

On predicting the ultimate maximum for exponential Lévy processes

Katsunori Ano* Roman V. Ivanov†

Abstract

We consider a problem of predicting of the ultimate maximum of the process over a finite interval of time. Mathematically, this problem relates to a particular optimal stopping problem. In this paper we discuss exponential Lévy processes. As the Lévy processes, we discuss α -stable Lévy processes, $0 < \alpha \leq 2$, and generalized hyperbolic Lévy processes. The method of solution uses the representations of these processes as time-changed Brownian motions with drift. Our results generalize results of papers [10] and [24].

Keywords: optimal stopping; exponential Lévy process; predicting; selling of asset; utility function.

AMS MSC 2010: 60G25; 60G51; 60G70.

Submitted to ECP on February 8, 2012, final version accepted on September 28, 2012.

1 Introduction

Throughout this paper we consider optimal stopping problems

$$\sup_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{S_\tau}{\max_{0 \leq t \leq T} S_t} \right) \quad (1.1)$$

and

$$\inf_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{\max_{0 \leq t \leq T} S_t}{S_\tau} \right), \quad (1.2)$$

where process $S = (S_t)_{t \leq T}$, $T < \infty$, is an exponential Lévy process

$$S_t = e^{H_t}.$$

These two problems were discussed primarily in papers [10] and [24] in connection with a problem of optimal stock liquidation. In both papers it is supposed that the stock price S is evaluated as a geometric Brownian motion,

$$dS_t = rS_t dt + \sigma S_t dB_t, \quad S_0 = 1, \quad t \leq T,$$

*Shibaura Institute of Technology, Tokyo, Japan.

E-mail: k-ano@shibaura-it.ac.jp

†Trapeznikov Institute of Control Sciences of RAS, Moscow, Russian Federation.

E-mail: roivanov@yahoo.com

where $B = (B_t)_{t \leq T}$ is a Brownian motion. In [24], the authors consider problem (1.1) in cases $r \geq \sigma^2/2$ and $r \leq 0$. In the first case, the solution of (1.1), i.e. a stopping time $0 \leq \tau^* \leq T$ such that

$$\sup_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{S_\tau}{\max_{0 \leq t \leq T} S_t} \right) = \mathbb{E} \left(\frac{S_{\tau^*}}{\max_{0 \leq t \leq T} S_t} \right),$$

is $\tau^* = T$, and the optimal liquidation strategy for the seller of the asset is "buy and hold". If $r \leq 0$, then $\tau^* = 0$ and the optimal strategy is "stop immediately". The authors of [10] discuss the case $0 < r < \sigma^2/2$ in (1.1) and prove that the solution is $\tau^* = 0$ there. Moreover, they solve (1.1) independently when $r \in [\sigma^2/2, \sigma^2)$ and consider problem (1.2) for all ratios between r and σ deriving that its solution $\tau_* = T$ if $r \geq \sigma^2$, $\tau_* = 0$ if $r \leq 0$ and proving that there exists an increasing boundary function which determines the optimal stopping time if $0 < r < \sigma^2$.

Concerning other works at the same direction, let the authors mention paper [12], where the problem of minimizing of square-quadratic error between a Brownian motion and its ultimate maximum is discussed, papers [4] and [13] in which for the logarithmic utility function and a Brownian motion with randomly changing drift the ultimate maximum of the process is detected, [9], where geometric and arithmetic averages are discussed instead of the maximum in problems (1.1) and (1.2), and works [7] and [8], in which authors solve an infinite time horizon problem of stopping as close as possible to the zero hitting time considering a mean-reverting diffusion process.

As we mentioned above, instead of a geometric Brownian motion, in this paper we discuss exponential Lévy processes, which are very popular as a model of dynamics of assets in mathematical finance (among others, see, for example, recent papers [1], [5], [15], [26] on pricing and hedging theory). Our results correspond to the exponentials of the Lévy processes, both problems (1.1) and (1.2), logarithmic and linear utilities. On empirical tests which support a suggestion that log-returns of financial assets have α -stable or generalized hyperbolic distributions we refer to papers [11], [17], [20].

The paper is constructed as follows. Section 2 is dedicated to α -stable Lévy processes H with drift and problem (1.1) is solved in Section 2 in case of positive drift. All $0 < \alpha \leq 2$ are discussed. Our result extends a result of [24], see Remark 2.1 and Theorem 2.3. In Section 3, we consider a time-changed Brownian motion. The results give full solution of problem (1.1) and extend results on (1.2) (when the optimal stopping time is 0 or T) which are obtained by [10] for geometric Brownian motion. Proofs are set in Section 4. In comparison with a recent work [16], the proofs do not depend on the particular change of time. The proof of Theorem 2.3 is based on well-known time-changed representation of α -stable Lévy processes and properties of an extra function introduced in the proof of Theorem 2.1 from [24]. The proof of Theorem 3.2 is simpler and straightly exploits the results of [10] and [24]. The paper is completed by a list of literature.

2 α -stable Lévy processes

Let $Z = (Z_t)_{t \leq T}$ be a symmetric α -stable Lévy process with characteristic function

$$\varphi_t(\theta) = \mathbb{E} e^{i\theta Z_t} = e^{-t|\theta|^\alpha}, \tag{2.1}$$

where $0 < \alpha < 2$.

If $X^{(\gamma)}$ is a positive random variable with Laplace transform

$$\mathbb{E} e^{-\lambda X^{(\gamma)}} = e^{-\lambda^\gamma}, \quad \lambda > 0,$$

$0 < \gamma < 1$, it is not difficult to prove that

$$Z_t = B_{\tilde{T}(t)}, \quad t \leq T, \quad (2.2)$$

where $\tilde{T}(t)$ an $\alpha/2$ -stable subordinator with

$$Law(\tilde{T}(1)) = Law(X^{(\alpha/2)}). \quad (2.3)$$

Remark 2.1. *The proof of decomposition (2.2) is given e.g. in [23] and consists in straight calculation of the characteristic function of $B_{\tilde{T}(t)}$ which appears to be equivalent to (2.1).*

Throughout this section, we model the price process of the asset S by the exponential Lévy process of the symmetric α -stable Lévy process with drift, i.e.

$$H_t = Z_t + \mu t, \quad \mu \in \mathbb{R}. \quad (2.4)$$

Such defined process H can be used as a model of evolution of log-returns of stock prices, see [20].

Remark 2.2. *Keeping in mind studying of work [24], one could observe that a geometric Brownian motion with $\sigma = 1$ is $S_t = e^{H_t}$, where*

$$H_t = B_t + \left(r - \frac{1}{2}\right)t.$$

Setting $\mu = r - \frac{1}{2}$, we are able to include the 2-stable symmetric Lévy process, a Brownian motion, in framework of (2.4) with $Z_t = B_t$.

Recalling proofs of results for a geometric Brownian motion ([10] and [24]), one could observe that the proofs are based on exploiting the closed form expression of the density of the maximum of the Brownian motion. We are able to obtain the next result on the exponentials of the stable Lévy processes without knowledge of the distribution of their maxima.

Theorem 2.3. *Assume that H is an α -stable symmetric Lévy process with $0 < \alpha \leq 2$ and drift μ . If $\mu \geq 0$, the solution of (1.1) is time T .*

Example 2.4. *If $\alpha = 2$, the 2-stable symmetric Lévy process is a Brownian motion $B = (B_t)_{t \leq T}$. As it is mentioned above, if the price of the asset is supposed to be a geometric Brownian motion, i.e.*

$$S_t = e^{\mu t + B_t}, \quad (2.5)$$

it was established (see [10] and [24]) that for $\mu \leq 0$ the optimal stopping moment is $\tau^* = 0$ and for $\mu \geq 0$ $\tau^* = T$ in (1.1). Therefore, the result of Theorem 2.3 extends that result of [10] and [24] if $\mu \geq 0$.

3 Time-changed Brownian motion

Let $H = (H_t)_{t \leq T}$ be a time-changed Brownian motion with drift, i.e.

$$H_t = \beta\gamma(t) + \sigma B_{\gamma(t)}, \tag{3.1}$$

where $\beta \in \mathbb{R}$, $\sigma > 0$ and random change of time (in sense of definition (a)–(b), p.109, [22]) γ is independent of B and satisfies condition

$$P(\gamma(T) < \infty) = 1.$$

Remark 3.1. Such defined time-changed Brownian motion determines a pure discontinuous Lévy process in many cases. This fact is obtained by straightforward calculation of its characteristic function for a particular change of time, see paper [2] and monograph [23] on generalized hyperbolic processes, and examples below. The proof of the next theorem is based on (3.1) and the results of [10] and [24].

Theorem 3.2. The solution of (1.1) is $\tau^* = T$ if $\beta \geq 0$ and $\tau^* = 0$ if $\beta \leq 0$. For problem (1.2), solution $\tau_* = T$ if $\beta \geq \sigma^2/2$ and $\tau_* = 0$ if $\beta \leq -\sigma^2/2$.

Example 3.3. Normal-inverse Gaussian process. A normal-inverse Gaussian distribution (NIG), introduced in [2] (see also [3] and [23]), is a normal variance-mean mixture where the mixing density is an independent inverse Gaussian distribution, i.e. the NIG random variable $H = H(\alpha, \beta, \delta)$ is defined as

$$H = \beta X + \sqrt{X} N,$$

where N is normally distributed and the density of X is

$$p_X(x) = \sqrt{\frac{b}{2\pi}} e^{\sqrt{ab}} \frac{1}{x^{3/2}} \exp\left(-\frac{1}{2}\left(ax + \frac{b}{x}\right)\right),$$

where $a = \alpha^2 - \beta^2$, $b = \delta^2$. Parameters α, β, δ are suggested to satisfy conditions

$$\alpha > 0, \quad 0 \leq |\beta| < \alpha, \quad \delta \geq 0.$$

The density function f of H is

$$f(x) = \frac{\alpha \delta K_1(\alpha \sqrt{\delta^2 + x^2})}{\pi \sqrt{\delta^2 + x^2}} e^{\delta \sqrt{\alpha^2 - \beta^2 + \beta x}}, \tag{3.2}$$

where K_1 is modified Bessel function of the second type. The NIG process $(H_t)_{t \geq 0}$ is defined as a Lévy process such that H_1 has density (3.2).

It is known, see for details [23], that for a Brownian motion $\tilde{B} = (\tilde{B}_s)_{s \geq 0}$, a change of time

$$\tilde{T}(t) = \inf\{s > 0 : \tilde{B}_s + \sqrt{as} \geq \sqrt{bt}\}$$

and an independent Brownian motion $B = (B_t)_{t \geq 0}$ process H_t can be represented in form

$$H_t = B_{\tilde{T}(t)} + \beta \tilde{T}(t).$$

Therefore, solutions of (1.1) and (1.2) for a NIG process do not depend on parameters α and δ , due to Theorem 3.2.

Example 3.4. *Variance-gamma process.* A variance-gamma (VG) process $Y = (Y_t)_{t \leq T}$ can be written (see e. g. [19]) as a time-changed Brownian motion $B = (B_t)_{t \geq 0}$, where the random time change follows a gamma process with unit mean $\Gamma(t; 1, \nu)$, $\nu > 0$, i.e.

$$Y_t = \beta\Gamma(t; 1, \nu) + \sigma B_{\Gamma(t; 1, \nu)}.$$

Despite the fact that parameters $\beta \in \mathbb{R}$, $\sigma > 0$ and ν reflect only indirectly such parameters of the VG distribution as variance, skewness and kurtosis (it can be shown by straightforward calculation of moments of Y), we immediately use such parametrization of the VG process as above since it is usually used in literature, see [14], [17], [19]).

Together with the NIG process, the VG process is often used as a model of distribution of market returns. The symmetric VG distribution was primarily studied in [17] and [18]. In [19], the general case of VG process with application to option pricing was discussed. For further investigations on the VG process, see [14] and [25].

Solutions of (1.1) and (1.2) for a VG process are determined by conditions of Theorem 3.2 on β and σ .

Remark 3.5. *In case of logarithmic utility, problems (1.1) and (1.2) can be rewritten as*

$$\sup_{0 \leq \tau \leq T} \mathbb{E}(H_\tau - \bar{H}_T)^q \quad \text{and} \quad \inf_{0 \leq \tau \leq T} \mathbb{E}(\bar{H}_T - H_\tau)^q,$$

respectively, with $q = 1$. For $q = 2$ these problems were discussed in [12] for a Brownian motion. Their result was extended to all $q > 0$ by [21]. For $q = 1$ and a Brownian motion with spontaneously changing drift, see [4] and [13].

Assume that H is a Lévy process which has decomposition

$$H_t = \mu t + \beta\varphi(t) + \sigma B_{\varphi(t)}, \tag{3.3}$$

where $\mu \in \mathbb{R}$, $\beta \in \mathbb{R}$, $\sigma > 0$ and stochastic change of time (in the sense of definition (a)–(b), p.109, [22]) φ satisfies condition

$$\mathbb{E}\sqrt{\varphi(T)} < \infty.$$

Since H is a Lévy process (3.3), it is submartingale if $\mathbb{E}H_t \geq 0$ and it is supermartingale if $\mathbb{E}H_t \leq 0$. Keeping in mind Hunt's stopping time theorem ((A.2), p.60, [22]) and Wald identity ((3.2.5), p.61, [22]), we conclude that solution of both problems (1.1) and (1.2) for logarithmic utility here is

$$\tau^* = \tau_* = T \quad \text{if } \mu \geq -\beta\mathbb{E}\varphi(1) \quad \text{and} \quad \tau^* = \tau_* = 0 \quad \text{if } \mu \leq -\beta\mathbb{E}\varphi(1).$$

In particular, for the VG process the solutions are time T if $\mu \geq -\beta$ and time 0 if $\mu \leq -\beta$.

4 Proofs

Proof of Theorem 2.3. Set for $t \geq 0$

$$\bar{H}_t = \max_{0 \leq u \leq t} H_u \quad \text{and} \quad \bar{S}_t = \max_{0 \leq u \leq t} S_u = e^{\bar{H}_t}. \tag{4.1}$$

Then problem (1.1) can be rewritten as

$$\sup_{0 \leq \tau \leq T} \mathbb{E}(S_\tau / \bar{S}_T).$$

Predicting the ultimate maximum

Due to (2.2) and (2.4) for any $\tau \leq T$

$$E(S_\tau/\bar{S}_T) = E\left(E\left(\frac{S_\tau}{\bar{S}_T} \middle| \tilde{T}(t), t \leq T\right)\right) \quad (4.2)$$

and

$$\begin{aligned} E\left(\frac{S_\tau}{\bar{S}_T} \middle| \tilde{T}(t), t \leq T\right) &= E\left(\exp(H_\tau - \bar{H}_T) \middle| \tilde{T}(t), t \leq T\right) = \\ &E\left(\exp\left(B_{\tilde{T}(\tau)} + \mu\tau - \max_{t \leq T} (B_{\tilde{T}(t)} + \mu t)\right) \middle| \tilde{T}(t), t \leq T\right). \end{aligned}$$

Observe that

$$\begin{aligned} &E\left(\exp\left(B_{\tilde{T}(\tau)} + \mu\tau - \max_{t \leq T} (B_{\tilde{T}(t)} + \mu t)\right) \middle| \tilde{T}(t), t \leq T\right) = \\ &E\left(\min\left\{\exp\left(B_{\tilde{T}(\tau)} + \mu\tau - \max_{t \leq \tau} (B_{\tilde{T}(t)} + \mu t)\right), \right. \right. \\ &\left. \left. \exp\left(-\max_{\tau \leq t \leq T} (B_{\tilde{T}(t)} + \mu t - B_{\tilde{T}(\tau)} - \mu\tau)\right)\right\} \middle| \tilde{T}(t), t \leq T\right). \end{aligned} \quad (4.3)$$

Set

$$\begin{aligned} G^\mu(s, x) &= E\left(\min\left\{\exp(-x), \right. \right. \\ &\left. \left. \exp\left(-\max_{s \leq t \leq T} (B_{\tilde{T}(t)} + \mu t - B_{\tilde{T}(s)} - \mu s)\right)\right\} \middle| \tilde{T}(t), t \leq T\right). \end{aligned} \quad (4.4)$$

Then we get from (4.3) and (4.4) that

$$\begin{aligned} &E\left(\exp(H_\tau - \bar{H}_T) \middle| \tilde{T}(t), t \leq T\right) = \\ &E\left(G^\mu\left(\tau, \max_{t \leq \tau} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(\tau)} - \mu\tau\right) \middle| \tilde{T}(t), t \leq T\right), \end{aligned} \quad (4.5)$$

and, clearly, at the same path $(\tilde{T}(t))_{t \leq T}$

$$G^\mu(s, x) \leq G(s, x),$$

where we imply that

$$\begin{aligned} &G(s, x) = \\ &E\left(\min\left\{\exp(-x), \exp\left(-\max_{s \leq t \leq T} (B_{\tilde{T}(t)} - B_{\tilde{T}(s)})\right)\right\} \middle| \tilde{T}(t), t \leq T\right). \end{aligned}$$

Next,

$$\begin{aligned} &G^\mu\left(T, \max_{t \leq T} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(T)} - \mu T\right) = \\ &\min\left\{\exp\left(B_{\tilde{T}(T)} + \mu T - \max_{t \leq T} (B_{\tilde{T}(t)} + \mu t)\right), 1\right\} \geq \\ &G\left(T, \max_{t \leq T} B_{\tilde{T}(t)} - B_{\tilde{T}(T)}\right), \end{aligned}$$

and hence

$$\begin{aligned} & \mathbb{E} \left\{ \mathbb{E} \left[G^\mu \left(T, \max_{t \leq T} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(T)} - \mu T \right) \right] \right. \\ & \quad \left. \max_{t \leq s} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(s)} - \mu s = x \right] - \\ & \quad \mathbb{E} \left[G \left(T, \max_{t \leq T} B_{\tilde{T}(t)} - B_{\tilde{T}(T)} \right) \right] \Big| \\ & \quad \left. \max_{t \leq s} B_{\tilde{T}(t)} - B_{\tilde{T}(s)} = x \right] \Big| \tilde{T}(u), u \leq T \Big\} \geq 0. \end{aligned}$$

It can be proved in the way of the proof of Proposition 5.1 from [24] that at the same path $(\tilde{T}(t))_{t \leq T}$

$$\begin{aligned} & \mathbb{E} \left\{ \left[G \left(T, \max_{t \leq T} B_{\tilde{T}(t)} - B_{\tilde{T}(T)} \right) \right] \Big| \max_{t \leq s} B_{\tilde{T}(t)} - B_{\tilde{T}(s)} = x \right] \Big| \tilde{T}(u), u \leq T \Big\} \\ & \geq G(s, x), \end{aligned}$$

and hence

$$\begin{aligned} & \mathbb{E} \left\{ \mathbb{E} \left[G^\mu \left(T, \max_{t \leq T} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(T)} - \mu T \right) \right] \right. \\ & \quad \left. \max_{t \leq s} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(s)} - \mu s = x \right] \Big| \tilde{T}(u), u \leq T \Big\} \geq G^\mu(s, x). \end{aligned} \tag{4.6}$$

Keeping in mind arbitrariness of s and x , we conclude from (4.6) that

$$\begin{aligned} & \mathbb{E} \left\{ \mathbb{E} \left[G^\mu \left(T, \max_{t \leq T} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(T)} - \mu T \right) \right] \right. \\ & \quad \left. \max_{t \leq \tau} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(\tau)} - \mu \tau \right] - \\ & \quad \left. G^\mu \left(\tau, \max_{t \leq \tau} (B_{\tilde{T}(t)} + \mu t) - B_{\tilde{T}(\tau)} - \mu \tau \right) \right] \Big| \tilde{T}(u), u \leq T \Big\} \geq 0, \end{aligned}$$

and taking into account (4.2) and (4.5), we conclude that $\tau^* = T$ is optimal. \square

Proof of Theorem 3.2. Let us consider the case of problem (1.2) and $\beta \leq -\sigma^2/2$. In designations (4.1) problem (1.2) has form

$$\inf_{0 \leq \tau \leq T} \mathbb{E}(\bar{S}_T/S_\tau). \tag{4.7}$$

Then because of (3.1) we have that

$$\inf_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{\bar{S}_T}{S_\tau} \right) \geq \mathbb{E} \left(\inf_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{\bar{S}_T}{S_\tau} \Big| \gamma(t), t \leq T \right) \right), \tag{4.8}$$

and

$$\begin{aligned} & \mathbb{E} \left(\frac{\bar{S}_T}{S_\tau} \Big| \gamma(t), t \leq T \right) = \\ & \mathbb{E} \left[\exp \left(\max_{t \leq T} (\beta \gamma(t) + \sigma B_{\gamma(t)}) - \beta \gamma(\tau) + \sigma B_{\gamma(\tau)} \right) \Big| \gamma(t), t \leq T \right]. \end{aligned}$$

Notice that $\gamma(\tau)$ is a stopping time again, due to definition of change of time (see p.109, [22])). Hence, if $\beta \leq -\sigma^2/2$, we have similarly to results of [10] that for any $0 \leq \tau \leq T$

$$\begin{aligned} & \mathbb{E} \left[\exp \left(\max_{t \leq T} (\beta\gamma(t) + \sigma B_{\gamma(t)}) - \beta\gamma(\tau) + \sigma B_{\gamma(\tau)} \right) - \right. \\ & \left. \exp \left(\max_{t \leq T} (\beta\gamma(t) + \sigma B_{\gamma(t)}) \right) \middle| \gamma(t), t \leq T \right] \geq 0, \end{aligned}$$

and therefore

$$\mathbb{E} \left(\inf_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{\bar{S}_T}{S_\tau} \middle| \gamma(t), t \leq T \right) \right) \geq \mathbb{E} \bar{S}_T \geq \inf_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{\bar{S}_T}{S_\tau} \right). \quad (4.9)$$

It immediately follows from (4.8) and (4.9) that $\tau_* = 0$ is optimal in (4.7).

In other cases we have similar proofs. For example, in case of $\beta \geq 0$ to problem (1.1) we use inequalities

$$\sup_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{S_\tau}{S_T} \right) \leq \mathbb{E} \left(\sup_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{S_\tau}{S_T} \middle| \gamma(t), t \leq T \right) \right)$$

and

$$\mathbb{E} \left(\sup_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{S_\tau}{S_T} \middle| \gamma(t), t \leq T \right) \right) \leq \mathbb{E} \frac{S_T}{S_T} \leq \sup_{0 \leq \tau \leq T} \mathbb{E} \left(\frac{S_\tau}{S_T} \right)$$

instead of (4.8) and (4.9). \square

Remark 4.1. *One can easily observe that the method of the proof of Theorem 3.2 is not suitable for Theorem 2.3. Indeed, it tends to problem (1.1) for Brownian motion with time-dependent drift instead of Brownian motion with constant drift, and there are no results in the way of results of papers [10] and [24] for it.*

References

- [1] Ano, K.: Multiple Stopping Problem for Jump Diffusion and Free-Boundary Problem (Financial Modeling and Analysis). *RIMS Kokyuroku* **1675**, (2010), 42–48.
- [2] Barndorff-Nielsen, O. E.: Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society. London. Ser. A. V* **353**, (1977), 401–419.
- [3] Barndorff-Nielsen, O. E.: Processes of normal inverse Gaussian type. *Finance and Stoch.* **2(1)**, (1998), 41–68. MR-1804664
- [4] Beibel, M., Lerche, H. R.: A New Look at Optimal Stopping Problems related to Mathematical Finance. *Statistica Sinica* **7**, (1997), 93–108. MR-1441146
- [5] Carr, P., Lee, R., Wu, L.: Variance swaps on time-changed Lévy processes. Accepted in *Finance Stoch.*, (2011), 21 p.
- [6] Dai, M., Jin, H., Zhong, Y., Zhou, X.: Buy low and sell high. *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen, C. Chiarella and A. Novikov Editors*, (2010) 317–334. MR-2732852
- [7] Elie, R., Espinosa, G. E.: Optimal stopping of a mean reverting diffusion: minimizing the relative distance to the maximum. (2011), preprint.
- [8] Espinosa, G. E., Touzi, N.: Detecting the maximum of a mean reverting scalar diffusion. (2010), preprint.
- [9] Dai, M., Zhong, Y.: Optimal Stock Selling/Buying with reference to the Ultimate Average. To appear in *Mathem. Finance*, (2009).

Predicting the ultimate maximum

- [10] du Toit, J., Peskir, G.: Selling a stock at the ultimate maximum. *Ann. Appl. Probab.* **19**, (2009), 983–1014. MR-2537196
- [11] Eberlein, E., Keller, U.: Hyperbolic Distribution in Finance. *Bernoulli*, **1(3)**, (1995), 281–299.
- [12] Graversen, S., Peskir, G., Shiryaev, A.N.: Stopping Brownian Motion without Anticipation as Close as Possible to its Ultimate Maximum. *Theor. Prob. Appl.* **45(1)**, (2001), 125–136. MR-1810977
- [13] Novikov, A., Shiryaev, A.N.: On a stochastic version of the trading rule “Buy and Hold”. *Statistics and Decisions* **26(4)**, (2009), 289–302.
- [14] Hirta, A., Madan, D.: Pricing American Options Under Variance Gamma. *Journal of Computational Finance* **7(2)**, (2003), 63–80.
- [15] Ivanov, R.V.: On the pricing of American options in exponential Lévy markets. *J. Appl. Probab.* **44(2)**, (2007), 409–419. MR-2340207
- [16] Ivanov, R.V.: On selling a stock at the ultimate maximum for the exponential of the normal-inverse Gaussian process. (2011), submitted in *Stochastics*.
- [17] Madan, D., Seneta, E.: The Variance Gamma (V.G.) Model for Share Market Returns. *Journal of Business* **63**, (1990), 511–524.
- [18] Madan, D., Milne, F.: Option pricing with VG martingale components. *Mathem. Finance* **1(4)**, (1991), 39–55.
- [19] Madan, D., Carr, P., Chang, E.: The Variance Gamma Process and Option Pricing. *European Finance Review* **2**, (1998), 79–105.
- [20] Mandelbrot, B., Taylor, H.: On the distribution of stock price difference. *Operations Research* **15(6)**, (1967), 1057–1062.
- [21] Pedersen, J. L.: Optimal Prediction of the Ultimate Maximum of Brownian Motion. *Stoch. Stoch. Reports* **75**, (2003), 205–219. MR-1994906
- [22] Peskir, G., Shiryaev, A.N.: Optimal Stopping and Free-Boundary Problems. *Birkhäuser*, (2006), Basel – Boston – Berlin. MR-2256030
- [23] Shiryaev, A.N.: Essentials of Stochastic Finance. Facts, Models, Theory. *World Scientific*, (1999), River Edge. MR-1695318
- [24] Shiryaev, A.N., Xu, Z., Zhou, X.Y.: Thou Shalt Buy and Hold. *Quant. Finance* **8**, (2008), 765–776. MR-2488734
- [25] Seneta, E.: The Early Years of the Variance–Gamma Process. In *M. Fu, R. Jarrow, J.-Y. Yen and R. Elliott eds. Advances in Mathematical Finance*, Birkhäuser, (2000), Boston. MR-2359359
- [26] Yip, W.Y., Stephens, D., Olhede, S.: Hedging Strategies and Minimal Variance Portfolios for European and Exotic Options in a Lévy Market. *Mathem. Finance* **20(4)**, (2010), 617–646. MR-2731410

Acknowledgments. The authors would like to thank professor A.N. Shiryaev and the anonymous referee for valuable remarks.