{T\|k\|_\infty}\|\sigma\|_\infty \left\|\left(\int_t^T |\nabla v(s,X_s)|^2 ds\right)^{\frac{1}{2}}\right\|_{L_p(\mathbb{P})}, \tag{4.15}$$

which follows from (4.1) and Proposition 2.7, and Lemma 4.6 for $s = 0$ we get that

$$d^0(t) \leq [1 + c_{(4.15)}]\left(\int_t^T [d^1(s)]^2 ds\right)^{\frac{1}{2}} \quad \text{and} \quad d^1(t) \leq d_1 + d_2\left(\int_0^t [d^2(r)]^2 dr\right)^{\frac{1}{2}}$$

with constants $d_1 := 1 + e^{\|k\|_\infty T}\left[\|K_0^X \nabla v(0,X_0)\|_{L_p(\mathbb{P})} + 2c_{(4.12)}\sqrt{dT}\|M_T\|_{L_p(\mathbb{P})}\right]$ and $d_2 := e^{\|k\|_\infty T} c_{(4.12)}\sqrt{d}$. Hence Lemma 4.7 and Remark 3.2(4) yield Theorem 3.1. □

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