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The Wronskian parametrises the class of diffusions with a given distribution at a random time

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Abstract

We provide a complete characterisation of the class of one-dimensional time-homogeneous diffusions consistent with a given law at an exponentially distributed time using classical results in diffusion theory. To illustrate we characterise the class of diffusions with the same distribution as Brownian motion at an exponentially distributed time.

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1 Introduction

The aim of this article is to characterise the class of one-dimensional time-homogeneous diffusions with a given law at an exponentially distributed time. We show, for instance, that there is a one-parameter family of diffusion processes started at 0 with the same law as Brownian motion at an exponentially distributed time. In general, given a probability distribution we find that consistent diffusions are parametrised by a choice of starting point and secondly by a choice of Wronskian.

We use classical results due to Dynkin [2] and Salminen [8] involving the h-transform (or Doob's h-transform) of a diffusion to provide necessary and sufficient conditions for a diffusion to have a given distribution at a random time. Previously, Cox, Hobson and Obłój [1] proved the existence of consistent diffusions when the first moment is finite. We recover the construction in [1] as a canonical choice from the class of consistent diffusions.

The analogue problem of constructing diffusions with a given distribution at a deterministic time is considered by Ekström, et al. in [3]. This article is also related to the inverse problem of constructing diffusions consistent with prices for perpetual American options or, more generally, with given value functions for perpetual horizon stopping problems, see Hobson and Klimmek [4]. As in this article, the underlying key idea in [4] is to construct consistent diffusions through the speed measure via the eigenfunctions of the diffusion.

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2 Generalised diffusions and the h-transform

Let $I \subseteq \mathbb{R}$ be a finite or infinite interval with a left endpoint a and right endpoint b. Let m be a non-negative, non-zero Borel measure on $\mathbb{R}$ with $I = \text{supp}(m)$. Let $s : I \to \mathbb{R}$ be a strictly increasing and continuous function. Let $x_0 \in I$ and let $B = (B_t)_{t\geq 0}$ be a Brownian motion started at $B_0 = s(x_0)$ supported on a filtration $\mathbb{F}^B = (\mathcal{F}_u^B)_{u\geq 0}$ with local time process $\{L_u^z; u \geq 0, z \in \mathbb{R}\}$. Define Γ to be the continuous, increasing, additive functional

$$\Gamma_u = \int_{\mathbb{R}} L_u^z m(dz),$$

and define its right-continuous inverse by

$$A_t = \inf\{u : \Gamma_u > t\}.$$

If $X_t = s^{-1}(B(A_t))$ then $X = (X_t)_{t\geq 0}$ is a one-dimensional regular diffusion started at x_0 with speed measure m and scale function s. Moreover, $X_t \in I$ almost surely for all $t \geq 0$.

Let $H_x = \inf\{u : X_u = x\}$. Then for $\lambda > 0$ (see e.g. [8]),

$$\xi_\lambda(x, y) = \mathbb{E}_x[e^{-\lambda H_y}] = \begin{cases} \frac{\varphi_\lambda(x)}{\varphi_\lambda(y)} & x \leq y \\ \frac{\phi_\lambda(x)}{\phi_\lambda(y)} & x \geq y, \end{cases} \tag{2.1}$$

where φ_λ and ϕ_λ are respectively a strictly increasing and a strictly decreasing solution to the differential equation

$$\frac{1}{2}\frac{d}{dm}\frac{d}{ds}f = \lambda f. \tag{2.2}$$

The two solutions are linearly independent with Wronskian $W_\lambda = \varphi_\lambda' \phi_\lambda - \phi_\lambda' \varphi_\lambda > 0$. Recall that if a diffusion $X = (X_t)_{t\geq 0}$ is in natural scale, then the Wronskian W_λ is a constant. In the smooth case, when m has a density ν so that $m(dx) = \nu(x)dx$ and s'' is continuous, (2.2) is equivalent to

$$\frac{1}{2}\sigma^2(x)f''(x) + \alpha(x)f'(x) = \lambda f(x), \tag{2.3}$$

where

$$\nu(x) = \sigma^{-2}(x)e^{M(x)}, \quad s'(x) = e^{-M(x)}, \quad M(x) = \int_{0-}^{x} 2\sigma^{-2}(z)\alpha(z)dz.$$

We will call the solutions to (2.2) the λ-eigenfunctions of the diffusion. We will scale the λ-eigenfunctions so that $\varphi_\lambda(X_0) = \phi_\lambda(X_0) = 1$.

The λ-eigenfunctions are well known to be λ-excessive. We recall that a Borel-measurable function $h : I \to \mathbb{R}^+$ is λ-excessive if for all $x \in I$ and $t \geq 0$, $\mathbb{E}_x[e^{-\lambda t}h(X_t)] \leq h(x)$ and if $\mathbb{E}_x[e^{-\lambda t}h(X_t)] \to h(x)$ pointwise as $t \to 0$.

Definition 2.1. *Let h be a λ-excessive function. The h-transform of a diffusion $X = (X_t)_{t\geq 0}$ is the diffusion $X^h = (X_t^h)_{t\geq 0}$ with transition function*

$$P^h(t; x, A) = \frac{1}{h(x)}\int_A e^{-\lambda t}p(t; x, y)h(y)m(dy),$$

where p is the transition density of X with respect to m.

By the following result due to Dynkin [2] (see also Salminen [8] (3.1)), any diffusion X can be transformed into a diffusion with a given law at an exponential killing time. Fix $\lambda > 0$ and let $X = (X_t)_{t\geq 0}$ be a diffusion with λ-eigenfunctions φ_λ and ϕ_λ. Let T be an exponentially distributed random variable with parameter λ, independent of X.

Theorem 2.2. *Given a probability measure μ on $[a,b]$ let*

$$h(x) = \int_{[a,b]} \frac{\xi_\lambda(x,y)}{\xi_\lambda(X_0,y)}\mu(dy). \tag{2.4}$$

Then $\mathbb{P}(X_T^h \in dx) = \mu(dx)$. Conversely, let h be a λ-excessive function with $h(X_0) = 1$ and let $\gamma_X^h(dy) = \mathbb{P}(X_T^h \in dx)$. Then h has the representation (2.4) with $\mu = \gamma_X^h$.

The measure γ_X^h in (2.4) is called *the representing measure for* h. It follows from Theorem 2.2 that we can start with any diffusion X on $[a,b]$ and construct a killed diffusion with a given representing measure via an h-transform. Thus, in principle, since the representing measure coincides with the law of X_T^h, Dynkin's result solves the inverse problem of constructing diffusions with a given law at an exponentially distributed (killing) time.

We will build on this observation to recover consistent diffusions using a characterisation of a representing measure in terms of the λ-eigenfunctions.

3 Characterising consistent diffusions

Without loss of generality, we will restrict the inverse problem to the class of diffusions in natural scale. Recall that results about a diffusion Y with a non-trivial scale function can be deduced from the corresponding results for $X = s(Y)$.

We make the trivial observation that $h \equiv 1$ is a λ-excessive function for any $\lambda > 0$. The h-transform corresponding to $h \equiv 1$ will be called the λ-transform. The λ-transform of X, denoted X^1, is equivalent to X up to the exponential time $T \sim \text{Exp}(\lambda)$ when X^1 is killed, while X remains on the state space I. Thus

$$X_t^1 = \begin{cases} X_t & t \le T \\ \Delta & t > T, \end{cases}$$

where Δ is the grave state of the killed diffusion X^1. Note that the transition density of X^1 with respect to m is given by $q(t;x,y) = e^{-\lambda t}p(t;x,y)$. Other fundamental quantities are related similarly, for instance $\mathbb{E}_{X_0}[e^{-\lambda H_x}] = \mathbb{P}_{X_0}(H_x < T) = \mathbb{P}_{X_0}(X^1 \text{ reaches } x)$.

We are now able to restate our inverse problem as follows. Given a probability measure μ on $[a,b]$, construct a diffusion $X = (X_t)_{t\ge 0}$ such that for all $x \in [a,b]$

$$1 = \int_{[a,b]} \frac{\xi_\lambda(x,y)}{\xi_\lambda(X_0,y)}\mu(dy), \tag{3.1}$$

whence by Theorem 2.2, $X_T^1 \sim \mu$. Since $X_T \equiv X_T^1 \sim \mu$, the idea is to construct the class of consistent diffusions via the λ-eigenfunctions for which (3.1) holds.

The following result is an elementary case ($h \equiv 1$) of Proposition (3.3) in Salminen [8].

Proposition 3.1. *Given a diffusion X, the representing measure $\gamma = \gamma_X^1$ is given by*

$$\gamma([a,x)) = \frac{\varphi_\lambda'(x-)}{W_\lambda}, \quad a < x \le X_0, \tag{3.2}$$

$$\gamma((x,b]) = \frac{-\phi_\lambda'(x+)}{W_\lambda}, \quad X_0 \le x < b, \tag{3.3}$$

where φ_λ (ϕ_λ) are the increasing (decreasing) λ-eigenfunctions of X and W_λ is the Wronskian.

Remark 3.2. *If a is accessible and $X_0 = a$ then the representing measure for $h = 1$ is given by $\gamma((x,b]) = \frac{-\phi_\lambda'(x+)}{W_\lambda}$ for $a \le x < b$. The case $X_0 = b$ with b accessible is analogous.*

The characterisation of the representing measure in Proposition 3.1 will be used to arrive at our main result. Suppose we are given a probability measure μ on $[a,b]$. Let $U_\mu(x) = \int_{[a,b]} |x-y|\mu(dy)$, $C_\mu(x) = \int_{[a,b]} (y-x)^+\mu(dy)$ and $P_\mu(x) = \int_{[a,b]} (x-y)^+\mu(dy)$. Let $X = (X_t)_{t\geq 0}$ be a one-dimensional diffusion in natural scale and let T be an independent exponentially distributed random variable with parameter $\lambda > 0$.

Theorem 3.3. *Suppose $X_0 \in (a,b)$. Then $X_T \sim \mu$ if and only if the speed measure of X satisfies*

$$m(dx) = \begin{cases} \frac{1}{2\lambda}\frac{\mu(dx)}{P_\mu(x)-P_\mu(X_0)+1/W_\lambda}, & a < x \leq X_0 \\ \frac{1}{2\lambda}\frac{\mu(dx)}{C_\mu(x)-C_\mu(X_0)+1/W_\lambda}, & X_0 \leq x < b. \end{cases}$$

where $W_\lambda > 0$ is the Wronskian of X.

Proof. Suppose first that $X_T \sim \mu$. Then since $X_T \equiv X_T^1$, by Theorem 2.2 μ is the representing measure for $h \equiv 1$. Differentiating both sides of (3.2) we find that for all points x such that $a < x \leq X_0$ and which are not atoms of μ,

$$\mu(dx) = \frac{1}{W_\lambda}\varphi_\lambda''(x)dx.$$

(If μ has an atom at x then $\mu(\{x\}) = \frac{1}{W_\lambda}(\varphi_\lambda''(x+) - \varphi_\lambda''(x-))$. The case $x \geq X_0$ is similar, with ϕ_λ replacing φ_λ.

On the other hand, integrating the two sides of (3.2) we have

$$\begin{aligned} P_\mu(x) + k_1 &= \frac{\varphi_\lambda(x)}{W_\lambda}, \quad x \leq X_0 \\ C_\mu(x) + k_2 &= \frac{\phi_\lambda(x)}{W_\lambda}, \quad x \geq X_0, \end{aligned}$$

for constants $k_1, k_2 \in \mathbb{R}$. Now using the fact that $\varphi_\lambda(X_0) = \phi_\lambda(X_0) = 1$ we find that $k_1 = 1/W_\lambda - P(X_0)$ and $k_2 = 1/W_\lambda - C(X_0)$. Since φ_λ and ϕ_λ are the λ-eigenfunctions for X and solutions to (2.2), the speed measure of X satisfies

$$m(dx) = \begin{cases} \frac{1}{2\lambda}\frac{\varphi_\lambda''(x)dx}{\varphi_\lambda(x)}, & a < x \leq X_0 \\ \frac{1}{2\lambda}\frac{\phi_\lambda''(x)dx}{\phi_\lambda(x)}, & X_0 \leq x < b. \end{cases}$$

Substituting for φ_λ and ϕ_λ we thus have

$$m(dx) = \begin{cases} \frac{1}{2\lambda}\frac{\mu(dx)}{P_\mu(x)+k_1}, & a < x \leq X_0 \\ \frac{1}{2\lambda}\frac{\mu(dx)}{C_\mu(x)+k_2}, & X_0 \leq x < b \end{cases}$$

as required.

Conversely suppose that X has the given speed measure on (a,b). Define a function $\eta : [a,b] \to \mathbb{R}^+$ as follows. Let $W_\lambda > 0$ be the Wronskian associated with X and set

$$\eta(x) = \begin{cases} W_\lambda(P_\mu(x) - P_\mu(X_0)) + 1, & a \leq x \leq X_0 \\ W_\lambda(C_\mu(x) - C_\mu(X_0)) + 1, & X_0 \leq x \leq b. \end{cases}$$

Then η solves (2.2) on the domain (a,b) and we therefore have

$$\eta(x) = \begin{cases} \varphi_\lambda(x), & a \leq x \leq X_0 \\ \phi_\lambda(x), & X_0 \leq x \leq b. \end{cases}$$

By Proposition 3.1 the representing measure for $h \equiv 1$ is given by

$$\gamma([a,x)) = \frac{\eta'(x-)}{W_\lambda} = \mu([a,x)), \quad a < x \le X_0$$
$$\gamma((x,b]) = \frac{-\eta'(x+)}{W_\lambda} = \mu((x,b]), \quad X_0 \le x < b,$$

and it follows that $X_T \sim \mu$. □

Remark 3.4. *If X is started at an accessible end-point, a say, then $X_T \sim \mu$ if and only if for all $x \in [a,b)$, $m(dx) = \frac{1}{2\lambda}\frac{\mu(dx)}{C_\mu(x)-C_\mu(a)+1/W_\lambda}$. The case $X_0 = b$ where b is accessible is analogous. Compare Remark 3.2.*

We have the following interpretation for the Wronskian.

Corollary 3.5. *If $X_T \sim \mu$ then the Wronskian satisfies*

$$\frac{W_\lambda}{2\lambda} = \left.\frac{m(dz)}{\mu(dz)}\right|_{z=X_0}.$$

Intuition for Corollary 3.5 is provided by the fact that $2/W_\lambda = \mathbb{E}_{X_0}[L^{X_0}_{A_T}]$ (see Lemma VI. 54.1 in Rogers and Williams [7]).

4 The Wronskian and the martingale property

Let $\tau \equiv \inf\{t \ge 0 : X_t \notin \mathrm{int}(I)\}$. It is well known (see for instance [7]) that $X^\tau = (X_{t\wedge\tau})_{t\ge 0}$ is a local martingale. We will say that X is a martingale diffusion whenever X^τ is a martingale. In this section we will see that when the first moment of the target law is finite, there exists a unique consistent martingale diffusion.

Let $\bar{x}_\mu = \int_{[a,b]} x\mu(dx)$. For the remainder of this section suppose that $\int_{[a,b]} |x|\mu(dx) < \infty$ and $X_0 = \bar{x}_\mu$. We then have the following corollary to Theorem 3.3.

Corollary 4.1. *$X_T \sim \mu$ if and only if for $a < x < b$,*

$$m(dx) = \frac{1}{\lambda}\frac{\mu(dx)}{U_\mu(x) - |x - X_0| - 2C_\mu(X_0) + 2/W_\lambda}. \tag{4.1}$$

Proof. For $x \ge X_0$, $U_\mu(x) - 2C_\mu(x) = C_\mu(x) + P_\mu(x) - 2C_\mu(x) = P_\mu(x) - C_\mu(x) = |x - X_0|$. Similarly for $x \le X_0$, $U_\mu(x) - 2P_\mu(x) = |x - X_0|$. Noting also that $P_\mu(X_0) = C_\mu(X_0)$ the result follows from Theorem 3.3. □

By inspection of (4.1), the most natural choice of W_λ is $W_\lambda = 1/C_\mu(\bar{x}_\mu)$ which, we note, also recovers the construction in [1]. By the following result, the diffusion corresponding to this choice of W_λ is in fact the unique martingale diffusion consistent with μ.

Theorem 4.2. *Suppose $X_0 = \bar{x}_\mu$ and $a = -\infty$ or $b = \infty$. Then X is a martingale diffusion consistent with μ if and only if $W_\lambda = 1/C_\mu(\bar{x}_\mu)$.*

The author would like to thank David Hobson for providing the proof used below that $\int^\infty \frac{xC''_\mu(x)}{C_\mu(x)}dx = \infty$.

Proof. We suppose $b = \infty$ (the case $a = \infty$ is analogous). Since m is positive, $W_\lambda \ge 1/C_\mu(\bar{x}_\mu)$. Suppose $W_\lambda > 1/C_\mu(\bar{x}_\mu)$ then

$$m(dx) = \frac{1}{\lambda}\frac{\mu(dx)}{U_\mu(x) - |x - \bar{x}_\mu| + c}$$

for some $c > 0$ and $\lim_{x\uparrow\infty} \dfrac{m(dx)}{\mu(dx)} = 1/\lambda c$. Thus $\int^\infty |x| m(dx) \propto \int^\infty |x|\mu(dx) < \infty$. It follows from Theorem 1 in Kotani [6] that X is not a martingale diffusion.

Conversely suppose that $W_\lambda = 1/C_\mu(\bar{x}_\mu)$. We will show that $\int^\infty \frac{xC''_\mu(x)}{C_\mu(x)} dx = \infty$. Write $h(x) = \frac{xC''(x)}{2C(x)}$. For fixed y and $x > y$, let $D(x) = \mathbb{E}^x \left[\exp\left(-\int_0^{H_y} \frac{h(B_s)}{B_s} ds \right) \right]$. Note that $D(y) = 1$ and D is positive and decreasing. Let $M_t = \exp\left(-\int_0^t \frac{h(B_s)}{B_s} \right) D(B_t)$. Then $M = (M_{t\wedge H_y})_{t\geq 0}$ is a bounded martingale. In particular, by Itô's formula, $\frac{1}{2} D''(B_s) = \frac{h(B_s)}{B_s} D(B_s)$, so that $D(x) = \frac{C_\mu(x)}{C_\mu(y)}$. It follows that $\lim_{x\to\infty} D(x) = 0$ and that

$$\lim_{\substack{x\uparrow\infty \\ B_0 = x}} \int_0^{H_y} \frac{h(B_s)}{B_s} ds = \infty$$

almost surely. Then we must have

$$\begin{aligned} \infty &= \lim_{\substack{x\uparrow\infty \\ B_0 = x}} \mathbb{E}\left[\int_0^{H_y} \frac{h(B_s)}{B_s} ds \right] \\ &= \lim_{x\uparrow\infty} \left\{ \int_y^x \frac{h(z)}{z}(z-y)dz + \int_x^\infty \frac{h(z)}{z}(x-y)dz \right\} \\ &= \int_y^\infty \frac{h(z)}{z}(z-y)dz, \end{aligned}$$

and thus $\int^\infty h(z)dz = \infty$. It follows by Theorem 1 in [6] that X is a martingale diffusion. □

Remark 4.3. *An alternative (less direct) proof of Theorem 4.2 is available using a result in Hulley and Platen [5]. By Theorem 1.2 and Proposition 2.2 in [5], X is a martingale diffusion if and only if $\lim_{x\uparrow\infty} \phi_\lambda(x) = 0$. Now recall that since X is consistent with μ we have $\phi_\lambda(x) = W_\mu C_\mu(x) - W_\mu C_\mu(X_0) + 1$ for $x \geq X_0$. Clearly $\lim_{x\uparrow\infty} \phi_\lambda(x) = 0$ if and only if $W_\mu = 1/C_\mu(X_0)$.*

5 Examples

Example 5.1. *Let $B = (B_t)_{t\geq 0}$ be Brownian motion, and $T \sim Exp(\lambda)$. Then we find that $B_T \sim \mu_\lambda$, where for $x > 0$*

$$\mu((x,\infty)) = \mu((-\infty,-x)) = \frac{1}{2} e^{-\sqrt{2\lambda} x}.$$

Let us recover the class of consistent diffusions started at $X_0 = 0$ with the same law at an exponential time as Brownian motion. The consistent diffusions have speed measures $m_W(x) = \nu_W(x)dx$, where

$$\nu_W(x) = \frac{e^{-\sqrt{2\lambda}|x|}}{e^{-\sqrt{2\lambda}|x|} - \sqrt{\lambda/2} + 2\lambda/W}.$$

The choice $W = 1/C(0) = 2\sqrt{2\lambda}$ corresponds to Brownian motion. Any choice of $W \in (0, 1/C(0))$ corresponds to a strict local martingale diffusion with the same marginal law.

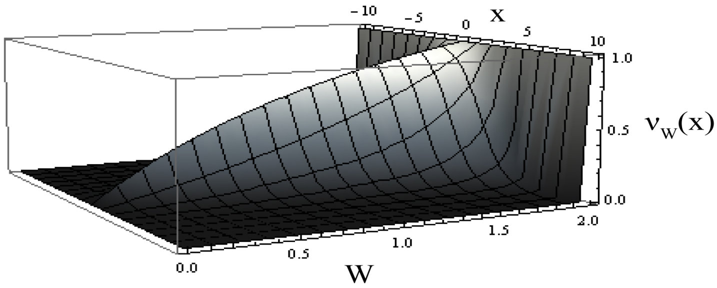

Figure 1: Plot of $\nu_W(x)$ for $\lambda = 1/2$ and $W \in (0,2]$. Note that $\nu_2(x) \equiv 1$ corresponds to Brownian motion which has Wronskian $W = 2\sqrt{2\lambda} = 2$.

Example 5.2. *Suppose that $a = -1, b = 1$ and we wish to recover diffusions started at $X_0 = 0$ that are uniformly distributed at an exponential time. We find that the consistent diffusions are parametrised by $W \in (0,4]$ with corresponding speed measures $m_W(dx) = \nu_W(x)dx$ given by*

$$1/\nu_W(x) = \begin{cases} \lambda(x^2 + 2x + 4/W), & -1 \le x \le 0 \\ \lambda(x^2 - 2x + 4/W), & 0 \le x \le 1 \\ \infty, & \text{otherwise.} \end{cases}$$

The canonical choice for W is $1/W = C(0) = 1/4$. Since $\nu_4(-1) = \nu_4(1) = \infty$ the boundary points are inaccessible whence $X_T^4 \sim U(-1,1)$.

For $W \in (0,4)$, the speed measure is finite on $[-1,1]$. The consistent diffusions reflect at the boundaries and $X_T^W \sim U[-1,1]$.

Now suppose instead that $X_0 = 1/2$. Then

$$1/\nu_W(x) = \begin{cases} \lambda(x^2 + 2x + 1/4 + 4/W), & -1 \le x \le 1/2 \\ \lambda(x^2 - 2x + 9/4 + 4/W), & 1/2 \le x \le 1 \\ \infty, & \text{otherwise.} \end{cases}$$

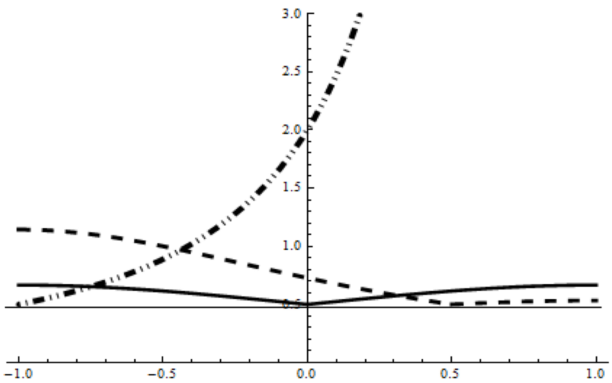

Figure 2: Plot of $\nu_W(x)$ for $\lambda = 1/2$ and $W = 1$ when $X_0 = 0$ (solid line) and $X_0 = 1/2$ (dashed line), and $X_0 = -1$ (alternating line). Note that $\nu_W(X_0) = \frac{W}{4\lambda} = 1/2$, see Corollary 3.5.

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