

# MAXIMA OF THE CELLS OF AN EQUIPROBABLE MULTINOMIAL

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## Abstract

Consider a sequence of multinomial random vectors with increasing number of equiprobable cells. We show that if the number of trials increases fast enough, the sequence of maxima of the cells after a suitable centering and scaling converges to the Gumbel distribution. While results are available for maxima of triangular arrays of independent random variables with certain types of distribution, such results in a dependent setup is new. We also prove that the maxima of a triangular sequence of appropriate Binomial random variables have the same limit distribution. An auxiliary large deviation result for multinomial distribution with increasing number of equiprobable cells may also be of independent interest.

## 1 Introduction and main result

Let  $(Y_{1n}, \dots, Y_{m_n n})_{n \geq 1}$  be a triangular sequence of random variables. Define the row maximum as  $M_n = \max\{Y_{1n}, \dots, Y_{m_n n}\}$ . The question of convergence in distribution of  $M_n$  with linear normalization has been addressed under a variety of conditions.

The classical case is when there is one sequence of i.i.d. random variables  $\{Y_i\}$  and  $M_n = \max\{Y_1, \dots, Y_n\}$ . In this case, necessary and sufficient conditions for the convergence are known. See for example, Fisher and Tippett (1928), Gnedenko (1943), de Haan (1970). In particular, it follows from these results that if  $\{Y_i\}$  are i.i.d. Poisson or i.i.d. binomial with fixed parameters, then  $M_n$  cannot converge to any non degenerate distribution under any linear normalization (cf. Leadbetter et al., 1983, pp 24–27). On the other hand (cf. Leadbetter et al.,

1983, Theorem 1.5.3), if  $Y_i$  are i.i.d. standard normal variables then

$$\lim_{n \rightarrow \infty} P[M_n \leq \alpha_n x + \beta_n] = \exp(-e^{-x}),$$

where

$$\alpha_n = \frac{1}{\sqrt{2 \log n}} \quad (1.1)$$

and

$$\beta_n = \sqrt{2 \log n} - \frac{\log \log n + \log(4\pi)}{2\sqrt{2 \log n}}. \quad (1.2)$$

General triangular schemes under various suitable conditions have been considered by several authors. The classical large deviation results due to Cramér (cf. Petrov, 1975, pg 218) play an important role in the proofs of these results.

Consider, for example, the case where  $Y_{m_n n} = (\sum_{1 \leq j \leq m_n} U_j - m_n \mu) / (\sigma m_n^{1/2})$  and  $U_j$  are i.i.d. with mean  $\mu$  and standard deviation  $\sigma$ . Assuming that  $U_j$  has a finite moment generating function in an open interval containing the origin and  $\log n = o(m_n^{(R+1)/(R+3)})$  for some integer  $R \geq 0$ , Anderson et al. (1997) showed that

$$\lim_{n \rightarrow \infty} P[M_n \leq \alpha_n x + \beta_n^{(R)}] = \exp(-e^{-x})$$

for  $\alpha_n$  as in (1.1) and some suitable sequences  $\beta_n^{(R)}$ .

They also consider the following case. Suppose  $m_n = n$  and for each  $n$ ,  $Y_{m_n n}$ , are independent Poisson with mean  $\lambda_n$  such that for some integer  $R \geq 0$ ,  $\log n = o(\lambda_n^{(R+1)/(R+3)})$ . Then again

$$\lim_{n \rightarrow \infty} P[M_n \leq \lambda_n + \lambda_n^{1/2}(\beta_n^{(R)} + \alpha_n x)] = \exp(-e^{-x}),$$

where  $\alpha_n$  and  $\beta_n^{(R)}$  are as before. In particular, in the above results, if  $R = 0$  then we can choose  $\alpha_n$  as in (1.1) and  $\beta_n^{(0)} = \beta_n$ , given by (1.2).

Nadarajah and Mitov (2002) consider the maximum of a triangular array of binomial, negative binomial and discrete uniform. The case of binomial triangular array is discussed with increasing number of trials  $m_n$  and fixed probability of success,  $p$ . The idea of the proof in this case is again similar to that of Anderson et al. (1997) and uses a large deviation result for binomial distribution.

In this paper we consider the following dependent situation. Suppose  $\mathbf{Y}_n = (Y_{1n}, \dots, Y_{nn})$  follow multinomial  $(m_n; 1/n, \dots, 1/n)$  distribution and define  $M_n = \max_{1 \leq i \leq n} Y_{in}$  to be the maximum of the  $n$  cell variables. If  $m_n$  tends to infinity fast enough, then the sequence  $M_n$  after a suitable linear normalization, converges to the Gumbel distribution. We summarize this result in the following theorem:

**Theorem 1.1.** *Suppose that  $\mathbf{Y}_n$  is distributed as multinomial  $(m_n; \frac{1}{n}, \dots, \frac{1}{n})$  and define as before  $M_n = \max_{1 \leq i \leq n} Y_{in}$ . If*

$$\lim_{n \rightarrow \infty} \frac{\log n}{m_n/n} = 0 \quad (1.3)$$

*holds, then, for  $x \in \mathbb{R}$ ,*

$$P \left[ \frac{M_n - (m_n/n) - \beta_n \sqrt{m_n/n}}{\alpha_n \sqrt{m_n/n}} \leq x \right] \rightarrow \exp(-e^{-x}), \quad (1.4)$$

where  $\alpha_n$  is as in (1.1) and  $\beta_n$  is the unique solution of

$$\log z + \frac{1}{2}z^2 + \frac{1}{2}\log(2\pi) - z^2 B\left(\frac{z}{\sqrt{m_n/n}}\right) = \log n \quad (1.5)$$

in the region  $\beta_n \sim \sqrt{2\log n}$ , where

$$B(z) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(i+1)(i+2)} z^i. \quad (1.6)$$

A similar result for the maximum of i.i.d. Binomial random variables is given below. Unlike Theorem 3 of Nadarajah and Mitov (2002), we do not require the probability of success to be constant.

**Proposition 1.1.** *Let  $\{Y_{in} : 1 \leq i \leq n, n \geq 1\}$  be a triangular array of independent Binomial random variables, with  $\{Y_{in} : 1 \leq i \leq n\}$  having i.i.d. Binomial  $(m_n; p_n)$  distribution for each  $n \geq 1$ . Define  $M_n = \max_{1 \leq i \leq n} Y_{in}$ . If we have*

$$\lim_{n \rightarrow \infty} \frac{\log n}{m_n p_n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} p_n (\log n)^r = 0,$$

for all  $r > 0$ , then we have

$$P\left[\frac{M_n - (m_n p_n) - \beta_n \sqrt{m_n p_n}}{\alpha_n \sqrt{m_n p_n}} \leq x\right] \rightarrow \exp(-e^{-x}),$$

where  $\alpha_n$  and  $\beta_n$  are chosen as in Theorem 1.1.

The large deviation results used by Anderson et al. (1997) or Nadarajah and Mitov (2002) are not directly applicable in our case. For our case, even though the random variables in each row of the array can be written as sum of independent variables, the distributions of the summands depend on the row. Our proof of the theorem is based on the following large deviation result. As we are unable to locate this particular large deviation result in the existing literature, we provide a detailed derivation in the next section.

**Theorem 1.2.** *Suppose  $\mathbf{Y}_n$  be distributed as multinomial  $(m_n; \frac{1}{n}, \dots, \frac{1}{n})$ , such that the condition (1.3)*

$$\lim_{n \rightarrow \infty} \frac{\log n}{m_n/n} = 0$$

holds. For any positive integer  $k$  and any sequence

$$v_n \sim \sqrt{2\log n}, \quad (1.7)$$

we have

$$P\left[\frac{\min_{1 \leq i \leq k} Y_{in} - m_n/n}{\sqrt{m_n/n}} > v_n\right] \sim \left[(1 - \Phi(v_n)) \exp\left(v_n^2 B\left(\frac{v_n}{\sqrt{m_n/n}}\right)\right)\right]^k, \quad (1.8)$$

where  $B(z)$  is given by (1.6) and  $\Phi$  is the univariate standard normal distribution function.

## 2 Proofs

For a real number  $x$ , denote

$$y_n = x\alpha_n\sqrt{m_n/n} + \beta_n\sqrt{m_n/n} + (m_n/n). \quad (2.1)$$

We prove Theorem 1.1 using the following lemma.

**Lemma 2.1.** *For each fixed  $k$ , and real number  $x$ , we have*

$$n^k P(\cap_{i=1}^k \{Y_{in} > y_n\}) \rightarrow e^{-kx}. \quad (2.2)$$

where  $y_n$  and  $x$  are related as in (2.1).

*Proof of Theorem 1.1.* For any fixed  $l$ , for sufficiently large  $n$ , using inclusion-exclusion principle and the identical distribution of the marginals from the multinomial distribution, we have,

$$\begin{aligned} & 1 - \sum_{k=1}^{2l-1} (-1)^{k+1} \frac{n(n-1)\cdots(n-k+1)}{k!} P(\cap_{i=1}^k \{Y_{in} > y_n\}) \\ & \leq P(\cap_{i=1}^n \{Y_{in} \leq y_n\}) \\ & \leq 1 - \sum_{k=1}^{2l} (-1)^{k+1} \frac{n(n-1)\cdots(n-k+1)}{k!} P(\cap_{i=1}^k \{Y_{in} > y_n\}). \end{aligned} \quad (2.3)$$

Hence using Lemma 2.1, we obtain from (2.2) and (2.3), for each fixed  $l$ ,

$$\begin{aligned} 1 - \sum_{k=1}^{2l-1} (-1)^{k+1} \frac{e^{-kx}}{k!} & \leq \liminf_{n \rightarrow \infty} P(\cap_{i=1}^n \{Y_{in} \leq y_n\}) \\ & \leq \limsup_{n \rightarrow \infty} P(\cap_{i=1}^n \{Y_{in} \leq y_n\}) \leq 1 - \sum_{k=1}^{2l} (-1)^{k+1} \frac{e^{-kx}}{k!}, \end{aligned}$$

which gives the desired result (1.4) since  $l$  is arbitrary.  $\square$

*Remark 2.1.* As pointed out by the referee, it can be easily seen, using negative dependence, that  $P(Y_{1n} \leq y_n)^n$  is another choice of the upper bound in (2.3) and, hence,

$$\limsup_{n \rightarrow \infty} P(\cap_{i=1}^n \{Y_{in} \leq y_n\}) \leq \lim_{n \rightarrow \infty} \exp(-nP(Y_{1n} > y_n)) = \exp(-e^{-x}).$$

However, there appears to be no easy way to obtain an appropriate lower bound from the existing literature.

Now we prove Lemma 2.1 using Theorem 1.2.

*Proof of Lemma 2.1.* Modifying Lemmas 1 and 2 of Anderson et al. (1997), we can find  $\alpha_n$  and  $\beta_n$ , so that, for  $x_n = \alpha_n x + \beta_n$ , we have

$$\log x_n + \frac{1}{2} \log(2\pi) + \frac{1}{2} x_n^2 - x_n^2 B\left(\frac{x_n}{\sqrt{m_n/n}}\right) - \log n \rightarrow x. \quad (2.4)$$

Note that the referred lemmas require a polynomial instead of a power series in the defining equation. However, the proofs work verbatim in our case due to the specific form of the coefficients. Also using (16) and (17) of the same reference, we have  $\alpha_n \sim 1/\sqrt{2\log n}$  and  $\beta_n$  is the unique solution of (1.5) satisfying  $\beta_n \sim \sqrt{2\log n}$ . Note that

$$x_n = \frac{y_n - m_n/n}{\sqrt{m_n/n}} = \alpha_n x + \beta_n \sim \sqrt{2\log n}.$$

Thus, using (1.8), we have,

$$\begin{aligned} n^k P [\cap_{i=1}^k Y_{in} > y_n] &= n^k P \left[ \frac{\min_{1 \leq i \leq k} Y_{in} - m_n/n}{\sqrt{m_n/n}} > x_n \right] \\ &\sim n^k \left[ (1 - \Phi(x_n)) \exp \left( x_n^2 B \left( \frac{x_n}{\sqrt{m_n/n}} \right) \right) \right]^k. \end{aligned}$$

Hence, using

$$1 - \Phi(t) \sim \exp(-t^2/2)/(t\sqrt{2\pi}) \text{ as } t \rightarrow \infty, \quad (2.5)$$

(cf. Feller, 1968, Lemma 2, Chapter VII), we have,

$$n^k P [\cap_{i=1}^k Y_{in} > y_n] \sim e^{-k \left[ \log x_n + \frac{1}{2} \log(2\pi) + \frac{1}{2} x_n^2 - x_n^2 B \left( \frac{x_n}{\sqrt{m_n/n}} \right) - \log n \right]} \rightarrow e^{-kx}.$$

The last step follows from (2.4).  $\square$

Now we prove the large deviation result given in Theorem 1.2.

*Proof of Theorem 1.2.* Let us consider a random vector  $(Z_0, Z_1, \dots, Z_k)$ , which has multinomial  $(1; \frac{n-k}{n}, \frac{1}{n}, \dots, \frac{1}{n})$  distribution. Denote by  $F_n$  the distribution of  $(Z_1 - \frac{1}{n}, \dots, Z_k - \frac{1}{n})$ . Note that  $F_n$  has mean vector  $\mathbf{0}$  and its covariance matrix is given by  $((a_{ij}))$ ,  $a_{ii} = 1/n - 1/n^2$ ,  $a_{ij} = -1/n^2$ ,  $i \neq j$ . Let  $\mathbf{U}_n^{(i)} = (U_{1n}^{(i)}, \dots, U_{kn}^{(i)})$ ,  $1 \leq i \leq m_n$ , be i.i.d.  $F_n$ . Define  $\mathbf{X}_n = (X_{1n}, \dots, X_{kn}) = \sum_{i=1}^{m_n} \mathbf{U}_n^{(i)}$ . We apply Esscher transform or exponential tilting on the distribution of  $\mathbf{X}_n$ . Let  $\Psi_n(t_1, \dots, t_k)$  be the cumulant generating function of  $F_n$ :

$$\Psi_n(t_1, \dots, t_k) = -\frac{t_1 + \dots + t_k}{n} + \log \left( 1 + \frac{e^{t_1} + \dots + e^{t_k} - k}{n} \right). \quad (2.6)$$

Let  $s_n$  be the unique solution of

$$m_n \partial_1 \Psi_n(s, \dots, s) = v_n \sqrt{m_n/n}. \quad (2.7)$$

Next we define the exponential tilting for the multivariate case as

$$dV_n(w_1, \dots, w_k) = e^{-\Psi_n(s_n, \dots, s_n)} e^{s_n(w_1 + \dots + w_k)} dF_n(w_1, \dots, w_k). \quad (2.8)$$

Then, the  $m_n$ -th convolution power of  $V_n$  is given by

$$dV_n^{*m_n}(w_1, \dots, w_k) = e^{-m_n \Psi_n(s_n, \dots, s_n)} e^{s_n(w_1 + \dots + w_k)} dF_n^{*m_n}(w_1, \dots, w_k).$$

Denote

$$u_n = e^{s_n} - 1. \quad (2.9)$$

Note that  $V_n$  has mean vector  $\mu_n \mathbf{1}_k$  and covariance matrix  $\Sigma_n = a_n I_k - b_n J_k$ , where  $\mathbf{1}_k$  is the  $k$ -vector with all coordinates 1,  $I_k$  is the  $k \times k$  identity matrix,  $J_k$  is the  $k \times k$  matrix with all entries 1 and  $\mu_n$ ,  $a_n$  and  $b_n$  are given as follows:

$$\begin{aligned} \mu_n &= \partial_1 \Psi_n(s_n, \dots, s_n) = -\frac{1}{n} + \frac{e^{s_n}}{n + k(e^{s_n} - 1)} \\ &= \frac{(n - k)(e^{s_n} - 1)}{n(n + k(e^{s_n} - 1))} = \frac{(n - k)u_n}{n(n + ku_n)}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} b_n &= -\partial_1 \partial_2 \Psi_n(s_n, \dots, s_n) = \frac{e^{2s_n}}{(n + k(e^{s_n} - 1))^2} \\ &= \left( \frac{1 + u_n}{n + ku_n} \right)^2, \end{aligned} \quad (2.11)$$

$$\begin{aligned} \tau_n^2 := a_n - b_n &= \partial_1^2 \Psi_n(s_n, \dots, s_n) = \frac{e^{s_n}(n - k + (k - 1)e^{s_n})}{(n + k(e^{s_n} - 1))^2} \\ &= \frac{(1 + u_n)(n - k + (k - 1)(1 + u_n))}{(n + ku_n)^2}, \end{aligned} \quad (2.12)$$

and we also denote

$$\gamma_n = \Psi_n(s_n, \dots, s_n). \quad (2.13)$$

With notations as above, the required probability becomes

$$\begin{aligned} P_n &= P \left[ \frac{\min_{1 \leq i \leq k} Y_{in} - m_n/n}{\sqrt{m_n/n}} > v_n \right] \\ &= P[X_{1n} > v_n \sqrt{m_n/n}, \dots, X_{kn} > v_n \sqrt{m_n/n}] \\ &= \int_{v_n \sqrt{m_n/n}}^{\infty} \dots \int_{v_n \sqrt{m_n/n}}^{\infty} dF_n^{*m_n}(w_1, \dots, w_k) \\ &= e^{m_n \gamma_n} \int_{m_n \mu_n}^{\infty} \dots \int_{m_n \mu_n}^{\infty} e^{-s_n(w_1 + \dots + w_k)} dV_n^{*m_n}(w_1, \dots, w_k). \end{aligned} \quad (2.14)$$

Now we replace  $V_n$  by a  $k$ -variate normal with mean vector  $\mu_n \mathbf{1}_k$  and covariance matrix  $\tau_n^2 I_k$  (i.e., independent coordinates). The result of this change of distribution leads to the approximation (for  $P_n$ ), given by

$$A_{s_n} = e^{m_n \gamma_n} \left[ \int_{m_n \mu_n}^{\infty} e^{-s_n y} \phi \left( \frac{y - m_n \mu_n}{\tau_n \sqrt{m_n}} \right) \frac{dy}{\tau_n \sqrt{m_n}} \right]^k \quad (2.15)$$

$$\begin{aligned} &= e^{m_n \gamma_n} \left[ \int_0^{\infty} \phi(z) e^{-s_n(m_n \mu_n + z \tau_n \sqrt{m_n})} dz \right]^k \\ &= e^{m_n(\gamma_n - k s_n \mu_n)} \rho^k(s_n \tau_n \sqrt{m_n}), \end{aligned} \quad (2.16)$$

where  $\rho(t) = \int_0^{\infty} e^{-zt} \phi(z) dz = e^{\frac{t^2}{2}} (1 - \Phi(t))$  and  $\phi$  and  $\Phi$  are the univariate standard normal density.

Then the proof follows combining the two following propositions.  $\square$

The first proposition shows that  $A_{s_n}$  has the correct asymptotic behavior and the second one shows that  $A_{s_n}$  is a good approximation for  $P_n$ .

**Proposition 2.1.** *If  $v_n$  satisfies (1.7), namely,  $v_n \sim \sqrt{2 \log n}$  and  $m_n$  satisfies (1.3) given by  $\log n = o(m_n/n)$ , then*

$$A_{s_n} \sim \left[ (1 - \Phi(v_n)) \exp \left( v_n^2 B \left( \frac{v_n}{\sqrt{m_n/n}} \right) \right) \right]^k. \quad (2.17)$$

**Proposition 2.2.** *If  $v_n$  satisfies (1.7), namely,  $v_n \sim \sqrt{2 \log n}$  and  $m_n$  satisfies (1.3) given by  $\log n = o(m_n/n)$ , then*

$$P_n \sim A_{s_n}. \quad (2.18)$$

The proofs of Propositions 2.1 and 2.2 depend on the following lemma about the rate of growth of  $u_n$ , where  $u_n$  is defined in (2.9).

**Lemma 2.2.** *If  $v_n$  satisfies (1.7) given by  $v_n \sim \sqrt{2 \log n}$  and if  $m_n$  satisfies (1.3) given by  $\log n = o(m_n/n)$ , we have*

$$u_n = \frac{v_n}{\sqrt{m_n/n}} \left( 1 + O \left( \frac{1}{n} \right) \right) \rightarrow 0. \quad (2.19)$$

*Proof.* Note that the first partial of  $\Psi_n$  is

$$\partial_1 \Psi_n(t_1, \dots, t_k) = -\frac{1}{n} + \frac{e^{t_1}}{e^{t_1} + \dots + e^{t_k} + n - k}.$$

Hence, using (2.7), we have

$$\frac{v_n}{\sqrt{m_n/n}} = \frac{(n-k)(e^{s_n} - 1)}{n + k(e^{s_n} - 1)} = \frac{(n-k)u_n}{n + ku_n}. \quad (2.20)$$

Solving, we get,

$$u_n = \left( 1 - \frac{k}{n} \right)^{-1} \left( 1 - \frac{k}{n-k} \frac{v_n}{\sqrt{m_n/n}} \right)^{-1} \frac{v_n}{\sqrt{m_n/n}}$$

and, the result follows using  $\frac{v_n}{\sqrt{m_n/n}} \sim \sqrt{\frac{2 \log n}{m_n/n}} \rightarrow 0$ , from (1.3) and (1.7).  $\square$

The following corollary regarding the asymptotic behavior of  $\mu_n$ ,  $b_n$  and  $\tau_n^2$  then follows immediately from (2.10)–(2.12).

**Corollary 2.1.** *Assume that  $v_n$  satisfies (1.7) given by  $v_n \sim \sqrt{2 \log n}$  and  $m_n$  satisfies (1.3) given by  $\log n = o(m_n/n)$ . If the tilted distribution  $V_n$  has mean vector  $\mu_n \mathbf{1}_k$  and covariance matrix  $\Sigma_n = a_n I_k - b_n J_k$ , then*

$$\mu_n \sim \frac{u_n}{n}, \quad b_n \sim \frac{1}{n^2}, \quad \text{and} \quad \tau_n^2 \sim \frac{1}{n}. \quad (2.21)$$

Now we prove Proposition 2.1 on the asymptotic behavior of  $A_{s_n}$ .

*Proof of Proposition 2.1.* We first treat the exponent in the first factor of the expression (2.16) for  $A_{s_n}$ . Since  $\gamma_n = -\frac{k}{n}s_n + \log(1 + \frac{k}{n}(e^{s_n} - 1)) = -\frac{k}{n}\log(1 + u_n) + \log(1 + \frac{k}{n}u_n)$  using (2.6), it follows from expression (2.10) for  $\mu_n$ ,

$$\begin{aligned}
& m_n(\gamma_n - ks_n\mu_n) \\
&= \frac{m_n}{n}n \log\left(1 + \frac{k}{n}u_n\right) - \frac{m_n}{n} \frac{k(1 + u_n) \log(1 + u_n)}{1 + \frac{k}{n}u_n} \\
&= \frac{m_n}{n} \left(1 + \frac{k}{n}u_n\right)^{-1} \left[ (n + ku_n) \log\left(1 + \frac{k}{n}u_n\right) - (k + ku_n) \log(1 + u_n) \right] \\
&= k \frac{m_n}{n} \left(1 + \frac{k}{n}u_n\right)^{-1} \sum_{r=2}^{\infty} \frac{(-1)^{r-1}}{r(r-1)} \left[ 1 - \left(\frac{k}{n}\right)^{r-1} \right] u_n^r \\
&= -\frac{k}{2} \frac{m_n u_n^2}{n} + \frac{k^2}{2n} \frac{m_n u_n^2}{n} \\
&\quad - k \frac{m_n u_n^2}{n} \sum_{i=1}^{\infty} (-1)^i \sum_{r=0}^i \frac{1}{(r+1)(r+2)} \left(\frac{k}{n}\right)^{i-r} \left[ 1 - \left(\frac{k}{n}\right)^{r+1} \right] u_n^i \\
&= -\frac{k}{2} \frac{m_n u_n^2}{n} + k \frac{m_n u_n^2}{n} B(u_n) + E_n^{(0)} + E_n^{(1)}, \tag{2.22}
\end{aligned}$$

where, using (2.19),

$$E_n^{(0)} = \frac{k^2}{2n} \frac{m_n u_n^2}{n} \sim k^2 \frac{\log n}{n} \rightarrow 0 \tag{2.23}$$

and

$$\begin{aligned}
E_n^{(1)} &= \frac{m_n u_n^2}{n} \sum_{i=1}^{\infty} (-1)^i u_n^i \left[ \sum_{r=0}^{i-1} \frac{1}{(r+1)(r+2)} \left(\frac{k}{n}\right)^{i-r} \left\{ 1 - \left(\frac{k}{n}\right)^{r+1} \right\} \right. \\
&\quad \left. - \frac{1}{(i+1)(i+2)} \left(\frac{k}{n}\right)^{i+1} \right].
\end{aligned}$$

Thus,  $|E_n^{(1)}| \leq S_1 + S_2$ , where

$$\begin{aligned}
S_1 &= \frac{m_n u_n^2}{n} \sum_{i=1}^{\infty} u_n^i \sum_{r=0}^{i-1} \frac{1}{(r+1)(r+2)} \left(\frac{k}{n}\right)^{i-r} \left\{ 1 - \left(\frac{k}{n}\right)^{r+1} \right\} \\
&\leq \frac{m_n u_n^2}{n} \sum_{i=0}^{\infty} \left(\frac{k}{n}u_n\right)^{i+1} \sum_{r=0}^i \frac{1}{(r+1)(r+2)} \left(\frac{k}{n}\right)^{-r} \left\{ 1 - \left(\frac{k}{n}\right)^{r+1} \right\} \\
&= \frac{m_n u_n^2}{n} \sum_{r=0}^{\infty} \frac{1}{(r+1)(r+2)} \frac{k}{n} \left\{ 1 - \left(\frac{k}{n}\right)^{r+1} \right\} u_n^{r+1} \left(1 - \frac{k}{n}u_n\right)^{-1} \\
&\sim 2k \frac{\log n}{n} u_n \sum_{r=0}^{\infty} \frac{1}{(r+1)(r+2)} \left\{ 1 - \left(\frac{k}{n}\right)^{r+1} \right\} u_n^r \rightarrow 0, \tag{2.24}
\end{aligned}$$

and

$$S_2 = \frac{m_n u_n^2}{n} \sum_{i=1}^{\infty} \frac{1}{(i+1)(i+2)} \left(\frac{k}{n}u_n\right)^i \leq \frac{m_n u_n^2}{n} \sum_{i=1}^{\infty} \left(\frac{k}{n}u_n\right)^i \sim 2 \log n \frac{k}{n} u_n \rightarrow 0, \tag{2.25}$$



since  $m_n u_n^2/n \sim v_n^2 \sim 2 \log n$ , using (2.19) and (1.3). Hence, we have

$$E_n^{(1)} \rightarrow 0. \quad (2.26)$$

Thus, using (2.22), (2.23) and (2.26), we have

$$m_n(\gamma_n - k s_n \mu_n) = -\frac{k}{2} \frac{m_n u_n^2}{n} + k \frac{m_n u_n^2}{n} B(u_n) + o(1)$$

and hence, we have,

$$e^{m_n(\gamma_n - k s_n \mu_n)} \sim \exp \left( -\frac{k}{2} \frac{m_n u_n^2}{n} + k \frac{m_n u_n^2}{n} B(u_n) \right). \quad (2.27)$$

Further, observe from (2.19) that,  $m_n u_n^2/n - v_n^2 = O(v_n^2/n) = O(\log n/n) \rightarrow 0$ , using (1.7). Also,  $B(u_n) \sim u_n/6 \rightarrow 0$  and

$$B(u_n) - B\left(\frac{v_n}{\sqrt{m_n/n}}\right) = O\left(u_n - \frac{v_n}{\sqrt{m_n/n}}\right) = O\left(\frac{v_n}{\sqrt{m_n/n}} \frac{1}{n}\right) = o(1/v_n^2),$$

using (2.19), (1.3) and (1.7). Hence,  $m_n u_n^2/n$  in (2.27) can be replaced by  $v_n$  giving

$$e^{m_n(\gamma_n - k s_n \mu_n)} \sim \exp \left( -\frac{k}{2} v_n^2 + k u_n^2 B\left(\frac{v_n}{\sqrt{m_n/n}}\right) \right). \quad (2.28)$$

Using the asymptotic expression (2.21) for  $\tau_n$ , the fact  $u_n = e^{s_n} - 1 \sim s_n$  and (2.19), we have

$$\tau_n s_n \sqrt{m_n} \sim u_n \sqrt{m_n/n} = v_n.$$

The proof is then completed using (2.5), which gives

$$\rho^k(s_n \tau_n \sqrt{m_n}) \sim \frac{1}{(v_n \sqrt{2\pi})^k}. \quad (2.29)$$

□

Next we prove Proposition 2.2.

*Proof of Proposition 2.2.* Let  $\Phi_{\mu, A}$  denote the  $k$ -variate normal distribution function with mean vector  $\mu$  and covariance matrix  $A$ . Using (2.14) and (2.15), we easily see that

$$\frac{P_n - A_{s_n}}{e^{m_n \gamma_n}} = \int_{m_n \mu_n}^{\infty} \dots \int_{m_n \mu_n}^{\infty} e^{-s_n(u_1 + \dots + u_k)} d(V_n^{\star m_n} - \Phi_{\mu_n \mathbf{1}_k, \tau_n^2 I_k}^{\star m_n})(u_1, \dots, u_k).$$

Denote the distribution function of the signed measure  $V_n^{\star m_n} - \Phi_{\mu_n \mathbf{1}_k, \tau_n^2 I_k}^{\star m_n}$  by  $H_n$ . Then, using Theorem 3.1 in Appendix and (2.16), we have

$$|P_n - A_{s_n}| \leq 2^k \|H_n\|_{\infty} e^{m_n(\gamma_n - k s_n \mu_n)} = 2^k A_{s_n} \rho^{-k}(s_n \tau_n \sqrt{m_n}) \|H_n\|_{\infty}, \quad (2.30)$$

where  $\|H_n\|_{\infty}$  is the sup norm. Hence, using (2.29) and the fact that  $z_n = \sqrt{m_n/n} u_n \sim v_n$ , using (2.19), we have

$$\frac{P_n}{A_{s_n}} = 1 + O(v_n^k \|H_n\|_{\infty}). \quad (2.31)$$

So, to complete the proof, we need to study  $\|H_n\|_\infty$ . We write  $H_n$  as sum of two signed measures by introducing the normal distribution with covariance matrix,  $\Sigma_n$ , same as that of  $V_n$ :

$$H_n = (\Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \tau_n^2 I_k}^{*m_n}) + (V_n^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n}). \quad (2.32)$$

We estimate the first part directly and the second part by Berry-Esseen theorem. Observe that

$$\|\Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \tau_n^2 I_k}^{*m_n}\|_\infty = \|\Phi_{\mathbf{0}, \tau_n^{-2} \Sigma_n} - \Phi_{\mathbf{0}, I_k}\|_\infty,$$

which is estimated easily using normal comparison lemma, attributed to Slepian, Berman and others (see, for example, Leadbetter et al., 1983, Theorem 4.2.1). Observe that  $\tau_n^{-2} \Sigma_n = \frac{a_n}{a_n - b_n} I_k - \frac{b_n}{a_n - b_n} J_k$ , using (2.11) and (2.12). Hence, from normal comparison lemma and asymptotic behavior of  $a_n$  and  $b_n$  in (2.21), we have

$$\begin{aligned} \|\Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \tau_n^2 I_k}^{*m_n}\|_\infty &= \|\Phi_{\mathbf{0}, \tau_n^{-2} \Sigma_n} - \Phi_{\mathbf{0}, I_k}\|_\infty \\ &\leq \frac{1}{2\pi} \frac{k(k-1)}{2} \frac{b_n}{\sqrt{a_n(a_n - 2b_n)}} \sim O(1/n). \end{aligned}$$

Hence, corresponding to the first term of (2.32), we have, using (1.7),

$$v_n^k \|\Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \tau_n^2 I_k}^{*m_n}\|_\infty = O(v_n^k/n) \rightarrow 0. \quad (2.33)$$

Next we study the second term of (2.32). Suppose  $\xi_j$  are i.i.d.  $V_n$  with mean  $\mu_n \mathbf{1}_k$ , covariance  $\Sigma_n$ . Then

$$\begin{aligned} V_n^{*m_n}(u_1, \dots, u_k) - \Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n}(u_1, \dots, u_k) \\ = P \left[ \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} (\xi_j - \mu_n \mathbf{1}_k) \leq \frac{\mathbf{u} - m_n \mu_n \mathbf{1}_k}{\sqrt{m_n}} \right] - \Phi_{\mathbf{0}, \Sigma_n} \left( \frac{\mathbf{u} - m_n \mu_n \mathbf{1}_k}{\sqrt{m_n}} \right), \end{aligned}$$

and hence, by multivariate Berry-Esseen theorem, (see, e.g., Bhattacharya and Ranga Rao, 1976, Corollary 17.2, pg. 165)

$$\begin{aligned} \|V_n^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n}\|_\infty &= \sup_{\mathbf{u}} \left| P \left[ \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} (\xi_j - \mu_n \mathbf{1}_k) \leq \mathbf{u} \right] - \Phi_{\mathbf{0}, \Sigma_n}(\mathbf{u}) \right| \\ &\leq \frac{C_3}{\sqrt{m_n}} \frac{\kappa_n}{\lambda_n^{3/2}}, \end{aligned} \quad (2.34)$$

where  $\kappa_n = E\|\xi_1 - \mu_n \mathbf{1}_k\|_2^3$ , (the norm being Euclidean one),

$$\lambda_n = a_n - kb_n \sim \frac{1}{n}, \quad (2.35)$$

by (2.11) and (2.12), is the smallest eigenvalue of  $\Sigma_n = a_n I - b_n J$ , and  $C_3$  is a universal constant. So, to complete the proof we need to estimate  $\kappa_n$ . Using the definition of  $V_n$ , (2.8), we have,

$$\kappa_n = e^{-m_n \gamma_n} \int \dots \int e^{s_n(u_1 + \dots + u_k)} (\sum_{j=1}^k (u_j - \mu_n)^2)^{3/2} dF_n(u_1, \dots, u_k).$$

Recall that  $F_n$  is the distribution of the last  $k$  coordinates of the centered multinomial  $(1; (n-k)/n, 1/n, \dots, 1/n)$  distribution, which puts mass  $1/n$  at each of the  $k$  vectors which have all coordinates  $-1/n$  except the  $i$ th one being  $(n-1)/n$ , for  $i = 1, \dots, k$ , and  $(n-k)/n$  at  $(-1/n, \dots, -1/n)$ . Thus,

$$e^{m_n \gamma_n} \kappa_n = \frac{n-k}{n} e^{-\frac{k s_n}{n}} k^{\frac{3}{2}} \left( \frac{1}{n} + \mu_n \right)^3 + \frac{k}{n} e^{\frac{(n-k)s_n}{n}} \left[ (k-1) \left( \frac{1}{n} + \mu_n \right)^2 + \left( 1 - \frac{1}{n} - \mu_n \right)^2 \right]^{\frac{3}{2}}.$$

Since, from (2.21) we have  $\mu_n \sim \frac{u_n}{n}$ , and by (2.19), we have  $s_n = \log(1 + u_n) \sim u_n \rightarrow 0$ ,

$$\kappa_n \sim e^{-m_n \gamma_n} \frac{k}{n}.$$

Thus, using (2.34) and (2.35), we have,

$$v_n^k \|V_n^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n}\|_\infty \leq k C_3 \frac{v_n^k}{\sqrt{\frac{m_n}{n}}} e^{-m_n \gamma_n}. \quad (2.36)$$

Also, from (2.6), we get, for fixed  $k$ ,

$$\begin{aligned} m_n \gamma_n &= m_n \Psi(s_n, \dots, s_n) = -\frac{k m_n s_n}{n} + m_n \log \left[ 1 + \frac{k(e^{s_n} - 1)}{n} \right] \\ &= \frac{m_n}{n} [-k \log(1 + u_n) + n \log(1 + \frac{k}{n} u_n)] \sim \frac{k}{2} \frac{m_n}{n} u_n^2 \sim \frac{k}{2} v_n^2 \rightarrow \infty \end{aligned}$$

using (2.19). Hence, from (2.36), we have

$$\lim_{n \rightarrow \infty} v_n^k \|V_n^{*m_n} - \Phi_{\mu_n \mathbf{1}_k, \Sigma_n}^{*m_n}\|_\infty = 0. \quad (2.37)$$

Combining (2.33) and (2.37), we get,

$$\lim_{n \rightarrow \infty} v_n^k \|H_n\|_\infty = 0$$

and the result follows from (2.31).  $\square$

Finally we prove Proposition 1.1.

*Proof of Proposition 1.1.* If  $p_n = 1/n$ , the condition  $p_n (\log n)^r \rightarrow 0$  holds for all  $r > 0$  and the result follows from (2.2) with  $k = 1$ , since

$$-\log(P[Y_{1n} \leq y_n])^n \sim nP[Y_{1n} > y_n] \rightarrow e^{-x}.$$

For general  $p_n$ , the argument is exactly same as that for derivation of (2.2). The extra condition is required to prove the convergence in (2.23)–(2.25) and (2.33). In (2.33), the bound is of the order  $v_n^k p_n \sim (2 \log n)^{k/2} p_n$ , necessitating the extra condition. In (2.23)–(2.25), the bound for  $E_n^{(0)}$  in (2.23) is of the highest order, which in this case becomes  $p_n \log n$ . It is still negligible under the assumption.  $\square$

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### 3 Appendix

Here we prove the result on integration by parts, which was used in the proof of Proposition 2.2, see (2.30). Let  $H$  be the distribution function of a finite signed measure on  $\mathbb{R}^k$ . For any subset  $I$  of  $\{1, \dots, k\}$  and  $a \in \mathbb{R}$ , define,

$$y_i^I = \begin{cases} a, & i \in I \\ y_i, & i \notin I \end{cases}, \quad \text{for } 1 \leq i \leq k,$$

$$H^I(a; y_1, \dots, y_k) = H(y_1^I, \dots, y_k^I)$$

and

$$H_y^I(y_i; i \in I) = H(y_1, \dots, y_k)$$

considered as a function in coordinates indexed by  $I$  only.

**Theorem 3.1.** *For  $1 \leq l \leq k$  and  $I \subseteq \{1, \dots, l\}$ , we have,*

$$\begin{aligned} \int_a^\infty \dots \int_a^\infty e^{-s(y_1 + \dots + y_k)} dH_{y_1, \dots, y_k}^{\{1, \dots, l\}}(y_1, \dots, y_k) \\ = \sum_{I \subseteq \{1, \dots, l\}} \int_a^\infty \dots \int_a^\infty (-1)^{|I|} s^l e^{-s(y_1 + \dots + y_k)} H^I(a; y_1, \dots, y_k) dy_1 \dots dy_k. \end{aligned} \quad (3.1)$$

The bound (2.30) then follows immediately by considering  $l = k$ .

*Proof.* We prove (3.1) by induction on  $l$ . For  $l = 1$ , (3.1) is the usual integration by parts formula. Assume (3.1) for  $l$ . Then

$$\begin{aligned} & \int_a^\infty \dots \int_a^\infty e^{-s(y_1 + \dots + y_{l+1})} H_{y_1, \dots, y_k}^{\{1, \dots, l+1\}}(dy_1, \dots, dy_{l+1}) \\ &= \sum_{I \subseteq \{1, \dots, l\}} (-1)^{|I|} \int_a^\infty \dots \int_a^\infty s^l e^{-s(y_1 + \dots + y_l)} \int_a^\infty e^{-sy_{l+1}} H_{y_1^I, \dots, y_k^I}^{\{l+1\}}(dy_{l+1}) dy_1 \dots dy_l \\ &= \sum_{I \subseteq \{1, \dots, l\}} (-1)^{|I|} \int_a^\infty \dots \int_a^\infty s^l e^{-s(y_1 + \dots + y_l)} \left[ e^{-sa} H^{I \cup \{l+1\}}(a; y_1, \dots, y_k) \right. \\ & \quad \left. + \int_a^\infty s e^{-sy_{l+1}} H^I(a; y_1, \dots, y_k) dy_{l+1} \right] dy_1 \dots dy_l \\ &= \sum_{I \subseteq \{1, \dots, l\}} \int_a^\infty \dots \int_a^\infty e^{-s(y_1 + \dots + y_{l+1})} s^{l+1} \times \\ & \quad \left[ (-1)^{|I|+1} H^{I \cup \{l+1\}}(a; y_1, \dots, y_k) + (-1)^{|I|} H^I(a; y_1, \dots, y_k) \right] dy_1 \dots dy_{l+1} \end{aligned}$$

where we use the induction hypothesis for the first step and the usual integration by parts for the second step, and the final step is the required sum, since any subset of  $\{1, \dots, l+1\}$  either contains  $l+1$  or does not and the remainder is a subset of  $\{1, \dots, l\}$ . This completes the inductive step and the proof of the theorem.  $\square$

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