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On a class of H–selfadjont random matrices with one eigenvalue of nonpositive type*

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Abstract

Large H–selfadjoint random matrices are considered. The matrix H is assumed to have one negative eigenvalue, hence the matrix in question has precisely one eigenvalue of nonpositive type. It is showed that this eigenvalue converges in probability to a deterministic limit. The weak limit of distribution of the real eigenvalues is investigated as well.

Keywords: Random matrix; Wigner matrix, eigenvalue; limit distribution of eigenvalues; Π_1–space..
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Introduction

The main object of this survey are non–symmetric random matrices with the structure of the entries arising from the theory of indefinite linear algebra. To specify the problem let us consider an invertible, hermitian–symmetric matrix $H \in \mathbb{C}^{n\times n}$. We say that $X \in \mathbb{C}^{n\times n}$ is H–selfadjoint if $X^*H = HX$. This is the same as to say that A is selfadjoint with respect to an inner product

$$[x, y]_H := y^*Hx, \quad x, y \in \mathbb{C}^n.$$

Note that this inner product is not positive definite if H has negative eigenvalues. In the literature the space $\mathbb{C}^n$ with the inner product $[\cdot,\cdot]_H$ is also called a Π_κ–space (where κ is the number of negative eigenvalues of H) or Pontryagin space. The infinite dimensional case is considered as well, recently spectra and pseudospectra of H–selfadjoint, infinite, random matrices were considered in [8, 9]. In the present paper the case when

$$H = \begin{bmatrix} -1 & 0 \\ 0 & I_N \end{bmatrix}, \tag{0.1}$$

with N converging to infinity, is considered. It is easy to check that for such H each H–selfadjoint matrix has the form

$$X = \begin{bmatrix} a & -b^* \\ b & C \end{bmatrix},$$

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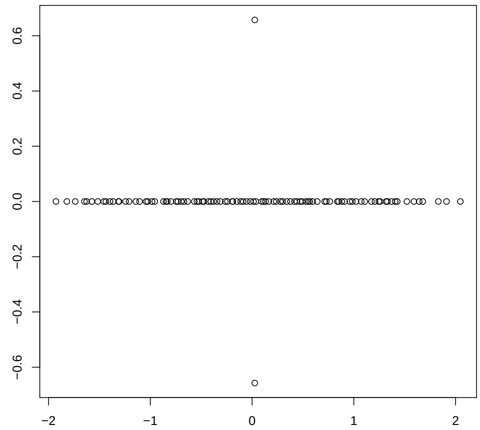

Figure 1: Eigenvalues of a random matrix X_{100} computed with R [28]

with $x \in \mathbb{R}$, $b \in \mathbb{C}^N$ and a hermitian–symmetric matrix $C \in \mathbb{C}^{N\times N}$. Due to the famous theorem of Pontryagin [27] the matrix X has precisely one eigenvalue β in the closed upper half–plane, for which the corresponding eigenvector x satisfies $[x,x]_H \leq 0$. The problem of tracking the nonpositive eigenvalue was considered for example in [11, 29]. In those papers the setting was non–random and X was in the family of one dimensional extensions of a fixed operator in an infinite dimensional Π_1–space. The aim of the present work is to investigate the behavior of β when X is a large random matrix. We show that the main method of [11, 29] – the use of Nevanlinna functions with one negative square – can be adapted to the random setting as well.

A classical result of Wigner [31] says that if the random variables y_{ij}, $0 \leq i \leq j < +\infty$ are real, i.i.d with mean zero and variance equal one, then the distribution of eigenvalues of a matrix

$$Y_N = \frac{1}{\sqrt{N}}[y_{ij}]_{ij=0}^N,$$

where $y_{ji} = y_{ij}$ for $j > i$, converge weakly in probability to the Wigner semicircle measure. Note that by multiplying the first row of Y_N by -1 we obtain a H–selfadjoint matrix X_N. A result of a preliminary numerical experiment with gaussian y_{ij} is plotted in Figure 1. Note that the spectrum of X_N is real, except two eigenvalues, lying symmetrically with respect to the real line. Although we pay a special attention to the above case, we study the behavior of the eigenvalue of nonpositive type in a more general setting. Namely, we assume that the random matrix X_N in $\mathbb{C}^{N\times N}$ is of a form

$$X = \begin{bmatrix} a_N & -b_N^* \\ b_N & C_N \end{bmatrix},$$

with a_N, b_N and C_N being independent. Furthermore, the vector b_N is a column of a Wigner matrix and a_N converges weakly to zero. The only assumption on C_N is that the limit distribution of its eigenvalues converge weakly in probability, see (R0)–(R3) for details. In Theorem 4.1 we prove that under these assumptions the non–real eigenvlaues converge in probability to a deterministic limit that can be computed knowing the limit distribution of eigenvalues of C_N. In the case when C_N is a Wigner matrix the nonreal eigenvalues converge to $\pm\mathrm{i}\sqrt{2}/2$, cf. Theorem 5.1. Furthermore, under a technical assumption of continuity of the entries of X_N, we show in Theorem 4.2 that the limit distribution of the real eigenvalues of X_N coincides with the limit distribution of eigenvalues for the matrices C_N. Again, in the case when C_N is a Wigner matrix we obtain a more precise result. Namely, in Theorem 5.1 we show that the real eigenvalues $\zeta_2^N, \dots, \zeta_N^N$ of X_N and the eigenvalues $\lambda_1^N, \dots, \lambda_N^N$ of C_N satisfy the following inequalities:

$$\lambda_1^N < \zeta_2^N < \lambda_2^N < \cdots < \lambda_{N-1}^N < \zeta_N^N < \lambda_N^N.$$

It shows that the nonreal eigenvalue of X_N plays an analogue role as the largest eigenvalue in one–dimensional, symmetric perturbations of Wigner matrices. This fact relates the present paper to the current work on finite dimensional perturbations of random matrices, see [2, 3, 4, 6, 7, 15, 17] and references therein. Also note that X_N is a product of a random and deterministic matrix, such products were already considered in the literature, see e.g. [30].

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1 Functions of class $\mathcal{N}_1$

The Nevanlinna functions with negative squares play a similar role for the class of H–selfadjoint matrices as the class of ordinary Nevanlinna function plays for hermitian–symmetric matrices. This phenomenon has its roots in in operator theory, we refer the reader to [10, 11, 18, 19, 20] and papers quoted therein for a precise description of a relation between $\mathcal{N}_\kappa$–functions and selfadjoint operators in Krein and Pontryagin spaces. We begin with a very general definition of the class $\mathcal{N}_\kappa$, but we immediately restrict ourselves to certain subclasses of those functions.

We say that Q is *a generalized Nevanlinna function of class* $\mathcal{N}_\kappa$ [18, 21] if it is meromorphic in the upper half–plane $\mathbb{C}^+$ and the kernel

$$N(z,w) = \frac{Q(z) - \overline{Q(w)}}{z - \bar{w}}$$

has precisely κ negative squares, that is for any finite sequence $z_1, \dots, z_k \in \mathbb{C}^+$ the hermitian–symmetric matrix

$$[N(z_i, z_j)]_{ij=1}^k$$

has not more then κ nonpositive eigenvalues and for some choice of $z_1, \dots, z_k$ it has precisely κ nonpositive eigenvalues. In the present paper we use this definition with $\kappa = 0, 1$.

The class $\mathcal{N}_0$ is the class of ordinary Nevanlinna functions, i.e. the functions that are holomorphic in $\mathbb{C}^+$ with nonnegative imaginary part. By $M_b^+(\mathbb{R})$ we denote the set of positive, bounded Borel measures on $\mathbb{R}$. For $\mu \in M_b^+(\mathbb{R})$ we define the Stieltjes transform as

$$\hat{\mu}(z) = \int_{\mathbb{R}} \frac{1}{t - z} dt, \quad z \in \mathbb{C} \setminus \operatorname{supp} \mu.$$

Clearly, $\hat{\mu}$ belongs to the class $\mathcal{N}_0$ and the values of $\hat{\mu}$ in the upper half–plane determine the measure uniquely by the Stieltjes inversion formula. Although not every function of class $\mathcal{N}_0$ is a Stieltjes transform of a Borel measure (cf. [13]), this subclass of $\mathcal{N}_0$ functions will be sufficient for present reasonings. Also, we will be interested in a special subclass of $\mathcal{N}_1$ functions, namely in the functions of the form (1.1) below. We refer the reader to the literature [10, 12] for representation theorems for $\mathcal{N}_\kappa$ functions.

Proposition 1.1. *If $\mu \in M_b^+(\mathbb{R})$, $a \in \mathbb{R}$ then*

$$Q(z) = \hat{\mu}(z) + a - z \tag{1.1}$$

is a holomorphic function in $\mathbb{C}^+$ and belongs to the class $\mathcal{N}_1$. Furthermore, there exists precisely one $z_0 \in \mathbb{C}$ such that either $z_0 \in \mathbb{C}^+$ and

$$Q(z_0) = 0, \tag{1.2}$$

or $z_0 \in \mathbb{R}$ and

$$\lim_{z \hat{\to} z_0} \frac{Q(z)}{z - z_0} \in (-\infty, 0]. \tag{1.3}$$

The symbol $\hat{\to}$ above denotes the non-tangential limit:

$$z \in \mathbb{C}^+, \quad z \to z_0, \quad \pi/2 - \theta \le \arg(z - z_0) \le \pi/2 + \theta,$$

with some $\theta \in (0, \pi/2)$. We call $z_0 \in \mathbb{C}^+ \cup \mathbb{R}$ the *generalized zero of nonpositive type* (*GZNT*) of $Q(z)$. The first part of the Proposition can be found e.g. in [19], while for the proof of the 'Furthermore' part in the general context[1] we refer the reader to [21, Theorem 3.1, Theorem 3.1']. In view of the above proposition we can define a function

$$G : M_b^+(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}^+$$

by saying that $G(\mu, a)$ is the GZNT of the function $\hat{\mu}(z) + a - z$. The following proposition plays a crucial role in our arguments.

Proposition 1.2. *The function G is jointly continuous with respect to the weak topology on $M_b^+(\mathbb{R})$ and the standard topology on $\mathbb{R}$.*

Proof. Assume that $(\mu_n)_n \subset M_b^+(\mathbb{R})$ converges weakly to $\mu \in M_b^+(\mathbb{R})$ and $a_n \in \mathbb{R}$ converges to $a \in \mathbb{R}$ with $n \to \infty$. Take a compact K in the open upper half–plane, with nonempty interior. Then $\hat{\mu}_n$ converges uniformly to $\hat{\mu}$ on the set K. Indeed, if $r = \sup_{t \in \mathbb{R},\ z \in K} 1/|t - z|$ then

$$\sup_{z \in K} |\hat{\mu}_n(z) - \hat{\mu}_0(z)| \le r|\mu_n - \mu_0|(\mathbb{R}),$$

the latter clearly converging to zero with $n \to \infty$. In consequence, $\hat{\mu}_n(z) + a_n - z$ converges to $\hat{\mu}(z) + a - z$ uniformly on K with $n \to \infty$. By [23] the GZNT of $\hat{\mu}_n(z) + a_n - z$ converges to the GZNT of $\hat{\mu}(z) + a - z$, which finishes the proof. $\square$

2 H–selfadjoint matrices

In this section we review basic properties of selfadjoint matrices in indefinite inner product spaces introducing the concept of a canonical form. For the infitnite–dimensional counterpart of the theory we refer the reader to [5, 22]. Let $H \in \mathbb{C}^{(n+1)\times(n+1)}$ ($n \in \mathbb{N}\backslash\{0\}$) be an invertible, Hermitian–symmetric matrix. We say that $X \in \mathbb{C}^{(n+1)\times(n+1)}$ is *H–selfadjoint* if $X^*H = HX$. Our main interest will lie in the matrix

$$H = \begin{bmatrix} -1 & 0 \\ 0 & I_n \end{bmatrix},$$

where I_n denotes the identity matrix of size $n \times n$. As it was already mentioned, each H–selfadjoint matrix has the form

$$X = \begin{bmatrix} a & -b^* \\ b & C \end{bmatrix}, \tag{2.1}$$

[1] For arbitrary $\mathcal{N}_1$ function $z_0 = \infty$ can be also the GZNT, in that case $\lim_{z \hat{\to} \infty} zQ(z) \in [0, \infty)$. However, this is clearly not possible for Q of the form (1.1).

with $a \in \mathbb{R}$, $b \in \mathbb{C}^n$ and hermitian–symmetric $C \in \mathbb{C}^{n\times n}$. Due to [16] there exists an invertible matrix S and a pair of matrices $H', S' \in \mathbb{C}^{(n+1)\times(n+1)}$ such that $X = S^{-1}X'S$ $H = S^*H'S$ and X', H' are of one of the following forms:

Case 1.
$$X' = \begin{bmatrix} \beta & 0 \\ 0 & \bar{\beta} \end{bmatrix} \oplus \operatorname{diag}(\zeta_2, \dots, \zeta_n), \quad H' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-1},$$
with $\beta \in \mathbb{C}^+$, $\zeta_2, \dots, \zeta_n \in \mathbb{R}$.

Case 2.
$$X' = [\beta] \oplus \operatorname{diag}(\zeta_1, \dots, \zeta_n), \quad H' = [-1] \oplus I_n,$$
with $\beta \in \mathbb{R}$, $\zeta_1, \dots, \zeta_n \in \mathbb{R}$.

Case 3.
$$X' = \begin{bmatrix} \beta & 1 \\ 0 & \beta \end{bmatrix} \oplus \operatorname{diag}(\zeta_2, \dots, \zeta_n), \quad H' = \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-1},$$
with $\beta \in \mathbb{R}$, $\zeta_2, \dots, \zeta_n \in \mathbb{R}$, $\gamma \in \{-1, 1\}$.

Case 4.
$$X' = \begin{bmatrix} \beta & 1 & 0 \\ 0 & \beta & 1 \\ 0 & 0 & \beta \end{bmatrix} \oplus \operatorname{diag}(\zeta_3, \dots, \zeta_n), \quad H' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \oplus I_{n-2},$$
with $\beta \in \mathbb{R}$, $\zeta_3, \dots, \zeta_n \in \mathbb{R}$.

It is easy to verify that in each case X' is H'-symmetric. The pair (X', H') is called *the canonical form* of (X, H). We refer the reader to [16] for the proof and for canonical forms for general H–symmetric matrices and to [5, 22] for the infinite–dimensional counterpart of the theory. At this point is enough to mention that the canonical form is uniquely determined (up to permutations of the numbers ζ_i) for each pair (X, H), where X is H–selfadjoint. Note that in each of the cases β is an eigenvalue of X and there exists a corresponding eigenvector $x \in \mathbb{C}^{n+1}$ satisfying $[x, x]_H \le 0$, furthermore, β is the only eigenvalue in $\mathbb{C}^+ \cup \mathbb{R}$ having this property. Therefore, we will call β the *eigenvalue of nonpositive type of X*.

Observe that the function
$$Q(z) = a - z + b^*(C - z)^{-1}b \tag{2.2}$$
is an $\mathcal{N}_1$–function. Indeed, if $D = UCU^* = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonalization of the hermitian–symmetric matrix C and $d = Ub$ then
$$Q(z) = a - z + \sum_{j=1}^{n} \frac{|d_j|^2}{\lambda_j - z} = a - z + \hat{\mu}(z), \quad \text{where} \quad \mu = \sum_{j=1}^{n} |d_j|^2 \delta_{\lambda_j},$$
and we may apply Proposition 1.1. The following lemma is a standard in the indefinite linear algebra theory. We present the proof for the reader's convenience.

Lemma 2.1. *Let X and Q be defined by* (2.1) *and* (2.2)*, respectively. A point $\beta \in \mathbb{C}^+ \cup \mathbb{R}$ is the eigenvalue of nonpositive type of X if and only if it is the GZNT of $Q(z)$. Furthermore, the algebraic multiplicity of β as an eigenvalue of X equals the order of β as a zero of $Q(z)$.*

Proof. First note that, due to the Schur complement formula[2],
$$-\frac{1}{Q(z)} = e^* H (X - z)^{-1} e,$$

[2] It is well known [19] that $-1/Q$ belongs to $\mathcal{N}_1$ provided that Q belongs to $\mathcal{N}_1$, however, this information is not essential for the proof.

where e denotes the first vector of the canonical basis of $\mathbb{C}^{n+1}$. Let (X', H') be the canonical form of (X, H) and let S be the appropriate transformation. Consequently,

$$-\frac{1}{Q(z)} = (Se)^* H'(X'-z)^{-1} Se. \tag{2.3}$$

Below we evaluate this expression in each of the Cases 1–4. Let $f = [f_0, \dots, f_n]^\top = Se$. Note that

$$f^* H' f = e^* H e = -1, \tag{2.4}$$

independently on the Case.

Case 1. Observe that $f_0 \bar{f}_1 \neq 0$, otherwise $f^* H' f \geq 0$, which contradicts (2.4). Due to (2.3) one has

$$-\frac{1}{Q(z)} = \frac{f_0 \bar{f}_1}{\beta - z} + \frac{f_1 \bar{f}_0}{\bar{\beta} - z} + \sum_{j=2}^{n} \frac{|f_j|^2}{\zeta_j - z}.$$

Hence, $\beta \in \mathbb{C}^+$ is a simple pole of $-1/Q$ and consequently it is the GZNT of Q and a simple zero of Q.

Case 2. Observe that $|f_0|^2 > \sum_{j=1}^{n} |f_j|^2$, otherwise $f^* H' f \geq 0$, which contradicts (2.4). Due to (2.3) one has

$$-\frac{1}{Q(z)} = \frac{-|f_0|^2}{\beta - z} + \sum_{j=1}^{n} \frac{|f_j|^2}{\zeta_j - z}.$$

Hence, the residue of $-1/Q$ in β is less then zero. Consequently $Q(\beta) = 0$, $Q'(\beta) < 0$ and β is the GZNT of Q.

Case 3. Observe that $|f_1|^2 > 0$, otherwise $f^* H' f \geq 0$, which contradicts (2.4). Due to (2.3) one has

$$-\frac{1}{Q(z)} = \frac{2\gamma \operatorname{Re} f_0 \bar{f}_1}{\beta - z} + \frac{-\gamma |f_1|^2}{(\beta - z)^2} + \sum_{j=2}^{n} \frac{|f_j|^2}{\zeta_j - z}.$$

Hence, β is pole of $-1/Q$ of order 2. Consequently, $Q(\beta) = Q'(\beta) = 0$, $Q''(\beta) \neq 0$ and β is the GZNT.

Case 4. Observe that $|f_2|^2 > 0$, otherwise $f^* H' f \geq 0$, which contradicts (2.4). Due to (2.3) one has

$$-\frac{1}{Q(z)} = \frac{2 \operatorname{Re} f_0 \bar{f}_2 + |f_1|^2}{\beta - z} + \frac{-2 \operatorname{Re} f_1 \bar{f}_2}{(\beta - z)^2} + \frac{|f_2|^2}{(\beta - z)^3} + \sum_{j=3}^{n} \frac{|f_j|^2}{\zeta_j - z},$$

Hence, β is pole of $-1/Q$ of order 3. Consequently, $Q(\beta) = Q'(\beta) = Q''(\beta) = 0$, $Q'''(\beta) \neq 0$ and β is the GZNT of Q. $\square$

3 Random H–selfadjoint matrices

By X_N, H_N we understand the following pair of a random and deterministic matrix in $\mathbb{C}^{(N+1)\times(N+1)}$

$$X_N = \begin{bmatrix} a_N & -b_N^* \\ b_N & C_N \end{bmatrix}, \quad H_N = \begin{bmatrix} -1 & 0 \\ 0 & I_N \end{bmatrix}, \tag{3.1}$$

where a_N is a real–valued random variable, b_N is a random vector in $\mathbb{C}^N$, and C_N is a hermitian–symmetric random matrix in $\mathbb{C}^{N\times N}$. Note that X_N is H_N–symmetric. By $\lambda_1^N \leq \cdots \leq \lambda_N^N$ we denote the eigenvalues of C_N and by ν_N we denote the random measure on $\mathbb{R}$

$$\nu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j^N}.$$

Recall that

$$\hat{\nu}_N(z) = \frac{\operatorname{tr}(C_N - z)^{-1}}{N}. \tag{3.2}$$

The assumptions on X_N are as follows:

(R0) The random variable a_N is independent on the entries of the vector b_N and on the entries of the matrix C_N for each $N > 0$, futhermore a_N converges with $N \to \infty$ to zero in probability.

(R1) The random vector b_N is of the form

$$b_N := \frac{1}{\sqrt{N}} [x_{j0}]_{j=1,\dots,N},$$

where $[x_{j0}]_{j>0}$ are i.i.d. random variables, independent on the entries of C_N for $N > 0$, of zero mean with $E|x_{j0}|^2 = s^2$ for $j > 0$.

(R2) The random measure ν_N converges with $N \to \infty$ to some non–random measure μ_0 weakly in probability

All the results below hold also in the case when all variables x_{j0} ($j > 0$) are real, in this situation b_N^* is just the transpose of b_N. The entries of C_N might be as well real or complex. In Section 5 we will consider two instances of the matrix C_N: a Wigner matrix and a diagonal matrix. In the case when C_N is a Wigner matrix the proposition below is a consequence of the isotropic semicircle law [14, 17]. We present below a simple proof of the general case, based on the ideas in [24].

Proposition 3.1. *Assume that* (R1) *and* (R2) *are satisfied. Then for each* $z \in \mathbb{C}^+$

$$b_N^*(C_N - z)^{-1} b_N \to s^2 \ \hat{\mu}_0(z) \quad (N \to \infty)$$

in probability.

By $\|y\|$ we denote the euclidean norm of $y \in \mathbb{C}^n$.

Proof. First we provide a proof in the case when

$$E|x_{0j}|^4 < \infty, \quad j = 1, \dots, N. \tag{3.3}$$

In the light of Chebyshev's inequality, (3.2) and assumption (R2) it is enough to show that

$$\lim_{N\to\infty} E \left| b_N^*(C_N - z)^{-1} b_N - s^2 \frac{\operatorname{tr}(C_N - z)^{-1}}{N} \right|^2 = 0. \tag{3.4}$$

Observe that

$$E \left| b_N^*(C_N - z)^{-1} b_N - s^2 \frac{\operatorname{tr}(C_N - z)^{-1}}{N} \right|^2 = \tag{3.5}$$

$$\left(E \left| b_N^*(C_N - z)^{-1} b_N \right|^2 - s^4 E \left| \frac{\operatorname{tr}(C_N - z)^{-1}}{N} \right|^2 \right) -$$

$$2 \operatorname{Re} E \left(s^2 \overline{\frac{\operatorname{tr}(C_N - z)^{-1}}{N}} \left(b_N^*(C_N - z)^{-1} b_N - s^2 \frac{\operatorname{tr}(C_N - z)^{-1}}{N} \right) \right). \tag{3.6}$$

First we prove that the summand (3.6) equals zero. Indeed, conditioning on the σ–algebra generated by the entries of the matrix C_N and setting

$$[c_{ij}]_{ij=1}^N = (C_N - z)^{-1}$$

one obtains

$$E\left(s^2\frac{\overline{\operatorname{tr}(C_N-z)^{-1}}}{N}\left(b_N^*(C_N-z)^{-1}b_N-s^2\frac{\operatorname{tr}(C_N-z)^{-1}}{N}\right)\right)=$$

$$E\left(s^2\sum_{i=1}^N\frac{\overline{c_{ii}}}{N}\left(\sum_{jk=1}^N c_{jk}\frac{x_{0j}\overline{x_{0k}}}{N}-s^2\sum_{j=1}^N\frac{c_{jj}}{N}\right)\right)=$$

$$E\left(s^2\sum_{i=1}^N\frac{\overline{c_{ii}}}{N}\left(\sum_{j=1}^N c_{jj}\frac{s^2}{N}-s^2\sum_{j=1}^N\frac{c_{jj}}{N}\right)\right)=0.$$

Next, observe that

$$E|b_N^*(C_N-z)^{-1}b_N|^2=E\sum_{ijkl=1}^N c_{ij}\overline{c_{kl}}\frac{x_{0i}\overline{x_{0j}}x_{0k}\overline{x_{0l}}}{N^2}=$$

$$s^4\sum_{ij=1}^N\frac{E(c_{ii}\overline{c_{jj}})}{N^2}+s^4\sum_{ij=1}^N\frac{E(c_{ij}\overline{c_{ij}})}{N^2}=s^4E\left|\frac{\operatorname{tr}(C_N-z)^{-1}}{N}\right|^2+s^4E\sum_{ij=1}^N\frac{c_{ij}\overline{c_{ij}}}{N^2}.$$

This allows us to estimate (3.5) by

$$s^4\left|E\sum_{ij=1}^N\frac{c_{ij}\overline{c_{ij}}}{N^2}\right|\le s^4E\frac{\left\|(C_N-z)^{-1}\right\|^2}{N}=\frac{s^4\operatorname{dist}(z,\sigma(C_N))^{-2}}{N}\le\frac{s^4}{(\operatorname{Im}z)^2N},$$

which finishes the proof of (3.4) in the case when the fourth moments of x_{0j} ($j=1,\dots,N$) are finite. To prove the general case one uses a standard truncation argument, setting for $r>0$

$$b_{Nr}=\frac{1}{\sqrt{N}}\left[x_{01}1_{\{x_{01}\le r\}},\dots,x_{0N}1_{\{x_{0N}\le r\}}\right]-\frac{1}{\sqrt{N}}E\left[x_{01}1_{\{x_{01}\le r\}},\dots,x_{0N}1_{\{x_{0N}\le r\}}\right].$$

Recall that by the first part of the proof for every $r>0$

$$E|b_{Nr}^*(C_N-z)^{-1}b_{Nr}-s^2\mu_0(z)|\to0,\quad N\to\infty. \tag{3.7}$$

Note that $\|b_N\|^2=N^{-1}\sum_{j=1}^N|x_{0j}|^2$ converges almost surely to s^2, by the strong law of large numbers. Furthermore,

$$\|b_{Nr}\|\le\|b_N\|+s.$$

Hence, the number

$$r_0:=\sup_{N,r}(\|b_N\|+\|b_{Nr}\|)$$

is almost surely finite. Consequently,

$$E|b_{Nr}^*(C_N-z)^{-1}b_{Nr}-b_N^*(C_N-z)^{-1}b_N|\le$$

$$E\left((\|b_{Nr}\|+\|b_N\|)\left\|(C_N-z)^{-1}\right\|\|b_{Nr}-b_N\|\right)\le\frac{r_0}{\operatorname{Im}z}E\left\|b_{Nr}-b_N\right\|.$$

Note that, due to (R1), one has

$$(E\left\|b_{Nr}-b_N\right\|)^2\le E\left\|b_{Nr}-b_N\right\|^2=E|x_{01}1_{x_{01}\ge r}|^2-(E|x_{01}1_{x_{01}\ge r}|)^2,$$

where both summands on the right hand side converge to zero with $r\to\infty$ and do not depend on N. This, together with (3.7), shows that

$$E|b_N^*(C_N-z)^{-1}b_N-s^2\mu_0(z)|\to0,\quad N\to\infty,$$

which completes the proof of the general case.

□

Let U_N be a unitary matrix, such that $U_N C_N U_N^*$ is diagonal and let $d_N = [d_1^N, \dots, d_N^N]^\top = U_N b_N$. Denote by μ_N the measure defined by

$$\mu_N = \sum_{j=1}^{N} |d_j^N|^2 \delta_{\lambda_j^N},$$

and observe that $\hat{\mu}_N(z) = b_N^*(C_N - z)^{-1} b_N$.

Proposition 3.2. *Assume that* (R1) *and* (R2) *are satisfied. Then the sequence of random measures μ_N converge weakly with $N \to \infty$ to μ_0 in probability.*

Proof. First note that almost surely $\mu_N(\mathbb{R}) \to s^2 \mu_0(\mathbb{R})$ with $N \to \infty$. Indeed,

$$\mu_N(\mathbb{R}) = \sum_{j=1}^{N} |d_j^N|^2 = \|d_N\|^2 = \|b_N\|^2 = \frac{1}{N} \sum_{j=1}^{N} |x_{0j}|^2,$$

which converges almost surely to s^2 by the strong law of large numbers. Furthermore, Proposition 3.1 shows that $\hat{\mu}_N(z)$ converges in probability to $\hat{\mu}_0(z)$ for every $z \in \mathbb{C}^+$. Repeating the proof of Theorem 2.4.4 of [1] we get the weak convegence of μ_N in probability. □

4 Main results

Theorem 4.1. *If* (R0) – (R2) *are satisfied then the eigenvalue of nonpositive type β_N of X_N converges in probability to the GZNT β_0 of the $\mathcal{N}_1$–function*

$$Q_0(z) = -z + s^2 \hat{\mu}_0(z).$$

Proof. Consider a sequence of $\mathcal{N}_1$–functions

$$Q_N(z) = a_N - z + \hat{\mu}_N(z). \tag{4.1}$$

Recall that each of those functions has precisely one GZNT which, by definition of μ_N and Lemma 2.1, is the eigenvalue of nonpositive type β_N of X_N. Recall that a_N converges to zero in probability by (R0) and μ_N converges to μ_0 in probability by Proposition 3.2. Let d be any metric that metrizises the topology of weak convergence on $M_b^+(\mathbb{R})$. Since β_N is a continuous function of μ_N and a_N (Proposition 1.2), for each $\varepsilon > 0$ one can find $\delta > 0$ such that for each $N > 0$ the event $\{|a_N| < \delta,\ d(\mu_N, \mu_0) < \delta\}$ is contained in $\{|\beta_N - \beta_0| < \varepsilon\}$. Using the assumed in (R0) independence of μ_N and a_N one obtains

$$P(|\beta_0 - \beta_N| \geq \varepsilon) \leq P(|a_N| \geq \delta) + P(d(\mu_N, \mu_0) \geq \delta) - P(|a_N| \geq \delta) \cdot P(d(\mu_N, \mu_0) \geq \delta).$$

Hence, β_N converges to β_0 in probability. □

As it was explained in Section 1.1, each matrix X_N has, besides the eigenvalue β_N of nonpositive type, a set of real eigenvaules $\zeta_{k_N}^N, \dots, \zeta_N^N$, where $k_N = 1$ in Case 1 and 3, $k_N = 2$ in Case 2 and $k_N = 3$ in Case 3. By τ_N we denote the empirical measure connected with these eigenvalues:

$$\tau_N = \frac{1}{N} \sum_{j=k_N}^{N} \delta_{\zeta_j}^N.$$

Theorem 4.2. *If* (R0)–(R2) *are saisfied, the random variables* $\{x_{0j} : j > 0\}$ *are continuous and the eigenvalues* $\lambda_1^N, \dots, \lambda_N^N$ *of* C_N *are almost surely distinct for large* N*, then the measure* τ_N *converges weakly in probability to* μ_0*.*

Proof. We use the notations U_N, d_N and μ_N from the previous section, let also Q_N be given by (4.1). Note that the set $\left\{y \in \mathbb{C}^N : (U_N y)_j = 0\right\}$ is of Lebesgue measure zero. Hence, with probability one $d_j^N \neq 0$ for $j = 1, \dots, N$, $N > 0$. Therefore, for large N the Stieltjes transform $\hat{\mu}_N(z)$ is a rational function almost surely with poles of order one in $\lambda_1^N < \dots < \lambda_N^N$. Furthermore,

$$Q_N(z) = \frac{(a_N - z)\ \prod_{j=1}^N(\lambda_j^N - z) + \sum_{i=1}^N |d_i^N|^2 \prod_{j\neq i}(\lambda_j^N - z)}{\prod_{j=1}^N(\lambda_j^N - z)}.$$

In consequence, Q_N has exactly $N+1$ zeros counting multiplicities, all of them different from $\lambda_1^N, \dots, \lambda_N^N$. Due to the Schur complement argument, each of those zeros is an eigenvalue of the matrix $X_N \in \mathbb{C}^{(N+1)\times(N+1)}$. Furthermore, due to Lemma 2.1 the algebraic multiplicity of β_N as eigenvalue of X_N equals the order of β_N as a zero of Q_N. In consequence, the spectrum of X_N coincides with the zeros of Q_N and β_N is the only zero of order possibly greater then one[3].

On the other hand, the function $\hat{\mu}_N$ is increasing on the real line with simple poles in $\lambda_1^N, \dots, \lambda_N^N$. Hence, in each of the intervals $(\lambda_j^N, \lambda_{j+1}^N)$ $(j = 1, \dots, N-1)$ there is an odd number of zeros of Q_N, counting multiplicities. Consequently, in each of the intervals $(\lambda_j^N, \lambda_{j+1}^N)$ $(j = 1, \dots, N-1)$ there is precisely one zero of Q_N, except possibly one interval that contains three zeros of Q_N. Out of these three zeros of Q_N either one or two of them belong to the set $\left\{\zeta_{k_N}^N, \dots, \zeta_N^N\right\}$, accordingly to the canonical form of X_N. Hence, in each of the intervals $(\lambda_j^N, \lambda_{j+1}^N)$ $(j = 1, \dots, N-1)$ there is precisely one of the eigenvalues $\zeta_{k_N}^N, \dots, \zeta_N^N$, except possibly one interval that contains two of the eigenvalues $\zeta_{k_N}^N, \dots, \zeta_N^N$. Consequently, the weak limit of τ_N in probability equals the weak limit of ν_N.

□

5 Two instances

In the present section we consider two instances of C_N: the Wigner matrix and the diagonal matrix. These both cases appear naturally as applications of main results. We refer the reader to [25] for a scheme joining both examples.

Consider an H–selfadjoint real Wigner matrix

$$X_N := \frac{1}{\sqrt{N}}\ H_N[x_{ij}]_{ij=0}^N, \tag{5.1}$$

with x_{ij} real, $x_{ij} = x_{ji}$ $(0 \leq i < j < \infty)$, i.i.d., of zero mean and variance equal to s^2, and let H_N be defined as in (3.1). Clearly X_N is H_N–selfadjoint and satisfies (R0)–(R2) with μ_0 equal to the Wigner semicircle measure σ. The Stieltjes transform of the σ equals

$$\hat{\sigma}(z) = \frac{-z + \sqrt{z^2 - 4s^2}}{2s^2}.$$

It is easy to check that $\beta_0 = \frac{\sqrt{2}}{2} s\,\mathrm{i}$ is a zero of $Q_0(z) = -z + s^2\hat{\sigma}(z)$. Hence, β_0 is the GZNT of Q_0 and we have proved the first part of the theorem below.

Theorem 5.1. *Let* X_N *be defined by* (5.1)*. Then*

[3] In other words: e is almost surely a cyclic vector of X_N.

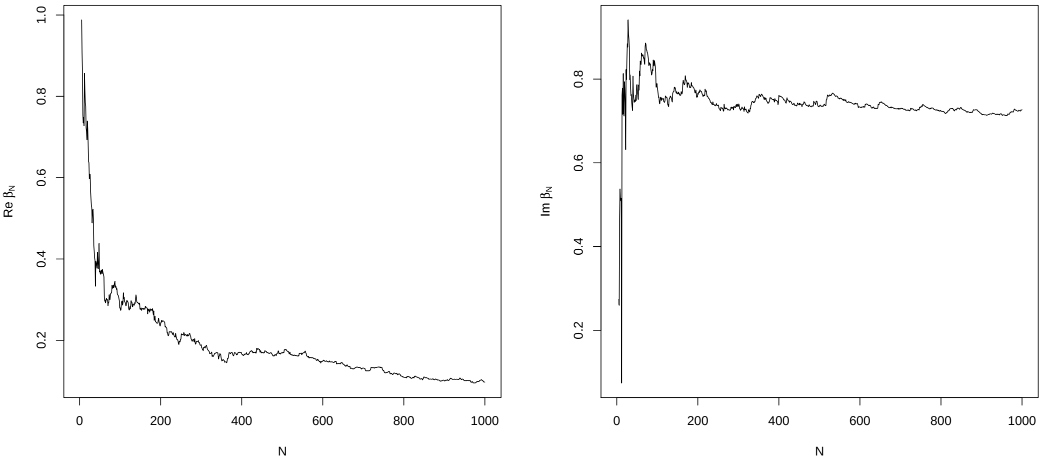

Figure 2: The real and imaginary part of β_N, with real, gaussian entries of X_N and $s^2 = 1$, computed with R [28].

(i) *β_N converges in probability to* $\beta_0 = \frac{\sqrt{2}}{2} s\,\mathrm{i}$;

(ii) *if, additionally, the random variables* b_{ij} $(0 \le i < j < \infty)$ *are continuous, then the probability of an event that there are precisely* $N-1$ *real eigenvalues* $\zeta_2^N < \cdots < \zeta_N^N$ *of* X_N *and the inequalities*

$$\lambda_1^N < \zeta_2^N < \lambda_2^N < \cdots < \lambda_{N-1}^N < \zeta_N^N < \lambda_N^N. \tag{5.2}$$

are satisfied, converges with N *to 1.*

Proof. (ii) Assume that

$$|\beta_N - \beta_0| \le \frac{\sqrt{2}}{4} s. \tag{5.3}$$

Then the canonical form of (X_N, H_N) is as in Case 1. In consequence, there are exactly $N-1$ real eigenvalues $\zeta_2^N, \ldots, \zeta_N^N$ of X_N. Let us recall now the arguments from proof of Theorem 4.2. The function $\hat{\mu}_N$ is increasing on the real line with simple poles in $\lambda_1^N < \cdots < \lambda_N^N$. In each of the intervals $(\lambda_j^N, \lambda_{j+1}^N)$ $(j = 1, \ldots, N-1)$ there at least one of the eigenvalues $\zeta_2^N, \ldots, \zeta_N^N$. Consequently, each of the intervals $(\lambda_j^N, \lambda_{j+1}^N)$ $(j = 1, \ldots, N-1)$ contains precisely one of the eigenvalues $\zeta_2^N, \ldots, \zeta_N^N$. To finish the proof it is enough to note that by point (i) for every $\varepsilon > 0$ there exists $N_0 > 0$ such that for $N > N_0$ the probability of (5.3) is greater then $1 - \varepsilon$. $\square$

The numerical simulations of values of $\mathrm{Re}\,\beta_N$ and $\mathrm{Im}\,\beta_N$ can be found in Figure 2. Note that β_0 lies in open upper half–plane and (1.2) is satisfied. We provide now an example when $\beta_0 \in \mathbb{R}$ and show that each number in $[0, \infty)$ can be the limit in (1.3). Let $a_N = 0$, x_{i0} $(i = 1, 2, , \ldots,)$ be independent real variables of zero mean and variance s^2 and let $C_N = \mathrm{diag}(c_1, \ldots, c_N)$, where the random variables $\{c_j : j = 1, \ldots\}$ are i.i.d. and independent on x_{i0} $(i = 1, 2, , \ldots,)$. Furthermore, let the law of c_j (which is simultaneously the limit measure μ_0) be given by a density

$$\phi(t) = \begin{cases} \frac{3t^2}{2} & : t \in [-1, 1] \\ 0 & : t \in \mathbb{R} \setminus [-1, 1] \end{cases}.$$

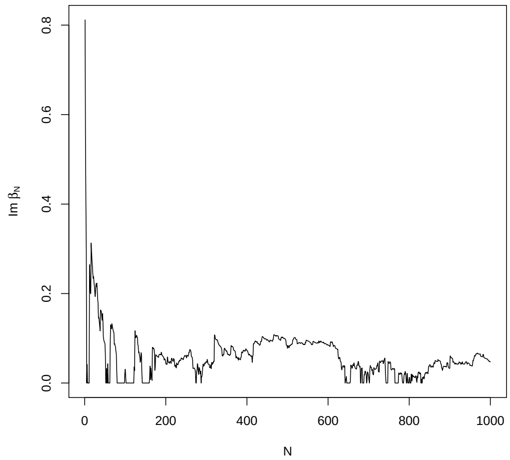

Figure 3: The imaginary part of β_N.

An easy calculation shows that

$$\lim_{z \hat{\to} 0} \frac{\hat{\mu}_0(z)}{z} = 3.$$

Hence,

$$\lim_{z \hat{\to} 0} \frac{-z + s^2 \hat{\mu}_0(z)}{z} = -1 + 3s^2$$

and the function

$$Q_0(z) = -z + \hat{\mu}_0(z) = -z + \int_{\mathbb{R}} \frac{\phi(t)}{t-z} dt$$

has a GZNT at $z = 0$ if $s^2 \leq 1/3$. Note that $\beta_0 = 0$ lies in the support of μ_0. The case $s^2 = 1/3$ is plotted in Figure 3. Only the imaginary part is displayed, since the numerical computation of the real part of β_N might be not reliable in case $\beta_N \in \mathbb{R}$. One may observe that the convergence of β_N is worse in Figure 2. Also, the canonical form of (X_N, H_N) changes with N, contrary to the case when $H_N X_N$ is a Wigner matrix. In the case $s^2 < 1/3$ in numerical simulations point β_N is real for all N.

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