

Gaussian multiplicative chaos for Gaussian Orthogonal and Symplectic Ensembles*

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Abstract

We study the characteristic polynomials of both the Gaussian Orthogonal and Symplectic Ensembles. We show that for both ensembles, powers of the absolute value of the characteristic polynomials converge in law to Gaussian multiplicative chaos measures after normalization for sufficiently small real powers. The main tool is a new asymptotic relation between the fractional moments of the absolute characteristic polynomials of Gaussian Orthogonal, Unitary, and Symplectic Ensembles.

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1 Introduction

The Gaussian Orthogonal/Unitary/Symplectic Ensembles are probability measures on the space of $N \times N$ Symmetric/Hermitian/Quaternionic-Hermitian matrices with density

$$\mathbb{P}_\beta(A) = \frac{1}{Z_{N,\beta}} \exp(-N\beta \text{Tr}(A^2)), \quad (1.1)$$

where here $\beta = 1/2/4$, respectively, and $Z_{N,\beta}$ is a normalization constant. We will write $A_{N,\beta}$ for a matrix sampled from the measure (1.1).

The focus of this paper will be the study of the (absolute) characteristic polynomial of matrices sampled from these ensembles in the large N limit. Our understanding of the behavior of such random characteristic polynomials for these (and other) matrix ensembles has seen much progress in recent years (see, for example [4] and the references therein). Much of this progress has been motivated by conjectures on these processes due to Fyodorov, Hiary, Keating, and Simm in [22, 19, 20], who predict a detailed picture of the extremal values of these processes by relating these characteristic polynomials to objects from the theory of log-correlated fields. Key to this picture is the

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foundational work of Fyodorov, Khoruzhenko, and Simm [21], which shows that for $\beta = 2$, the normalized logarithm of the characteristic polynomial converges (in a suitable sense) to a log-correlated Gaussian random field, expanding the picture derived by Hughes, Keating, and O’Connell [23] in the Unitary case. More explicitly, in [21] they construct a centered Gaussian field X on $(-1, 1)$ with covariance kernel

$$\mathbb{E}X(x)X(y) = -\frac{1}{2} \log |2(x - y)|.$$

Due to the divergence of this covariance kernel on the diagonal, X does not exist as a random function, but instead, they show that it may be constructed as a random variable valued in a suitable space of distributions. They then proceed to show that the normalized logarithm of $|\det(A_{N,2} - xI)|$ converges in law to X with respect to the topology given by a suitable Sobolev norm.

A key tool used to understand the geometric properties of log-correlated fields (such as X) is their associated (GMC) Gaussian multiplicative chaos measures, introduced by Kahane [26], which are roughly given by regularizing the exponential of the field. For more background on log-correlated fields and GMC measures, the reader is invited to consult [36].

In view of the results of [21, 23], one thus expects that the asymptotic behavior of the characteristic polynomials of these ensembles and their powers should be described by a family of GMC measures, and indeed it is from this perspective that [22, 20] were able to formulate their conjectures. Much progress has been achieved in establishing this picture rigorously. In particular, for $\beta = 2$, Berestycki, Webb, and Wong [4] establish that small enough powers of the absolute value of the characteristic polynomial can be described in the large N limit by a Gaussian multiplicative chaos (GMC) measure. Explicitly, they construct a family of random measures μ_α , for $\alpha \in (0, 2)$, which may be heuristically written as

$$\mu_\alpha(dx) = \exp\left(\alpha X(x) - \frac{\alpha^2}{2} \mathbb{E}[X(x)^2]\right) dx. \tag{1.2}$$

They then show that when viewed as random measures on $(-1, 1)$, the sequence

$$\frac{|\det(A_{N,2} - xI)|^\alpha}{\mathbb{E}[|\det(A_{N,2} - xI)|^\alpha]} dx$$

converges in law to μ_α with respect to the topology of weak convergence of measures. Our main result will be an analog of this result for the Gaussian Orthogonal and Symplectic ensembles (i.e., $\beta = 1, 4$).

Theorem 1.1. *For $\alpha \in (0, 1/\sqrt{2})$, we have as $N \rightarrow \infty$ that*

$$\frac{|\det(A_{2N,1} - xI)|^\alpha}{\mathbb{E}[|\det(A_{2N,1} - xI)|^\alpha]} dx \Rightarrow \mu_{\sqrt{2}\alpha}(dx)$$

in law with respect to the topology of weak convergence of measures.

Similarly, for $\alpha \in (0, \sqrt{2})$ we have as $N \rightarrow \infty$ that

$$\frac{|\det(A_{N,4} - xI)|^\alpha}{\mathbb{E}[|\det(A_{N,4} - xI)|^\alpha]} dx \Rightarrow \mu_{\alpha/\sqrt{2}}(dx)$$

in law with respect to the topology of weak convergence of measures.

We note that some analogous results are known outside of the case of Gaussian ensembles. In particular, Webb [40] showed that suitably small powers (more precisely powers in the L^2 -regime) of the characteristic polynomial of a Haar distributed random

unitary matrix converges in law to a certain GMC measure on the unit circle, after a suitable normalization. This result was later extended by Nikula, Saksman, and Webb [34] to the larger L^1 -regime, which is believed to be optimal. Additionally, Forkel and Keating [16] have shown a similar result for the characteristic polynomials of matrices sampled from the Haar measures on both the orthogonal and symplectic groups.

In all of these cases though, the results rely on precise asymptotics for the fractional moments of characteristic polynomials (for example, see Sections 3 and 4 of [4]). Depending on the case, this becomes equivalent to obtaining asymptotics for certain Toeplitz, Hankel, or Toeplitz+Hankel determinants associated with certain measures with Fischer-Hartwig singularities (see [18] for further background). These asymptotics have been computed in many cases through the use of Riemann-Hilbert methods (see, for example, [29]). These methods though, are specific to the case of $\beta = 2$, with no clear generalization to the $\beta = 1, 4$ cases.

Here we take a different approach to prove Theorem 1.1. Instead of calculating the fractional moments directly (which appear to still be inaccessible to current methods), we instead relate fractional moments of $|\det(A_{2N,1} - xI)|$ and $|\det(A_{N,4} - xI)|$ to those of $|\det(A_{2N,2} - xI)|$. For notational convenience in stating our result, here and elsewhere, we will denote

$$D_{N,\beta}(x) = \det(A_{N,\beta} - xI).$$

Our main tool in proving Theorem 1.1 is then the following asymptotic relation, which may also be of independent interest.

Theorem 1.2. *Let $m \geq 1$, and for each $1 \leq i \leq m$, let us take $\alpha_i > 0$ and $\lambda_i \in (-1, 1)$ such that $-1 < \lambda_m < \dots < \lambda_1 < 1$. Let \mathcal{W} be a smooth, compactly-supported function that coincides with a polynomial on a neighborhood of $[-1, 1]$. Then we have that*

$$\begin{aligned} & \mathbb{E} \left[e^{\text{Tr}(\mathcal{W}(A_{2N,1}))} \prod_{i=1}^m |D_{2N,1}(\lambda_i)|^{\alpha_i} \right] \times \mathbb{E} \left[e^{\text{Tr}(2\mathcal{W}(A_{N,4}))} \prod_{i=1}^m |D_{N,4}(\lambda_i)|^{2\alpha_i} \right] \\ &= \mathbb{E} \left[e^{\text{Tr}(2\mathcal{W}(A_{2N,2}))} \prod_{i=1}^m |D_{2N,2}(\lambda_i)|^{2\alpha_i} \right] (1 + O(N^{-1/6})). \end{aligned}$$

Moreover the error term is uniform over compact subsets of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$.

Remark 1.3. Theorem 1.2 may be compared to the following relation given by Baik and Rains [3] (also noted in [18]). Denote by O_N, U_N , and S_N , a random matrix sampled from the Haar measure on $SO(N), U(N)$, and $Sp(N)$, respectively. Then for a large class of functions g (including integrable functions), we have the following relation

$$\mathbb{E}[\det(g(O_{2N+2}))]\mathbb{E}[\det(g(S_{2N}))] = \mathbb{E}[|\det(g(U_{2N}))|^2].$$

For other occurrences of GMC measures in random matrix theory, see [31] as well as the references therein. We also mention the work of Lambert, Ostrovsky, and Simm [32], where they establish the convergence of a regularized version of $\log(|D_{N,\beta}(x)|)$ to a log-correlated field for arbitrary $\beta > 0$. Additionally, we also mention the recent work of Bourgade, Mody, and Pain [6], where they establish normalized fluctuations of $\log(|D_{N,\beta}(x)|)$ at a finite collection of points. In the work of Claeys, Fahs, Lambert, and Webb [9], a similar convergence to a GMC measure is shown for the exponential of the eigenvalue counting function, from which they derive optimal bounds on eigenvalue fluctuations. Lastly, for some beautiful relationships between log-correlated fields, random matrices, and the behavior of special functions in number theory, one may consult [28, 20, 22], and the references therein.

Lastly, we also remark that the restriction to the even case for $\beta = 1$ in Theorem 1.1 appears to be a technical artifact of our method of proof, which relies crucially on Theorem 1.2. Similarly, we note that the parameter ranges for α in Theorem 1.1 amount to only a strict subset of the L^2 -phase, which occurs for $\alpha < \sqrt{\beta}$. This restriction as well is a limitation of our method and is due to having to rely on the convergence of the $\beta = 2$ at larger powers, as is explained below.

1.1 Construction of GMC and sketch of the proof of Theorem 1.1 from Theorem 1.2

We now outline our method to obtain Theorem 1.1 from Theorem 1.2. Details are given in Section 3. We begin by recalling the basics of the construction of the GMC measure (1.2). We first recall a sequence of Gaussian processes on $(-1, 1)$, denoted X_M , such that a.s. $X_M \Rightarrow X$ as $M \rightarrow \infty$ in the distributional sense. This decomposition comes from the identity (see, for example, Appendix C of [35]) that for $x, y \in (-1, 1)$, we have that

$$-\frac{1}{2} \log(2|x - y|) = \sum_{k=1}^{\infty} \frac{1}{k} T_k(x) T_k(y),$$

where T_k is k -th Chebyshev polynomial of the first kind defined by the relation $T_k(\cos(\theta)) = \cos(k\theta)$ for $\theta \in (0, 2\pi]$. The polynomial T_k is also the k -th orthogonal polynomial with respect to the measure μ_{as} , where $\mu_{as}(x) = \frac{2}{\pi}(1 - x^2)^{-1/2}I(|x| < 1)$ denotes the (shifted) arcsine distribution. In particular, from this, we have the following (formal) distributional identity

$$X(x) = \sum_{k=1}^{\infty} \frac{Z_k}{\sqrt{k}} T_k(x),$$

where $(Z_k)_k$ is a sequence of i.i.d standard Gaussians random variables. We define

$$X_M(x) = \sum_{k=1}^M \frac{Z_k}{\sqrt{k}} T_k(x),$$

which is the formal projection of X onto polynomials of degree- M in $L^2(\mu_{as})$.

If we define $\mu_{M,\alpha}(dx) = \exp(\alpha X_M(x) - \frac{\alpha^2}{2} \mathbb{E}[X_M(x)^2])dx$, it is shown in [4] that for $\alpha \in (0, 2)$, the measures $\mu_{M,\alpha}$ converge weakly in law to a nontrivial limiting measure, which will be our μ_α . In addition, for $\alpha \in (0, \sqrt{2})$, it is shown that $\mu_{M,\alpha}$ converges to μ_α in the L^2 -norm as well.

We now define

$$\mu_{N,\beta,\alpha}(dx) = \frac{|D_{N,\beta}(x)|^\alpha}{\mathbb{E}[|D_{N,\beta}(x)|^\alpha]} dx.$$

In addition, we introduce processes $X_{N,M,\beta}$, similar to X_M , but relating instead to $\log(|D_{N,\beta}(x)|)$. As the eigenvalues of $A_{N,\beta}$ may live outside of $(-1, 1)$, we first must smoothly extend T_k to a compactly-supported function. In particular, let us fix some $\epsilon > 0$ and choose, for each k , a function $\tilde{T}_k(x)$ which is smooth and of compact support, and such that $\tilde{T}_k(x) = T_k(x)$ on $(-1 - \epsilon, 1 + \epsilon)$. We define

$$X_{N,M,\beta}(x) = - \sum_{k=1}^M \frac{2}{k} \text{Tr}(\tilde{T}_k(A_{N,\beta})) T_k(x), \quad \mu_{N,M,\beta,\alpha}(dx) = \frac{e^{\alpha X_{N,M,\beta}(x)}}{\mathbb{E} e^{\alpha X_{N,M,\beta}(x)}} dx.$$

For a measure μ on $(-1, 1)$, and a function f , let us denote $\mu(f) := \int_{-1}^1 f(x)\mu(dx)$. We note that by a standard argument (see Section 4 of [27]), there is no loss of generality to replace the weak convergence in Theorem 1.1 with vague convergence. That is, it suffices

to show for any fixed continuous compactly-supported function $\varphi : (-1, 1) \rightarrow [0, \infty)$, and for $\alpha < \sqrt{\beta/2}$, that we have that

$$\mu_{N,\beta,\alpha}(\varphi) \Rightarrow \mu_{\alpha\sqrt{2/\beta}}(\varphi) \tag{1.3}$$

in law as $N \rightarrow \infty$ for $\beta = 1, 4$ (where N only runs over even numbers if $\beta = 1$).

For $\beta = 1, 2, 4$ and $\alpha > 0$ and M fixed we have that

$$\mu_{N,M,\beta,\alpha}(\varphi) \Rightarrow \mu_{M,\alpha\sqrt{2/\beta}}(\varphi)$$

in law as $N \rightarrow \infty$. This result is a consequence of the classical asymptotics for linear statistics given in [25]. In particular, this follows immediately by repeating the proof of the $\beta = 2$ case given in Section 6 of [4], as the results of [25] hold for all $\beta = 1, 2, 4$. Noting the inequality,

$$\begin{aligned} \frac{1}{4} \mathbb{E}(\mu_{N,\beta,\alpha}(\varphi) - \mu_{\alpha\sqrt{2/\beta}}(\varphi))^2 &\leq \mathbb{E}(\mu_{N,\beta,\alpha}(\varphi) - \mu_{N,M,\beta,\alpha}(\varphi))^2 \\ &+ \mathbb{E}(\mu_{N,M,\beta,\alpha}(\varphi) - \mu_{M,\alpha\sqrt{2/\beta}}(\varphi))^2 + \mathbb{E}(\mu_{M,\alpha\sqrt{2/\beta}}(\varphi) - \mu_{\alpha\sqrt{2/\beta}}(\varphi))^2, \end{aligned}$$

we see that when taking the limit $N \rightarrow \infty$ and then the limit $M \rightarrow \infty$, the second and third terms on the right-hand side vanish, so to demonstrate (1.3) it suffices to show that for $\beta = 1, 4$, we have that

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\mu_{N,\beta,\alpha}(\varphi) - \mu_{N,M,\beta,\alpha}(\varphi))^2 = 0, \tag{1.4}$$

where again N only ranges over even numbers if $\beta = 1$.

One should note that in the case of $\beta = 2$, the left-hand side of (1.4) is shown to vanish in [4]. The key step will then be to define an additional pair of measures

$$\begin{aligned} \nu_{N,\alpha}(dx) &= \frac{|D_{2N,1}(x)|^\alpha |D_{N,4}(x)|^{2\alpha}}{\mathbb{E}[|D_{2N,1}(x)|^\alpha] \mathbb{E}[|D_{N,4}(x)|^{2\alpha}]} dx, \\ \nu_{N,M,\alpha}(dx) &= \frac{e^{\alpha X_{2N,M,1}(x)} e^{2\alpha X_{N,M,4}(x)}}{\mathbb{E}e^{\alpha X_{2N,M,1}(x)} \mathbb{E}e^{2\alpha X_{N,M,4}(x)}} dx, \end{aligned}$$

where $A_{2N,1}$ and $A_{N,4}$ are sampled independently. We observe that $\mathbb{E}_{A_{2N,1}}[\nu_{N,\alpha}(\varphi)] = \mu_{N,4,2\alpha}(\varphi)$ and $\mathbb{E}_{A_{N,4}}[\nu_{N,\alpha}(\varphi)] = \mu_{2N,1,\alpha}(\varphi)$, where $\mathbb{E}_{A_{2N,1}}$ and $\mathbb{E}_{A_{N,4}}$ denote expectation with respect to $A_{2N,1}$ and $A_{N,4}$, respectively. A similar result holds for $\nu_{N,M,\alpha}$. A key observation is that Jensen's inequality implies that

$$\mathbb{E}(\mu_{N,4,\alpha}(\varphi) - \mu_{N,M,4,\alpha}(\varphi))^2 = \mathbb{E}(\mathbb{E}_{A_{2N,1}}(\nu_{N,\alpha}(\varphi) - \nu_{N,M,\alpha}(\varphi)))^2 \leq \mathbb{E}(\nu_{N,\alpha}(\varphi) - \nu_{N,M,\alpha}(\varphi))^2.$$

With a similar observation at $\beta = 1$, this implies that to show (1.4), it suffices to show that for $\alpha < 1/\sqrt{2}$

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\nu_{N,\alpha}(\varphi) - \nu_{N,M,\alpha}(\varphi))^2 = 0.$$

In addition, it will suffice to show that

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\nu_{N,\alpha}(\varphi) - \nu_{N,M,\alpha}(\varphi))^2 = \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\mu_{2N,2,2\alpha}(\varphi) - \mu_{2N,M,2,2\alpha}(\varphi))^2, \tag{1.5}$$

as the latter is shown to vanish in Proposition 2.9 of [4]. Expanding both sides of (1.5), we will see that all the integrands coincide pointwise as $N \rightarrow \infty$ by Theorem 1.2, reducing the remaining work to showing that the error term is suitably uniform. Details of this are given in Section 3.

1.2 Outline of the proof of Theorem 1.2

We now outline the method of proof of Theorem 1.2. Let us choose some $m \geq 1$ and $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ satisfying the conditions of Theorem 1.2. We will denote the polynomial coinciding with \mathcal{W} on $(-1, 1)$ as \mathcal{W}_0 , and denote the difference as $\mathcal{E} = \mathcal{W} - \mathcal{W}_0$. We will omit the data $(N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ from the notation when such a choice is clear from the context. We will reduce the proof of Theorem 1.2 to the computation of the asymptotics of the determinant of an N -by- N matrix $\Delta_N := \Delta_N(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$, defined in (2.3). More specifically, we show that

$$\frac{\mathbb{E}[e^{\text{Tr}(\mathcal{W}(A_{2N,1}))} \prod_{i=1}^m |D_{2N,1}(\lambda_i)|^{\alpha_i}] \mathbb{E}[e^{\text{Tr}(2\mathcal{W}(A_{N,4}))} \prod_{i=1}^m |D_{N,4}(\lambda_i)|^{2\alpha_i}]}{\mathbb{E}[e^{\text{Tr}(2\mathcal{W}(A_{2N,2}))} \prod_{i=1}^m |D_{2N,2}(\lambda_i)|^{2\alpha_i}]} = \sqrt{\det(\Delta_N)}. \tag{1.6}$$

This reduces the proof of Theorem 1.2 to the computation of $\det(\Delta_N)$.

This representation was essentially observed as Remark 2.4 of [37] and will essentially follow (up to normalization constants) from the representations presented in [2]. This matrix Δ_N originally appeared in [41], and was used to express the correlation functions of Orthogonal and Symplectic ensembles, and is shown there to be quite sparse. In particular in the case $\mathcal{W} = 0$, we have that Δ_N coincides with the identity outside of an $(m + 1)$ -by- $(m + 1)$ block. In the case of $\mathcal{W} \neq 0$ we will show that Δ_N is the identity outside a $(m + \max(\deg(\mathcal{W}_0) - 1, 1))$ -by- $(m + \max(\deg(\mathcal{W}_0) - 1, 1))$ block, up to an error term which we show is exponentially small in Section 5.

In either case, the entries of this distinguished block may be expressed explicitly in terms of the orthogonal polynomials with respect to the measure

$$w_N(x) := w(x; N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) = e^{-2Nx^2} e^{2\mathcal{W}(x)} \prod_{i=1}^m |x - \lambda_i|^{2\alpha_i}. \tag{1.7}$$

The asymptotics of the orthogonal polynomials with respect to this measure are known in the case of $\mathcal{W} = 0$ by [29], and $\mathcal{W} \neq 0$ by [4], and are recalled in detail in Section 4. With this information, we are able to compute the asymptotics of this block matrix to sufficient order to compute the first-order behavior of $\det(\Delta_N)$. This is done in Section 6.

We remark that computations of Δ_N for other measures form a key step in the proof of universality for the correlations functions of various symmetric and symplectic ensembles [12, 13, 11, 37].

1.3 Structure of the paper

In Section 2, we prove (1.6) and reduce the asymptotic computation of $\det(\Delta_N)$ to Propositions 2.6 and 2.9. In addition, we provide a modification of Theorem 1.2, Proposition 2.10, which applies in the case of (subcritical) merging singularities and reduce its proof to Proposition 2.13. In Section 3 we complete the proof of Theorem 1.1, which by the argument given above is reduced to demonstrating (1.5). The bulk of this statement will follow from Theorem 1.2 and Proposition 2.10. In Section 4, we provide asymptotics for the orthogonal polynomials with respect to w_N , while in Section 5 we employ these asymptotics to prove Proposition 2.6, which effectively allows us to neglect all but finitely many entries of $\det(\Delta_N)$. In Section 6, we prove Proposition 2.9, which consists of computing the highest order behavior of certain integrals of orthogonal polynomials, by employing classical methods of steepest descent to the asymptotics of Section 4. Furthermore, in Section 7, we will provide a proof of Proposition 2.13. The methods of this section will be similar to those of Section 6, and indeed the cases of Proposition 2.9 and Proposition 2.13 have significant overlap.

2 Preliminary results

In this section, we will prove (1.6). In addition, we will analyze the terms of Δ_N to reduce the computation of $\det(\Delta_N)$ to a sequence of integrals and quantities given in terms of the orthogonal polynomials of w_N .

We will assume henceforth that $N > m + \deg(\mathcal{W}_0)$. We define the unnormalized expected characteristic polynomial as

$$\Phi_{N,\beta}(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) = \frac{1}{N!} \int_{\mathbb{R}^N} |\Delta_N(x)|^\beta \prod_{i=1}^N (e^{-N\beta x_i^2 + \mathcal{W}(x_i)} \prod_{j=1}^m |x_i - \lambda_j|^{\alpha_j}) dx^N,$$

where here $\{\lambda_i\}_i = (\lambda_1, \dots, \lambda_m)$, $\{\alpha_i\}_i = (\alpha_1, \dots, \alpha_m)$, and $\Delta_N(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$ denotes the Vandermonde determinant. We note that (see Chapter 1 of [17])

$$\Phi_{N,\beta}(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) = Z_{N,\beta} \mathbb{E} \left[e^{\text{Tr}(\mathcal{W}(A_{N,\beta}))} \prod_{i=1}^m |\det(A_{N,\beta} - \lambda_i I)|^{\alpha_i} \right].$$

We will need the following relation.

Lemma 2.1. *For all $N \geq 1$, we have that $Z_{2N,1} Z_{N,4} = 2^{2N} Z_{2N,2}$.*

Proof. The values of $Z_{N,\beta}$ are known exactly as

$$Z_{N,\beta} = (2N)^{-\beta \frac{N(N-1)}{4} - N/2} (N!)^{-1} \beta^{-\frac{N}{2} - \beta \frac{N(N-1)}{4}} (2\pi)^{N/2} \prod_{j=1}^N \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}.$$

This follows from equation 1.163 of [17], up to rescaling. In view of this, we may rewrite

$$\begin{aligned} \frac{Z_{2N,1} Z_{N,4}}{Z_{2N,2}} &= \frac{(4N)^{-\frac{2N(2N-1)}{4} - N} (2N)^{-N(N-1) - N/2} 4^{-N/2 - N(N-1)} \Gamma(2)^{2N}}{(4N)^{-\frac{2N(2N-1)}{2} - N} 2^{-N - N(2N-1)} \Gamma(3)^N} \\ &\quad \times \frac{(2\pi)^{N/2} \prod_{j=1}^{2N} \Gamma(1 + j/2) \prod_{j=1}^N \Gamma(1 + 2j)}{\Gamma(3/2)^{2N} N! \prod_{j=1}^{2N} \Gamma(1 + j)}. \end{aligned} \tag{2.1}$$

We observe that

$$\frac{(4N)^{-\frac{2N(2N-1)}{4} - N} (2N)^{-N(N-1) - N/2}}{(4N)^{-\frac{2N(2N-1)}{2} - N}} = 2^{N^2 - \frac{N}{2}}, \quad \frac{4^{-N/2 - N(N-1)} \Gamma(2)^{2N}}{2^{-N - N(2N-1)} \Gamma(3)^N} = 1.$$

We may also obtain that

$$\begin{aligned} \frac{\prod_{j=1}^{2N} \Gamma(1 + j/2) \prod_{j=1}^N \Gamma(1 + 2j)}{\prod_{j=1}^{2N} \Gamma(1 + j)} &= N! \prod_{j=1}^N \frac{\Gamma(j + 1/2) \Gamma(j)}{\Gamma(2j)} = \prod_{j=1}^N \frac{\Gamma(j + 1/2) \Gamma(1 + j)}{\Gamma(2j)} \\ &= N! \pi^{N/2} \prod_{j=1}^N 2^{1-2j} = N! \pi^{N/2} 2^{-N^2}, \end{aligned}$$

where in the second to last equality we have employed the Legendre duplication formula (see 6.1.18 of [1])

$$\Gamma(z) \Gamma(z + 1/2) / \Gamma(2z) = 2^{1-2z} \pi.$$

In particular we may rewrite (2.1) as

$$2^{-N/2} \pi^{N/2} \frac{(2\pi)^{N/2}}{\Gamma(3/2)^{2N}} = \frac{\pi^N}{\Gamma(3/2)^{2N}} = 2^{2N},$$

where in the last step we used $\Gamma(3/2) = \frac{1}{2} \sqrt{\pi}$. As (2.1) coincides with $Z_{2N,1} Z_{N,4} / Z_{2N,2}$ we obtain the desired result. \square

From this, we see that

$$\begin{aligned} & \frac{\mathbb{E}[e^{\text{Tr}(\mathcal{W}(A_{2N,1}))} \prod_{i=1}^m |D_{2N,1}(\lambda_i)|^{\alpha_i}] \mathbb{E}[e^{\text{Tr}(2\mathcal{W}(A_{N,4}))} \prod_{i=1}^m |D_{N,4}(\lambda_i)|^{2\alpha_i}]}{\mathbb{E}[e^{\text{Tr}(2\mathcal{W}(A_{2N,2}))} \prod_{i=1}^m |D_{2N,2}(\lambda_i)|^{2\alpha_i}]} \\ &= 2^{2N} \frac{\Phi_{2N,1}(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) \Phi_{N,4}(2\mathcal{W}, \{\lambda_i\}_i, \{2\alpha_i\}_i)}{\Phi_{2N,2}(2\mathcal{W}, \{\lambda_i\}_i, \{2\alpha_i\}_i)}. \end{aligned} \tag{2.2}$$

We now recall a classical representation of these integrals. For simplicity, for the remainder of the section, we will fix $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ and denote, as above,

$$w_N(x) = w(x; N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i).$$

We give families of N -by- N matrices defined for $1 \leq i, j \leq N$ as

$$\begin{aligned} [M_N^2(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)]_{ij} &= \int x^{i+j-2} w_N(x) dx, \\ [M_N^4(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)]_{ij} &= \frac{1}{2} \int ((x^{i-1})'(x^{j-1}) - (x^{i-1})(x^{j-1})') w_N(x) dx, \\ [M_N^1(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)]_{ij} &= \frac{1}{2} \int \int x^{j-1} y^{i-1} \epsilon(x-y) \sqrt{w_N(x)w_N(y)} dx dy, \end{aligned}$$

where here and elsewhere ϵ refers to the sign function.

We note that the latter two matrices are antisymmetric, while the first is symmetric. The relationship between these matrices and the various $\Phi_{N,\beta}$ is classical.

Lemma 2.2. *We have for all $N \geq 1$, that*

$$\begin{aligned} \det(M_N^2(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)) &= \Phi_{2,N}(2\mathcal{W}, \{\lambda_i\}_i, \{2\alpha_i\}_i), \\ \text{pf}(M_{2N}^1(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)) &= 2^{-N} \Phi_{1,2N}(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i), \\ \text{pf}(M_{2N}^4(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)) &= 2^{-N} \Phi_{4,N}(2\mathcal{W}, \{\lambda_i\}_i, \{2\alpha_i\}_i), \end{aligned}$$

where here pf denotes the Pfaffian.

Proof. This follows immediately from Propositions 5.2.1, 6.1.8, and 6.3.4 of [17]. □

From this we may further write (2.2) as

$$\frac{\text{pf}(M_{2N}^1(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)) \text{pf}(M_{2N}^4(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i))}{\det(M_{2N}^2(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i))}.$$

To relate the various M_N^β further we introduce operators $J_N^{\pm 1} := J^{\pm 1}(N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ by

$$\begin{aligned} J_N f(x) &= w_N(x)^{-1/2} \frac{d}{dx} w_N(x)^{1/2} f(x) = f'(x) - 2N x f(x) + \sum_{i=1}^m \frac{\alpha_i}{x - \lambda_i} f(x) + \mathcal{W}'(x) f(x), \\ J_N^{-1} f(x) &= w_N(x)^{-1/2} \frac{1}{2} \int \epsilon(x-y) f(y) w_N(y)^{1/2} dy. \end{aligned}$$

We note that as long as f is continuously differentiable and of sub-exponential growth, then

$$(J_N J_N^{-1} f)(x) = w_N(x)^{-1/2} \frac{d}{dx} \frac{1}{2} \int \epsilon(x-y) f(y) w_N(y)^{1/2} dy = f(x),$$

and similarly $(J_N J_N^{-1} f)(x) = f(x)$. As this will always be the case in this work, we will use these relations without mention. The utility of these operators is clear from the following classical relations.

Lemma 2.3. *Let f and g be polynomials. Then*

$$\int (J_N^{-1}f)(x)g(x)w_N(x)dx = \frac{1}{2} \int \int g(x)f(y)\epsilon(x-y)\sqrt{w_N(x)w_N(y)}dxdy,$$

$$\int (J_Nf)(x)g(x)w_N(x)dx = \frac{1}{2} \int (f'(x)g(x) - f(x)g'(x))w_N(x)dx.$$

We note that in view of this lemma, we may express, for even N ,

$$[M_N^1(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)]_{ij} = \int (J_N^{-1}x^{i-1})x^{j-1}w_N(x)dx,$$

$$[M_N^4(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)]_{ij} = \int (J_Nx^{i-1})x^{j-1}w_N(x)dx.$$

To further simplify these, it will be useful to recall the sequence of orthogonal polynomials related to w_N . Namely, let $p_i(x) := p_i(x; N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ denote the unique sequence of polynomials, with p_i of degree- i , such that

$$\int p_i(x)p_j(x)w_N(x)dx = \delta_{ij},$$

and such that if we denote the leading coefficient of p_i as $\kappa_i := \kappa_i(N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$, we have that $\kappa_i > 0$. It is important to note that each p_i is N -dependant. As we will keep N fixed throughout the remainder of this section, we will omit N , as well $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$, from the notation for the p_i below.

We introduce the Christoffel-Darboux kernel

$$K_N(x, y) = \sum_{i=0}^{N-1} p_i(x)p_i(y),$$

as well as the corresponding operator

$$(\Pi_N f)(x) = \int K_N(x, y)f(y)w_N(y)dy.$$

Let us denote the space of polynomials of degree less than m as \mathcal{P}_m . We will denote the space of polynomials as \mathcal{P} . Both of these inherit the inner-product $(f, g)_{w, N} = \int f(x)g(x)w_N(x)dx$. With respect to this inner-product Π_N is the orthogonal projection of \mathcal{P} onto \mathcal{P}_N . We will also use the notation $\Pi_N^\perp = I - \Pi_N$. We introduce an $N \times N$ matrix given for $1 \leq i, j \leq N$ as

$$[\Delta_N]_{ij} = [\Delta_N(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)]_{ij} = \int (J_N^{-1}\Pi_N J_N p_{i-1})(x)p_{j-1}(x)w_N(x)dx. \tag{2.3}$$

We observe that the matrix Δ_N is the representation of the operator $\Pi_N J_N^{-1} \Pi_N J_N \Pi_N$ with respect basis of \mathcal{P}_N given by the orthogonal polynomials. We will often refer to $\Pi_N J_N^{-1} \Pi_N J_N \Pi_N$ as Δ_N when no confusion may arise. We have the following formula.

Lemma 2.4. *For N even, we have that*

$$\left[\frac{\text{pf}(M_N^1(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i))\text{pf}(M_N^4(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i))}{\det(M_N^2(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i))} \right]^2 = \det(\Delta_N(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)).$$

Proof. For simplicity, we will omit $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ from the notation for M_N^β for this proof. By changing to the basis of \mathcal{P}_N given by orthogonal polynomials, we see that $\det(M_N^2) = \prod_{i=0}^{N-1} \kappa_i^{-2}$. In addition, we see that if we define for $1 \leq i, j \leq N$,

$$[M_N^1]_{ij} = \int (J_N^{-1}p_{i-1})(x)p_{j-1}(x)w_N(x)dx, \quad [M_N^4]_{ij} = \int (J_N p_{i-1})(x)p_{j-1}(x)w_N(x)dx,$$

then by again changing the basis, we have that

$$\det(M_{2N}^1) \det(M_N^4) = \left(\prod_{i=0}^{N-1} \kappa_i^{-2} \right)^2 \det(M_N^1) \det(M_N^4).$$

On the other hand, we see that

$$\begin{aligned} [M_N^1 (M_N^4)^T]_{ij} &= \sum_{k=0}^{N-1} \left(\int (J_N^{-1} p_{i-1})(x) p_k(x) w_N(x) dx \right) \left(\int (J_N p_{j-1})(y) p_k(y) w_N(y) dy \right) \\ &= \int (J_N^{-1} p_{i-1})(x) \left(\int K_N(x, y) (J_N p_{j-1})(y) w_N(y) dy \right) w_N(x) dx \\ &= \int (J_N^{-1} p_{i-1})(x) (\Pi_N J_N p_{j-1})(x) w_N(x) dx \\ &= - \int p_{i-1}(x) (J_N^{-1} \Pi_N J_N p_{j-1})(x) w_N(x) dx. \end{aligned}$$

In particular

$$\left[\frac{\text{pf}(M_N^1) \text{pf}(M_N^4)}{\det(M_N^2)} \right]^2 = \det(M_N^1) \det(M_N^4) = (-1)^N \det(\Delta_N).$$

As N is even, this completes the proof. □

These lemmas combined complete the verification of (1.6). To proceed, we must reduce Δ_N to a more manageable form. In view of the identity,

$$\Pi_N J_N^{-1} \Pi_N J_N \Pi_N = \Pi_N - \Pi_N J_N^{-1} [J_N, \Pi_N] \Pi_N \tag{2.4}$$

we see that it suffices to understand the action of $J_N^{-1} [J_N, \Pi_N]$ on \mathcal{P}_N . To do so, we first compute the commutator of $[J_N, \Pi_N]$ on \mathcal{P}_N . For this, we recall the Christoffel-Darboux formula (see [38]):

$$K_N(x, y) = \frac{\kappa_{N-1}}{\kappa_N} \left(\frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y} \right); \quad x \neq y, \tag{2.5}$$

$$K_N(x, x) = \frac{\kappa_{N-1}}{\kappa_N} (p'_N(x)p_{N-1}(x) - p'_{N-1}(x)p_N(x)), \tag{2.6}$$

as well as the classical three-term recurrence

$$xp_j(x) = b_{j-1}p_{j-1}(x) + a_jp_j(x) + b_jp_{j+1}(x), \quad b_j = \frac{\kappa_j}{\kappa_{j+1}}, \quad a_j = \beta_j - \beta_{j+1}, \tag{2.7}$$

where β_j is defined so that $p_j(x)\kappa_j^{-1} = x^j + \beta_jx^{j-1} + \dots$. Additionally, we denote by

$$H_N(f)(y) = \text{p.v} \int \frac{f(x)}{x - y} w_N(x) dx$$

the w_N -weighted Hilbert transform of f , where here p.v denotes the Cauchy principal value integral taken at y . We define, for $1 \leq i \leq m$,

$$\ell_i(x) = e^{N\lambda_i^2} N^{\alpha_i} \frac{\kappa_{N-1}}{\kappa_N} \frac{p_N(x)H_N(p_{N-1})(\lambda_i) - H_N(p_N)(\lambda_i)p_{N-1}(x)}{x - \lambda_i}, \tag{2.8}$$

$$q_i(x) = e^{-N\lambda_i^2} N^{-\alpha_i} K_N(x, \lambda_i). \tag{2.9}$$

The utility of these functions comes from the following decomposition.

Proposition 2.5. For $f \in \mathcal{P}_N$, we have that

$$\begin{aligned} [J_N, \Pi_N]f &= - \sum_{k=1}^m e^{-N\lambda_k^2} N^{-\alpha_k} \alpha_k \ell_k f(\lambda_k) - 2N \Pi_N^\perp(xf(x)) + \Pi_N^\perp(\mathcal{W}'(x)f(x)) \\ &= - \sum_{k=1}^m \alpha_k \ell_k \int q_k(x) f(x) w_N(x) dx - 2N \Pi_N^\perp(xf(x)) + \Pi_N^\perp(\mathcal{W}'(x)f(x)). \end{aligned}$$

Proof of Proposition 2.5. Observing that $\Pi_N f = f$ we see that

$$\begin{aligned} [J_N, \Pi_N] &= \left[\frac{d}{dx}, \Pi_N \right] f - 2N [x, \Pi_N] f + \sum_k \alpha_k \left[\frac{1}{x - \lambda_k}, \Pi_N \right] f + [\mathcal{W}', \Pi_N] f \\ &= \Pi_N^\perp(f') - 2N \Pi_N^\perp(xf) + \Pi_N^\perp(\mathcal{W}'f) + \sum_k \alpha_k \left[\frac{1}{x - \lambda_k}, \Pi_N \right] f. \end{aligned}$$

We note that $\Pi_N^\perp(f') = 0$ so it suffices to evaluate $[(x - \lambda_k)^{-1}, \Pi_N]$. We compute that

$$\begin{aligned} &\left(\left[\frac{1}{x - \lambda_k}, \Pi_N \right] f \right) (x) \\ &= \int \left(\frac{1}{x - \lambda_k} - \frac{1}{y - \lambda_k} \right) \frac{k_{N-1} p_N(x) p_{N-1}(y) - p_N(y) p_{N-1}(x)}{k_N (x - \lambda_k)(y - \lambda_k)} f(y) w_N(y) dy \\ &= - \int \frac{k_{N-1} p_N(x) p_{N-1}(y) - p_N(y) p_{N-1}(x)}{k_N (x - \lambda_k)(y - \lambda_k)} f(y) w_N(y) dy \\ &= - \int \frac{k_{N-1} p_N(x) p_{N-1}(y) - p_N(y) p_{N-1}(x)}{k_N (x - \lambda_k)} \left(\frac{f(y) - f(\lambda_k)}{y - \lambda_k} + \frac{f(\lambda_k)}{y - \lambda_k} \right) w_N(y) dy \\ &= - \int \frac{k_{N-1} p_N(x) p_{N-1}(y) - p_N(y) p_{N-1}(x)}{k_N (x - \lambda_k)(y - \lambda_k)} f(\lambda_k) w_N(y) dy \\ &= -e^{-N\lambda_k^2} N^{-\alpha_k} \ell_k(x) f(\lambda_k), \end{aligned}$$

where in second to last equality we have used that fact that $(f(y) - f(\lambda_k))/(y - \lambda_k)$ is a polynomial of degree less than $N - 1$, so that

$$\int p_N(y) \frac{f(y) - f(\lambda_k)}{y - \lambda_k} w_N(y) dy = \int p_{N-1}(y) \frac{f(y) - f(\lambda_k)}{y - \lambda_k} w_N(y) dy = 0.$$

Together, these statements establish the first equality. To obtain the second equality, we observe that

$$\int q_i(x) f(x) w_N(x) dx = e^{-N\lambda_i^2} N^{-\alpha_i} f(\lambda_i). \tag{2.10}$$

□

We now introduce a modification of Δ_N that will be more computable. For $f \in \mathcal{P}_N$ let

$$\begin{aligned} \Delta_N^0 f &= f + \Pi_N J_N^{-1} \left(\sum_{k=1}^m \alpha_k \ell_k \int q_k(y) f(y) w_N(y) dy \right. \\ &\quad \left. + 2N \Pi_N J_N^{-1} \Pi_N^\perp(xf(x)) - \Pi_N J_N^{-1} \Pi_N^\perp(\mathcal{W}'_0(x)f(x)) \right), \end{aligned} \tag{2.11}$$

and let us define the difference $\Delta_N^1 = \Delta_N - \Delta_N^0$. In sight of Proposition 2.5 and (2.4) only the final terms in the expressions for Δ_N and Δ_N^0 differ, and we see that

$$\Delta_N^1 = -\Pi_N J_N^{-1} \Pi_N^\perp \mathcal{W}'(x) + \Pi_N J_N^{-1} \Pi_N^\perp \mathcal{W}'_0 = -\Pi_N J_N^{-1} \Pi_N^\perp \mathcal{E}', \tag{2.12}$$

where $\mathcal{E} = \mathcal{W} - \mathcal{W}_0$, as before. It is clear that when $\mathcal{W} = 0$, we have that $\Delta_N^0 = \Delta_N$. In general, we shall show that this difference is negligible on the exponential scale.

Proposition 2.6. *Let $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ be as in Theorem 1.2. There exists a choice of $c > 0$ such that*

$$\det(\Delta_N) = \det(\Delta_N^0) + O(e^{-Nc}).$$

Moreover for any choice of compact subset of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$, we may choose $c > 0$ small enough that the error term is uniform over the chosen compact subset.

This proposition will be proven in Section 5. We now turn our attention to the variant Δ_N^0 . Let us write $d = \max(1, \deg(\mathcal{W}_0) - 1)$, and let us define two subspaces of \mathcal{P}_N :

$$\mathcal{P}_{N,2} = \{f \in \mathcal{P}_{N-d} : f(\lambda_1) = \dots = f(\lambda_m) = 0\},$$

and $\mathcal{P}_{N,1} = \mathcal{P}_{N,2}^\perp$, the subspace of \mathcal{P}_N which is orthogonal to $\mathcal{P}_{N,2}$ with respect to the inner-product product of $(*, *)_{w,N}$. We first show a preliminary result about these subspaces.

Lemma 2.7. *If $N > m + d + 1$, then $\mathcal{B}_N = (q_1, \dots, q_m, p_{N-d}, \dots, p_{N-1})$ is a basis of $\mathcal{P}_{N,1}$. Additionally for $f \in \mathcal{P}_{N,2}$, we have that $\Delta_N^0 f = f$.*

Proof. By the Euclidean algorithm for polynomials, we conclude that $\dim(\mathcal{P}_{N,2}) = N - m - d$, so that $\dim(\mathcal{P}_{N,1}) = m + d$. In view of (2.10) we see that $\mathcal{B}_N \subseteq \mathcal{P}_{N,1}$. Thus to show that \mathcal{B}_N is a basis for $\mathcal{P}_{N,1}$, it suffices to show that it is linearly independent. For this, we observe that for $1 \leq i, j \leq m$, and $1 \leq l \leq d$

$$\int q_i(x) \prod_{k=1, k \neq j}^m (x - \lambda_k) w_N(x) dx = \delta_{ij} e^{-N\lambda_i^2} N^{-\alpha_i} \prod_{k=1, k \neq i}^m (\lambda_i - \lambda_k),$$

$$\int p_{N-l}(x) \prod_{k=1, k \neq j}^m (x - \lambda_k) w_N(x) dx = 0.$$

We will derive linear independence from these relations. For this let us assume we have an arbitrary relation

$$\sum_{k=1}^m x_k q_k + \sum_{k=1}^d y_k p_{N-k} = 0, \tag{2.13}$$

for some $x_k, y_k \in \mathbb{R}$. Integrating this expression against $\prod_{k=1, k \neq i}^m (x - \lambda_k) w_N(x)$, then yields

$$x_i \left(e^{-N\lambda_i^2} N^{-\alpha_i} \prod_{k=1, k \neq i}^m (\lambda_i - \lambda_k) \right) = 0.$$

As the second term on the left is non-zero, we conclude that $x_i = 0$ for all $1 \leq i \leq m$. In particular, (2.13) now reads $\sum_{k=1}^d y_k p_{N-k} = 0$. As the p_i are independent by construction, we conclude that $y_k = 0$ for $1 \leq k \leq d$. This demonstrates the desired linear independence.

For the second claim, we observe that if $f \in \mathcal{P}_{N,2}$, then $(2Nx - \mathcal{W}'_0(x))f(x) \in \mathcal{P}_N$, so $\Pi_N^\perp (2Nx - \mathcal{W}'_0(x))f(x) = 0$. We have as well that

$$\sum_{k=1}^m \alpha_k \ell_k \int q_k(y) f(y) w_N(y) dy = \sum_{k=1}^m \alpha_k \ell_k e^{-N\lambda_k^2} N^{-\alpha_k} f(\lambda_k) = 0,$$

so indeed, we see that for $f \in \mathcal{P}_{N,2}$, only the identity term is nonzero, which completes the proof of the second claim. \square

Let us now write the block-decomposition of the operator Δ_N^0 on \mathcal{P}_N with respect to $(\mathcal{P}_{N,1}, \mathcal{P}_{N,2})$ as

$$\Delta_N^0 = \begin{bmatrix} [\Delta_N^0]_{11} & [\Delta_N^0]_{12} \\ [\Delta_N^0]_{21} & [\Delta_N^0]_{22} \end{bmatrix},$$

and similarly for other operators on \mathcal{P}_N . From the second claim of Lemma 2.7, we see that $[\Delta_N^0]_{12} = 0$ and $[\Delta_N^0]_{22} = I$. In particular, we have that $\det(\Delta_N^0) = \det([\Delta_N^0]_{11})$.

We now focus our attention on understanding $\det([\Delta_N^0]_{11})$. To proceed we observe that for $f \in \mathcal{P}_N$, we have that $(2Nx - \mathcal{W}'_0(x))f(x) \in \mathcal{P}_{N+d}$. Noting that for $g \in \mathcal{P}_{N+d}$,

$$(\Pi_N^\perp g)(x) = \sum_{i=1}^d p_{N+i-1}(x) \int p_{N+i-1}(y)g(y)w_N(y)dy,$$

we see that

$$\begin{aligned} & (\Pi_N^\perp (2Ny - \mathcal{W}'_0(y))f(y))(x) \\ &= \sum_{i=1}^d p_{N+i-1}(x) \int p_{N+i-1}(y)(2Ny - \mathcal{W}'_0(y))f(y)w_N(y)dy \\ &= \sum_{i=1}^d \sum_{k=0}^{N-1} p_{N+i-1}(x) \int (2Nz - \mathcal{W}'_0(z))p_{N+i-1}(z)p_k(z)w_N(z)dz \int p_k(y)f(y)w_N(y)dy \\ &= \sum_{i=1}^d \sum_{j=1}^d p_{N+i-1}(x) \int (2Nz - \mathcal{W}'_0(z))p_{N+i-1}(z)p_{N-j}(z)w_N(z)dz \int p_{N-j}(y)f(y)w_N(y)dy, \end{aligned} \tag{2.14}$$

where we have employed the relation $\Pi_N f = f$ in the second equality, and in the third we have observed that $(2Nz - \mathcal{W}'_0(z))p_k(z) \in \mathcal{P}_{k+d+1}$ so that if $k < N - d$ and $i > 0$

$$\int (2Nx - \mathcal{W}'_0(x))p_{N+i-1}(x)p_k(x)w_N(x)dx = 0.$$

If we define d -by- d matrices $M_{N,0}, M_{N,1}$ by setting for $1 \leq i, j \leq d$,

$$\begin{aligned} [M_{N,0}]_{ij} &= \int 2Nz p_{N+i-1}(z)p_{N-j}(z)w_N(z)dz, \\ [M_{N,1}]_{ij} &= - \int \mathcal{W}'_0(z)p_{N+i-1}(z)p_{N-j}(z)w_N(z)dz, \end{aligned}$$

then we may rewrite (2.14) as

$$\Pi_N^\perp (2Ny - \mathcal{W}'_0(y))f(y)(x) = \sum_{i,j=1}^d [M_{N,0} + M_{N,1}]_{ij} p_{N+i-1}(x) \int p_{N-j}(y)f(y)w_N(y)dy. \tag{2.15}$$

To understand the other terms in Δ_N^0 we further define an $(d+m)$ -by- $(d+m)$ matrix, M_N , by

$$M_N = \left(\begin{array}{c|ccc} M_{N,0} + M_{N,1} & & & \\ \hline & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_m \end{array} \right),$$

and define an $(d+m)$ -by- $(d+m)$ inner-product matrix by

$$D_N = \left(\begin{array}{c|c} \int J_N^{-1} p_{N+j-1}(x)p_{N-i}(x)w_N(x)dx & \int J_N^{-1} \ell_l(x)p_{N-i}(x)w_N(x)dx \\ \hline \int J_N^{-1} p_{N+j-1}(x)q_k(x)w_N(x)dx & \int J_N^{-1} \ell_l(x)q_k(x)w_N(x)dx \end{array} \right)$$

where $1 \leq i, j \leq d$ and $1 \leq k, l \leq m$. We now have the following identification.

Lemma 2.8. For $N > d + m + 1$, we have that $\det(\Delta_N^0) = \det(I + D_N M_N)$.

Proof. Let us denote $v_i = p_{N-i}$ for $1 \leq i \leq d$ and $v_i = q_{i-d}$ for $d < i \leq m + d$. By Lemma 2.7 this is a basis for $\mathcal{P}_{N,1}$. We shall denote the matrix for the operator $[\Delta_N^0]_{11}$ with respect to this basis as $\Delta_{N,\mathcal{B}}^0$, so that by definition $\det([\Delta_N^0]_{11}) = \det(\Delta_{N,\mathcal{B}}^0)$. We will also denote $u_i = p_{N+i-1}$ for $1 \leq i \leq d$, and $u_i = \ell_{i-d}$ for $d < i \leq m + d$. Now with (2.15), and recalling the notation $(f, g)_{w,N} = \int f(x)g(x)w_N(x)dx$, we see that for $f \in \mathcal{P}_N$

$$\Delta_N^0 f = f + \sum_{i,j=1}^{d+m} (\Pi_N J_N^{-1} u_i) M_{ij}(v_j, f)_{w,N}. \tag{2.16}$$

Let us define an $(m+d)$ -by- $(m+d)$ matrix O by setting for $1 \leq i, j \leq m+d$, $O_{ij} = (v_j, v_i)_{w,N}$. This matrix is invertible as $(v_i)_{i=1}^{d+m}$ is a linearly independent set. We also observe that

$$(\Delta_N^0 v_j, v_i)_{w,N} = ([\Delta_N^0]_{11} v_j, v_i)_{w,N} = \sum_{k=1}^{d+m} ([\Delta_{N,\mathcal{B}}^0]_{kj} v_k, v_i)_{w,N} = [O \Delta_{N,\mathcal{B}}^0]_{ij}.$$

On the other hand noting that $[D_N]_{ij} = (J_N^{-1} u_j, v_i)_{w,N}$, we see by (2.16) that

$$(\Delta_N^0 v_j, v_i)_{w,N} = O_{ij} + (D_N M_N O)_{ij},$$

so that finally

$$\Delta_{N,\mathcal{B}}^0 = O^{-1}(I + D_N M_N)O.$$

Taking the determinant of both sides completes the proof. □

We will later derive that (see Lemma 4.5) for fixed $k \in \mathbb{Z}$ we have that $b_{N-k}, a_{N-k} = O(1)$. Repeatedly applying (2.7) we may thus conclude that for $1 \leq i, j \leq d$, $[M_{N,1}]_{ij} = O(1)$ and $[M_{N,0}]_{ij} = 2N b_{N-1} \delta_{ij} \delta_{j1} = O(N) \delta_{ij} \delta_{j1}$. Our understanding of D_N comes from the following proposition.

Proposition 2.9. For N even, $1 \leq l, k \leq m$, and $1 \leq i, j \leq d$ we have that

$$\begin{aligned} \int J_N^{-1} p_{N+j-1}(x) p_{N-i}(x) w_N(x) dx &= O(N^{-1}) \\ \int J_N^{-1} \ell_l(x) p_{N-i}(x) w_N(x) dx &= O(N^{-1} \log(N)), \\ \int J_N^{-1} p_{N+j-1}(x) q_k(x) w_N(x) dx &= O(N^{-1/2}), \\ \int J_N^{-1} \ell_l(x) q_k(x) w_N(x) dx &= O(N^{-1/2}) \\ \int J_N^{-1} p_N(x) p_{N-1}(x) w_N(x) dx &= O(N^{-7/6}). \end{aligned}$$

Moreover each error term is uniform over compact subsets of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$.

The proof of this will be carried out in Section 6. We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. As noted above, it suffices to show that $\det(\Delta_N) = 1 + O(N^{-1/6})$. By Proposition 2.6, we see that it further suffices to show that $\det(\Delta_N^0) = \det(I + D_N M_N) = 1 + O(N^{-1/6})$. If we write D_N as a block matrix with square blocks of size 1 by $(d - 1)$ by m , we see that Proposition 2.9 shows that

$$D_N = \begin{bmatrix} O(N^{-7/6}) & O(N^{-1}) & O(N^{-1} \log(N)) \\ O(N^{-1}) & O(N^{-1}) & O(N^{-1} \log(N)) \\ O(N^{-1/2}) & O(N^{-1/2}) & O(N^{-1/2}) \end{bmatrix},$$

where all errors are taken entrywise in each block. Similarly, the above observations for M_N show that

$$M_N = \begin{bmatrix} O(N) & O(1) & 0 \\ O(1) & O(1) & 0 \\ 0 & 0 & O(1) \end{bmatrix}.$$

In particular, we see that

$$I + D_N M_N = \begin{bmatrix} 1 + O(N^{-1/6}) & O(N^{-1}) & O(N^{-1} \log(N)) \\ O(1) & I + O(N^{-1}) & O(N^{-1} \log(N)) \\ O(N^{1/2}) & O(N^{-1/2}) & I + O(N^{-1/2}) \end{bmatrix}.$$

If we denote the diagonal matrix $\Lambda_N = \text{diag}(N^{-2/3}, 1, \dots, 1)$ we see that

$$\begin{aligned} \Lambda_N^{-1}(I + D_N M_N)\Lambda_N &= I + \Lambda_N^{-1} D_N M_N \Lambda_N \\ &= \begin{bmatrix} 1 + O(N^{-1/6}) & O(N^{-1/3}) & O(N^{-1/3} \log(N)) \\ O(N^{-2/3}) & I + O(N^{-1}) & O(N^{-1} \log(N)) \\ O(N^{-1/6}) & O(N^{-1/2}) & I + O(N^{-1/2}) \end{bmatrix}. \end{aligned}$$

Taking the determinant of this expression we see that

$$\det(I + D_N M_N) = \det(\Lambda_N^{-1}(I + D_N M_N)\Lambda_N) = 1 + O(N^{-1/6}).$$

Noting as well that all of the error estimates we have used are uniform in the sense of Theorem 1.2, we obtain the desired statement. \square

2.1 Case of merging singularities

As mentioned above, to prove Theorem 1.1, we will need to supplement Theorem 1.2 with an analogous result in the case in which $m = 2$, $\mathcal{W} = 0$, and $\alpha_1 = \alpha_2 = \alpha$, but where λ_1, λ_2 are allowed to depend on N and $\lambda_1 - \lambda_2 \rightarrow 0$. We will assume this set-up for the remainder of this subsection. We have the following analog of Theorem 1.2.

Proposition 2.10. *Let us fix a choice $\alpha > 0$, $\epsilon > 0$, and $0 < \gamma < 1$. Then for any $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$ (possibly N -dependant) such that $(\lambda_1 - \lambda_2) > N^{-\gamma}$, we have that*

$$\begin{aligned} &\mathbb{E} \left[\prod_{i=1}^2 |D_{2N,1}(\lambda_i)|^\alpha \right] \mathbb{E} \left[\prod_{i=1}^2 |D_{N,4}(\lambda_i)|^{2\alpha} \right] \\ &= \mathbb{E} \left[\prod_{i=1}^2 |D_{2N,2}(\lambda_i)|^{2\alpha} \right] (1 + O(\max(N^{-1/6}, N^{\gamma-1} \log(N)))). \end{aligned} \tag{2.17}$$

Moreover the error term is uniform over all available choices of $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$ with $(\lambda_1 - \lambda_2) > N^{-\gamma}$.

Remark 2.11. We observe that as we have allowed the choice of (λ_1, λ_2) to be N -dependant, the uniformity of the error claimed in Proposition 2.10 follows immediately from the pointwise statement of (2.17).

Remark 2.12. We note that the case of $m = 2$ and $(\lambda_1 - \lambda_2) = 0$ coincides with Theorem 1.2 in the case of $m = 1$ with $\lambda_1 = \lambda_2$ and $\alpha \mapsto 2\alpha$. In particular, by Theorem 1.2, $|\det(\Delta_N)| = 1 + O(N^{-1/6})$. As such, we expect that the apparent divergence of the error-term in Proposition 2.10 when $N(\lambda_1 - \lambda_2) = O(1)$ is an artifact of our method of proof, which avoids using more refined asymptotics for the singularities in the merging case.

All of the constructions of the above section work without modification. We also note that an analog of Proposition 2.6 is not required as $\mathcal{W} = 0$ in this case. We will show in Proposition 4.16 below, that the conclusions of Lemma 4.5 still hold, so that $[M_{N,0}]_{11} = 2Nb_{N-1} = O(N)$ uniformly in the choice of (λ_1, λ_2) . We also have that $M_{N,1} = 0$. Finally, we will need the following modification of Proposition 2.9.

Proposition 2.13. *With the assumptions of Proposition 2.10 and N even, we have for $1 \leq l, k \leq 2$ that*

$$\begin{aligned} \int J_N^{-1} p_N(x) p_{N-1}(x) w_N(x) dx &= O(N^{-1-\min(1-\gamma, 1/6)}), \\ \int J_N^{-1} \ell_l(x) p_{N-1}(x) w_N(x) dx &= |\lambda_1 - \lambda_2|^\alpha O(\max(N^{-1/2-(1-\gamma)}, N^{-1} \log(N))), \\ \int J_N^{-1} p_N(x) q_k(x) w_N(x) dx &= |\lambda_1 - \lambda_2|^{-\alpha} O(N^{-1/2}), \\ \int J_N^{-1} \ell_l(x) q_k(x) w_N(x) dx &= O(\max(N^{-1/2}, N^{\gamma-1} \log(N))). \end{aligned}$$

Moreover all error terms are uniform over all available choices of $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$ with $(\lambda_1 - \lambda_2) > N^{-\gamma}$.

The proof of this result will be given in Section 7.

Proof of Proposition 2.10. As above, it will suffice to show that

$$\det(I + D_N M_N) = 1 + O(N^{-\min(1-\gamma, 1/6)} \log(N)).$$

If we let $\delta = \min(1 - \gamma, 1/6)$ and define the diagonal matrix

$$\Lambda_N = \text{diag}(N^{-1/2-\delta} |\lambda_1 - \lambda_2|^\alpha, 1, 1),$$

then as above, Proposition 2.13 shows that if we write the 3-by-3 matrix $\Lambda_N^{-1} D_N M_N \Lambda_N$ in terms of square blocks of size 1 and 2 we have that

$$I + \Lambda_N^{-1} D_N M_N \Lambda_N = \begin{bmatrix} 1 + O(N^{-\delta}) & O(\max(N^{-(1-\gamma)+\delta}, N^{-1/2+\delta} \log(N))) \\ O(N^{-\delta}) & I + O(\max(N^{-1/2}, N^{\gamma-1} \log(N))) \end{bmatrix}. \quad (2.18)$$

We observe that $\max(N^{-(1-\gamma)+\delta}, N^{-1/2+\delta} \log(N)) \leq 1$ and $\max(N^{-1/2}, N^{\gamma-1} \log(N)) \leq N^{-\delta} \log(N)$ so that (2.18) implies the weaker bound

$$I + \Lambda_N^{-1} D_N M_N \Lambda_N = \begin{bmatrix} 1 + O(N^{-\delta}) & O(1) \\ O(N^{-\delta}) & I + O(N^{-\delta} \log(N)) \end{bmatrix}.$$

Taking a cofactor expansion in the first column we see that

$$\det(I + \Lambda_N^{-1} D_N M_N \Lambda_N) = 1 + O(N^{-\delta} \log(N)),$$

implies the desired claim. □

3 Proof of Theorem 1.1

In this section, we will complete the proof of Theorem 1.1 given in the introduction, assuming Theorem 1.2 and Proposition 2.10. As explained in the introduction, it will suffice to show that the quantity

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\nu_{N,\alpha}(\varphi) - \nu_{N,M,\alpha}(\varphi))^2$$

coincides with the quantity

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\mu_{2N,2,2\alpha}(\varphi) - \mu_{2N,M,2,2\alpha}(\varphi))^2$$

which vanishes for $0 < 2\alpha < \sqrt{2}$ by Proposition 2.9 of [4]. We begin by noting that by classical asymptotics for smooth, compactly-supported linear statistics (see [25]), we have that

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\nu_{N,M,\alpha}(\varphi)^2) = \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}(\mu_{2N,M,2,2\alpha}(\varphi)^2).$$

For the remaining terms, we have that

$$\begin{aligned} & \mathbb{E}(\nu_{N,\alpha}(\varphi)^2) \\ &= \int \int \varphi(x)\varphi(y) \frac{\mathbb{E}|D_{2N,1}(x)|^\alpha |D_{2N,1}(y)|^\alpha \mathbb{E}|D_{N,4}(x)|^{2\alpha} |D_{N,4}(y)|^{2\alpha}}{\mathbb{E}|D_{2N,1}(x)|^\alpha \mathbb{E}|D_{2N,1}(y)|^\alpha \mathbb{E}|D_{N,4}(x)|^{2\alpha} \mathbb{E}|D_{N,4}(y)|^{2\alpha}} dx dy. \end{aligned} \tag{3.1}$$

$$\begin{aligned} & \mathbb{E}(\nu_{N,\alpha}(\varphi)\nu_{N,M,\alpha}(\varphi)) \\ &= \int \int \varphi(x)\varphi(y) \frac{\mathbb{E}|D_{2N,1}(x)|^\alpha e^{\alpha X_{2N,M,1}(y)} \mathbb{E}|D_{N,4}(x)|^{2\alpha} e^{2\alpha X_{N,M,4}(y)}}{\mathbb{E}|D_{2N,1}(x)|^\alpha \mathbb{E}e^{\alpha X_{2N,M,1}(y)} \mathbb{E}|D_{N,4}(x)|^{2\alpha} \mathbb{E}e^{2\alpha X_{N,M,4}(y)}} dx dy. \end{aligned} \tag{3.2}$$

Working first with the latter, we may apply Theorem 1.2 to the integrand of (3.2) to obtain that

$$\begin{aligned} & \mathbb{E}(\nu_{N,\alpha}(\varphi)\nu_{N,M,\alpha}(\varphi)) \\ &= \int \int \varphi(x)\varphi(y) \frac{\mathbb{E}|D_{2N,2}(x)|^{2\alpha} e^{2\alpha X_{2N,M,2}(y)}}{\mathbb{E}|D_{2N,2}(x)|^{2\alpha} \mathbb{E}e^{2\alpha X_{2N,M,2}(y)}} (1 + O(N^{-1/6})) dx dy \\ &= \int \int \varphi(x)\varphi(y) \frac{\mathbb{E}|D_{2N,2}(x)|^{2\alpha} e^{2\alpha X_{2N,M,2}(y)}}{\mathbb{E}|D_{2N,2}(x)|^{2\alpha} \mathbb{E}e^{2\alpha X_{2N,M,2}(y)}} dx dy (1 + O(N^{-1/6})) \\ &= \mathbb{E}(\mu_{2N,2,2\alpha}(\varphi)\mu_{2N,M,2,2\alpha}(\varphi))(1 + O(N^{-1/6})), \end{aligned}$$

where we have used the uniformity of the $O(N^{-1/6})$ -term to obtain the second equality. Thus we are left to understand (3.1). This case is complicated by the non-uniformity of the error of Theorem 1.2 around the diagonal. We will proceed similarly to the proof of Proposition 6.4 of [4]. For any $\epsilon > 0$ and $\beta \in (2\alpha^2, 1)$, we may decompose the integral of (3.1) into three integrals over $\{|x - y| > \epsilon\}$, $\{\epsilon > |x - y| > 2N^{-\beta}\}$, and $\{2N^{-\beta} > |x - y|\}$. We denote these integrals by $A_{1,\epsilon}^N$, $A_{2,\epsilon}^N$ and A_3^N , respectively. Identically to the above case, we can show that for fixed $\epsilon > 0$

$$A_{1,\epsilon}^N = \int \int_{|x-y|>\epsilon} \varphi(x)\varphi(y) \frac{\mathbb{E}|D_{2N,2}(x)|^{2\alpha} |D_{2N,2}(y)|^{2\alpha}}{\mathbb{E}|D_{2N,2}(x)|^{2\alpha} \mathbb{E}|D_{2N,2}(y)|^{2\alpha}} dx dy (1 + O(N^{-1/6})).$$

In view of the proof of Proposition 6.4 of [4], we have that

$$\lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} A_{1,\epsilon}^N = \lim_{N \rightarrow \infty} \mathbb{E}(\mu_{2N,2,2\alpha}(\varphi)^2),$$

so that it suffices to show that $\lim_{N \rightarrow \infty} A_3^N = 0$ and that $\lim_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} A_{2,\epsilon}^N = 0$. To bound A_3^N , we observe by the Cauchy-Schwarz inequality we have that A_3^N is less than

$$\int \int_{2N^{-\beta} > |x-y|} \varphi(x)\varphi(y) \frac{\sqrt{\mathbb{E}|D_{2N,1}(x)|^{2\alpha} \mathbb{E}|D_{N,4}(x)|^{4\alpha} \mathbb{E}|D_{2N,1}(y)|^{2\alpha} \mathbb{E}|D_{N,4}(y)|^{4\alpha}}}{\mathbb{E}|D_{2N,1}(x)|^\alpha \mathbb{E}|D_{N,4}(x)|^{2\alpha} \mathbb{E}|D_{2N,1}(y)|^\alpha \mathbb{E}|D_{N,4}(y)|^{2\alpha}} dx dy.$$

By repeatedly applying Theorem 1.2 we may write the right-hand side as

$$\int \int_{2N^{-\beta} > |x-y|} \varphi(x)\varphi(y) \frac{\sqrt{\mathbb{E}|D_{2N,2}(x)|^{4\alpha} \mathbb{E}|D_{2N,2}(y)|^{4\alpha}}}{\mathbb{E}|D_{2N,2}(x)|^{2\alpha} \mathbb{E}|D_{2N,2}(y)|^{2\alpha}} dx dy (1 + O(N^{-1/6})).$$

By applying the asymptotics of [29] (see also the proof of Proposition 6.4 in [4]) we see that

$$\frac{\sqrt{\mathbb{E}|D_{2N,2}(x)|^{4\alpha}\mathbb{E}|D_{2N,2}(y)|^{4\alpha}}}{\mathbb{E}|D_{2N,2}(x)|^{2\alpha}\mathbb{E}|D_{2N,2}(y)|^{2\alpha}} = O(N^{2\alpha^2-\beta}),$$

uniformly on the support of φ . As $\beta > 2\alpha^2$, we see that $\lim_{N \rightarrow \infty} A_3^N = 0$.

We finally show that $\lim_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} A_{2,\epsilon}^N = 0$. By repeatedly applying Proposition 2.10 as before, we have that

$$A_{2,\epsilon}^N = \int \int_{2N^{-\beta} < |x-y| < \epsilon} \varphi(x)\varphi(y) \frac{\mathbb{E}|D_{2N,2}(x)|^{2\alpha}|D_{2N,2}(y)|^{2\alpha}}{\mathbb{E}|D_{2N,2}(x)|^{2\alpha}\mathbb{E}|D_{2N,2}(y)|^{2\alpha}} dx dy \times (1 + O(N^{-\min(1-\beta, 1/6)} \log(N))).$$

It is shown in the last line of the proof of Proposition 6.4 of [4] that

$$\lim_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} \int \int_{2N^{-\beta} < |x-y| < \epsilon} \varphi(x)\varphi(y) \frac{\mathbb{E}|D_{2N,2}(x)|^{2\alpha}|D_{2N,2}(y)|^{2\alpha}}{\mathbb{E}|D_{2N,2}(x)|^{2\alpha}\mathbb{E}|D_{2N,2}(y)|^{2\alpha}} dx dy = 0.$$

As $\beta < 1$, together these imply that $\lim_{\epsilon \rightarrow 0} \limsup_{N \rightarrow \infty} A_{2,\epsilon}^N = 0$, which completes the proof of Theorem 1.1.

4 Asymptotics of orthogonal polynomials and related quantities

In this section, we state the leading order asymptotics for orthogonal polynomials with respect to the measure w_N , as well as those for some related quantities. Due to the complexity of the expressions of this asymptotics (especially in the regions around λ_i), we will state our expressions as entries in a matrix product. These asymptotics were obtained in our setting by [4], building upon the work of [29]. In both cases, they are obtained via asymptotic analysis of an associated Riemann-Hilbert problem.

We note that away from the points $\{\lambda_i\}_{i=1}^m$, these asymptotics are closely related to the classical Plancherel-Rotach asymptotics for Hermite polynomials (see [38]). In particular, Propositions 4.7 and 4.10 coincide with the corresponding classical expressions for Hermite polynomials except with a modification to the $O(1)$ -term. Around the points $\{\lambda_i\}_{i=1}^m$ though (where the asymptotics are described by Proposition 4.13), they are described in terms of Bessel functions. Some related expressions appear in [30], where they give a description of the asymptotics of the Christoffel-Darboux kernel around these points in terms of the Bessel kernel.

The structure of this section will be as follows. We will begin this section by recalling some relevant background and notation to present Proposition 4.2, which is essentially Theorem 4.37 of [4] restricted to \mathbb{R} . This expression gives (uniform) asymptotics for our desired orthogonal polynomials in terms of a variety of parametric. To make use of these asymptotics, we must recall and compute the behavior of these parametrices on \mathbb{R} . The result of this translation will be our main asymptotic results, which are given by Proposition 4.7, 4.10, and 4.13. After this, we will give asymptotics for ℓ_i and q_i in Proposition 4.14.

Finally, we will provide the modifications of these results needed in the merging case in Subsection 4.1. These results will essentially follow from the results of [8, 10]. The main results are Propositions 4.16 and 4.17. As we are only interested in the subcritical merging regime (i.e., $N(\lambda_1 - \lambda_2) \rightarrow \infty$) our asymptotic expressions will essentially coincide with those in the non-merging case, with the only significant difference being that the bounds on the error terms are weaker, and the domains in which they are valid are smaller.

We will fix for this section a choice of $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ as in Theorem 1.2. We begin by recalling an analytic matrix-valued function on $\mathbb{C} - \mathbb{R}$, $Y_N(z) := Y(z; N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$, defined as

$$Y_N(z) = \begin{bmatrix} \kappa_N^{-1} p_N(z) & \kappa_N^{-1} (2\pi i)^{-1} \int \frac{p_N(y)}{y-z} w_N(y) dy \\ -2\pi i \kappa_{N-1} p_{N-1}(z) & -\kappa_{N-1} \int \frac{p_{N-1}(y)}{y-z} w_N(y) dy \end{bmatrix},$$

where w_N is defined in (1.7). This function occurs as the solution to the Riemann-Hilbert problem defined by the measure w_N , in the sense of [15] (see Proposition 3.4 of [4]). The asymptotics of Y_N are then computed by using the method of nonlinear steepest descent, pioneered in [14]. Our primary purpose is to specialize these asymptotics to \mathbb{R} . To ease the reader in translation of the results of [4], we note that we are in their case of $t = s = 1$ and their $(k, V, \mathcal{T}, \{x_i\}_{i=1}^k, \{\beta_i\}_{i=1}^k)$ coincides with our $(m, 2x^2, 2\mathcal{W}, \{\lambda_{m-i}\}_{i=1}^m, \{2\alpha_i\}_{i=1}^m)$. It is well known in the quadratic case that $\ell_V = \ell$ where $\ell = -1 - 2 \log(2)$, as can easily be verified from (2.3) and (2.4) in [4], and that $d_1(\lambda) = 2/\pi$.

When $x \in \mathbb{R}$ we will use the notation $f_+(x) = \lim_{z \rightarrow x} f(z)$ where the limit is taken along sequences in the upper-half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, within the domain of f , not tangential to \mathbb{R} . We are interested in understanding the asymptotics of $(Y_N)_+$. The asymptotics of Y_N are stated in terms of a related matrix-valued function S_N (defined in (4.5) and (4.11) of [4]). To understand the relation between S_N and Y_N , we recall the 3rd and 1st Pauli matrices defined as

$$\sigma = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then for $z \in \mathbb{H}$, such that $\text{Re}(z) \in \mathbb{R} \setminus \{\lambda_1, \dots, \lambda_m\}$, and $\text{Im}(z)$ is sufficiently small (depending only on $\text{Re}(z)$), we have that

$$Y_N(z) = e^{N \frac{\ell}{2} \sigma} S_N(z) e^{N(g_1(z) - \ell/2)\sigma} \quad \text{for } |\text{Re}(z)| \geq 1, \tag{4.1}$$

$$Y_N(z) = e^{N \frac{\ell}{2} \sigma} S_N(z) \begin{bmatrix} 1 & 0 \\ f_1(z)^{-1} e^{-N h_1(z)} & 1 \end{bmatrix} e^{N(g_1(z) - \ell/2)\sigma} \quad \text{for } |\text{Re}(z)| < 1, \tag{4.2}$$

where $g_1(z)$, $h_1(z)$ and $f_1(z)$ are defined in (4.3), (4.9) and (4.13) of [4]. These follow from Definitions 4.1 and 4.9 of [4]. We will not need the exact form of these functions, though we will need their behavior of \mathbb{R} . For this, we define the functions

$$s(x) = \begin{cases} 2 \int_x^1 \sqrt{1-y^2} dy; & |x| < 1 \\ 2 \int_1^{|x|} \sqrt{y^2-1} dy; & |x| \geq 1 \end{cases}, \quad \omega(x) = \prod_{i=1}^m |x - \lambda_i|^{2\alpha_i}.$$

We observe as well that

$$s(x) = \begin{cases} |x| \sqrt{x^2-1} - \text{arcosh}(|x|); & |x| \geq 1 \\ -x \sqrt{1-x^2} + \arccos(x); & |x| < 1 \end{cases}.$$

We may now state the behavior of the above functions on the real line.

Lemma 4.1. *We have that for $x \in \mathbb{R}$*

$$(g_1)_+(x) = x^2 + \ell/2 + (h_1)_+(x)/2, \quad (h_1)_+(x) = \begin{cases} -2s(x); & x \geq 1 \\ 2\pi i - 2s(x); & x \leq -1 \\ i2s(x); & |x| < 1 \end{cases},$$

and $(f_1)_+(x) = e^{2\mathcal{W}(x)} \omega(x)$.

Proof. The statements for $(h_1)_+$ and $(f_1)_+$ follow routinely from their definitions (i.e. (4.9) and (4.13) of [4]). For the results on $(g_1)_+$, combining the results of (4.6) and (4.7) of [4], we see that for $x \in (-1, 1)$,

$$(g_1)_+(x) = x^2 + \ell/2 + h_+(x)/2 = x^2 + \ell/2 + is(x).$$

Finally, we see by direct computation that for $x \geq 1$,

$$(g_1)_+(x) = \int_{-1}^1 \frac{2}{\pi} \sqrt{1 - \lambda^2} \log(x - \lambda) d\lambda = x^2 + \ell/2 - s(x),$$

and $(g_1)_+(-x) = (g_1)_+(x) + i\pi$. □

By applying the results of Lemma 4.1 to (4.1-4.2) we see that for $x \in (-1, 1) \setminus \{\lambda_1, \dots, \lambda_m\}$

$$(Y_N)_+(x) = e^{N\frac{\ell}{2}\sigma} (S_N)_+(x) \begin{bmatrix} 1 & 0 \\ e^{-2Nis(x)-2\mathcal{W}(x)}\omega(x)^{-1} & 1 \end{bmatrix} e^{Nis(x)\sigma} e^{Nx^2\sigma}, \tag{4.3}$$

and for $|x| \geq 1$ that

$$(Y_N)_+(x) = e^{N\frac{\ell}{2}\sigma} (S_N)_+(x) e^{-Ns(x)\sigma} e^{Nx^2\sigma} (-1)^{NI(x<0)\sigma}. \tag{4.4}$$

We are primarily interested in the first column of Y_N , so noting that $w_N(x) = \omega(x)e^{-2Nx^2+2\mathcal{W}(x)}$, we see that for $x \in (-1, 1) \setminus \{\lambda_1, \dots, \lambda_m\}$

$$\begin{aligned} e^{-N\frac{\ell}{2}\sigma} \begin{bmatrix} [(Y_N)_+]_{11}(x) \\ [(Y_N)_+]_{21}(x) \end{bmatrix} &= (S_N)_+(x) \begin{bmatrix} e^{Nis(x)} \\ e^{-Nis(x)-2\mathcal{W}(x)}\omega(x)^{-1} \end{bmatrix} e^{Nx^2\sigma} \\ &= (S_N)_+(x) e^{Nis(x)\sigma} e^{\mathcal{W}(x)\sigma} \omega(x)^{\sigma/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_N(x)^{-1/2}, \end{aligned} \tag{4.5}$$

and that for $|x| \geq 1$

$$e^{-N\frac{\ell}{2}\sigma} \begin{bmatrix} [(Y_N)_+]_{11}(x) \\ [(Y_N)_+]_{21}(x) \end{bmatrix} = ((S_N)_+(x)e_1) e^{N(x^2-s(x))\sigma} (-1)^{NI(x<0)}. \tag{4.6}$$

Now Theorem 4.37 of [4] implies the following asymptotic result, which shows that up to a small error term R_N , S_N may be described in terms of various explicit parametrices, whose behavior on \mathbb{R} we will specify below. Here and elsewhere, we will use the notation $\lambda_0 = 1$ and $\lambda_{m+1} = -1$.

Proposition 4.2. *There exists $\delta_0 > 0$, such that for $0 < \delta < \delta_0$, we may write*

$$(S_N)_+(x) = (I + R_N(x))(P_N)_+(x),$$

for some matrix-valued function $R_N(x) := R(x; \delta, N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha\}_i)$, and

$$(P_N)_+(x) = \begin{cases} (P_{\lambda_i})_+(x); & x \in (\lambda_i - \delta, \lambda_i + \delta) \\ (P_{\pm 1})_+(x); & x \in (\pm 1 - \delta, \pm 1 + \delta) \\ (P_\infty)_+(x); & \text{otherwise} \end{cases},$$

where here P_∞ , P_{λ_i} , and $P_{\pm 1}$ are certain matrix-valued functions defined in Definition 4.12, 4.21, and 4.28 of [4], respectively. Then, for any $l \geq 0$, we have that $R_N^{(l)}(x) = O(N^{-1})$ uniformly for $x \in \mathbb{R} \setminus \bigcup_{i=0}^{m+1} \{\lambda_i - \delta, \lambda_i + \delta\}$. Moreover for any choice of compact subset of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$, we may choose $\delta > 0$ small enough that $R_N^{(l)}(x) = O(N^{-1})$ uniformly in both $x \in \mathbb{R} \setminus \bigcup_{i=0}^{m+1} \{\lambda_i - \delta, \lambda_i + \delta\}$ and $\{\lambda_i\}_{i=1}^m$ in our chosen compact set.

Remark 4.3. In the statement of Theorem 4.37 in [4], the error bounds are only stated for R_N and R'_N . On the other hand, the method of proof used to establish the error bounds on R'_N extends trivially to higher derivatives.

We note as well that all of these parametrices are N -dependant, though as we will not use them in our final asymptotic expressions, we omit this from the notation. We will compute the values of P_∞ , $P_{\pm 1}$, and P_{λ_i} on \mathbb{R} below, though before beginning this computation we explain how to obtain results for the full range of polynomials p_{N-k} for $k \in \mathbb{Z}$ fixed from Proposition 4.2 instead of only (p_N, p_{N-1}) . As the polynomials $p_{N-k}(x) := p_{N-k}(x; N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ themselves depend on the N -dependant measure w_N , we observe that the $(N - k)$ -case of Proposition 4.2 does not precisely describe an asymptotic for p_{N-k} . On the other hand, we observe that if we define $\eta_k = \eta_{N,k} = \sqrt{(N - k)/N}$, then for $N > k$, and $\mathcal{A} = \sum_{i=1}^m \alpha_i$, the polynomial

$$\eta_k^{A+1/2} p_{N-k}(\eta_k x; N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$$

coincides with the $(N - k)$ -th orthogonal polynomial with respect to the rescaled measure

$$w_{N,k}(x) = w_{N,k}(x; \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) = w_N(\eta x) \eta^{-2A} = e^{2\mathcal{W}(\eta_k x)} e^{-2(N-k)x^2} \prod_{i=1}^m |x - \eta_k^{-1} \lambda_i|^{2\alpha_i}.$$

If we denote $\mathcal{W}_{N,k}(x) = \mathcal{W}(\eta_k x)$, then we see that

$$w_{N,k}(x; \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) = w_{N-k}(x; \mathcal{W}_{N,k}, \{\eta_k^{-1} \lambda_i\}_i, \{\alpha_i\}_i).$$

In particular, we may apply Proposition 4.2 to $w_{N,k}$ to obtain asymptotics for p_{N-k} defined with respect to w_N , though with a number of scaling factors of η_k appearing. For clarity of the notation, we will reserve the notations P_{λ_i} , P_∞ , $P_{\pm 1}$ for the parametrices resulting from the choice of parameters $(N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ and will take care to explicitly specify when objects are considered which are defined with respect to the measure $w_{N,k}$.

Lastly, to account for the factors of η_k appearing in the following asymptotics, we will have repeated use for the following result, which follows from an application of Taylor's Theorem.

Lemma 4.4. For fixed $k \in \mathbb{Z}$, we define $s_{N,k}(x) = s(\eta_k^{-1} x)$. Then if one defined functions r_N such that

$$\begin{aligned} (N - k)s_{N,k}(x) &= Ns(x) - k \arccos(x) + r_N(x) \text{ for } |x| < 1, \\ (N - k)s_{N,k}(x) &= Ns(x) + k \operatorname{arcosh}(|x|) + r_N(x) \text{ for } |x| > 1, \end{aligned}$$

then for any $l \geq 0$, we have that $r_N^{(l)}(x) = O(N^{-1})$ uniformly over compact subsets of $x \in \mathbb{R} \setminus \{\pm 1\}$.

We are now ready to begin stating our asymptotic expressions, beginning with those for some leading-order coefficients of p_{N-k} specified in (2.7).

Lemma 4.5. For fixed $k \in \mathbb{Z}$, we have that

$$\kappa_{N-k}^2 = \pi^{-1} e^{-(N-k)\ell+k} D_\infty^{-2} (1 + O(N^{-1})), \quad \beta_{N+1-k} = O(1), \tag{4.7}$$

where here $D_\infty = 2^{-A} \exp\left(\frac{1}{\pi} \int_{-1}^1 \frac{\mathcal{W}(u)}{\sqrt{1-u^2}} du\right)$. Moreover the error terms are uniform over compact subsets of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$.

Remark 4.6. We observe from Lemma 4.5 that

$$D_\infty^{-\sigma} e^{-N\frac{\ell}{2}\sigma} \begin{bmatrix} \kappa_N^{-1} p_N(x) \\ -2\pi i \kappa_{N-1} p_{N-1}(x) \end{bmatrix} = (I + O(N^{-1})) \begin{bmatrix} \pi^{1/2} p_N(x) \\ -i\pi^{1/2} p_{N-1}(x) \end{bmatrix}, \tag{4.8}$$

where the error term in (4.8) is independent of x and uniform in $\{\lambda_i\}_{i=1}^m$ in the same sense as Lemma 4.5.

For notational clarity, we will delay the proof of Lemma 4.5 to after the statement of Proposition 4.7 below. Now we define functions

$$D(x) = \begin{cases} \exp(i[-\mathcal{A}\arccos(x) + \frac{\sqrt{1-x^2}}{\pi} \text{p.v} \int_{-1}^1 \frac{\mathcal{W}(u)}{\sqrt{1-u^2}} \frac{1}{x-u} du]); & |x| < 1 \\ \exp([- \mathcal{A}\text{arcosh}(x) + \frac{\sqrt{x^2-1}}{\pi} \int_{-1}^1 \frac{\mathcal{W}(u)}{\sqrt{1-u^2}} \frac{1}{x-u} du]); & x > 1 \\ \exp([- \mathcal{A}\text{arcosh}(-x) - \frac{\sqrt{x^2-1}}{\pi} \int_{-1}^1 \frac{\mathcal{W}(u)}{\sqrt{1-u^2}} \frac{1}{x-u} du]); & x < -1 \end{cases} \quad (4.9)$$

We introduce a fixed matrix-valued function $A(x)$, defined for $|x| < 1$ by

$$A(x) = \frac{1}{2(1-x^2)^{1/4}} \begin{bmatrix} e^{-i\pi/4}\sqrt{1+x} + e^{i\pi/4}\sqrt{1-x} & i[e^{-i\pi/4}\sqrt{1+x} - e^{i\pi/4}\sqrt{1-x}] \\ -i[e^{-i\pi/4}\sqrt{1+x} - e^{i\pi/4}\sqrt{1-x}] & e^{-i\pi/4}\sqrt{1+x} + e^{i\pi/4}\sqrt{1-x} \end{bmatrix},$$

and on $|x| > 1$ by

$$A(x) = \frac{1}{2(x^2-1)^{1/4}} \begin{bmatrix} \sqrt{|1+x|} + \sqrt{|1-x|} & i[\sqrt{|1+x|} - \sqrt{|1-x|}] \\ -i[\sqrt{|1+x|} - \sqrt{|1-x|}] & \sqrt{|1+x|} + \sqrt{|1-x|} \end{bmatrix}. \quad (4.10)$$

We now define

$$T_\infty(x) = A(x)D(x)^{-\sigma}. \quad (4.11)$$

Lastly we define an additional function

$$T_{N,k,\infty}(x) = T_\infty(\eta_k^{-1}x; \mathcal{W}_{N,k}, \{\eta_k^{-1}\lambda_i\}_i, \{\alpha_i\}_i).$$

We may now state the asymptotics away from the points $\{\pm 1, \lambda_1, \dots, \lambda_m\}$.

Proposition 4.7. *For $\delta > 0$ small enough, and any choice of $k \in \mathbb{Z}$, we may define $R_N(x)$ such that for any choice of $0 \leq i \leq m$, and $x \in (\lambda_{i+1} + \delta, \lambda_i - \delta)$ we have that*

$$\begin{bmatrix} \pi^{1/2}p_{N-k}(x) \\ -i\pi^{1/2}p_{N-k-1}(x) \end{bmatrix} w_N(x)^{1/2} = (I + R_N(x))T_\infty(x)e^{[-ik\arccos(x) - i\pi \sum_{k=1}^i \alpha_k]\sigma} \begin{bmatrix} e^{iNs(x)} \\ e^{-iNs(x)} \end{bmatrix}, \quad (4.12)$$

and for $|x| > 1 + \delta$ we have that

$$\begin{bmatrix} \pi^{1/2}p_{N-k}(x) \\ -i\pi^{1/2}p_{N-k-1}(x) \end{bmatrix} w_N(x)^{1/2} = [(I + R_N(x))T_{N,k,\infty}(x)e_1]e^{-Ns_{N,k}(x)}(-1)^{(N-k)I(x<0)}. \quad (4.13)$$

Then for any $l \geq 0$ and any choice of compact subset of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$, we may choose $\delta_0 > 0$ small enough that for $0 < \delta < \delta_0$ we have that $R_N^{(l)}(x) = O(N^{-1})$ uniformly in both $x \in \mathbb{R} \setminus \bigcup_{i=0}^{m+1} (\lambda_i - \delta, \lambda_i + \delta)$ and $\{\lambda_i\}_{i=1}^m$ in our chosen compact set.

In the proof of both of these results, we will need to understand the parametric P_∞ given in Definition 4.12 of [4]. While we will not recall the definitions for each of their functions, we will recall some of their real limits for the ease of the reader. Namely the functions r , a , and \mathcal{D}_1 defined in (4.19), (4.20) and (4.21) satisfy for $x \in \mathbb{R}$

$$\begin{aligned} r_+(x) &= \begin{cases} \epsilon(x)\sqrt{x^2-1}; & |x| \geq 1 \\ i\sqrt{1-x^2}; & |x| < 1 \end{cases}, \\ a_+(x) &= \frac{|x-1|}{|x+1|} \begin{cases} 1 & |x| \geq 1 \\ e^{i\pi/4}; & |x| < 1 \end{cases}, \\ (\mathcal{D}_1)_+(x) &= D(x)\omega(x)^{1/2} \begin{cases} 1; & |x| \geq 1 \\ e^{\mathcal{W}(x)}e^{-\pi i \sum_{i=1}^m \alpha_i I(x < \lambda_i)}; & |x| < 1 \end{cases}, \end{aligned}$$

where in the last case for \mathcal{D}_1 we have used the Sokhotski–Plemelj formula: for smooth f and $a < 0 < b$ we have that

$$\lim_{\epsilon \rightarrow 0^+} \int_a^b \frac{f(x)}{x + i\epsilon} dx = -i\pi f(0) + \text{p.v.} \int_a^b \frac{f(x)}{x} dx.$$

From this we can see that for $x \in (-1, 1) \setminus \{\lambda_1, \dots, \lambda_m\}$,

$$(P_\infty)_+(x) = D_\infty^\sigma T_\infty(x) e^{-i\pi \sum_{k=1}^m I(x < \lambda_k) \alpha_k \sigma} e^{-\mathcal{W}(x)\sigma} \omega(x)^{-\sigma/2}, \tag{4.14}$$

and for $x \in \mathbb{R} \setminus (-1, 1)$ we have that

$$(P_\infty)_+(x) = D_\infty^\sigma T_\infty(x) \omega(x)^{-\sigma/2}. \tag{4.15}$$

Proof of Lemma 4.5. The proof will be similar to the proof of the $\mathcal{W} = 0$ case given in Section 4 of [29] (one may also see the proof of this case given in Section 5 of [7]). We will first provide a proof in the special case of $k = 1$. We observe that we may write

$$\kappa_{N-1}^2 = \frac{1}{-2\pi i} \lim_{x \rightarrow \infty} \frac{[(Y_N)_+]_{21}(x)}{x^{N-1}}, \quad \beta_N = \lim_{x \rightarrow \infty} \frac{[(Y_N)_+]_{11}(x) - x^N}{x^{N-1}}.$$

Now employing Proposition 4.2 in view of (4.6) and (4.15), we see that for any small $\delta > 0$, $x > 1 + \delta$, and $j = 1, 2$

$$e^{N\ell/2(-1)^j} [(Y_N)_+]_{j1}(x) = [(I + R_N(x)) D_\infty^\sigma T_\infty(x) \omega(x)^{-\sigma/2}]_{j1} e^{N(x^2 - s(x))}, \tag{4.16}$$

where here $R_N(x) = O(N^{-1})$ uniformly in the sense of Theorem 4.2. Routine application of Taylor’s Theorem shows that

$$\begin{aligned} s(x) &= x\sqrt{x^2 - 1} - \text{arcosh}(x) = x^2 + \ell/2 - \log(x) + O(x^{-2}), \\ \omega(x)^{-1/2} &= x^{-A} \left(1 + \sum_{i=1}^m \frac{\alpha_i \lambda_i}{x} + O(x^{-2}) \right), \\ A(x) &= I + \frac{1}{2x} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} + O(x^{-2}), \\ D(x) &= x^{-A} D_\infty \left(1 + \frac{1}{\pi x} \int_{-1}^1 \frac{\mathcal{W}(u)u}{\sqrt{1-u^2}} du + O(x^{-2}) \right). \end{aligned}$$

From these results, we see that for $j = 1, 2$,

$$\begin{aligned} & [D_\infty^\sigma T_\infty(x) \omega(x)^{-\sigma/2}]_{j1} e^{N(x^2 - s(x))} \\ &= D_\infty^{-(1)^j} [A(x)]_{j1} D(x)^{-1} \omega(x)^{-1/2} e^{N(x^2 - s(x))} \\ &= D_\infty^{-(1)^j-1} \left[I \left(1 + \frac{1}{x} \left[\sum_{i=1}^m \alpha_i \lambda_i - \frac{1}{\pi} \int_{-1}^1 \frac{\mathcal{W}(u)u}{\sqrt{1-u^2}} du \right] \right) + \frac{1}{2x} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} + O(x^{-2}) \right]_{j1} x^N e^{-N\frac{\ell}{2}} \\ &= e^{-N\frac{\ell}{2}} D_\infty^{-(1)^j-1} \left[1 + x^{-1} \left(\sum_{i=1}^m \alpha_i \lambda_i - \frac{1}{\pi} \int_{-1}^1 \frac{\mathcal{W}(u)u}{\sqrt{1-u^2}} du \right) \right]_{j1} x^N + O(x^{N-2}). \end{aligned} \tag{4.17}$$

We now recall from (4.58) of [4] that $R_N(x) = O(|x|^{-1})$. In fact, we additionally have that $R_N(x) = O(N^{-1}|1 + x|^{-1})$, uniformly in $x \in \mathbb{R}$, and compact subsets of $\{\lambda_i\}_{i=1}^m$, as in Proposition 4.2. This is not explicitly stated in [4], but follows from a standard contour deformation argument using the uniform decay of the jump matrix ($\Delta = J_R - I$ in [4]) for large $|z|$, as in the $\mathcal{W} = 0$ case discussed in Section 4 of [29] (one also may consult the detailed argument given in the proof of Proposition 7.5 of [9] in a related case).

Employing this asymptotic, (4.17), and (4.16), we see that

$$\begin{aligned} \kappa_{N-1}^2 &= \frac{1}{-2\pi i} \lim_{x \rightarrow \infty} \frac{[(Y_N)_+]_{21}(x)}{x^{N-1}} = \frac{1}{4\pi} e^{-N\ell} D_\infty^{-2} (1 + O(N^{-1})) \\ &= \pi^{-1} e^{-(N-1)\ell+1} D_\infty^{-2} (1 + O(N^{-1})). \end{aligned}$$

Similarly, we derive that

$$\beta_N = \lim_{x \rightarrow \infty} \frac{[(Y_N)_+]_{11}(x) - x^N}{x^{N-1}} = \sum_{i=1}^m \alpha_i \lambda_i - \frac{1}{\pi} \int_{-1}^1 \frac{\mathcal{W}(u)u}{\sqrt{1-u^2}} du + O(N^{-1}) = O(1).$$

This completes the proof of the $k = 1$ case. To complete the general case, we see that

$$\begin{aligned} \kappa_{N-k-1}(N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) &= \eta_k^{-\mathcal{A}-1/2-(N-k-1)} \kappa_{N-k-1}(N-k, \mathcal{W}_{N,k}, \{\eta_k^{-1}\lambda_i\}_i, \{\alpha_i\}_i), \\ \beta_{N-k}(N, \mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i) &= \eta_k \beta_{N-k}(N-k, \mathcal{W}_{N,k}, \{\eta_k^{-1}\lambda_i\}_i, \{\alpha_i\}_i). \end{aligned}$$

Observing that $\eta_k^{-\mathcal{A}-1/2-(N-k-1)} = e^{k/2}(1 + O(N^{-1}))$ and that

$$D_\infty(\mathcal{W}_{N,k}, \{\eta_k^{-1}\lambda_i\}_i, \{\alpha_i\}_i) = D_\infty(1 + O(N^{-1})),$$

we obtain the general case by applying the $k = 1$ case to $w_{N,k}$. □

Before continuing on to the proof of Proposition 4.7, we record the following useful remark.

Remark 4.8. We observe as $T_\infty(x)$ is smooth on $\mathbb{R} \setminus \{\pm 1\}$ and independent of $\{\lambda_i\}_{i=1}^m$ and N , we may show that for fixed $k \in \mathbb{Z}$, if we define $\bar{R}_N(x)$ so that $T_{N,k,\infty}(x) = T_\infty(x) + \bar{R}_N(x)$, then for $l \in \mathbb{N}$, $\bar{R}_N^{(l)}(x) = O(N^{-1})$ uniformly over both compact subsets of $x \in \mathbb{R} \setminus \{\pm 1\}$ (and of course still independently of $\{\lambda_i\}_{i=1}^m$). On the other hand, taking $R_N(x) = \bar{R}_N(x)T_\infty^{-1}(x)$, we may write

$$T_{N,k,\infty}(x) = (I + R_N(x))T_\infty(x).$$

Now as one may check that $\det(T_\infty(x)) = 1$, we see that the entries of $T_\infty^{-1}(x)$ are also smooth on $\mathbb{R} \setminus \{\pm 1\}$, so we see that R_N satisfies the same estimates as \bar{R}_N .

Proof of Proposition 4.7. We will take $\delta = \delta_0/2$, where δ_0 is the constant given in Proposition 4.2. We will begin with the proof in the case that $k = 0$. By Proposition 4.2, (4.14), and (4.5) we see that for $x \in (\lambda_{i+1} + \delta, \lambda_i - \delta)$

$$\begin{aligned} &D_\infty^{-\sigma} e^{-N\frac{\ell}{2}\sigma} \begin{bmatrix} \kappa_N^{-1} p_N(x) \\ -2\pi i \kappa_{N-1} p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2} \\ &= D_\infty^{-\sigma} (I + \bar{R}_N(x))(P_\infty)_+(x) e^{Nis(x)\sigma} e^{\mathcal{W}(x)\sigma} \omega(x)^{\sigma/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (I + R_N(x))T_\infty(x) e^{-i\pi \sum_{k=1}^i \alpha_k \sigma} \begin{bmatrix} e^{Nis(x)} \\ e^{-Nis(x)} \end{bmatrix}, \end{aligned}$$

where $\bar{R}_N(x)$ is the error term from Proposition 4.2 and $R_N(x) = D_\infty^{-\sigma} \bar{R}_N(x) D_\infty^\sigma$. By Remark 4.6, we see that modifying $R_N(x)$ by the constant matrix $I + O(N^{-1})$ appearing in (4.8), we obtain (4.12) in the $k = 0$ case. Now using the fact that the coefficients D_∞ are N -independent and independent of $\{\lambda_i\}_i$, we see that the function R_N satisfies the same estimates as \bar{R}_N , as stated in Proposition 4.2. In particular, $R_N^{(l)}(x) = O(N^{-1})$ uniformly in the sense needed in Proposition 4.7. Similarly, using (4.15) and (4.6) and the same choice of R_N , we obtain (4.13) in the case of $k = 0$. Together these complete the $k = 0$ case.

To establish the case of $k \neq 0$, with notation as above, we see that by Remark 4.8, Remark 4.6, and the $k = 0$ case applied to the measure $w_{N,k}$, we see for $x \in (\lambda_{i+1} + \delta, \lambda_i - \delta)$,

$$\begin{aligned} D_\infty^{-\sigma} e^{-N \frac{\ell}{2} \sigma} & \left[\begin{array}{c} \kappa_N^{-1} p_N(x) \\ -2\pi i \kappa_{N-1} p_{N-1}(x) \end{array} \right] w_N(x)^{1/2} \\ & = (I + R_N(x)) T_\infty(x) e^{-i\pi \sum_{k=1}^i \alpha_k \sigma} \left[\begin{array}{c} e^{i(N-k)s_{N,k}(x)} \\ e^{-i(N-k)s_{N,k}(x)} \end{array} \right], \end{aligned}$$

where R_N satisfies asymptotics as before.

Now recalling $r_N(x)$ from Lemma 4.4 we see that

$$\begin{aligned} T_\infty(x) e^{-i\pi \sum_{k=1}^i \alpha_k \sigma} & \left[\begin{array}{c} e^{i(N-k)s_{N,k}(x)} \\ e^{-i(N-k)s_{N,k}(x)} \end{array} \right] \\ & = (I + \hat{R}_N(x)) T_\infty(x) e^{[-ik \arccos(x) - i\pi \sum_{k=1}^i \alpha_k] \sigma} \left[\begin{array}{c} e^{iNs(x)} \\ e^{-iNs(x)} \end{array} \right], \end{aligned}$$

where $\hat{R}_N(x) = (I + R_N(x)) T_\infty(x) e^{r_N(x)\sigma} T_\infty(x)^{-1} - I$. As $r_N(x) = O(N^{-1})$, $T_\infty(x)$ is smooth on $(-1, 1)$, and $\det(T_\infty(x)) = 1$, we see from the estimates on R_N that $\hat{R}_N^{(l)}(x) = O(N^{-1})$ for $l \geq 0$. Together, these results complete the proof of (4.12). The general case of (4.13) follows more simply from Remark 4.6 and a similar appeal to the $k = 0$ case. \square

Remark 4.9. for any $\delta > 0$, we may find $c > 0$, such that $s(x) > 2cx^2$ for $x \in \mathbb{R} \setminus [-1 - \delta, 1 + \delta]$. Noting that T_∞ grows only polynomially in x , we see that the asymptotic (4.15) implies that for any choice of $k \in \mathbb{Z}$, $l \geq 0$, and $|x| > 1 + \delta$, we have that

$$p_{N-k}^{(l)}(x) w_N(x)^{1/2} = O\left(e^{-Ncx^2}\right),$$

and additionally, for any compact subset of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$, we may choose $c > 0$ small enough that this error bound is uniform in $|x| > 1 + \delta$ and our chosen compact subset of $\{\lambda_i\}_{i=1}^m$.

We will now turn to describe the asymptotics of the orthogonal polynomials around the points ± 1 . We define functions

$$f_1(x) = \epsilon(x-1) \left(\frac{3}{2}|s(x)|\right)^{2/3}, \quad f_{-1}(x) = f_1(-x). \tag{4.18}$$

It is routine to verify that $f_{\pm 1}$ are defined and smooth in a neighborhood around ± 1 (see also Definition 4.25 of [4]). Moreover, one may verify that $f_{\pm 1}'(\pm 1) = \pm 2$ and that $f_{\pm 1}(\pm 1) = 0$. We also define $f_{N,k,\pm 1}(z) = f_{\pm 1}(\eta_k^{-1}z)$ as before. We will also need additional matrix valued functions

$$T_1(x) = \sqrt{\pi} e^{-i\frac{\pi}{4}} T_\infty(x) e^{\mathcal{W}(x)I(x \geq 1)\sigma} e^{i\frac{\pi}{4}\sigma} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{i\sigma \frac{\pi}{4} I(x < 1)} |f_1(x)|^{\sigma/4}, \tag{4.19}$$

$$T_{-1}(x) = \sqrt{\pi} e^{-i\frac{\pi}{4}} T_\infty(x) e^{\mathcal{W}(x)I(x \leq -1)\sigma} e^{-i\pi \mathcal{A}\sigma} e^{i\frac{\pi}{4}\sigma} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} e^{i\sigma \frac{\pi}{4} I(x > -1)} |f_{-1}(x)|^{\sigma/4}.$$

We are now ready to state the asymptotics around ± 1 . In the following proposition, Ai denotes the Airy function.

Proposition 4.10. For $\delta > 0$ small enough, and any choice of $k \in \mathbb{Z}$, we may find $R_N(x)$ such that for $x \in (1 - \delta, 1 + \delta)$ we have that

$$\begin{aligned} & \left[\begin{array}{c} \pi^{1/2} p_{N-k}(x) \\ -i\pi^{1/2} p_{N-k-1}(x) \end{array} \right] w_N(x)^{1/2} \\ & = (I + R_N(x)) T_1(x) \left[\begin{array}{c} (N-k)^{1/6} \text{Ai}((N-k)^{2/3} f_{N,k,1}(x)) \\ (N-k)^{-1/6} \text{Ai}'((N-k)^{2/3} f_{N,k,1}(x)) \end{array} \right], \end{aligned} \tag{4.20}$$

and for $x \in (-1 - \delta, -1 + \delta)$ we have that

$$\begin{aligned} & \begin{bmatrix} \pi^{1/2} p_{N-k}(x) \\ -i\pi^{1/2} p_{N-k-1}(x) \end{bmatrix} w_N(x)^{1/2} \\ &= (I + R_N(x)) T_{-1}(x) \begin{bmatrix} (N-k)^{1/6} \text{Ai}((N-k)^{2/3} f_{N,k,-1}(x)) \\ -(N-k)^{-1/6} \text{Ai}'((N-k)^{2/3} f_{N,k,-1}(x)) \end{bmatrix} (-1)^{N-k}. \end{aligned} \quad (4.21)$$

Additionally, for any $l \geq 0$, and any choice of compact subset of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$, we may choose $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ we have that $R_N^{(l)}(x) = O(N^{-1})$ uniformly in both $x \in (-1 - \delta, -1 + \delta) \cup (1 - \delta, 1 + \delta)$ and $\{\lambda_i\}_{i=1}^m$ in our chosen compact set.

To prove this, we will have to recall the form of $P_{\pm 1}$. We begin with the case of P_1 , which is given in Definition 4.28 of [4]. We observe that for the function $\xi_1(z)$ defined in (4.42) of [4], we have that $(\xi_1)_+(x) = N^{2/3} f_1(x)$, and additionally we have that the function defined in (4.28) of [4] has $(\phi_1)_+(x) = (h_1)_+(x)/2$. Employing these observations and Lemma 4.1 we derive from Definition 4.28 of [4] that for $x \in (1 - \delta, 1 + \delta)$

$$(P_1)_+(x) = F_+(x) Q_+(N f_1(x)) e^{(I(x \geq 1) - iI(x < 1))s(x)\sigma} \omega(x)^{-\sigma/2} e^{-\mathcal{W}(x)\sigma}, \quad (4.22)$$

$$F_+(x) = (P_\infty)_+(x) e^{\mathcal{W}(x)\sigma} \omega(x)^{\sigma/2} e^{i\frac{\pi}{4}\sigma} \sqrt{\pi} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{i\sigma \frac{\pi}{4} I(x < 1)} |f_1(x)|^{\sigma/4} N^{\sigma/6} e^{-i\pi/12},$$

$$Q_+(\zeta) = \begin{bmatrix} \text{Ai}(\zeta) & \text{Ai}(\omega^2 \zeta) \\ \text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta) \end{bmatrix} e^{-\pi i \sigma / 6}; \quad \zeta > 0,$$

$$Q_+(\zeta) = \begin{bmatrix} \text{Ai}(\zeta) & \text{Ai}(\omega^2 \zeta) \\ \text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta) \end{bmatrix} e^{-\pi i \sigma / 6} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; \quad \zeta < 0,$$

where here $\omega = e^{2\pi i/3}$. Now employing (4.14) and (4.15) we see that

$$\begin{aligned} F_+(x) &= D_\infty^\sigma T_\infty(x) e^{\mathcal{W}(x)I(x \geq 1)} e^{i\frac{\pi}{4}\sigma} \sqrt{\pi} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{i\sigma \frac{\pi}{4} I(x < 1)} |f_1(x)|^{\sigma/4} N^{\sigma/6} e^{-i\pi/12} \\ &= D_\infty^\sigma e^{i\pi/6} T_1(x) N^{\sigma/6}. \end{aligned} \quad (4.23)$$

We now proceed to the -1 case. While not defined in [4], the definition is a similar modification of the equation obtained in the $\mathcal{W} = 0$ case considered in (80) of [29] (or similarly, from the $+1$ case for the reversed measure $w_N(-x, \mathcal{W}(-x), \{-\lambda_i\}_{i=1}^m, \{\alpha_i\}_{i=1}^m)$. In particular, we obtain by a similar argument that for $x \in (-1 - \delta, -1 + \delta)$

$$\begin{aligned} (P_{-1})_+(x) &= \hat{F}_+(x) \sigma Q_-(N f_{-1}(x)) \sigma e^{(I(x \leq -1) - iI(x > -1))s(x)\sigma} \\ &\quad \times \omega(x)^{-\sigma/2} e^{-\mathcal{W}(x)\sigma} (-1)^{N\sigma}, \end{aligned} \quad (4.24)$$

$$\bar{F}_+(x) = D_\infty^\sigma e^{i\frac{\pi}{6}} T_{-1}(x) N^{\sigma/6}, \quad (4.25)$$

$$\sigma Q_-(\zeta) \sigma = \begin{bmatrix} \text{Ai}(\zeta) & \omega^2 \text{Ai}(\omega \zeta) \\ -\text{Ai}'(\zeta) & -\text{Ai}'(\omega^2 \zeta) \end{bmatrix} e^{-\pi i \sigma / 6}; \quad \zeta \geq 0,$$

$$\sigma Q_-(\zeta) \sigma = \begin{bmatrix} \text{Ai}(\zeta) & \omega^2 \text{Ai}(\omega \zeta) \\ -\text{Ai}'(\zeta) & -\text{Ai}'(\omega^2 \zeta) \end{bmatrix} e^{-\pi i \sigma / 6} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; \quad \zeta < 0.$$

Remark 4.11. We observe that $F(z)$ ($\bar{F}(z)$) is analytic in a neighborhood around 1 (-1) respectively (see Lemma 4.30 of [4]). In particular, from (4.23) and (4.25) we see that $T_{\pm 1}(z)$ is as well. We note that $T_{\pm 1}$ is independent of $\{\lambda_i\}_i$ and N . As we also have $|\det(T_{\pm 1}(x))| = 4\pi$, the same argument as in Remark 4.8 shows that for $k \in \mathbb{Z}$, there is $R_N(x)$ such that for $x \in (\pm 1 - \delta, \pm 1 + \delta)$

$$T_{\pm 1}(\eta_k^{-1} x; \mathcal{W}_{N,k}, \{\eta_k^{-1} \lambda_i\}_i, \{\alpha_i\}_i) = (I + R_N(x)) T_{\pm 1}(x),$$

where for any $\ell \geq 0$, $R_N^{(\ell)}(x) = O(N^{-1})$ uniformly in both $x \in (\pm 1 - \delta, \pm 1 + \delta)$ (and again independently of the choice of $\{\lambda_i\}_{i=1}^m$).

Proof of Proposition 4.10. The proof is essentially the same as that of Proposition 4.7. In particular, employing Proposition 4.2, (4.14), (4.5), (4.6) and Remark 4.6 as in the proof of Proposition 4.7, we obtain the $k = 0$ case routinely from the expansions for $P_{\pm 1}$ given by (4.22), (4.24), (4.23), and (4.25). The proof of the arbitrary k follows similarly by employing Remark 4.11. \square

Finally, we will state the asymptotics in the region around the points $\{\lambda_1, \dots, \lambda_m\}$. Define for $x \in (-1, 1)$, the function

$$f_x(z) = s(x) - s(z) = 2 \int_x^z \sqrt{1 - y^2} dy. \tag{4.26}$$

This function is smooth and invertible in a neighborhood of x , and we see that $f'_x(x) = 2\sqrt{1 - x^2}$. As before, we define $f_{N,k,x}(z) = f_{\eta_k^{-1}x}(\eta_k^{-1}z)$. We define a matrix-valued function for $1 \leq j \leq m$ by

$$T_{N,k,\lambda_j}(x) = e^{-i\frac{\pi}{4}} T_\infty(x) e^{i\frac{\rho_{N,j,k}}{2}\sigma} e^{-i\frac{\pi}{4}\sigma} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \tag{4.27}$$

where $\rho_{N,j,k}$ denotes

$$\rho_{N,j,k} = 2(N - k)s_{N,k}(\lambda_j) - \pi\alpha_j - 2\pi \sum_{i=1}^{j-1} \alpha_i.$$

For simplicity, we will denote $T_{N,0,\lambda_j}(x) = T_{N,\lambda_j}(x)$ and $\rho_{N,j,0} = \rho_{N,j}$.

Remark 4.12. Unlike $T_\infty(x)$ and $T_{\pm 1}(x)$, the function $T_{N,k,\lambda_j}(x)$ depends on N . On the other hand, this is only through the factor $\exp(i\rho_{N,j,k}\sigma/2)$, whose norm is N -independent. Thus by Remark 4.8 we see that $T_{N,k,\lambda_j}(x)$ is smooth on $(-1, 1)$, and that for any $\delta > 0$, and any $l \geq 0$, $T_{N,k,\lambda_j}^{(l)}(x)$ is uniformly bounded jointly in N , $x \in (\lambda_j - \delta, \lambda_j + \delta)$ and any chosen compact subset of $\{\lambda_i\}_{i=1}^m \in (-1, 1)^m$. Observing that $|\det(T_{N,k,\lambda_j}(x))| = 1$, we see additionally that the same is true for $T_{N,k,\lambda_j}^{-1}(x)$. Lastly, combining these observations with Lemma 4.4, we see that for any $k \in \mathbb{Z}$, there is $R_N(x)$ such that

$$T_{N,k,\lambda_j}(x) = (I + R_N(x))T_{N,\eta_k^{-1}\lambda_j}(\eta_k^{-1}x; \mathcal{W}_{N,k}, \{\eta_k^{-1}\lambda_i\}_i, \{\alpha_i\}_i),$$

where for any $l \geq 0$, $R_N^{(l)}(x) = O(N^{-1})$ uniformly both $x \in (\lambda_j - \delta, \lambda_j + \delta)$ and compact subsets of $\{\lambda_i\}_{i=1}^m \in (-1, 1)^m$.

Lastly, we define the following vector of special functions

$$\mathcal{J}_\alpha(x) = \begin{bmatrix} \epsilon(x)\sqrt{\pi|x|}J_{\alpha+1/2}(|x|) \\ \sqrt{\pi|x|}J_{\alpha-1/2}(|x|) \end{bmatrix}, \tag{4.28}$$

where here J_ν denotes the Bessel function of parameter ν .

Proposition 4.13. For $\delta > 0$ small enough, and any choice of $k \in \mathbb{Z}$, we may find $R_N(x)$ such that for any choice of $1 \leq i \leq m$ and $x \in (\lambda_i - \delta, \lambda_i + \delta)$ we have that

$$\begin{bmatrix} \pi^{1/2}p_{N-k}(x) \\ -i\pi^{1/2}p_{N-k-1}(x) \end{bmatrix} w_N(x)^{1/2} = (I + R_N(x))T_{N,k,\lambda_i}(x)\mathcal{J}_\alpha((N - k)f_{N,k,\lambda_i}(x)). \tag{4.29}$$

Additionally, for any $l \geq 0$, and any choice of compact subset of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$, we may choose $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ we have that $R_N^{(l)}(x) = O(N^{-1})$ uniformly in both $x \in \bigcup_{k=1}^m (\lambda_k - \delta, \lambda_k + \delta)$ and $\{\lambda_i\}_{i=1}^m$ in our chosen compact set.

As before, this result will follow from analyzing the parametrix P_{λ_j} defined in Definition 4.21 of [4]. We observe that the function defined in (4.27) of [4] satisfies

$$(W_j)_+(x) = e^{\mathcal{W}(x)} \omega(x)^{1/2} e^{-i\pi\alpha_j \epsilon(x-\lambda_j)}.$$

Employing this we see that for $x \in (\lambda_j - \delta, \lambda_j + \delta)$

$$(P_{\lambda_j})_+(x) = (E_j)_+(x) (\Psi_{\alpha_j})_+(Nf_{\lambda_j}(x)) e^{-\mathcal{W}(x)\sigma} \omega(x)^{-\sigma/2} e^{i\pi\alpha_j \epsilon(x-\lambda_j)\sigma} e^{-Nis(x)\sigma}, \quad (4.30)$$

$$(E_j)_+(x) = (P_\infty)_+(x) e^{\mathcal{W}(x)\sigma} \omega(x)^{\sigma/2} e^{-i\pi\alpha_j \epsilon(x-\lambda_j)\sigma/2} e^{iNs(\lambda_j)\sigma} e^{-i\frac{\pi}{4}\sigma} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix},$$

and where Ψ_α is a fixed matrix-valued function (defined in (4.26-4.33) of [39]) and satisfying

$$(\Psi_\alpha)_+(\zeta) = \frac{\sqrt{\pi}}{2} \sqrt{\zeta} \begin{bmatrix} H_{\alpha+1/2}^{(2)}(\zeta) & -iH_{\alpha+1/2}^{(1)}(\zeta) \\ H_{\alpha-1/2}^{(2)}(\zeta) & -iH_{\alpha-1/2}^{(1)}(\zeta) \end{bmatrix} e^{-(\alpha+1/4)\pi i\sigma}; \quad \zeta > 0,$$

$$(\Psi_\alpha)_+(\zeta) = \frac{\sqrt{\pi}}{2} \sqrt{-\zeta} \begin{bmatrix} iH_{\alpha+1/2}^{(1)}(-\zeta) & -H_{\alpha+1/2}^{(2)}(-\zeta) \\ -iH_{\alpha-1/2}^{(1)}(-\zeta) & H_{\alpha-1/2}^{(2)}(-\zeta) \end{bmatrix} e^{(\alpha+1/4)\pi i\sigma}; \quad \zeta < 0,$$

where here $H_\beta^{(1)}$ and $H_\beta^{(2)}$ denote the Hankel functions of the first and second kind at parameter β , respectively. Employing (4.14) we see that we may write

$$(E_j)_+(x) = D_\infty^\sigma e^{i\frac{\pi}{4}} T_{N,\lambda_j}(x). \quad (4.31)$$

Proof of Proposition 4.13. As we have that $H_\beta^{(1)}(x) + H_\beta^{(2)}(x) = 2J_\beta(x)$ (see Section 9.1 of [1]), we see that for $\zeta \in \mathbb{R}$

$$(\Psi_\alpha)_+(\zeta) e^{i\pi\alpha\epsilon(\zeta)\sigma} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-i\pi/4} \mathcal{J}_\alpha(\zeta).$$

Combining this with (4.30), (4.31), and observing that $\epsilon(x - \lambda_j) = \epsilon(Nf_{\lambda_j}(x))$, we see that

$$(P_{\lambda_j})_+(x) e^{Nis(x)\sigma} e^{\mathcal{W}(x)\sigma} \omega(x)^{\sigma/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = D_\infty^\sigma e^{i\frac{\pi}{4}} T_{N,\lambda_j}(x) e^{-i\frac{\pi}{4}} \mathcal{J}_{\alpha_j}(Nf_{\lambda_j}(x))$$

$$= D_\infty^\sigma T_{N,\lambda_j}(x) \mathcal{J}_{\alpha_j}(Nf_{\lambda_j}(x)).$$

Now employing Proposition 4.2, (4.5), and Remark 4.6 we obtain the $k = 0$ case of (4.29). The proof for arbitrary k follows from Remark 4.12. \square

Together these results will allow us to uniformly describe the behavior of p_{N-k} on \mathbb{R} . What remains is to understand the behavior of ℓ_i and q_i introduced in (2.8) and (2.9) for $1 \leq i \leq m$. To do so, we introduce the following matrix

$$V_{N,\lambda_i} = e^{N\lambda_i^2\sigma} N^{\alpha_i\sigma} \begin{bmatrix} -\kappa_{N-1} H(p_{N-1})(\lambda_i) & -\kappa_N^{-1} (2\pi i)^{-1} H(p_N)(\lambda_i) \\ 2\pi i \kappa_{N-1} p_{N-1}(\lambda_i) & \kappa_N^{-1} p_N(\lambda_i) \end{bmatrix}.$$

The utility of this matrix comes from the following relation

$$(x - \lambda_i) \begin{bmatrix} -\ell_i(x) \\ 2\pi i q_i(x) \end{bmatrix} = V_{N,\lambda_i} \begin{bmatrix} \kappa_N^{-1} p_N(x) \\ -2\pi i \kappa_{N-1} p_{N-1}(x) \end{bmatrix}. \quad (4.32)$$

In particular, in view of the above understanding of p_N and p_{N-1} , to understand ℓ_i and q_i , we only need to provide asymptotics for V_{N,λ_i} .

Proposition 4.14. For $1 \leq j \leq m$, we have that

$$V_{N,\lambda_j} = c_j^{-\sigma} e^{i\frac{\pi}{4}} e^{-\frac{\pi i}{4}\sigma} \sigma_1 T_{N,\lambda_j}(\lambda_j)^{-1} D_\infty^{-\sigma} (I + O(N^{-1})) e^{-N\frac{\ell}{2}\sigma},$$

where here we denote

$$c_j = \frac{\sqrt{2\pi}(1 - \lambda_j^2)^{\alpha_j/2}}{\Gamma(\alpha_j + 1/2)} e^{-W(\lambda_j)} \prod_{k \neq j} |\lambda_k - \lambda_j|^{-\alpha_k}.$$

Moreover the error term is uniform over compact subsets of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$.

Remark 4.15. We observe that by combining Remark 4.6, Proposition 4.14, and (4.32) we see that for $1 \leq i \leq m$ there are constants $C_{k,N}$ for $1 \leq k \leq 4$, such that

$$\begin{aligned} \ell_i(x) &= C_{1,N} \frac{p_N(x)}{x - \lambda_i} + C_{2,N} \frac{p_{N-1}(x)}{x - \lambda_i}, \\ q_i(x) &= C_{3,N} \frac{p_N(x)}{x - \lambda_i} + C_{4,N} \frac{p_{N-1}(x)}{x - \lambda_i}, \end{aligned}$$

where for all i , we have that $C_{i,N} = O(1)$ uniformly in compact subsets of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$.

Proof. By the Christoffel-Darboux formula (2.5), we see that $\det(Y_N(z)) = 1$. From this, we observe that for $1 \leq j \leq m$

$$V_{N,\lambda_j} = e^{N\lambda_j^2\sigma} N^{\alpha_j\sigma} (Y_N)_+(\lambda_j)^{-1}. \tag{4.33}$$

Additionally, we observe the matrix relation

$$e^{-W(x)\sigma} \omega(x)^{-\sigma/2} e^{-Nis(x)\sigma} \begin{bmatrix} 1 & 0 \\ e^{-2Nis(x)-2W(x)} \omega(x)^{-1} & 1 \end{bmatrix} e^{N(is(x)+x^2)\sigma} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} w_N(x)^{-\sigma/2}.$$

Using this relation and Proposition 4.2, (4.3), (4.30) and (4.31), we see that for $x \in (\lambda_j, \lambda_j + \delta)$

$$(Y_N)_+(x) = e^{N\frac{\ell}{2}\sigma} (I + R_N(x)) D_\infty^\sigma e^{i\frac{\pi}{4}} T_{N,\lambda_j}(x) (\Psi_{\alpha_j})_+(Nf_{\lambda_j}(x)) e^{i\pi\alpha_j\sigma} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} w_N(x)^{-\sigma/2}, \tag{4.34}$$

where R_N is the remainder term from Proposition 4.2. We recall from Section 4.3 of [39] (more precisely, the analytic continuation of their (4.21)) that for $\zeta > 0$:

$$(\Psi_\alpha)_+(\zeta) \begin{bmatrix} 1 & 0 \\ e^{-2\pi i\alpha} & 1 \end{bmatrix} = \sqrt{\zeta} \begin{bmatrix} \sqrt{\pi} I_{\alpha+1/2}(-i\zeta) & -\frac{1}{\sqrt{\pi}} K_{\alpha+1/2}(-i\zeta) \\ -i\sqrt{\pi} I_{\alpha-1/2}(-i\zeta) & -\frac{i}{\sqrt{\pi}} K_{\alpha-1/2}(-i\zeta) \end{bmatrix} e^{-i\frac{\alpha}{2}\pi\sigma}, \tag{4.35}$$

where here I_α and K_α are modified Bessel functions of the first and second kind, respectively, where both are taken with respect to the principal branch given by $\arg(z) \in (-\pi, \pi)$. The asymptotics of the right-hand side of (4.35) as $\zeta \rightarrow 0$ are computed in (112) of [29] as

$$e^{i\frac{\pi}{4}} \left(\begin{bmatrix} 0 & -1 \\ -i & 0 \end{bmatrix} + C_\alpha(\zeta) \right) \left(\frac{\sqrt{2\pi}\zeta^\alpha}{\Gamma(\alpha + 1/2)2^\alpha} \right)^\sigma e^{-i\pi\alpha\sigma},$$

where $C_\alpha(\zeta) = O(\zeta \log(|\zeta|))$ entrywise as $\zeta \rightarrow 0$. We note as well that

$$e^{-i\pi\alpha\sigma} \begin{bmatrix} 1 & 0 \\ e^{-2\pi i\alpha} & 1 \end{bmatrix}^{-1} e^{i\pi\alpha\sigma} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = I.$$

Recalling that both $R_N(x)$ and $T_{N,\lambda_j}(x)$ are continuous at λ_j , we see by (4.34) that

$$\begin{aligned} (Y_N)_+(\lambda_j) &= e^{N\frac{\epsilon}{2}\sigma}(I + R_N(\lambda_j))D_\infty^\sigma e^{i\frac{\pi}{4}}T_{N,\lambda_j}(\lambda_j)e^{i\frac{\pi}{4}} \begin{bmatrix} 0 & -1 \\ -i & 0 \end{bmatrix} \\ &\quad \times \left(\frac{\sqrt{2\pi}(Nf'_{\lambda_j}(\lambda_j))^{\alpha_j}}{2^{\alpha_j}\Gamma(\alpha_j + 1/2)} \right)^\sigma \left(\prod_{k \neq j} |\lambda_j - \lambda_k|^{-\alpha_k\sigma} \right) e^{-\mathcal{W}(\lambda_j)\sigma} e^{N\lambda_j^2\sigma} \\ &= e^{N\frac{\epsilon}{2}\sigma}(I + R_N(\lambda_j))D_\infty^\sigma T_{N,\lambda_j}(\lambda_j) \begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix} c_j^\sigma N^{\alpha_j\sigma} e^{N\lambda_j^2\sigma}. \end{aligned}$$

where in the last step we have used that $f'_x(x) = 2\sqrt{1-x^2}$. Combining this with (4.33) and observing that

$$\begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}^{-1} = e^{i\frac{\pi}{4}} e^{-\frac{\pi i}{4}\sigma} \sigma_1$$

completes the proof. □

4.1 Modifications for merging singularities

We will conclude this section by considering the modifications that need to be made when we are in the case of merging singularities required in Proposition 2.10. In particular, for the rest of this section assume that we are in the case of $m = 2$, $\mathcal{W} = 0$, $\alpha_1 = \alpha_2 = \alpha > 0$, and are allowing λ_1, λ_2 to depend on N . Analysis of the Riemann-Hilbert problem with such merging singularities is undertaken in [8], where they modify the above parametrices to obtain asymptotics in this case. We will primarily be reliant on the results on merging singularities in [8, 10], as well as the discussion in [4]. As we only require the supercritical case for merging singularities (i.e., we consider $|\lambda_1 - \lambda_2|^{-1} = O(N^\gamma)$ where $\gamma < 1$), all the above results will still hold only with worse bounds on the error terms and different domains.

Away from the points $\{\lambda_1, \lambda_2\}$ though, the above asymptotics hold essentially as stated.

Proposition 4.16. *Let $\epsilon > 0$. Then there is $\delta_0 > 0$, such that for $0 < \delta < \delta_0$, and any choice of $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$ (possibly N -dependant) with $\delta > \lambda_1 - \lambda_2 > 0$, there is an error term R_N , such that for $l \geq 0$, we have that $R_N^{(l)}(x) = O(N^{-1})$ uniformly in both $x \in \mathbb{R} \setminus \{\pm 1 - \delta, \pm 1 + \delta\} \cup [\lambda_2 - \delta, \lambda_1 + \delta]$ and the choice (λ_1, λ_2) , and such that the following asymptotics hold: For $x \in (-1 + \delta, 1 - \delta) \setminus (\lambda_2 - \delta, \lambda_1 + \delta)$, we have (4.12), for $|x| \geq 1 + \delta$, we have (4.13), for $x \in (1 - \delta, 1 + \delta)$, we have (4.20), and for $x \in (-1 - \delta, -1 + \delta)$, we have (4.21). Lastly, (4.7) and thus (4.8) hold as well, with error term uniform in the choice of (λ_1, λ_2) as well.*

For the remaining regions around λ_1 and λ_2 , the asymptotics of Proposition 4.13 and Proposition 4.12 occur in a smaller region and have a worse error bound.

Proposition 4.17. *Let $\epsilon > 0$ and $0 < \gamma < 1$. Then there is $\delta_0 > 0$, such that for $0 < \delta < \delta_0$, and any choice of $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$ (possibly N -dependant) with $\delta > \lambda_1 - \lambda_2 > N^{-\gamma}$, the following holds. Denoting $r_N = 2(\lambda_1 - \lambda_2)$, there is an error term R_N , such that for $l \geq 0$, we have that $R_N^{(l)}(x) = O(r_N^{-1})$ uniformly in $x \in (\lambda_2 - \delta, \lambda_1 + \delta)$ and the choice of (λ_1, λ_2) , and such that the following asymptotics hold: For $x \in (\lambda_2 - \delta, \lambda_2 - \delta r_N^{-1}) \cup (\lambda_2 + \delta r_N^{-1}, \lambda_1 - \delta r_N^{-1}) \cup (\lambda_1 + \delta r_N^{-1}, \lambda_1 + \delta)$, we have (4.12), and for $1 \leq i \leq 2$, and $x \in (\lambda_i - \delta r_N^{-1}, \lambda_i + \delta r_N^{-1})$, we have that (4.29) holds. Additionally, the asymptotics of Proposition 4.14 holds with error term now of order $O(N^{\gamma-1})$, but now uniform in the choice of (λ_1, λ_2) .*

We will now begin to establish the prerequisites for the proofs for these statements. For the remainder of the section we will denote $u = (\lambda_1 - \lambda_2)^2/4$ and $v = (\lambda_1 + \lambda_2)/2$. As

before, this will essentially follow from recalling and simplifying various parametrices. More specifically, in [8] they give asymptotics of Y_N in the merging case in terms of a related matrix-valued function “ S ”, which we denote by S_N^M to avoid confusion with the above S_N . To aid the reader in the translation of their notation, their (n, t, V, a, b) coincides with our $(N, u, -2(x - v)^2, 1 - v, -1 - v)$. The definition of S_N^M in [8] is given in (4.10) and (5.1), only employing Fig. 8 rather than Fig. 7, as explained in their Section 6.1. The construction is almost identical to the one in [4] (which we used above) with the only non-notational change being that for $x \in (\lambda_1, \lambda_2)$ the function S_N^M is defined by (4.4) rather than (4.3) (see the difference between Figure 1 of [4] and Figure 8 of [8]). In particular, the function $(S_N^M)_+$ may be given in terms of the above $(S_N)_+$ as follows: For $x \notin [\lambda_1, \lambda_2]$, $(S_N^M)_+(x) = (S_N)_+(x)$, and for $x \in (\lambda_1, \lambda_2)$, we have that

$$(S_N^M)_+(x) = (S_N)_+(x) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{4.36}$$

They then, as above, solve this by employing a sequence of parametrices. We are only concerned with the case that $t > 0$, which is dealt with in Section 6 of [8]. They construct parametrices N , $P^{(a)}$, $P^{(b)}$, and P , where $P^{(a)}$ is defined in a neighborhood of 1, $P^{(b)}$ is defined in a neighborhood of -1 , P is defined in a neighborhood of $[\lambda_2, \lambda_1]$, and N is defined in the remainder. These first three though are defined as $P^{(\infty)}$, $P^{(1)}$, and $P^{(-1)}$ in [4], which coincide with our P_∞ , P_1 , and P_{-1} . In particular, they prove the following result.

Proposition 4.18. *Let us fix $\epsilon > 0$. Then there is $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, and any choice of $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$ (possibly N -dependant) with $\delta > (\lambda_1 - \lambda_2) > 0$. Then there is R_N such that*

$$(S_N^M)_+(x) = (I + R_N(x))(P_\delta)_+(x),$$

where here

$$(P_\delta)_+(x) = \begin{cases} (P_{u,v})_+(x); & x \in (\lambda_2 - \delta, \lambda_1 + \delta) \\ (P_{\pm 1})_+(x); & x \in (\pm 1 - \delta, \pm 1 + \delta) \\ (P_\infty)_+(x); & \text{otherwise} \end{cases},$$

and where for any $l \geq 0$, we have that $R_N^{(l)}(x) = O(N^{-1})$ uniformly in $x \in \mathbb{R} \setminus \{\pm 1 - \delta, \pm 1 + \delta, \lambda_2 - \delta, \lambda_1 + \delta\}$ and the choice of (λ_1, λ_2) .

This result for R_N essentially follows from Section 6.5 of [8], though they only state the error bounds for R_N . On the other hand, they actually establish that R_N satisfies a small norm Riemann-Hilbert problem, with N -independent contours, whose jump matrices are $O(N^{-1})$ uniformly in a N -independent neighborhood of the contours. In particular, applying a contour deformation argument and the Cauchy integral formula as in the proof of Theorem 4.37 in [4] (see also the discussion in Section 4 of [29]), we obtain the desired error bounds on $R_N^{(l)}$ for all $l \geq 0$. In addition, while the results of [8] establish uniformly only in u , the extension to uniformity in v is shown in Appendix G of [4].

As the parametrices P_∞ and $P_{\pm 1}$ here coincide with those considered in the previous subsection exactly, replacing Proposition 4.2 with Proposition 4.18 as appropriate in the proofs in the previous subsection gives a proof of Proposition 4.16 (noting that $(S_N)_+ = (S_N^M)_+$ in all regions considered in this proposition).

On the other hand, the parametrix $P_{u,v}$, which is defined in Section 6.4 of [8], does not directly coincide with any of the parametrices considered above. It is primarily reliant on the special family of parametrices studied in [10], which involve more complicated special functions than those considered above. Moreover, when one lets $(\lambda_1 - \lambda_2) = O(N^{-1})$, they

appear to be quite different from a simple combination of the parametrices considered above.

Luckily though, we do not need to contend with these parametrices directly, as the asymptotics of the parametrix family of [10] is already considered in [10]. Up to a variety of translations, these relate the parametrix $P_{u,v}$, to those considered above, up to an error term now dependent on $(\lambda_1 - \lambda_2)$. In particular, the result of this translation will be the following result, whose proof will comprise the majority of the remainder of the section.

Lemma 4.19. *With the set-up of Proposition 4.18, there is $\delta_1 > 0$, such that if $(\lambda_1 - \lambda_2) > \delta_1 N^{-1}$, then there is $R_N(x)$ such that*

$$\begin{aligned} (P_{u,v})_+(x) &= (I + R_N(x))(P_\infty)_+(x); & x \notin (\lambda_2 - \sqrt{u}\delta, \lambda_1 + \sqrt{u}\delta), \\ (P_{u,v})_+(x) &= (I + R_N(x))(P_\infty)_+(x) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; & x \in (\lambda_2 + \sqrt{u}\delta, \lambda_1 - \sqrt{u}\delta), \\ (P_{u,v})_+(x) &= (I + R_N(x))(P_{\lambda_1})_+(x); & x \in (\lambda_1, \lambda_1 + \sqrt{u}\delta), \\ (P_{u,v})_+(x) &= (I + R_N(x))(P_{\lambda_1})_+(x) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; & x \in (\lambda_1 - \sqrt{u}\delta, \lambda_1), \\ (P_{u,v})_+(x) &= (I + R_N(x))(P_{\lambda_2})_+(x); & x \in (\lambda_2 - \sqrt{u}\delta, \lambda_2), \\ (P_{u,v})_+(x) &= (I + R_N(x))(P_{\lambda_2})_+(x) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; & x \in (\lambda_2, \lambda_2 + \sqrt{u}\delta), \end{aligned}$$

where for any $l \geq 0$, we have that $R_N^{(l)}(x) = O(u^{-1/2}N^{-1})$ uniformly in both x in these regions, and in the choice of (λ_1, λ_2) .

Recalling (4.36) for $x \in [\lambda_1, \lambda_2]$, we see that the combined asymptotics of this lemma and Proposition 4.19 essentially coincide with those in Proposition 4.2 around $\{\lambda_1, \lambda_2\}$, except that now that choice of δ is now N -dependant and the error terms are now of order $O(N^{-1}u^{-1/2})$ instead of $O(N^{-1})$. Thus repeating the proofs given above, replacing $O(N^{-1})$ by $O(N^{-1}u^{-1/2}) = O(N^{-1}r_N)$ when necessary is sufficient to prove Proposition 4.17.

The remainder of this subsection will consist of the proof of Lemma 4.19. As previously remarked, this Lemma is essentially a special case of the asymptotics considered in Section 5 of [10], though we will have to trace a variety of notations from [8, 10] and perform some simple manipulations to obtain this. For the convenience of the reader, we remark that a similar translation is given in Appendix G of [4], whose notation we will borrow when possible.

In preparation, let us define for $x \in \mathbb{R}$

$$\lambda(x) = N \left(\frac{s(\lambda_1) + s(\lambda_2)}{2} - s(x) \right), \quad s_N = 2Ni(s(\lambda_2) - s(\lambda_1)).$$

We recall a form for the parametrix $P_{u,v}$, which is defined in Section 6.4 of [8]. For $x \in (v - \delta_0, v + \delta_0)$

$$\begin{aligned} (P_{u,v})_+(x) &= E_+(x)\Psi_+^{(2)}(\lambda(x); s_N)\omega(x)^{\sigma/2}e^{Nis(x)\sigma}(\sigma\sigma_1), \\ E_+(x) &= D_\infty^\sigma T_\infty(x)(\sigma\sigma_1)^{-1}e^{i\pi\alpha\sigma}e^{-iN\frac{s(\lambda_1)+s(\lambda_2)}{2}\sigma}, \end{aligned}$$

where $\Psi^{(2)}(\lambda, s)$ is a fixed matrix-valued function defined in Section 3 of [8], which depends only on α and s . We will need the asymptotics of $\Psi_+^{(2)}$ as $s \rightarrow \infty$. To get these, we note that in view of (3.11) and (3.5) of [8], we may write

$$\Psi^{(2)}(x, s) = \Psi_{CK} \left(\frac{-4x}{|s|}i; s \right) \begin{cases} e^{-\epsilon(x)\pi i\alpha\sigma}; & |x| > |s|/4 \\ I; & |x| < |s|/4 \end{cases}, \quad (4.37)$$

where Ψ_{CK} is the solution to the model Riemann-Hilbert problem introduced in Section 3 of [10] with parameters $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = 0$. In Section 5 of [10] uniform asymptotics for Ψ_{CK} in the case of $|s| \rightarrow \infty$ are derived. To demonstrate these, the Riemann-Hilbert problem for Ψ_{CK} is solved using elementary transformations and yet another auxiliary Riemann-Hilbert problem M . We note for the convenience of the reader that in the case relevant to us (i.e., $\beta = 0$), M is expressed explicitly in terms of Hankel and modified Bessel functions in (2.47) of [8]. In particular, we see that

$$M_+(x) = (\sigma\sigma_1)e^{-i\frac{\pi}{4}\sigma} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} (\Psi_\alpha)_+(x)(\sigma\sigma_1)^{-1}e^{-i\frac{\pi}{2}\alpha i\sigma}. \tag{4.38}$$

The following asymptotics of $\Psi^{(2)}(x; s)$ follow from those in Section 5 of [10] and (4.37).

Lemma 4.20. *There exists $s_0 > 0$ and $\delta_0 > 0$ such that for any $s \geq s_0$ and $\delta_0 > \delta > 0$, there is \bar{R}_s , such that*

$$\begin{aligned} \Psi_+^{(2)}(x; s) &= (I + \bar{R}_s(x))e^{ix\sigma}e^{-i\alpha\pi\sigma\epsilon(x)}; \quad |x| > \frac{|s|}{4} + |s|\delta \\ \Psi_+^{(2)}(x; s) &= (I + \bar{R}_s(x))e^{ix\sigma} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \quad |x| < \frac{|s|}{4} - |s|\delta \\ \Psi_+^{(2)}(x; s) &= (I + \bar{R}_s(x))e^{-i\pi\alpha/2\sigma}e^{-i|s|/4\sigma}M(x)e^{\frac{3}{2}\alpha i\pi\sigma}; \quad \frac{|s|}{4} \leq x \leq \frac{|s|}{4} + |s|\delta \\ \Psi_+^{(2)}(x; s) &= (I + \bar{R}_s(x))e^{-i\pi\alpha/2\sigma}e^{-i|s|/4\sigma}M(x)e^{-\frac{1}{2}\alpha i\pi\sigma} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \quad \frac{|s|}{4} - |s|\delta \leq x \leq \frac{|s|}{4} \\ \Psi_+^{(2)}(x; s) &= (I + \bar{R}_s(x))e^{-\frac{3i\pi}{2}\alpha\sigma}e^{i|s|/4\sigma}M(x)e^{-\frac{1}{2}\alpha\pi\sigma}; \quad \frac{|s|}{4} \leq -x \leq \frac{|s|}{4} + |s|\delta \\ \Psi_+^{(2)}(x; s) &= (I + \bar{R}_s(x))e^{-\frac{3i\pi}{2}\alpha\sigma}e^{i|s|/4\sigma}M(x)e^{\frac{3}{2}\alpha i\pi\sigma} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}; \quad \frac{|s|}{4} - |s|\delta \leq -x \leq \frac{|s|}{4}, \end{aligned}$$

and such that for $l \geq 0$, we have that $\bar{R}_s^{(l)}(x) = O(s^{-l}(s + |x|)^{-1})$ uniformly in $s \in (s_0, \infty)$ and x in the regions above.

Remark 4.21. The factor of s^{-l} occurring in the error term in Lemma 4.20 is due to the s^{-1} -scaling in (4.37). Indeed if one denotes by \hat{R}_s the error term for the corresponding error term in the expansion of $\Psi_{CK}(x; s)$, one has that for any $l \geq 0$, $\hat{R}_s^{(l)}(x) = O(s^{-1}(1 + |x|)^{-1})$ (see (5.25) of [10]).

Proof of Lemma 4.19. We recall that $\sigma_1\sigma = -\sigma\sigma_1$, so that $a^\sigma\sigma_1 = \sigma a^{-\sigma}$. We also note that

$$(\sigma\sigma_1)^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} (\sigma\sigma_1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

With these results and (4.38), all of these expressions follow by routine computation. For example, we observe for $x \in (\lambda_1, \lambda_1 + \sqrt{u}\delta)$ that (potentially changing δ) $\lambda(x) \in (|s|/4, |s|/4 + \delta|s|)$ so that

$$\begin{aligned} (P_{u,v})_+(x) &= D_\infty^\sigma T_\infty(x)(\sigma\sigma_1)^{-1}e^{i\pi\alpha\sigma}e^{-iN\frac{s(\lambda_1)+s(\lambda_2)}{2}\sigma} \\ &\quad \times (I + \bar{R}_s(\lambda(x)))e^{iN\frac{s(\lambda_1)+s(\lambda_2)}{2}\sigma}e^{-i\alpha\pi\sigma}\omega(x)^{\sigma/2}(\sigma\sigma_1). \end{aligned}$$

Noting that $\bar{R}_{s_N}(\lambda(x)) = O(u^{-1/2}N^{-1})$ we see that conjugating \bar{R}_{s_N} as above

$$(P_{u,v})_+(x) = (I + O(N^{-1}u^{-1/2}))D_\infty^\sigma T_\infty(x).$$

All claims follow from similar computations. □

5 Proof of Proposition 2.6

In this section, we will prove Proposition 2.6. As \mathcal{E} vanishes on a neighborhood of $[-1, 1]$, the integrals which compose the entries of Δ_N^1 are supported on regions where the integrand is exponentially small (see Remark 4.9). Combining this with some rough estimates for the elements of Δ_N^0 (see Proposition 5.1), we will be able to obtain Proposition 2.6 by employing bounds for the determinant under perturbation.

Now, as before we fix a choice of $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ satisfying the hypotheses of Theorem 1.2. For a linear operator A on an inner-product space, let $\|A\|_F = \sqrt{\text{Tr}(A^*A)}$ denote the Frobenius norm. We recall the operators Δ_N^1 and Δ_N^0 (defined in (2.11) and (2.12)) of Section 2, which act on \mathcal{P}_N with the inner-product $(f, g)_{w,N} = \int f(x)g(x)w_N(x)dx$. The main technical result of this section is the following proposition.

Proposition 5.1. *There are $C, c > 0$, such that*

$$\|\Delta_N^1\|_F = O(e^{-Nc}), \quad \|\Delta_N^0\|_F = O(N^C).$$

Moreover for any choice of compact subset of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$, we may choose C, c such that these estimates are uniform on the chosen compact set.

We will now give the proof of Proposition 2.6 assuming Proposition 5.1.

Proof of Proposition 2.6. We recall from Lemma 2.7 that $[\Delta_N^0]_{22} = I$, so that $[\Delta_N]_{22} = I + [\Delta_N^1]_{22}$. We observe that by Proposition 5.1, we have that $\|[\Delta_N^1]_{22}\|_F = O(e^{-Nc})$. Thus by the Neumann series for the inverse, we see that for sufficiently large N $[\Delta_N]_{22}$ is invertible and that $\|[\Delta_N]_{22}^{-1} - I\|_F = O(e^{-Nc})$. By the Schur complement formula, we have that

$$\det(\Delta_N) = \det([\Delta_N]_{22}) \det([\Delta_N]_{11} - [\Delta_N]_{12}[\Delta_N]_{22}^{-1}[\Delta_N]_{21}). \tag{5.1}$$

Now recall that $[\Delta_N^0]_{12} = 0$ so that $[\Delta_N]_{12} = [\Delta_N^1]_{12}$. Thus we have that

$$\begin{aligned} \|[\Delta_N^1]_{11} - [\Delta_N]_{12}[\Delta_N]_{22}^{-1}[\Delta_N]_{21}\|_F &= \|[\Delta_N^1]_{11} - [\Delta_N^1]_{12}[\Delta_N]_{22}^{-1}[\Delta_N]_{21}\|_F \\ &\leq \|[\Delta_N^1]_{11}\|_F + \|[\Delta_N^1]_{12}\|_F \|[\Delta_N]_{22}^{-1}\|_F \|[\Delta_N]_{21}\|_F. \end{aligned}$$

The work above shows that $\|[\Delta_N]_{22}^{-1}\|_F = \|I\|_F + O(e^{-Nc}) = O(N)$, and by Proposition 5.1 we have that $\|[\Delta_N^1]_{11}\|_F, \|[\Delta_N^1]_{12}\|_F = O(e^{-Nc})$ and $\|[\Delta_N]_{21}\|_F = O(N^c)$. Combining these with the above inequality, we see that

$$\|[\Delta_N^1]_{11} - [\Delta_N]_{12}[\Delta_N]_{22}^{-1}[\Delta_N]_{21}\|_F = O(e^{-Nc/2}). \tag{5.2}$$

We recall the bound on ℓ -by- ℓ matrices A and B (see Theorem 2.12 of [24])

$$|\det(A) - \det(B)| \leq \ell \|A - B\|_F (\max(\|A\|_F, \|B\|_F))^{\ell-1}.$$

Recalling that $\dim(\mathcal{P}_{N,1}) = m + d$, we see that by applying this inequality, the bound of (5.2), and Proposition 5.1

$$|\det([\Delta_N]_{11} - [\Delta_N]_{12}[\Delta_N]_{22}^{-1}[\Delta_N]_{21}) - \det([\Delta_N^0]_{11})| = O(e^{-Nc/4}). \tag{5.3}$$

We will further need the bound on ℓ -by- ℓ matrices A (see Corollary 2.14 of [24])

$$|\det(I + A) - 1| \leq (\ell \|A\|_F + 1)^\ell - 1.$$

Employing the formula $[\Delta_N]_{22} = I + [\Delta_N^1]_{22}$, as well as Proposition 5.1, this bound yields that $|\det([\Delta_N]_{22}) - 1| = O(e^{-Nc/2})$. Using this, (5.1), and (5.3), we have that

$$\det(\Delta_N) = \det([\Delta_N^0]_{11})(1 + O(e^{-Nc/2})) + O(e^{-Nc/4}).$$

Finally applying Proposition 5.1 to see that $\det([\Delta_N^0]_{11}) = O(e^{Nc/4})$, and recalling that $\det([\Delta_N^0]_{11}) = \det(\Delta_N^0)$ completes the proof. \square

The remainder of the section will be spent on the proof of Proposition 5.1. We first note that by Proposition 2.5 we have that

$$\|\Delta_N^1\|_F = \|\Pi_N J_N^{-1} \Pi_N^\perp \mathcal{E}' \Pi_N\|_F \leq \|\Pi_N J_N^{-1} \mathcal{E}' \Pi_N\|_F + \|\Pi_N J_N^{-1} \Pi_N \mathcal{E}' \Pi_N\|_F, \tag{5.4}$$

$$\|\Delta_N^0\|_F \leq \|I\|_F + 2N \|\Pi_N J_N^{-1} \Pi_N x \Pi_N\|_F + \sum_{i=1}^m |\alpha_i| \|\Pi_N J_N^{-1} \ell_i q_i^T \Pi_N\|_F, \tag{5.5}$$

where here, the transpose f^T is taken with respect to the inner-product on \mathcal{P}_N inherited from $(*, *)_{w,N}$. Thus it suffices to compute the order of all terms on the right. The terms $\|\Pi_N J_N^{-1} \mathcal{E}' \Pi_N\|_F$, $\|\Pi_N J_N^{-1} \Pi_N \mathcal{E}' \Pi_N\|_F$ and $\|\Pi_N J_N^{-1} \Pi_N x \Pi_N\|_F$ will all follow from the same method, which we now illustrate. Let A be a symmetric operator on \mathcal{P}_N with Schwartz kernel $Af(x) = \int G(x, y) f(y) w_N(y) dy$. Then evaluating the Frobenius norm with respect to the basis given by orthogonal polynomials, we see that

$$\begin{aligned} \|A\|_F^2 &= \sum_{i=0}^{N-1} \int \int \int G(x, y) G(x, z) p_i(y) p_i(z) w_N(x) w_N(y) w_N(z) dx dy dz \\ &= \int \int \int G(x, y) G(x, z) K_N(y, z) w_N(x) w_N(y) w_N(z) dx dy dz \end{aligned} \tag{5.6}$$

where K_N is the Christoffel-Darboux kernel. Let us denote

$$\mathcal{K}_N(x, y) = w_N(x)^{1/2} w_N(y)^{1/2} K_N(x, y).$$

In the sense above, the operator $\Pi_N J_N^{-1} \mathcal{E}' \Pi_N$ has kernel

$$G(x, y) = w_N(x)^{-1/2} \left(\frac{1}{2} \int \mathcal{K}_N(x, z) \epsilon(z - y) dz \right) \mathcal{E}'(y) w_N^{-1/2}(y)$$

on \mathcal{P}_N . We may employ (5.6) to obtain

$$\begin{aligned} &\|\Pi_N J_N^{-1} \mathcal{E}' \Pi_N\|_F^2 \\ &= \frac{1}{4} \int \cdots \int \mathcal{K}_N(x_1, x_2) \epsilon(x_2 - x_3) \mathcal{E}'(x_3) \mathcal{K}_N(x_1, x_4) \epsilon(x_4 - x_5) \mathcal{E}'(x_5) \mathcal{K}_N(x_3, x_5) \prod_{i=1}^5 dx_i. \end{aligned}$$

By taking the absolute value of the integrand, we may upper-bound this quantity by

$$\frac{1}{4} \left(\int \int \int |\mathcal{K}_N(x_1, x_2) \mathcal{K}_N(x_1, x_4)| dx_1 dx_2 dx_4 \right) \left(\int \int |\mathcal{E}'(x_3) \mathcal{E}'(x_5) \mathcal{K}_N(x_3, x_5)| dx_3 dx_5 \right). \tag{5.7}$$

By the Cauchy-Schwarz inequality we have that

$$|\mathcal{K}_N(x, y)| \leq \mathcal{K}_N(x, x)^{1/2} \mathcal{K}_N(y, y)^{1/2}.$$

Repeatedly applying this bound to (5.7), and observing that $\int \mathcal{K}_N(x, x) dx = N$, we see that $\|\Pi_N J_N^{-1} \mathcal{E}' \Pi_N\|_F^2 \leq \frac{N}{4} I_1^2 I_2^2$, where

$$I_1 = \int \mathcal{K}_N(w, w)^{1/2} dw, \quad I_2 = \int \mathcal{K}_N(w, w)^{1/2} |\mathcal{E}'(w)| dw.$$

To compute these integrals let us denote $\varphi_{N,1}(x) = p_N(x) w_N(x)^{1/2}$ and $\varphi_{N,2}(x) = p_{N-1}(x) w_N(x)^{1/2}$. We note that by Remark 4.9 for any $\delta > 0$ there is $c > 0$ small enough so that for $|x| > 1 + \delta$

$$\frac{d}{dx} \varphi_{N,i}(x), \varphi_{N,i}(x) = O(\exp(-cNx^2)).$$

By the Christoffel-Darboux formula (2.6) and Lemma 4.5, we see that $\mathcal{K}_N(x, x) = O(\exp(-cNx^2))$ uniformly on $|x| > 1 + \delta$. Now applying the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} I_1 &= \int_{|w|>1+\delta} \mathcal{K}_N(w, w)^{1/2} dw + \int_{|w|<1+\delta} \mathcal{K}_N(w, w)^{1/2} \\ &\leq \int_{|w|>1+\delta} \mathcal{K}_N(w, w)^{1/2} dw + (2 + 2\delta)^{1/2} \left(\int \mathcal{K}_N(w, w) dw \right)^{1/2}. \end{aligned} \tag{5.8}$$

The first integral on the right-hand side of (5.8) is seen to be of order $O(e^{-Nc/4})$ using the above asymptotic, and the second is exactly $(2 + 2\delta)^{1/2} N^{1/2}$, so that $I_1 = O(N^{1/2})$. When applying the same trick to I_2 , the first term is $O(e^{-Nc/4})$, and the latter vanishes, so that $I_2 = O(e^{-Nc/4})$, so that $\|\Pi_N J_N^{-1} \mathcal{E}' \Pi_N\|_F^2 = O(e^{-Nc/8})$. The same method works to show that $\|\Pi_N J_N^{-1} \Pi_N \mathcal{E}' \Pi_N\|_F = O(e^{-Nc/4})$ and that $\|\Pi_N J_N^{-1} \Pi_N x \Pi_N\|_F = O(N^5)$.

We now focus our attention to $\|\Pi_N J_N^{-1} \ell_i q_i^T \Pi_N\|_F$. In this case we see that, recalling (2.10) and applying again the inequality $|\mathcal{K}_N(x, y)| \leq \mathcal{K}_N(x, x)^{1/2} \mathcal{K}_N(y, y)^{1/2}$,

$$\begin{aligned} &\|\Pi_N J_N^{-1} \ell_i q_i^T \Pi_N\|_F^2 \\ &= e^{-2N\lambda_i^2} N^{-2\alpha_i} K_N(\lambda_i, \lambda_i) \int \int K_N(x_1, x_2) J_N^{-1} \ell_i(x_1) J_N^{-1} \ell_i(x_2) w_N(x_1) w_N(x_2) dx_1 dx_2 \\ &\leq e^{-2N\lambda_i^2} N^{-2\alpha_i} K_N(\lambda_i, \lambda_i) \left(\int \mathcal{K}_N(x, x)^{1/2} \left| \epsilon(\ell_i w_N^{1/2})(x) \right| dx \right)^2, \end{aligned}$$

where here

$$\epsilon(f)(x) = \frac{1}{2} \int \epsilon(x - y) f(y) dy.$$

To understand these quantities we will first employ Proposition 4.29 to write, for $x \in (\lambda_i - \delta, \lambda_i + \delta)$ and $k = 1, 2$

$$\varphi_{N,k}(x) = G_{N,k}(x) \mathcal{J}_{\alpha_i}(Nf_{\lambda_i}(x)), \quad G_{N,k}(x) = i^{I(k=2)} \pi^{-1/2} e_k^T (I + R_N(x)) T_{N,\lambda_i}(x), \tag{5.9}$$

where e_k is the k -th standard coordinate vector. We also recall some classical asymptotics of Bessel functions (see Chapter 9 of [1]): for $\nu > -1$ and $x \in (0, \infty)$, we have that

$$x^{-\nu} J_\nu(x), \quad \frac{d}{dx} \left(x^{-\nu} J_\nu(x) \right) = O \left((1+x)^{-\nu-1/2} \right),$$

where both the errors are uniform in $x \in (0, \infty)$. From this, and recalling the definition of $\mathcal{J}_\nu(x)$ from (4.28) we see that for $\nu > -1/2$ and for $x \in \mathbb{R}$,

$$x^{-\nu} \mathcal{J}_\nu(x), \quad \frac{d}{dx} \left(x^{-\nu} \mathcal{J}_\nu(x) \right) = O \left((1+|x|)^{-\nu+1/2} \right), \tag{5.10}$$

where again, the error term is uniform in $x \in \mathbb{R}$. We will denote $\hat{\varphi}_{N,k}(x) = \varphi_{N,k}(x) |x - \lambda_i|^{-\alpha_i}$, which removes the singular part of $\varphi_{N,k}$ around λ_i coming from $w_N(x)^{1/2}$. Now employing (5.10) and Remark 4.12 to the representation in (5.9), we see that for $x \in (\lambda_i - \delta, \lambda_i + \delta)$ and $k = 1, 2$, we have that,

$$\hat{\varphi}_{N,k}(x), \quad \frac{d}{dx} \hat{\varphi}_{N,k}(x) = O(N^{\alpha_i+2}), \tag{5.11}$$

again uniformly in x and compact subsets of $\{\lambda_i\}_{i=1}^m \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$. Applying this to the Christoffel-Darboux formula (2.6), we obtain that

$$e^{-2N\lambda_i^2} N^{-2\alpha_i} K_N(\lambda_i, \lambda_i) = O(N^8). \tag{5.12}$$

We now focus on the term $\epsilon(\ell_i w_N^{1/2})(x)$. We note that

$$\left| \int \epsilon(x-y) \frac{\varphi_{N,k}(y) dy}{y-\lambda_i} \right| \leq \frac{1}{\delta} \int_{|y-\lambda_i|>\delta} |\varphi_{N,k}(y)| dy + \left| \int_{|y-\lambda_i|<\delta} \epsilon(x-y) \frac{\varphi_{N,k}(y)}{y-\lambda_i} dy \right|. \tag{5.13}$$

Using that

$$\int \varphi_{N,k}^2(y) dy = \int p_{N-k+1}(y)^2 w_N(x) dx = 1$$

and proceeding as in (5.8), we may see that the first integral on the right-hand side of (5.13) is $O(1)$, leading us to bound the second. We have that

$$\begin{aligned} \int_{|y-\lambda_i|<\delta} \epsilon(x-y) \frac{\varphi_{N,k}(y)}{y-\lambda_i} dy &= \int_{|y-\lambda_i|<\delta} \epsilon(x-y) \frac{(\hat{\varphi}_{N,k}(y) - \hat{\varphi}_{N,k}(\lambda_i)) |y-\lambda_i|^{\alpha_i}}{y-\lambda_i} dy \\ &\quad + \hat{\varphi}_{N,k}(\lambda_i) \int_{|y-\lambda_i|<\delta} \epsilon(x-y) \frac{|y-\lambda_i|^{\alpha_i}}{y-\lambda_i} dy. \end{aligned}$$

The latter term is $O(N^{\alpha_i+2})$ by (5.11). Additionally, applying again (5.11) and the mean value theorem to $\varphi_{N,k,i}$, we see that

$$\int_{|y-\lambda_i|<\delta} \epsilon(x-y) \frac{(\hat{\varphi}_{N,k}(y) - \hat{\varphi}_{N,k}(\lambda_i)) |y-\lambda_i|^{\alpha_i}}{y-\lambda_i} dy = O(N^{\alpha_i+2}).$$

These bounds are additionally uniform in y , so that in particular we may conclude that for $k = 1, 2$ that $\epsilon(\varphi_{N,k})(x) = O(N^{\alpha_i+2})$, uniformly in $x \in \mathbb{R}$. By Remark 4.15, we see that $\epsilon(\ell_i w_N^{1/2})(x) = O(N^{\alpha_i+2})$. Again proceeding as in (5.8), we obtain that

$$\int \mathcal{K}_N(x, x)^{1/2} \epsilon(\ell_i w_N^{1/2})(x) dx = O(N^{\alpha_i+4}). \tag{5.14}$$

Combining (5.12) and (5.14) we see that

$$e^{-2N\lambda_i^2} K_N(\lambda_i, \lambda_i) \left(\int \mathcal{K}_N(x, x)^{1/2} \epsilon(\ell_i w_N^{1/2})(x) dx \right)^2 = O(N^{16(1+\alpha_i)}).$$

In total these results show that the right-hand side of (5.4) is of order $O(e^{-Nc/4})$, and that the right-hand side of (5.5) is of order $O(N^{16(1+\sum_{i=1}^m \alpha_i)})$, which together completes the proof of Proposition 5.1.

6 Proof of Proposition 2.9

In this section, we will prove Proposition 2.9. The evaluation of these integrals is a key technical step in this paper. Away from the points $\{\pm 1\}$ and $\{\lambda_i\}_{i=1}^m$ this computation will follow by standard methods for oscillatory integrals (see for example Lemma 6.3). The remainder of the work is then spent understanding the contributions around the points $\{\pm 1\}$ and $\{\lambda_i\}_{i=1}^m$. This will follow from applications of the classical asymptotics for the Bessel and Airy functions and careful analysis to compare these integrals with those in the remainder of the bulk. There is some similarity with methods used in [12, 11, 13], particularly for the asymptotics away from the points $\{\lambda_i\}_{i=1}^m$.

To begin, we first note that we may write

$$\int J_N^{-1} f(x) g(x) w_N(x) dx = \frac{1}{2} \int \int \epsilon(x-y) g(x) f(y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy.$$

We observe that if functions f and g are supported on disjoint intervals, then we have that

$$\int \int \epsilon(x - y)g(x)f(y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy = \pm \int g(x)w_N(x)^{1/2}dx \int f(y)w_N(y)^{1/2}dy, \tag{6.1}$$

where $\pm = +$ if the interval supporting g lies to the right of the interval supporting f in \mathbb{R} and $\pm = -$ if this interval instead lies to the left.

This will be an important observation. With this identity and a partition of unity, we will reduce the integrals into a sequence of simpler integrals supported over regions where a single asymptotic expression holds. We again fix $(\mathcal{W}, \{\lambda_i\}_i, \{\alpha_i\}_i)$ satisfying the conditions of Proposition 1.2, and in all results below, the error will be uniform in a neighborhood of $\{\lambda_i\} \in \{(\lambda_1, \dots, \lambda_m) \in (-1, 1)^m : \lambda_1 > \dots > \lambda_m\}$. We will employ, as before, the notation $\lambda_0 = 1$ and $\lambda_{m+1} = -1$.

Lemma 6.1. *Assume that ϕ is a smooth function on \mathbb{R} of sub-exponential growth. Assume that there is some $\delta > 0$ such that ϕ vanishes on $(\lambda_i - \delta, \lambda_i + \delta)$ for all $0 \leq i \leq m + 1$. Then for each k and any $c \in \mathbb{N}$, we have that*

$$\int \phi(x)p_{N-k}(x)w_N(x)^{1/2}dx = O(N^{-c}).$$

Proof. For notational clarity, we will assume that $k = 0$. The proof will proceed by applying the results of Section 4 on each region. Using the assumptions on the support of ϕ , we see that

$$\int \phi(x)p_N(x)w_N(x)^{1/2}dx = \int_{|x|>1+\delta} \phi(x)p_N(x)w_N(x)^{1/2}dx + \int_{|x|<1} \phi(x)p_N(x)w_N(x)^{1/2}dx.$$

We note that by Remark 4.9 we have that

$$\int_{|x|>1+\delta} \phi(x)p_N(x)w_N(x)^{1/2}dx = O\left(e^{-NC/2}\right).$$

We now focus on the computation of $\int_{|x|<1} \phi(x)p_N(x)w_N(x)^{1/2}dx$. If ϕ is supported on $(\lambda_{i+1} + \delta, \lambda_i - \delta)$, for some $0 \leq i \leq m$, we may apply Proposition 4.7 to rewrite

$$\int_{|x|<1} \phi(x)p_N(x)w_N(x)^{1/2}dx = \frac{1}{\pi^{1/2}} \left[\int \phi(x)[(I + R_N(x))T_\infty(x)]_{11} e^{-i\pi \sum_{k=1}^i \alpha_k} e^{iNs(x)} dx + \int \phi(x)[(I + R_N(x))T_\infty(x)]_{12} e^{i\pi \sum_{k=1}^i \alpha_k} e^{-iNs(x)} dx \right], \tag{6.2}$$

where R_N is the error term of Proposition 4.7. We now recall that if f is a smooth function of compact support, then for any $c \in \mathbb{N}$, we have that

$$\left| \int e^{-iNx} f(x) dx \right| \leq \frac{1}{N^c} \int |f^{(c)}(x)| dx. \tag{6.3}$$

This inequality, and a change of variables, will be sufficient to complete the proof. Indeed, let us consider the first term of (6.2) and denote for $l \in \{1, 2\}$

$$f_{N,l}(x) = \phi(x)[(I + R_N(x))T_\infty(x)]_{1l} e^{(-1)^l i\pi \sum_{k=1}^i \alpha_k}.$$

As we have that for any $n \geq 0$, $R_N^{(n)}(x) = O(N^{-1})$ uniformly on the support of ϕ , and recalling that by Remark 4.8, $T_\infty(x)$ is N -independent and smooth, we see that for any

$c \in \mathbb{N}$, we have that $f_{N,l}^{(c)}(x) = O(1)$ uniformly for $x \in \mathbb{R}$. By (6.3) we see that

$$\begin{aligned} \left| \int f_{N,l}(x) e^{\pm iNs(x)} dx \right| &= \left| \int (s^{-1})'(y) f_{N,l}(s^{-1}(y)) e^{\pm iNy} dy \right| \\ &\leq \frac{1}{N^c} \int \left| \frac{d^c}{dy^c} ((s^{-1})'(y) f_{N,l}(s^{-1}(y))) \right| dy. \end{aligned}$$

Recalling the definition of s from (4.3), we see that s is smooth and non-degenerate on the support of $f_{N,l}$, we see that the integrand of the right-hand side is uniformly $O(1)$, and supported inside of the compact set $[s(1), s(-1)] = [0, \pi]$, so that terms of (6.2) are of order $O(N^{-c})$. Applying this argument to each region completes the proof. \square

Lemma 6.2. *Let ϕ and φ be smooth functions of subexponential growth on \mathbb{R} . Assume that there is some $\delta > 0$ such that both ϕ and φ vanishes on $(\lambda_i - \delta, \lambda_i + \delta)$ for all $0 \leq i \leq m + 1$. Then for each n and m , we have that*

$$\begin{aligned} &\frac{1}{2} \int \int \phi(x) \varphi(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x) \varphi(x)}{(1-x^2)} \sin((m-n) \arccos(x)) dx + O(N^{-2}). \end{aligned}$$

Proof. Applying a partition of unity and linearity, we may assume we are in the case where ϕ is supported on $(\lambda_{i+1} + \delta, \lambda_i - \delta)$ for some i , or on $(1 + \delta, \infty) \cup (-\infty, -1 - \delta)$, and similarly for ψ . In the case where ψ and ϕ have disjoint support, we see by (6.1) that

$$\begin{aligned} &\left| \int \int \phi(x) \varphi(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \right| \\ &= \left| \int \phi(x) p_{N-n}(x) w_N(x)^{1/2} dx \int \varphi(y) p_{N-m}(y) w_N(y)^{1/2} dy \right|. \end{aligned}$$

By Lemma 6.1, both of these integrals on the right-hand side are $O(N^{-2})$. Thus we may assume that we are in the case where either both ψ and ϕ are supported on $(\lambda_{i+1} + \delta, \lambda_i - \delta)$ for some i , or on $(1 + \delta, \infty) \cup (-\infty, -1 - \delta)$. In the latter case, by the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} &\left| \int \int_{\min(|x|, |y|) > 1 + \delta} \phi(x) \varphi(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \right| \\ &\leq \left(\int_{|x| > 1 + \delta} p_{N-n}(x)^2 \phi(x)^2 w_N(x) dx \int_{|y| > 1 + \delta} p_{N-m}(y)^2 \varphi(y)^2 w_N(y) dy \right)^{1/2}. \end{aligned} \tag{6.4}$$

Then applying Remark 4.9 as in the proof of Lemma 6.1, we see that the right-hand side of (6.4) is $O(e^{-NC/4})$.

Thus we may assume that ϕ and φ are both supported on $(\lambda_{i+1} + \delta, \lambda_i - \delta)$ for some i . To compute this integral, we will use the following lemma (see Chapter 8.4 of [5]).

Lemma 6.3. *Let h be a smooth, compactly supported function on \mathbb{R} . Then we have that*

$$\begin{aligned} &\frac{1}{2} \int \int h(x, y) e^{\pm iN(x+y)} \epsilon(x-y) dx dy = O(N^{-2}) \\ &\frac{1}{2} \int \int h(x, y) e^{\pm iN(x-y)} \epsilon(x-y) dx dy = \pm \frac{i}{N} \int h(x, x) dx + O(N^{-2}). \end{aligned}$$

More specifically, the error may be bounded by

$$\frac{1}{N^2} \left(\int \int \|\nabla h(x, y)\| dx dy + 4 \int \left| \frac{d}{dx} h(x, x) \right| dx \right).$$

Using the asymptotics of Proposition 4.12, we may write

$$\begin{aligned} & \frac{1}{2} \int \int \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ &= \frac{1}{2\pi} \sum_{j,k=1,2} \int \int f_{N,j,k}(x,y)e^{-(-1)^j Nis(x)-(-1)^k Nis(y)}\epsilon(x-y)dxdy, \end{aligned}$$

where here

$$\begin{aligned} f_{N,j,k}(x,y) &= \phi(x)\varphi(y)[(I + R_N(x))T_\infty(x)]_{1j}[(I + \bar{R}_N(y))T_\infty(y)]_{1k} \\ &\quad \times \exp\left((-1)^j i(\text{narccos}(x) + \pi \sum_{l=1}^i \alpha_l) + (-1)^k i(\text{marccos}(y) + \pi \sum_{l=1}^i \alpha_l)\right), \end{aligned}$$

and R_N and \bar{R}_N denote the error terms in Proposition 4.7 for the case of n and m , respectively. Recalling Remark 4.8 as before, as well as the bound $R_N^{(c)}(x), \bar{R}_N^{(c)}(y) = O(N^{-1})$ for any $c \in \mathbb{N}$, we see that for $j, k \in \{1, 2\}$, we have that $f_{N,j,k}(x, y) = O(1)$ and $\|\nabla f_{N,j,k}(x, y)\| = O(1)$, both uniformly in $x, y \in \mathbb{R}$. From this, we see that

$$\|\nabla((s^{-1})'(x)(s^{-1})'(y)f_{N,j,k}(s^{-1}(x), s^{-1}(y)))\| = O(1)$$

uniformly in $x, y \in \mathbb{R}$. Thus applying Lemma 6.3 and a change of variables with respect to s , and recalling that $s'(x) < 0$ for $x \in (-1, 1)$, so that $\epsilon(s(x) - s(y)) = -\epsilon(x - y)$, we see that

$$\begin{aligned} & \frac{1}{2\pi} \sum_{j,k=1,2} \int \int f_{N,j,k}(x,y)e^{-(-1)^j Nis(x)-(-1)^k Nis(y)}\epsilon(x-y)dxdy \\ &= \frac{-1}{2\pi} \sum_{j,k=1,2} \int \int (s^{-1})'(x)(s^{-1})'(y)f_{N,j,k}(s^{-1}(x), s^{-1}(y))e^{-(-1)^j Nix-(-1)^k Niy}\epsilon(x-y)dxdy \\ &= -\frac{i}{\pi N} \int \frac{f_{N,1,2}(x,x)}{|s'(x)|}dx + \frac{i}{\pi N} \int \frac{f_{N,2,1}(x,x)}{|s'(x)|}dx + O(N^{-2}). \end{aligned} \tag{6.5}$$

We now observe that as $R_N(x), \bar{R}_N(x) = O(N^{-1})$, we have that

$$\begin{aligned} f_{N,j,k}(x,y) &= \phi(x)\varphi(y)[T_\infty(x)]_{1j}[T_\infty(y)]_{1k} \\ &\quad \times e^{(-1)^j i(\text{narccos}(x) + \pi \sum_{l=1}^i \alpha_l) + (-1)^k i(\text{marccos}(y) + \pi \sum_{l=1}^i \alpha_l)} + O(N^{-1}). \end{aligned}$$

Thus we may express the right-hand side of (6.5) as

$$\frac{i}{\pi N} \int \int \frac{\phi(x)\varphi(x)}{|s'(x)|} [T_\infty(x)]_{11} [T_\infty(x)]_{12} \left(e^{i(n-m)\text{arccos}(x)} - e^{-i(n-m)\text{arccos}(x)} \right) dx + O(N^{-2}).$$

Recalling the definition of T_∞ from (4.11) and s from (4.3), we may derive the identities $[T_\infty(x)]_{11}[T_\infty(x)]_{12} = \frac{1}{2(1-x^2)^{1/2}}$ and $s'(x) = -2\sqrt{1-x^2}$. With these we obtain that

$$\begin{aligned} & \frac{i}{4\pi N} \int_{-1}^1 \frac{\phi(x)\varphi(x)}{(1-x^2)} \left(e^{i(n-m)\text{arccos}(x)} - e^{-i(n-m)\text{arccos}(x)} \right) dx \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\text{arccos}(x))dx + O(N^{-2}). \end{aligned}$$

Considered altogether, these results yield the desired claim. □

We will now begin to compute integrals of functions supported around λ_i for $1 \leq i \leq m$. It will be convenient to prove a result to match the behavior of the asymptotics of Proposition 4.7 and 4.13 over their overlapping regions.

We recall the asymptotics for the Bessel function (see Chapter 9 of [1]):

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1}) \right]. \tag{6.6}$$

By employing this expansion, we obtain that with \mathcal{J}_α defined as in (4.28), we have that,

$$\mathcal{J}_\alpha(x) = (I + O(x^{-1}))\mathcal{I}_\alpha(x), \quad \mathcal{I}_\alpha(x) = \begin{bmatrix} \epsilon(x)\sqrt{2} \sin(|x| - \frac{\alpha\pi}{2}) \\ \sqrt{2} \cos(|x| - \frac{\alpha\pi}{2}) \end{bmatrix}, \tag{6.7}$$

uniformly in $x \in \mathbb{R}$. We will make use of the following matching computation.

Lemma 6.4. *For any $q \in \mathbb{R}$ and $x \in (-1, 1)$, we have that*

$$\begin{aligned} & T_{N,k,\lambda_j}(x)\mathcal{I}_{\alpha_j+q}((N-k)(s_{N,k}(\lambda_j) - s_{N,k}(x))) \\ &= T_\infty(x)e^{-i\pi \sum_{k=1}^{j-1} \alpha_k - i\pi\alpha_j I(x < \lambda_j)} \sigma e^{i\epsilon(x-\lambda_j)\frac{\pi}{2}q\sigma} \begin{bmatrix} e^{i(N-k)s_{N,k}(x)} \\ e^{-i(N-k)s_{N,k}(x)} \end{bmatrix}. \end{aligned} \tag{6.8}$$

Proof. We observe that as $s_{N,k}$ is monotonically decreasing, if we denote $\pm = \epsilon(x - \lambda_j) = \epsilon(s_{N,k}(\lambda_j) - s_{N,k}(x))$ then

$$\mathcal{I}_{\alpha_j+q}((N-k)(s_{N,k}(\lambda_j) - s_{N,k}(x))) = \begin{bmatrix} \sqrt{2} \sin((N-k)(s_{N,k}(\lambda_j) - s_{N,k}(x)) \mp (\alpha_j + q)\frac{\pi}{2}) \\ \sqrt{2} \cos((N-k)(s_{N,k}(\lambda_j) - s_{N,k}(x)) \mp (\alpha_j + q)\frac{\pi}{2}) \end{bmatrix}.$$

We also note that for any $a, b \in \mathbb{R}$, we have that

$$e^{-i\frac{\pi}{4}} e^{-i\frac{\pi}{4}\sigma} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \sin(b-a) \\ \sqrt{2} \cos(b-a) \end{bmatrix} = e^{-i\frac{\pi}{4}} e^{-i\frac{\pi}{4}\sigma} \begin{bmatrix} ie^{i(a-b)} \\ e^{-i(a-b)} \end{bmatrix} = e^{ia\sigma} \begin{bmatrix} e^{-ib} \\ e^{ib} \end{bmatrix},$$

so that we see that

$$\begin{aligned} & e^{-i\frac{\pi}{4}} e^{-i\frac{\pi}{4}\sigma} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \mathcal{I}_{\alpha_j+q}((N-k)(s_{N,k}(\lambda_j) - s_{N,k}(x))) \\ &= e^{\pm i\frac{\pi}{2}(\alpha_j+q)\sigma} e^{-i(N-k)s_{N,k}(\lambda_j)\sigma} \begin{bmatrix} e^{i(N-k)s_{N,k}(x)} \\ e^{-i(N-k)s_{N,k}(x)} \end{bmatrix}. \end{aligned} \tag{6.9}$$

Multiplying both sides of (6.9) by $T_\infty(x)e^{i\frac{\rho_{N,k,j}}{2}\sigma}$, and recalling the definition for T_{N,k,λ_j} given in (4.27), we see that the left-hand and right-hand sides simplify to the left-hand and right-hand sides of (6.8). \square

We will also need the following elementary result, which will follow by integration by parts.

Lemma 6.5. *Let $f, g : [0, \infty) \rightarrow [0, \infty)$ be monotone increasing smooth bijections, with everywhere positive derivatives on $(0, \infty)$. Let h, l be smooth functions of compact support and let F, G be continuous functions, all defined on $[0, \infty)$. Then we have that*

$$\frac{1}{2} \int_0^\infty \int_0^\infty h(x)l(y)F(f(x))G(g(y))\epsilon(x-y)dx dy = I + II + III, \tag{6.10}$$

where here

$$\begin{aligned} I &= \int_0^\infty F(f(x)) \left(\int_0^{g(x)} G(z)dz \right) \frac{h(x)l(x)}{g'(x)} dx, \\ II &= \int_0^\infty \left(\int_0^{f(x)} F(z)dz \right) \left(\int_0^{g(x)} G(z)dz \right) \frac{h(x)}{f'(x)} \frac{d}{dx} \left(\frac{l(x)}{g'(x)} \right) dx, \\ III &= \frac{1}{2} \int_0^\infty \int_0^\infty \left(\int_0^{f(x)} F(z)dz \right) \left(\int_0^{g(y)} G(z)dz \right) \frac{d}{dx} \left(\frac{h(x)}{f'(x)} \right) \frac{d}{dy} \left(\frac{l(y)}{g'(y)} \right) \epsilon(x-y) dx dy. \end{aligned}$$

Proof. We introduce $\bar{h}(x) = h(f^{-1}(x))(f^{-1})'(x)$ and $\bar{l}(x) = l(g^{-1}(x))(g^{-1})'(x)$, so that we may write the left-hand side of (6.10) as

$$\frac{1}{2} \int_0^\infty \int_0^\infty \bar{h}(u)\bar{l}(v)F(u)G(v)\epsilon(f^{-1}(u) - g^{-1}(v))dudv. \tag{6.11}$$

We observe that by monotonicity of g we have that $\epsilon(f^{-1}(u) - g^{-1}(v)) = \epsilon(g(f^{-1}(u)) - v)$, and furthermore by integration by parts we see that for $u > 0$

$$\begin{aligned} & \frac{1}{2} \int_0^\infty \bar{l}(v)G(v)\epsilon(g(f^{-1}(u)) - v)dv \\ &= \frac{1}{2} \int_0^{g(f^{-1}(u))} \bar{l}(v)G(v)dv - \frac{1}{2} \int_{g(f^{-1}(u))}^\infty \bar{l}(v)G(v)dv \\ &= \bar{l}(g(f^{-1}(u))) \left(\int_0^{g(f^{-1}(u))} G(z)dz \right) - \frac{1}{2} \int_0^\infty \bar{l}'(v) \left(\int_0^v G(z)dz \right) \epsilon(g(f^{-1}(u)) - v)dv. \end{aligned}$$

Applying this, we see that (6.11) may be rewritten as the sum $I' + II'$ where

$$\begin{aligned} I' &= \int_0^\infty \bar{h}(u)F(u)\bar{l}(g(f^{-1}(u))) \left(\int_0^{g(f^{-1}(u))} G(z)dz \right) du, \\ II' &= -\frac{1}{2} \int_0^\infty \int_0^\infty \bar{h}(u)F(u)\bar{l}'(v) \left(\int_0^v G(z)dz \right) \epsilon(g(f^{-1}(u)) - v)dudv. \end{aligned}$$

Again applying a change of variables with respect to f , and observing that $\bar{l}(g(y)) = l(y)/g'(y)$, we see that $I = I'$. Similarly, noting that

$$\bar{l}'(g(y)) = \frac{l'(y)}{(g'(y))^2} - \frac{l(y)g''(y)}{(g'(y))^3} = \frac{1}{g'(y)} \frac{d}{dy} \left(\frac{l(y)}{g'(y)} \right)$$

we see that

$$II' = -\frac{1}{2} \int_0^\infty \int_0^\infty F(f(x)) \left(\int_0^{g(y)} G(z)dz \right) h(x) \frac{d}{dy} \left(\frac{l(y)}{g'(y)} \right) \epsilon(x - y)dx dy.$$

By applying essentially the same argument, integrating by parts with respect to x instead of y , we may show that $II' = II + III$. More specifically, we may observe that if we swap the dummy-variables (x, y) in II' , then II' is just the left-hand side of (6.10) with a modified set of functions, so repeating the above argument, one may indeed confirm that $II' = II + III$. \square

We are now able to state the appropriate generalizations of the above lemmas to the case of functions supported around λ_i .

Lemma 6.6. *For each $1 \leq i \leq m$, there exists $\delta > 0$, such that if ϕ and φ are smooth functions supported on $(\lambda_i - \delta, \lambda_i + \delta)$, then we have for any n and m that*

$$\begin{aligned} & \int \phi(x)p_{N-n}(x)w_N(x)^{1/2}dx = O(N^{-1}), \\ & \frac{1}{2} \int \int \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x - y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x)\varphi(x)}{(1 - x^2)} \sin((m - n)\arccos(x))dx + O(N^{-2}). \end{aligned}$$

Proof. For convenience, let us denote $\lambda = \lambda_i$ and $\alpha = \alpha_i$ for this proof. We choose $\delta > 0$ so the asymptotics of Proposition 4.13 hold on $(\lambda - 2\delta, \lambda + 2\delta)$. We recall the following lemma due to [33].

Lemma 6.7. *Let h be a smooth, compactly supported function, and assume that $\mu + \nu > -1$. Then we have that*

$$\int_0^\infty J_\nu(Nt)t^\mu h(t)dt - D(\nu, \mu) \frac{h(0)}{N^{\mu+1}} = O(N^{-\mu-2}), \quad D(\nu, \mu) = 2^\mu \frac{\Gamma((\mu + \nu + 1)/2)}{\Gamma((\nu - \mu + 1)/2)}.$$

More specifically, the error is bounded by:

$$\frac{C_{\mu,\nu}}{N^{\mu+2}} \left(\sum_{i=0}^{\lceil \mu+3/2 \rceil} \int_0^\infty |h^{(i)}(x)|dx \right)$$

for some absolute constant $C_{\mu,\nu} > 0$.

From Proposition 4.13, and recalling $f_{N,n,\lambda}$ from (4.26), we may write

$$\begin{aligned} \phi(x)p_{N-n}(x)w_N(x)^{1/2} &= h_{N,1}(x)\epsilon(x-\lambda)\sqrt{\pi|(N-n)f_{N,n,\lambda}(x)|}J_{\alpha+1/2}((N-n)f_{N,n,\lambda}(x)) \\ &\quad + h_{N,2}(x)\sqrt{\pi|(N-n)f_{N,n,\lambda}(x)|}J_{\alpha-1/2}((N-n)f_{N,n,\lambda}(x)), \end{aligned}$$

where here $h_{N,k}(x) = \phi(x) \frac{1}{\pi^{1/2}} [(I + R_N(x))T_{N,n,\lambda}(x)]_{1k}$. We recall from Remark 4.12 that for $c \in \mathbb{N}$ and $i, j \in \{1, 2\}$, we have that $[T_{N,n,\lambda}^{(c)}(x)]_{ij} = O(1)$ uniformly in $x \in (\lambda - \delta, \lambda + \delta)$. As we also have that $R^{(c)}(x) = O(N^{-1})$ uniformly, we see that $h_{N,k}^{(c)}(x) = O(1)$ for $c \in \mathbb{N}$. From this, we see from Lemma 6.7 and a change of coordinates that there is C , only dependant on α , such that

$$\begin{aligned} &\left| \int_\lambda^{\lambda+\delta} \phi(x)p_{N-n}(x)w_N(x)^{1/2} dx \right| \\ &\leq N^{-1}\pi^{1/2}|(f_{N,n,\lambda}^{-1})'(0)| |D(\alpha + 1/2, 1/2)|h_{N,1}(\lambda) + D(\alpha - 1/2, 1/2)|h_{N,2}(\lambda)| \\ &\quad + \frac{C}{N^2} \sum_{k=1,2} \sum_{l=0}^2 \int_0^\infty \left| \frac{d^l}{dx^l} \left((f_{N,n,\lambda}^{-1})'(x)h_{N,k}(f_{N,n,\lambda}^{-1}(x)) \right) \right| dx. \end{aligned} \tag{6.12}$$

We see as above that the integrands on the right-hand side of (6.12) are $O(1)$, so we see that

$$\int_\lambda^{\lambda+\delta} \phi(x)p_{N-n}(x)w_N(x)^{1/2} dx = O(N^{-1}), \tag{6.13}$$

which establishes the first claim.

The case of double integrals will require more care. We note that by (6.1) and (6.13), we have that

$$\frac{1}{2} \int_\lambda^{\lambda+\delta} \int_\lambda^{\lambda+\delta} \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx dy = O(N^{-2}),$$

as we may split this double integral into a product of the single variable integrals dealt with above. Thus by breaking the double-integral into four regions, we see that

$$\begin{aligned} &\frac{1}{2} \int \int \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx dy \\ &= \frac{1}{2} \int_\lambda^\infty \int_\lambda^\infty \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx dy \\ &\quad + \frac{1}{2} \int_{-\infty}^\lambda \int_{-\infty}^\lambda \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx dy + O(N^{-2}). \end{aligned} \tag{6.14}$$

We compute the first integral, with the other case being identical. For notational ease, we will denote $g_{N,k}(x) = (N - k)f_{N,k,\lambda}(x)$. In view of Proposition 4.13, we further denote $h_{N,i}(x) = \phi(x)[(I + R_N(x))T_{N,n,\lambda}(x)]_{1i}$ and $l_{N,i}(x) = \varphi(x)[(I + \bar{R}_N(x))T_{N,m,\lambda}(x)]_{1i}$, where R_N and \bar{R}_N denote the error terms in Proposition 4.13 for case of $k = n$ and $k = m$, respectively. Thus by employing Proposition 4.13, we may rewrite the first integral on the right-hand side of (6.14) as

$$\frac{1}{2\pi} \sum_{k,l=1,2} \int_{\lambda}^{\infty} \int_{\lambda}^{\infty} h_{N,k}(x)[\mathcal{J}_{\alpha}(g_{N,n}(x))]_k l_{N,l}(y)[\mathcal{J}_{\alpha}(g_{N,m}(y))]_l \epsilon(x - y) dx dy. \tag{6.15}$$

We now observe that $f'_{N,k,\lambda}(x) = 2\eta_k^{-1} \sqrt{1 - \eta_k^{-2} x^2}$, and that $f_{N,k,\lambda}(\lambda) = 0$. Thus by Lemma 6.5 we see that for k fixed and N sufficiently large, we may rewrite (6.15) as $I + II + III$ where

$$\begin{aligned} I &= \frac{1}{\pi} \sum_{k,l=1,2} \int_{\lambda}^{\infty} [\mathcal{J}_{\alpha}(g_{N,n}(x))]_k \left(\int_0^{g_{N,m}(x)} [\mathcal{J}_{\alpha}(z)]_l dz \right) \frac{h_{N,k}(x) l_{N,l}(x)}{g'_{N,m}(x)} dx, \\ II &= \frac{1}{\pi} \sum_{k,l=1,2} \int_{\lambda}^{\infty} \left(\int_0^{g_{N,n}(x)} [\mathcal{J}_{\alpha}(z)]_k dz \right) \left(\int_0^{g_{N,m}(x)} [\mathcal{J}_{\alpha}(z)]_l dz \right) \\ &\quad \times \frac{h_{N,k}(x)}{g'_{N,n}(x)} \frac{d}{dx} \left(\frac{l_{N,l}(x)}{g'_{N,m}(x)} \right) dx, \\ III &= \frac{1}{2\pi} \sum_{k,l=1,2} \int_{\lambda}^{\infty} \int_0^{\infty} \left(\int_0^{g_{N,n}(x)} [\mathcal{J}_{\alpha}(z)]_k dz \right) \left(\int_0^{g_{N,m}(y)} [\mathcal{J}_{\alpha}(z)]_l dz \right) \\ &\quad \times \left(\frac{d}{dx} \frac{h_{N,k}(x)}{g'_{N,n}(x)} \right) \left(\frac{d}{dy} \frac{l_{N,l}(y)}{g'_{N,m}(y)} \right) \epsilon(x - y) dx dy. \end{aligned}$$

We now recall another important asymptotic for understanding these terms. Assuming that $\ell + \nu > -1$, we may define

$$J_{\nu,\ell}(x) = \int_0^x y^{\ell} J_{\nu}(y) dy, \quad I_{\nu,\ell}(x) = D(\nu, \ell) - J_{\nu,\ell}(x).$$

We have the following asymptotics (see Chapter 2 of [33])

$$I_{\nu,\ell}(z) = z^{\ell-1/2} \sqrt{\frac{2}{\pi}} \cos\left(z - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) + O(z^{\ell-3/2}), \quad z > 1; \quad I_{\nu,\ell}(z) = O(1), \quad z \leq 1. \tag{6.16}$$

From these asymptotics, and recalling the definition of \mathcal{J}_{α} from (4.28), we see that

$$\int_0^{g_{N,n}(x)} [\mathcal{J}_{\alpha}(z)]_k dz = \sqrt{\pi} J_{\alpha - (-1)^k/2, 1/2}(g_{N,n}(x)) = O(1). \tag{6.17}$$

As $h_{N,k}(x), l_{N,l}(x), h'_{N,k}(x), l'_{N,l}(x) = O(1)$ uniformly, and as $g_{N,n}(x) = (N - k)f_{N,n,\lambda}(x)$, we thus see that $II, III = O(N^{-2})$.

We may further rewrite I as the sum of two terms

$$\begin{aligned} I' &= - \sum_{k,l=1,2} \int_{\lambda}^{\infty} \sqrt{g_{N,n}(x)} J_{\alpha - (-1)^k/2, 1/2}(g_{N,n}(x)) I_{\alpha - (-1)^l/2, 1/2}(g_{N,m}(x)) \frac{h_{N,k}(x) l_{N,l}(x)}{g'_{N,m}(x)} dx \\ II' &= \sum_{k,l=1,2} D(\alpha - (-1)^l/2, 1/2) \int_{\lambda}^{\infty} J_{\alpha - (-1)^k/2, 1/2}(g_{N,n}(x)) \frac{h_{N,k}(x) l_{N,l}(x)}{g'_{N,m}(x)} dx. \end{aligned}$$

We have that $II' = O(N^{-2})$ by Lemma 6.7. Finally, employing the asymptotics for J_μ and $I_{\mu,\ell}$ given by (6.16) and (6.7) one derives that

$$\begin{aligned} I' &= -\frac{2}{\pi} \sum_{k,l=1,2} \int_\lambda^\infty \cos\left(g_{N,n}(x) - \frac{\alpha\pi}{2} + \frac{((-1)^k - 1)\pi}{4}\right) \\ &\quad \times \cos\left(g_{N,m}(x) - \frac{\alpha\pi}{2} + \frac{((-1)^l + 1)\pi}{4}\right) \frac{h_{N,k}(x)l_{N,l}(x)}{g'_{N,m}(x)} dx + O(N^{-2}) \\ &= -\sum_{k,l=1,2} \frac{1}{\pi} \int_\lambda^\infty [\mathcal{I}_\alpha(g_{N,n}(x))]_k [\mathcal{I}_{\alpha-1}(g_{N,m}(x))]_l \frac{h_{N,k}(x)l_{N,l}(x)}{g'_{N,m}(x)} dx + O(N^{-2}). \end{aligned}$$

From the error bounds on R_N and \bar{R}_N , we see that $h_{N,i}(x) = \phi(x)[T_{N,n,\lambda}(x)]_{1i} + O(N^{-1})$ and $l_{N,i}(x) = \varphi(x)[T_{N,m,\lambda}(x)]_{1i} + O(N^{-1})$, uniformly. Applying these, we see that

$$\begin{aligned} I' &= -\frac{1}{\pi N} \sum_{k,l=1,2} \int_\lambda^\infty [\mathcal{I}_\alpha(g_{N,n}(x))]_k [\mathcal{I}_{\alpha-1}(g_{N,m}(x))]_l \\ &\quad \times \frac{\phi(x)\varphi(x)[T_{N,n,\lambda}(x)]_{1k}[T_{N,m,\lambda}(x)]_{1l}}{f'_{N,m,\lambda}(x)} dx + O(N^{-2}). \end{aligned}$$

Recalling that $g_{N,k}(x) = (N - k)f_{N,k,\lambda}(x) = (N - k)(s_{N,k}(\lambda) - s_{N,k}(x))$, we see that by Lemma 6.4

$$\begin{aligned} I' &= -\frac{1}{\pi N} \int_\lambda^\infty [T_\infty(x)e^{-i\pi \sum_{p=1}^{j-1} \alpha_p \sigma} \begin{bmatrix} e^{i(N-n)s_{N,n}(x)} \\ e^{-i(N-n)s_{N,n}(x)} \end{bmatrix}]_1 \\ &\quad \times [T_\infty(x)e^{-i\pi \sum_{p=1}^{j-1} \alpha_p \sigma} e^{-i\frac{\pi}{2}\sigma} \begin{bmatrix} e^{i(N-m)s_{N,m}(x)} \\ e^{-i(N-m)s_{N,m}(x)} \end{bmatrix}]_1 \frac{\phi(x)\varphi(x)}{|s'_{N,m}(x)|} dx + O(N^{-2}). \end{aligned} \tag{6.18}$$

Employing Lemma 4.4, we may now expand the integral on the right-hand side of (6.18) as

$$-\frac{i}{\pi N} \sum_{k,l=1,2} \int_\lambda^\infty e^{-(-1)^k Nis(x) - (-1)^l Nis(x)} \frac{f_{k,l}(x)}{|s'(x)|} dx + O(N^{-2}), \tag{6.19}$$

where here

$$\begin{aligned} f_{k,l}(x) &= \phi(x)e^{(-1)^k i \arccos(x)} [T_\infty(x)]_{1k} e^{(-1)^k i\pi \sum_{p=1}^{j-1} \alpha_p} \\ &\quad \times \varphi(x)e^{(-1)^l i \arccos(x)} [T_\infty(x)]_{1l} e^{(-1)^l i\pi \sum_{p=1}^{j-1} \alpha_p} (-1)^l. \end{aligned}$$

We see that by integration by parts (6.3) that the terms with $l = k$ in (6.19) are $O(N^{-2})$. Thus (6.19) may further be written as

$$\begin{aligned} -\frac{i}{\pi N} \int_\lambda^\infty \left(\frac{f_{1,2}(x)}{|s'(x)|} + \frac{f_{2,1}(x)}{|s'(x)|} \right) dx &= \frac{-i}{\pi N} \int_\lambda^\infty \phi(x)\varphi(x) \frac{[T_\infty(x)]_{11}[T_\infty(x)]_{12}}{|s'(x)|} dx \\ &\quad \times \left(e^{-i(n-m)\arccos(x)} - e^{i(n-m)\arccos(x)} \right) + O(N^{-2}). \end{aligned}$$

The identities at the end of Lemma 6.2 now show that this coincides with

$$I' = \frac{1}{2\pi N} \int_\lambda^\infty \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx + O(N^{-2})$$

This completes the evaluation of the first integral on the right-hand side of (6.14). The second integral is similarly given by

$$\frac{1}{2\pi N} \int_{-\infty}^\lambda \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx + O(N^{-2}).$$

Together these computations complete the proof of the second claim. \square

To compute the integrals supported around $\{\pm 1\}$ we will need a lemma similar to Lemma 6.4. We recall that for $x > 0$ we have that (Chapter 10 of [1])

$$\begin{aligned} \text{Ai}(-x) &= \frac{1}{\sqrt{\pi x^{1/4}}} \left(\sin\left(\frac{2}{3}x^{3/2} + \pi/4\right) + O\left(x^{-3/2}\right) \right) \\ \text{Ai}'(-x) &= -\frac{x^{1/4}}{\sqrt{\pi}} \left(\cos\left(\frac{2}{3}x^{3/2} + \pi/4\right) + O\left(x^{-3/2}\right) \right). \end{aligned}$$

Similarly to Lemma 6.4, we have the following matching formula.

Lemma 6.8. *We have for $|x| \leq 1$ that*

$$\frac{1}{\sqrt{\pi}} T_1(x) \begin{bmatrix} |f_1(x)|^{-1/4} \sin(a + \pi/4) \\ -|f_1(x)|^{1/4} \cos(a + \pi/4) \end{bmatrix} = T_\infty(x) \begin{bmatrix} e^{ia} \\ e^{-ia} \end{bmatrix},$$

and that

$$\frac{1}{\sqrt{\pi}} T_{-1}(x) \begin{bmatrix} |f_{-1}(x)|^{-1/4} \sin(a + \pi/4) \\ |f_{-1}(x)|^{1/4} \cos(a + \pi/4) \end{bmatrix} = T_\infty(x) e^{-i\pi \sum_{j=1}^m \alpha_j \sigma} \begin{bmatrix} e^{ia} \\ -e^{-ia} \end{bmatrix}.$$

Proof. We observe that

$$e^{-i\frac{\pi}{4}} e^{i\frac{\pi}{4}\sigma} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} e^{i\frac{\pi}{4}\sigma} \begin{bmatrix} \sin(a + \pi/4) \\ -\cos(a + \pi/4) \end{bmatrix} = e^{i\frac{\pi}{4}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \begin{bmatrix} \sin(a + \pi/4) \\ -\cos(a + \pi/4) \end{bmatrix} = \begin{bmatrix} e^{ia} \\ e^{-ia} \end{bmatrix}.$$

Applying T_∞ and recalling the definition of T_1 given in (4.19) then achieves the first equality. The proof of the result for T_{-1} follows similarly. \square

We will also need the following computation.

Lemma 6.9. *We have that*

$$[T_1(1)]_{11} = \sqrt{2\pi}, \quad [T_{-1}(-1)]_{11} = \sqrt{2\pi}.$$

Proof. With D as in (4.9), we see that $D(x) = \exp(\mathcal{W}(x))(1 + O(|x - 1|^{1/2}))$ (see as well the proof of Lemma 4.15 in [4]). Recalling f_1 and T_1 from (4.18) and (4.19), we see that $f_1'(1) = 2$, and in addition that for $x > 1$

$$T_1(x) = \sqrt{\pi} e^{-i\frac{\pi}{4}} A(x) e^{i\frac{\pi}{4}\sigma} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} 2^{\sigma/4} (x - 1)^{\sigma/4} + O((x - 1)^{1/4}).$$

Recalling as well A from (4.10), we note that for $x > 1$ we have that

$$A(x) = \frac{1}{2^{3/4}(x - 1)^{1/4}} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} + \frac{(x - 1)^{1/4}}{2^{5/4}} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} + O((x - 1)^{3/4}).$$

From this, one may routinely obtain that

$$\begin{aligned} \sqrt{\pi} e^{-i\frac{\pi}{4}} A(x) e^{i\frac{\pi}{4}\sigma} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} 2^{\sigma/4} &= \sqrt{2\pi} (x - 1)^{-1/4} \begin{bmatrix} 1 & 0 \\ -i & 0 \end{bmatrix} \\ &+ i \sqrt{\frac{\pi}{2}} (x - 1)^{1/4} \begin{bmatrix} 0 & -1 \\ 0 & -i \end{bmatrix} + O((x - 1)^{3/4}). \end{aligned}$$

Together these are sufficient to establish that $[T_1(1)]_{11} = \sqrt{2\pi}$. The case of -1 is similar. \square

With these results established, we are now able to compute the relevant integrals in the regions around $\{\pm 1\}$. This result will follow from similar methods to Lemma 6.6, though we will contend with a variety of complications as the integrand switches from being oscillatory to exponentially decaying. Particularly, this transition will cause a variety of boundary terms to appear, which requires us to carefully analyze each term carefully to get the exact contribution. In addition, the non-smooth nature of many of the functions composing $T_{\pm 1}$ will require us to carefully Taylor expand many quantities to show cancellation.

We note that we will only need this level of detail for one integral of Proposition 2.9, which is recalled as (6.37) below, and showing simply that (6.21) is simply $O(N^{-1})$ is significantly simpler.

Lemma 6.10. *There is $\delta > 0$, such that if ϕ and φ are smooth functions supported on $(\pm 1 - \delta, \pm 1 + \delta)$, then we have for any n and m that*

$$\int \phi(x)p_{N-n}(x)w_N(x)^{1/2}dx = \frac{1}{(2N)^{1/2}}(\pm 1)^{N-n}\phi(\pm 1) + O(N^{-5/6}), \tag{6.20}$$

$$\begin{aligned} & \frac{1}{2} \int \int \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x))dx + O(N^{-7/6}). \end{aligned} \tag{6.21}$$

Proof. Fix δ as in Lemma 6.6. We demonstrate the case of $+1$, the case of -1 being identical. We first recall that there is C such that $|\text{Ai}(x)|, |\text{Ai}'(x)| \leq Ce^{-x}$ (see Chapter 10 of [1]). From the asymptotics of Proposition 4.10 we see that

$$\begin{aligned} \int_1^\infty \phi(x)p_{N-n}(x)w_N(x)^{1/2}dx &= \frac{1}{\pi^{1/2}} \int_0^\infty (N-n)^{1/6}h_{N,1}(x)\text{Ai}((N-n)^{2/3}x)dx \\ &+ \frac{1}{\pi^{1/2}} \int_0^\infty (N-n)^{-1/6}h_{N,2}(x)\text{Ai}'((N-n)^{2/3}x)dx, \end{aligned} \tag{6.22}$$

where here $h_{N,i}(x) = (f_{N,n,1}^{-1})'(x)[(I + R_N(f_{N,n,1}^{-1}(x)))T_1(f_{N,n,1}^{-1}(x))]_{1i}\phi(f_{N,n,1}^{-1}(x))$. Noting that $h_{N,2}$ is uniformly bounded on $(1, \infty)$, we see that by the exponential bound on Ai' the second integral on the right-hand side of (6.22) is $O(N^{-5/6})$, so we may focus on the first integral on the right-hand side. As $h_{N,1}(x)$ is smooth and of compact support, we see that there is C such that $|h_{N,1}(x) - h_{N,1}(0)| \leq C|x|$. In particular,

$$\left| \int_0^\infty (h_{N,1}(x) - h_{N,1}(0))\text{Ai}((N-n)^{2/3}x)dx \right| \leq C \int_0^\infty |x\text{Ai}((N-n)^{2/3}x)|dx.$$

Using that Ai is subexponential, we see there is $C > 0$ such that

$$\int_0^\infty |x\text{Ai}((N-n)^{2/3}x)|dx \leq C(N-n)^{-4/3} \int_0^\infty xe^{-x}dx = O(N^{-4/3}),$$

so that

$$\int_0^\infty h_{N,1}(x)\text{Ai}((N-n)^{2/3}x)dx = h_{N,1}(0) \int_0^\infty \text{Ai}((N-n)^{2/3}x)dx + O(N^{-4/3}).$$

As $f_1'(1) = 2$, we see that $h_{N,1}(0) = 2^{-1}\phi(1)[T_1(1)]_{11} + O(N^{-1})$, and thus (6.22) may be rewritten as

$$\frac{1}{2\pi^{1/2}}N^{-1/2}\phi(1)[T_1(1)]_{11} \int_0^\infty \text{Ai}(x)dx + O(N^{-5/6}).$$

Applying Lemma 6.9 and the fact that $\int_0^\infty \text{Ai}(x)dx = \frac{1}{3}$ (see (10.4.82) of [1]) we see that we may further reduce this to

$$\phi(1) \frac{1}{(2N)^{1/2}} \frac{1}{3} + O(N^{-5/6}). \tag{6.23}$$

To compute the integral in the region $(1 - \delta, 1)$ we observe that we have that

$$\begin{aligned} \text{Ai}(-x) &= \frac{\sqrt{x}}{3} \left(J_{1/3} \left(\frac{2}{3}x^{3/2} \right) + J_{-1/3} \left(\frac{2}{3}x^{3/2} \right) \right), \\ \text{Ai}'(-x) &= \frac{x}{3} \left(J_{2/3} \left(\frac{2}{3}x^{3/2} \right) - J_{-2/3} \left(\frac{2}{3}x^{3/2} \right) \right), \end{aligned}$$

which occur as (10.4.15) and (10.4.17) of [1]. By applying Lemma 6.7 to the terms in this expression, we obtain that for smooth, compactly-supported h , we have that

$$\begin{aligned} \int_0^\infty h(t)\text{Ai}(-N^{2/3}t)dt &= \frac{h(0)}{N^{2/3}} \frac{2}{3} + O(N^{-4/3}), \\ \int_0^\infty h(t)\text{Ai}'(-N^{2/3}t)dt &= \frac{h(0)}{3^{2/3}\Gamma(2/3)N^{2/3}} + O(N^{-4/3}), \end{aligned} \tag{6.24}$$

where the error can similarly be expressed in terms of the derivatives of h . Proceeding similarly to the case of $(1, 1 + \delta)$, and noting that

$$\int_{-\infty}^1 \phi(x)p_{N-n}(x)w_N(x)^{1/2}dx = \frac{\phi(1)}{(2N)^{1/2}} \frac{2}{3} + O(N^{-5/6}). \tag{6.25}$$

Together these yield (6.20).

We now proceed to derive the double integral result. We observe that in view of (6.25), (6.23) and (6.1), we have that

$$\begin{aligned} &\int_{\mp\infty}^1 \int_1^{\pm\infty} \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ &= \pm \frac{\phi(1)\varphi(1)}{9N} + O(N^{-4/3}), \end{aligned}$$

so that splitting the domain of the double integral into four regions, we see that

$$\begin{aligned} &\int \int \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ &= \int_1^\infty \int_1^\infty \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ &\quad + \int_{-\infty}^1 \int_{-\infty}^1 \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy + O(N^{-4/3}). \end{aligned}$$

To evaluate these integrals, we will denote $h_{N,i}(x) = [(I + R_N(x))T_1(x)]_{1i}\phi(x)$ and $l_{N,i}(x) = [(I + \bar{R}_N(x))T_1(x)]_{1i}\varphi(x)$, where R_N and \bar{R}_N denote the error terms in Proposition 4.10 for case of $k = n$ and $k = m$, respectively. Let us denote

$$\mathcal{A}_{N,k}(x) = \begin{bmatrix} (N-k)^{1/6}\text{Ai}(x) \\ (N-k)^{-1/6}\text{Ai}'(x) \end{bmatrix},$$

and $g_{N,k}(x) = N^{2/3}f_{N,k,1}(x)$. With this notation, we may write the integral over the region $(1, \infty)^2$ as

$$\frac{1}{2\pi} \sum_{i,j=1,2} \int_1^\infty \int_1^\infty h_{N,i}(x)l_{N,j}(y)[\mathcal{A}_{N,n}(g_{N,n}(x))]_i[\mathcal{A}_{N,m}(g_{N,m}(y))]_j\epsilon(x-y)dxdy. \tag{6.26}$$

We note that for $l, k \in \{1, 2\}$, there is $C, c > 0$ such that for $x \geq 1$,

$$|[\mathcal{A}_{N,i}(g_{N,m}(x))]_k| \leq CN^{-(1)^k/6} e^{-cN^{2/3}(x-1)}.$$

Applying this and noting that $h_{N,i}, l_{N,i} = O(1)$, we see that we see that (i, j) -th term in (6.26) is of order $O(N^{-4/3 - (-1)^i/6 - (-1)^j/6})$, so that it suffices to treat the $i = j = 1$ term. Arguing with Taylor's theorem as in the single integral case, we see that the integral

$$\int_1^\infty \int_1^\infty h_{N,1}(x) \left(g_{N,1}(y) - \frac{g_{N,1}(1)}{h_{N,1}(1)} h_{N,1}(y) \right) \times [\mathcal{A}_{N,n}(g_{N,n}(x))]_1 [\mathcal{A}_{N,m}(g_{N,m}(y))]_1 \epsilon(x-y) dx dy$$

is of order $O(N^{-4/3})$. Noting that $|g_{N,n}(x) - g_{N,m}(x)| = O(N^{-1/3})$, we see by the above estimate on Ai' , as well as Taylor's theorem again, that there is $C, c > 0$ such that for $x \in (1, 1 + \delta)$

$$|[\mathcal{A}_{N,n}(g_{N,n}(x))]_1 - [\mathcal{A}_{N,m}(g_{N,m}(x))]_1| \leq CN^{-1/6} e^{-cN^{2/3}(x-1)},$$

so that

$$\int_1^\infty \int_1^\infty h_{N,1}(x) h_{N,1}(y) [\mathcal{A}_{N,n}(g_{N,n}(x))]_1 ([\mathcal{A}_{N,m}(g_{N,n}(y))]_1 - [\mathcal{A}_{N,m}(g_{N,m}(y))]_1) \epsilon(x-y) dx dy$$

is of order $O(N^{-4/3})$. Now noting that by symmetry, we have that

$$\frac{1}{2} \int_1^\infty \int_1^\infty h_{N,1}(x) h_{N,1}(y) [\mathcal{A}_{N,n}(g_{N,n}(x))]_1 [\mathcal{A}_{N,n}(g_{N,n}(y))]_1 \epsilon(x-y) dx dy = 0,$$

we see combining these results that the $i = j = 1$ term in (6.26) is of order $O(N^{-4/3})$, so that in total we have that

$$\int_1^\infty \int_1^\infty \phi(x) \varphi(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy = O(N^{-4/3}).$$

Combining these to show (6.21), we see that it suffices to show that

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^1 \int_{-\infty}^1 \phi(x) \varphi(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x) \varphi(x)}{(1-x^2)} \sin((m-n) \arccos(x)) dx + O(N^{-7/6}). \end{aligned} \tag{6.27}$$

As above, we write the integral over $(-\infty, 1)^2$ as

$$\frac{1}{2\pi} \sum_{i,j=1,2} \int_{-\infty}^1 \int_{-\infty}^1 h_{N,i}(x) l_{N,j}(y) [\mathcal{A}_{N,n}(g_{N,n}(x))]_i [\mathcal{A}_{N,m}(g_{N,m}(y))]_j \epsilon(x-y) dx dy. \tag{6.28}$$

Except for the case of $i = j = 1$, all of these terms may be dealt with essentially as in the proof of Lemma 6.6. On the other hand, the $i = j = 1$ term will require an amount of technical care, as did for the integral over $(1, \infty)^2$. For this reason, instead of employing Lemma 6.5 directly, we will need to modify its proof by integrating by parts using \int_{-y}^∞ rather than \int_0^y . This will be convenient as otherwise, each integral in Lemma 6.5 would still be of leading order for the $i = j = 1$ term.

To begin, we note that by integration by parts, we may write

$$\begin{aligned} & \frac{1}{2} \int_{-\infty}^1 l_{N,j}(y) [\mathcal{A}_{N,m}(g_{N,m}(y))]_j \epsilon(x-y) dy \\ &= \frac{l_{N,j}(x)}{g'_{N,m}(x)} \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) - \frac{1}{2} \frac{l_{N,j}(1)}{g'_{N,m}(1)} \left(\int_{-g_{N,m}(1)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) \\ & \quad - \frac{1}{2} \int_{-\infty}^1 \left(\int_{-g_{N,m}(y)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) \frac{d}{dy} \left(\frac{l_{N,j}(y)}{g'_{N,m}(y)} \right) \epsilon(x-y) dy. \end{aligned}$$

Employing this, we see that we may write the (i, j) -th term of (6.28) as the sum $I_{ij} + II_{ij} + III'_{ij}$ where

$$\begin{aligned} I_{ij} &= \frac{1}{\pi} \int_{-\infty}^1 [\mathcal{A}_{N,n}(g_{N,n}(x))]_i \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) \frac{h_{N,i}(x) l_{N,j}(x)}{g'_{N,m}(x)} dx, \\ II_{ij} &= -\frac{1}{2\pi} \left(\int_{-\infty}^1 [\mathcal{A}_{N,n}(g_{N,n}(x))]_i h_{N,i}(x) dx \right) \frac{l_{N,j}(1)}{g'_{N,m}(1)} \int_{-g_{N,m}(1)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz, \\ III'_{ij} &= -\frac{1}{2\pi} \int_{-\infty}^1 \int_{-\infty}^1 [\mathcal{A}_{N,n}(g_{N,n}(x))]_i \\ & \quad \times \left(\int_{-g_{N,m}(y)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) h_{N,i}(x) \frac{d}{dy} \left(\frac{l_{N,j}(y)}{g'_{N,m}(y)} \right) \epsilon(x-y) dx dy. \end{aligned}$$

By a similar integration by parts we may write III'_{ij} itself as a sum $III_{ij} + IV_{ij} + V_{ij}$ with

$$\begin{aligned} III_{ij} &= \frac{1}{\pi} \int_{-\infty}^1 \left(\int_{-g_{N,n}(x)}^{\infty} [\mathcal{A}_{N,n}(-z)]_i dz \right) \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) \\ & \quad \times \frac{h_{N,i}(x)}{g'_{N,n}(x)} \frac{d}{dx} \left(\frac{l_{N,j}(x)}{g'_{N,m}(x)} \right) dx, \\ IV_{ij} &= -\frac{1}{2\pi} \frac{h_{N,i}(1)}{g'_{N,n}(1)} \int_{-g_{N,n}(1)}^{\infty} [\mathcal{A}_{N,n}(-z)]_i dz \\ & \quad \times \int_{-\infty}^1 \frac{d}{dy} \left(\frac{l_{N,j}(y)}{g'_{N,m}(y)} \right) \left(\int_{-g_{N,m}(y)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) dy, \\ V_{ij} &= \frac{1}{2\pi} \int_{-\infty}^1 \int_{-\infty}^1 \left(\int_{-g_{N,n}(x)}^{\infty} [\mathcal{A}_{N,n}(-z)]_i dz \right) \left(\int_{-g_{N,m}(y)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) \\ & \quad \times \frac{d}{dx} \left(\frac{h_{N,i}(x)}{g'_{N,n}(x)} \right) \frac{d}{dy} \left(\frac{l_{N,j}(y)}{g'_{N,m}(y)} \right) \epsilon(x-y) dx dy. \end{aligned}$$

We now collect the following asymptotics for $x \in (1, \infty)$ (see Chapter 10 of [1])

$$\begin{aligned} \sqrt{\pi} \text{Ai}(-x) &= \frac{1}{x^{1/4}} \sin \left(\frac{2}{3} x^{3/2} + \pi/4 \right) + O(x^{-7/4}) \\ \sqrt{\pi} \text{Ai}'(-x) &= -x^{1/4} \cos \left(\frac{2}{3} x^{3/2} + \pi/4 \right) + O(x^{-5/4}), \\ \sqrt{\pi} \int_x^{\infty} \text{Ai}(-y) dy &= -\frac{1}{x^{3/4}} \sin \left(\frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) + O(x^{-9/4}), \\ \sqrt{\pi} \int_x^{\infty} \text{Ai}'(-y) dy &= -\frac{1}{x^{1/4}} \cos \left(\frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) + O(x^{-7/4}). \end{aligned} \tag{6.29}$$

For now we only take from these that these functions are bounded, so that for fixed $k \in \mathbb{Z}$ and $x \in [1 - \delta, 1]$

$$[\mathcal{A}_{N,k}(x)]_l = O(N^{-(-1)^l/6}), \quad \int_{-g_{N,k}(x)}^{\infty} [\mathcal{A}_{N,k}(-z)]_l dz = O(N^{-(-1)^l/6}). \quad (6.30)$$

Employing these bounds, and recalling that $g_{N,k}(x) = O(N^{2/3})$, we see that for any $i, j = 1, 2$ the integrals III_{ij} , IV_{ij} and V_{ij} are of order $O(N^{-4/3-(-1)^i/6-(-1)^j/6})$. Further employing (6.24) we see that

$$\int_{-\infty}^1 [\mathcal{A}_{N,n}(g_{N,n}(x))]_i h_{N,i}(x) dx = O(N^{-(-1)^i/6-2/3}).$$

Thus we see as well that II_{ij} has order $O(N^{-4/3-(-1)^i/6-(-1)^j/6})$. In particular, if $(i, j) \neq (1, 1)$ all integrals except for I_{ij} are $O(N^{-4/3})$.

If $i = j = 1$ though, none of these bounds are sufficient. Instead we will need to note that for $k \in \mathbb{Z}$ and $x \in (1 - \delta, 1]$

$$[\mathcal{A}_{N,k}(x)]_1 = O((1-x)^{-1/4}), \quad \int_{-g_{N,k}(x)}^{\infty} [\mathcal{A}_{N,k}(-z)]_1 dz = O(N^{-1/3}(1-x)^{-3/4}). \quad (6.31)$$

Employing (6.30) for $x \in (1 - N^{-2/3}, 1]$ and (6.31) elsewhere we see that there is C such that

$$|III_{11}| \leq \frac{C}{N^{4/3}} \left(\frac{1}{N^{2/3}} \int_{N^{-2/3}}^{\infty} \frac{dx}{x^{6/4}} + N^{1/3} \int_0^{N^{-2/3}} dx \right) = \frac{3C}{N^{5/3}}.$$

Similarly we see that IV_{11}, V_{11} are $O(N^{-4/3})$.

Lastly, by applying (6.24) to II_{11} , we see that

$$II_{11} = -\frac{N^{1/3}}{2\pi} \frac{h_{N,1}(1)}{g'_{N,n}(1)} \frac{l_{N,1}(1)}{g'_{N,m}(1)} \left(\frac{2}{3}\right)^2 + O(N^{-4/3}) = -\frac{\phi(1)\varphi(1)}{18\pi N} [T_1(1)]_{11}^2 + O(N^{-4/3}).$$

Now we are left to consider I_{ij} . As before, the case of $i = j = 1$ will present an additional subtlety. Let us denote

$$\begin{aligned} \mathcal{S}_N(x) &= \left[\begin{array}{l} |f_{N,n}(x)|^{-1/4} \sin((N-n)s_{N,n}(x) + \pi/4) \\ -|f_{N,n}(x)|^{1/4} \cos((N-n)s_{N,n}(x) + \pi/4) \end{array} \right], \\ \mathcal{C}_N(x) &= \left[\begin{array}{l} -|f_{N,m}(x)|^{-1/4} \sin((N-m)s_{N,m}(x) - \pi/4) \\ -|f_{N,m}(x)|^{1/4} \cos((N-m)s_{N,m}(x) - \pi/4) \end{array} \right]. \end{aligned}$$

We observe that by applying the asymptotics of (6.29), for $1 - \delta \leq x \leq 1 - N^{-2/3}$ and $i = 1, 2$

$$\begin{aligned} [\mathcal{A}_{N,n}(g_{N,n}(x))]_i &= [\mathcal{A}_{N,n}((N-n)^{2/3} f_{N,n,1}(x))]_i \\ &= \pi^{-1/2} [\mathcal{S}_N(x)]_i + O(N^{-1}(1-x)^{-7/4}), \\ \int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_i dz &= (N-m)^{1/2} |f_{N,m,1}(x)|^{1/2} \int_{-(N-m)^{2/3} f_{N,m,1}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_i dz \\ &= \pi^{-1/2} (N-m)^{-1/3} |f_{N,m,1}(x)|^{-1/2} [\mathcal{C}_N(x)]_i \\ &\quad + O(N^{-4/3}(1-x)^{-9/4}). \end{aligned} \quad (6.32)$$

The subtlety that arises is that direct substitution of these asymptotic into I_{11} , even neglecting the error terms, leads to a quantity that is not integrable. To deal with this, we

will need to treat the integral of I_{ij} over the regions $[1 - N^{-1/3}, 1]$ and $[1 - \delta, 1 - N^{-1/3}]$ separately. In the latter integral, we will be able to apply (6.32), while the integral over the prior will be negligible unless $i = j = 1$, where it will surprisingly cancel the contribution from II_{11} up to lower order terms.

To begin, we note that by (6.32) for $i, j = 1, 2$ and $1 - \delta \leq x \leq 1 - N^{-2/3}$

$$\begin{aligned} & [\mathcal{A}_{N,n}(g_{N,n}(x))]_i \int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \\ &= \pi^{-1} N^{-1/3} |f_{N,m,1}(x)|^{-1/2} [\mathcal{S}_N(x)]_i [\mathcal{C}_N(x)]_j + O(N^{-7/3}(1-x)^{-4}). \end{aligned} \tag{6.33}$$

We now note that

$$N^{-7/3} \int_{N^{-1/3}}^{\infty} x^{-4} dx = O(N^{-4/3}).$$

Employing this and (6.33) we see that

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{1-N^{-1/3}} [\mathcal{A}_{N,n}(g_{N,n}(x))]_i \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) \frac{h_{N,i}(x)l_{N,j}(x)}{g'_{N,m}(x)} dx \\ &= \frac{1}{\pi^2(N-m)} \int_{-\infty}^{1-N^{-1/3}} [\mathcal{S}_N(x)]_i [\mathcal{C}_N(x)]_j \frac{h_{N,i}(x)l_{N,j}(x)}{f_{N,m,1}(x)^{1/2} f'_{N,m,1}(x)} dx + O(N^{-4/3}). \end{aligned}$$

Now if $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$ then for $1 - \delta \leq x \leq 1 - N^{-2/3}$

$$[\mathcal{A}_{N,n}(g_{N,n}(x))]_i \int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz = O(N^{-1/3} x^{-1/2}),$$

and $O(1)$ for $1 - N^{-2/3} \leq x \leq 1$. Combining both of these, we see that

$$\frac{1}{\pi} \int_{1-N^{-1/3}}^1 [\mathcal{A}_{N,n}(g_{N,n}(x))]_i \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_j dz \right) \frac{h_{N,i}(x)l_{N,j}(x)}{g'_{N,m}(x)} dx = O(N^{-4/3}). \tag{6.34}$$

The case of $(i, j) = (2, 2)$ follows more simply by just employing (6.30) to obtain (6.34). Now we will focus on the case of $(i, j) = (1, 1)$. We note that by Taylor's Theorem $f_{N,n,1}(x) - f_{N,m,1}(x) = O(N^{-1})$ uniformly for $x \in (1 - \delta, 1 + \delta)$. Moreover we note that by the above asymptotic for Ai' there is C such that for $x \in (-\infty, 0)$ we have that $|Ai'(x)| \leq C(1 + |x|^{1/4})$. Thus by again applying Taylor's Theorem we see that for $1 - N^{-1/3} \leq x \leq 1$,

$$|[\mathcal{A}_{N,n}(g_{N,n}(x))]_1 - [\mathcal{A}_{N,m}(g_{N,m}(x))]_1| = O(N^{-1/6} + (1-x)^{1/4}).$$

From this, we see that there is C such that

$$\begin{aligned} & \left| \int_{1-N^{-1/3}}^1 ([\mathcal{A}_{N,n}(g_{N,n}(x))]_1 - [\mathcal{A}_{N,m}(g_{N,m}(x))]_1) \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_1 dz \right) \right. \\ & \times \left. \frac{h_{N,1}(x)l_{N,1}(x)}{g'_{N,m}(x)} dx \right| \leq CN^{-1} \int_{1-N^{-1/3}}^1 \frac{N^{-1/6} + (1-x)^{1/4}}{(1-x)^{3/4}} dx = O(N^{-7/6}). \end{aligned}$$

In addition, we note that by integration by parts we may write

$$\frac{1}{\pi} \int_{1-N^{-1/3}}^1 [\mathcal{A}_{N,m}(g_{N,m}(x))]_1 \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_1 dz \right) \frac{h_{N,1}(x)l_{N,1}(x)}{g'_{N,m}(x)} dx = I' + II',$$

where here

$$I' = \frac{1}{2\pi} \left[\frac{h_{N,1}(x)l_{N,1}(x)}{g'_{N,m}(x)g'_{N,m}(x)} \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_1 dz \right)^2 \right]_{1-N^{-1/3}}^1,$$

$$II' = -\frac{1}{2\pi} \int_{1-N^{-1/3}}^1 \frac{d}{dx} \left(\frac{h_{N,1}(x)l_{N,1}(x)}{g'_{N,m}(x)g'_{N,m}(x)} \right) \left(\int_{-g_{N,m}(x)}^{\infty} [\mathcal{A}_{N,m}(-z)]_1 dz \right)^2.$$

Using (6.30) we see that $II' = O(N^{-4/3})$ and using (6.31) that

$$I' = \frac{\phi(1)\varphi(1)}{18\pi N} [T_1(1)]_{11}^2 + O(N^{-4/3}).$$

We observe that this is the negation of asymptotic for II_{11} obtained above. In particular, combining all of these results, we see that the left-hand side of (6.27) may be rewritten as

$$\frac{1}{\pi^2(N-m)} \sum_{i,j=1,2} \int_{-\infty}^{1-N^{-1/3}} [S_N(x)]_i [C_N(x)]_j \frac{h_{N,i}(x)l_{N,j}(x)}{f_{N,m,1}(x)^{1/2} f'_{N,m,1}(x)} dx + O(N^{-7/6}). \tag{6.35}$$

Combining the relation $s'_{N,m}(x) = f_{N,m,1}(x)^{1/2} f'_{N,m,1}(x)$ (recalling $f_{N,m,1}$ from (4.18)) Lemma 6.8, and the bounds on R_N, \bar{R}_N , as in the proof of Lemma 6.6, we see that the integral of (6.35) may further be rewritten as

$$\begin{aligned} & \frac{1}{\pi(N-m)} \int_{-\infty}^{1-N^{-1/3}} \left[T_{\infty}(x) \begin{bmatrix} e^{i(N-n)s_{N,n}(x)} \\ e^{-i(N-n)s_{N,n}(x)} \end{bmatrix} \right]_1 \\ & \times \left[T_{\infty}(x) e^{-i\frac{\pi}{2}\sigma} \begin{bmatrix} e^{i(N-m)s_{N,m}(x)} \\ e^{-i(N-m)s_{N,m}(x)} \end{bmatrix} \right]_1 \frac{\phi(x)\varphi(x)}{s'_{N,m}(x)} dx + O(N^{-7/6}) \\ & = \frac{-i}{\pi(N-m)} \sum_{k,l=1,2} \int_{-\infty}^{1-N^{-1/3}} [T_{\infty}(x)]_{1k} [T_{\infty}(x)]_{l1} \\ & \times \exp \left(-(-1)^k i(N-n)s_{N,n}(x) - (-1)^l i(N-m)s_{N,m}(x) \right) (-1)^l \frac{\phi(x)\varphi(x)}{s'_{N,m}(x)} dx \tag{6.36} \end{aligned}$$

We will show that the terms in this equation when $k = l$ are negligible. For this, we observe that by integration by parts, we may write the $(k, l) = (1, 1)$ term in (6.36) as the sum $I'' + II''$ where

$$\begin{aligned} I'' &= \left[\frac{-i}{\pi(N-m)} \left(\frac{[T_{\infty}(x)]_{11}^2 \phi(x)\varphi(x)}{s'_{N,m}(x)((N-n)s'_{N,n}(x) + (N-m)s'_{N,m}(x))} \right) \right. \\ & \quad \left. \times \exp(i((N-n)s_{N,n}(x) + (N-m)s_{N,m}(x))) \right]_{-\infty}^{1-N^{-1/3}} \\ II'' &= \frac{i}{\pi(N-m)} \int_{-\infty}^{1-N^{-1/3}} \frac{d}{dx} \left(\frac{[T_{\infty}(x)]_{11}^2 \phi(x)\varphi(x)}{s'_{N,m}(x)((N-n)s'_{N,n}(x) + (N-m)s'_{N,m}(x))} \right) \\ & \quad \times \exp(i((N-n)s_{N,n}(x) + (N-m)s_{N,m}(x))) dx. \end{aligned}$$

Observing that $[T_{\infty}(x)]_{11} = O((1-x)^{-1/4})$ and that $s'(x) = O((1-x)^{1/2})$, we see that there is C such that

$$|I'| \leq N^{-2} C [x^{-3/2}]_{N^{-1/3}}^{\infty} = CN^{-3/2}, \quad |II''| \leq \frac{C}{N^2} \int_{N^{-1/3}}^{\infty} \frac{dx}{x^{5/2}} \leq CN^{-3/2}.$$

Together these show that the $(k, l) = (1, 1)$ term in (6.36) is $O(N^{-3/2})$. Showing that the term $(k, l) = (2, 2)$ is $O(N^{-3/2})$ is similar. Now lastly, we will need a sharp form

of Lemma 4.4. For this we note that for $|x| < 1$, $s'(x) = -\sqrt{1-x^2}$, and so for $x \in (1-\delta, 1-N^{-1/3})$ we have that $s''(x) = O(N^{1/6})$. From this, we see that for fixed k and $x \in (1-\delta, 1-N^{-1/3})$, we have that

$$(N-k)s_{N,k}(x) = Ns(x) - k\arccos(x) + O(N^{-5/6}).$$

Employing this and proceeding as in the proof of Lemma 6.6, we may show now that (6.36) coincides with

$$\begin{aligned} & \frac{1}{2\pi N} \int_{-1}^{1-N^{-1/3}} \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx + O(N^{-7/6}) \end{aligned}$$

Altogether, these statements complete the proof of (6.27). □

Taking a partition of unity with respect to all of the above regions, we may conclude the following proposition.

Proposition 6.11. *Let ϕ and φ be smooth functions of subexponential growth. Then we have that for any n and m that*

$$\begin{aligned} & \frac{1}{2} \int \int \phi(x)\varphi(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx \\ &= \frac{1}{2\pi N} \left[\int_{-1}^1 \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx \right] \\ &+ \frac{1}{4N} [(-1)^{N-m}\phi(1)\varphi(-1) - (-1)^{N-n}\phi(-1)\varphi(1)] + O(N^{-7/6}). \end{aligned}$$

We see that as

$$\int_{-1}^1 \frac{\sin(\arccos(x))}{(1-x^2)} dx = \pi,$$

Proposition 6.11 shows that when N is even that

$$\int J_N^{-1}(p_N)(x)p_{N-1}(x)w_N(x) dx = O(N^{-7/6}). \tag{6.37}$$

For n and m arbitrary, we additionally see that

$$\int J_N^{-1}(p_{N-n})(x)p_{N-m}(x)w_N(x) dx = O(N^{-1}).$$

Together these demonstrate two of the integrals required by Proposition 2.9. Before proceeding, we also observe the following corollary of Lemma 6.1, and the single integral cases of Lemmas 6.6 and 6.10.

Corollary 6.12. *Assume that ϕ is a smooth function on \mathbb{R} of sub-exponential growth. Then for fixed k , we have that*

$$\int \phi(x)p_{N-k}(x)w_N(x)^{1/2} dx = O(N^{-1/2}).$$

To complete the proof of Proposition 2.9, we must now focus our attention on the integrals involving ℓ_i and q_i (defined in (2.8) and (2.9) above). We will begin with some single integral computations.

Lemma 6.13. Assume that ϕ is a smooth function of subexponential growth. Then for any $1 \leq i \leq m$, we have that

$$\int \phi(x)\ell_i(x)w_N(x)^{1/2} dx = O(N^{-1/2}), \quad \int \phi(x)q_i(x)w_N(x)^{1/2} dx = O(1).$$

Proof. We adopt the notation $\lambda = \lambda_i$ and $\alpha = \alpha_i$ as before. In view of Proposition 4.14 and (4.32) we see that

$$\int \phi(x) \begin{bmatrix} -\ell_i(x) \\ 2\pi i q_i(x) \end{bmatrix} w_N(x)^{1/2} dx = c_i^{-\sigma} e^{-i\frac{\pi}{4}} e^{-\frac{\pi i}{4}\sigma} \sigma_1 T_{N,\lambda}(\lambda)^{-1} D_\infty^\sigma (I + O(N^{-1})) e^{-N\frac{\ell}{2}\sigma} \times \left(\int \frac{\phi(x)}{x-\lambda} \begin{bmatrix} \kappa_N^{-1} p_N(x) \\ -2\pi i \kappa_{N-1} p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2} dx \right).$$

From Remark 4.6 we see that the right-hand side may further be written as

$$c_i^{-\sigma} e^{-i\frac{\pi}{4}} e^{-\frac{\pi i}{4}\sigma} \sigma_1 T_{N,\lambda}(\lambda)^{-1} (I + O(N^{-1})) \left(\int \frac{\phi(x)}{x-\lambda} \begin{bmatrix} \pi^{1/2} p_N(x) \\ -i\pi^{1/2} p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2} dx \right).$$

Thus we see that it suffices to show that

$$\int \frac{\phi(x)}{x-\lambda} \begin{bmatrix} \pi^{1/2} p_N(x) \\ -i\pi^{1/2} p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2} dx = T_{N,\lambda}(\lambda) \begin{bmatrix} O(1) \\ 0 \end{bmatrix} + O(N^{-1/2}). \tag{6.38}$$

Let us pick $\delta_0 > 0$ such that asymptotics of Proposition 4.13 hold. By applying Corollary 6.12 we see that it suffices to assume that ϕ is supported on $(\lambda - \delta_0, \lambda + \delta_0)$. With the asymptotics of Proposition 4.13, and a change of coordinates by f_λ , (and recalling the definition of \mathcal{J}_α from 4.28) we may write

$$\int \frac{\phi(x)}{x-\lambda} \begin{bmatrix} \pi^{1/2} p_N(x) \\ -i\pi^{1/2} p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2} dx = \int \frac{G_{N,1}(x)}{x} \epsilon(x) \sqrt{\pi|Nx|} J_{\alpha+1/2}(N|x|) dx + \int \frac{G_{N,2}(x)}{x} \sqrt{\pi|Nx|} J_{\alpha-1/2}(N|x|) dx = I + II,$$

where here $G_{N,i} = (f_\lambda^{-1})'(x)\phi(f_\lambda^{-1}(x))[(I + R_N(f_\lambda^{-1}(x)))T_{N,\lambda}(f_\lambda^{-1}(x))]e_i$. We observe that $G_{N,i}^{(c)}(x) = O(1)$ uniformly for all $c \in \mathbb{N}$ and $x \in \mathbb{R}$. From this, we see by Lemma 6.7 that

$$I = 2\sqrt{\pi}G_{N,1}(0)D(\alpha + 1/2, -1) + O(N^{-1}) \\ = \sqrt{\pi}(f_\lambda^{-1})'(0)\phi(\lambda)[T_{N,\lambda}(\lambda)]e_1 D(\alpha + 1/2, -1/2) + O(N^{-1}).$$

The case of II is similar except that the first-order contribution of the integrals on $(0, \infty)$ and $(-\infty, 0)$ cancel so that $II = O(N^{-1})$. Together these computations show that

$$\int \frac{\phi(x)}{x-\lambda} \begin{bmatrix} \pi^{1/2} p_N(x) \\ -i\pi^{1/2} p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2} dx = T_{N,\lambda}(\lambda) \begin{bmatrix} \phi(\lambda)\sqrt{\pi}2D(\alpha + 1/2, -1/2) \\ 0 \end{bmatrix} + O(N^{-1}),$$

which is more than sufficient to verify (6.38). □

We recall from Remark 4.15 that for fixed $1 \leq i \leq m$ there are constants $C_{1,N}, C_{2,N}$ with $C_{k,N} = O(1)$ and such that

$$\ell_i(x) = C_{1,N} \frac{p_N(x)}{x-\lambda_i} + C_{2,N} \frac{p_{N-1}(x)}{x-\lambda_i}, \tag{6.39}$$

and similarly for q_i . Let us take δ sufficiently small as above. We fix for, each $1 \leq l \leq m$, a choice of smooth function ϕ_l supported on $(\lambda_l - \delta/2, \lambda_l + \delta/2)$, such that $\phi_l(x) = 1$ for

$x \in (\lambda_l - \delta/4, \lambda_l + \delta/4)$. In addition, we choose φ_l supported on $(\lambda_l - \delta, \lambda_l + \delta)$, such that $\varphi_l(x) = 1$ for $x \in (\lambda_l - \delta/2, \lambda_l + \delta/2)$.

With these selected, we may write

$$\int J_N^{-1} \ell_i(x) p_{N-n}(x) w_N(x) dx = \int (J_N^{-1} \phi_i \ell_i)(x) p_{N-n}(x) w_N(x) dx + \int (J_N^{-1} (1 - \phi_i) \ell_i)(x) p_{N-n}(x) w_N(x) dx.$$

Employing (6.39), we may write the second integral as

$$\sum_{k=0,1} \frac{C_{k+1,N}}{2} \int \int p_{N-n}(x) p_{N-k}(y) \frac{(1 - \phi_i(y))}{(y - \lambda_i)} w_N(x) w_N(y) dx dy. \tag{6.40}$$

Observing that the function $(1 - \phi_i(x))/(x - \lambda_i)$ is smooth and bounded, we may apply Proposition 6.11 to see that the terms in (6.40) are of order $O(N^{-1})$. From this, we see that

$$\int J_N^{-1} \ell_i(x) p_{N-n}(x) w_N(x) dx = \int (J_N^{-1} \phi_i \ell_i)(x) p_{N-n}(x) w_N(x) dx + O(N^{-1}).$$

We now also write

$$\int (J_N^{-1} \phi_i \ell_i)(x) p_{N-n}(x) w_N(x) dx = \int (J_N^{-1} \phi_i \ell_i)(x) p_{N-n}(x) \varphi_i(x) w_N(x) dx + \int (J_N^{-1} (\phi_i \ell_i))(x) (1 - \varphi_i(x)) p_{N-n}(x) w_N(x) dx.$$

We note that as $\phi_i \ell_i$ and $(1 - \varphi_i)$ have disjoint support, we may expand J_N^{-1} and employ (6.1) to rewrite the second integral on the right-hand side as

$$\frac{1}{2} \int \phi_i(y) \ell_i(y) w_N(y)^{1/2} dy \int (1 - \varphi_i(x)) \epsilon(x - \lambda_i) p_{N-n}(x) w_N(x)^{1/2} dx$$

which is $O(N^{-1})$ by employing Lemma 6.13 and Corollary 6.12. Altogether we see that

$$\int J_N^{-1} \ell_i(x) p_{N-n}(x) w_N(x) dx = \int J_N^{-1} (\phi_i \ell_i)(x) \varphi_i(x) p_{N-n}(x) w_N(x) dx + O(N^{-1}).$$

Similarly, we obtain that

$$\int J_N^{-1} p_{N-n}(x) q_i(x) w_N(x) dx = \int J_N^{-1} (\phi_i p_{N-n})(x) \varphi_i(x) q_i(x) w_N(x) dx + O(N^{-1/2}),$$

and that

$$\int J_N^{-1} \ell_i(x) q_j(x) w_N(x) dx = \delta_{ij} \int J_N^{-1} (\phi_i \ell_i)(x) \varphi_i(x) q_i(x) w_N(x) dx + O(N^{-1/2}).$$

From these computations and (6.39), we see that to complete the proof of Proposition 2.9, it suffices to show the following.

Lemma 6.14. *There is $\delta > 0$ such that if ϕ and φ are smooth functions supported on $(\lambda_j - \delta, \lambda_j + \delta)$ for some $1 \leq j \leq m$, then for any choice of fixed n and m we have that*

$$\frac{1}{2} \int \int \frac{\phi(x) p_{N-n}(x)}{x - \lambda_j} \varphi(y) p_{N-m}(y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy = O(N^{-1} \log(N)). \tag{6.41}$$

Moreover, if $|n - m| \leq 1$, then we additionally have that

$$\frac{1}{2} \int \int \frac{\phi(x) p_{N-n}(x)}{x - \lambda_j} \frac{\varphi(y) p_{N-m}(y)}{y - \lambda_j} w_N(x) w_N(y) dx dy = O(N^{-1} \log(N)). \tag{6.42}$$

Proof. Again, for convenience, we write $\lambda = \lambda_j$ and $\alpha = \alpha_j$, and choose δ as before. We begin by proving the first statement. With notation as in Lemma 6.6, we may rewrite the left-hand side of (6.41) as in (6.15) as

$$\frac{1}{2\pi} \sum_{k,l=1,2} \int \int \frac{1}{x-\lambda} h_{N,k}(x) [\mathcal{J}_\alpha(g_{N,n}(x))]_k l_{N,l}(y) [\mathcal{J}_\alpha(g_{N,m}(y))]_l \epsilon(x-y) dx dy. \tag{6.43}$$

We claim that the summand for each choice of (k, l) is $O(N^{-1} \log(N))$. To begin, we define, for $i = 1, 2$, the functions $I_i(x) = \frac{1}{2} \int \epsilon(x-y) [\mathcal{J}_\alpha(y)]_i y^{-1} dy$. We note that by performing integration by parts in x (see the proof of Lemma 6.5) we may write (6.43) as the sum $I + II$ where

$$I = \frac{1}{\pi} \sum_{k,l=1,2} \int [\mathcal{J}_\alpha(g_{N,m}(x))]_l I_k(g_{N,n}(x)) \frac{h_{N,k}(x) l_{N,l}(x) g_{N,n}(x)}{g'_{N,n}(x)(x-\lambda)} dx,$$

$$II = -\frac{1}{2\pi} \sum_{k,l=1,2} \int \int [\mathcal{J}_\alpha(g_{N,m}(x))]_l I_k(g_{N,n}(y)) l_{N,l}(y) \frac{d}{dx} \left(\frac{h_{N,k}(x) g_{N,n}(x)}{g'_{N,n}(x)(x-\lambda)} \right) \epsilon(x-y) dx dy.$$

We observe that as $g_{N,n}(\lambda) = 0$ we have that $g_{N,n}(x)/((x-\lambda)g'_{N,n}(x))$ as well as its derivatives are $O(1)$ for $x \in (\lambda - \delta, \lambda + \delta)$. We note that $I_1(x)$ ($I_2(x)$) is anti-symmetric (symmetric), respectively. Moreover, for $x > 0$, we have the relations

$$I_1(x) = \int_0^x [\mathcal{J}_\alpha(y)]_1 y^{-1} dy = \sqrt{\pi} J_{\alpha+1, -1/2}(x),$$

$$I_2(x) = \int_x^\infty [\mathcal{J}_\alpha(y)]_2 y^{-1} dy = \sqrt{\pi} \int_x^\infty y^{-1/2} J_{\alpha+1/2}(y) dy.$$

Applying the asymptotics (6.16) and (5.10) above, it follows that for $i = 1, 2$ we have that $I_i(x) = O((1 + |x|)^{-1})$ for $x \in \mathbb{R}$. Applying these bounds pointwise, and noting that $h_{N,k}(x), l_{N,l}(x) = O(1)$, we see that the integrand of I is uniformly $O(N^{-1}|x|^{-1})$ for $|x| \geq 1/N$, and $O(1)$ for $|x| \leq 1/N$. Thus contribution of the integral over $\mathbb{R} \setminus (\lambda - N^{-1}, \lambda + N^{-1})$ is of order $O(N^{-1} \log(N))$ and the contribution over $(\lambda - N^{-1}, \lambda + N^{-1})$ is of order $O(N^{-1})$, so that $I = O(N^{-1} \log(N))$. Similarly we see that $II = O(N^{-1} \log(N))$. Together these establish (6.41).

We now establish (6.42). The key difference, in this case, is that we may employ Proposition 4.13 only once to obtain the asymptotics of both p_{N-n} and p_{N-m} . For notational ease, we will only prove the case when $n = 0$ and $m = 0, 1$. We write $h_{N,k}(x) = \phi(x)[(I + R_N(x))T_{N,\lambda}(x)]_{1k}$ and $l_{N,k}(x) = i^{l(m=1)} \varphi(x)[(I + R_N(x))T_{N,\lambda}(x)]_{(1+m)k}$, where R_N is as in Proposition 4.13. This is where we use the assumption that $|n - m| \leq 1$ so that we may write both terms with the same asymptotic. Using this, we may rewrite the left-hand side of (6.42) as

$$\frac{1}{2\pi} \sum_{k,l=1,2} \int \int \frac{h_{N,k}(x) l_{N,l}(y)}{(x-\lambda)(y-\lambda)} [\mathcal{J}_\alpha(Nf_\lambda(x))]_k [\mathcal{J}_\alpha(Nf_\lambda(y))]_l \epsilon(x-y) dx dy.$$

If we define

$$\bar{h}_{N,i}(x) = h_{N,i}(f_\lambda^{-1}(x)) \frac{(f_\lambda^{-1})'(x)x}{(f_\lambda^{-1}(x) - \lambda)},$$

and similarly for $\bar{l}_{N,i}$, we may further rewrite (6.42) as

$$\frac{1}{2\pi} \sum_{k,l=1,2} \int \int \left(\frac{\bar{h}_{N,k}(x) \bar{l}_{N,l}(y)}{xy} \right) [\mathcal{J}_\alpha(Nx)]_k [\mathcal{J}_\alpha(Ny)]_l \epsilon(x-y) dx dy.$$

Let us fix $\chi(x)$, a smooth, even, function of compact support, with $\chi(x) = 1$ for $x \in (-\delta/2, \delta/2)$. As before, we have that $\bar{h}_{N,l}$ and $\bar{l}_{N,l}$, as well as their derivatives, are both uniformly $O(1)$ for $x \in (\lambda - \delta, \lambda + \delta)$, and vanish elsewhere. Applying Taylor's theorem, we may write

$$\bar{h}_{N,l}(x) = \bar{h}_{N,l}(\lambda)\chi(x - \lambda) + (x - \lambda)\hat{h}_{N,l}(x),$$

where $\hat{h}_{N,l}$ is a smooth function, with values and derivatives uniformly $O(1)$, and which is supported on $(\lambda - \delta, \lambda + \delta)$. Likewise, we may define such a function $\hat{l}_{N,l}$.

Inserting these two decompositions into (6.42) yields four integrals. However, to treat the integrals involving either $(x - \lambda)\hat{h}_{N,l}(x)$ or $(y - \lambda)\hat{l}_{N,l}(y)$, one may cancel either the common factor of $(x - \lambda)$ or $(y - \lambda)$, and show the resulting integral is $O(N^{-1} \log(N))$ in the same way as (6.41). Thus we may conclude that (6.42) is

$$\sum_{k,l=1,2} \frac{\bar{h}_{N,k}(\lambda)\bar{l}_{N,l}(\lambda)}{2\pi} \int \int \frac{\chi(x)\chi(y)}{xy} [\mathcal{J}_\alpha(Nx)]_k [\mathcal{J}_\alpha(Ny)]_l \epsilon(x - y) dx dy + O(N^{-1} \log(N)).$$

In particular, it suffices to show that that for all $1 \leq i, j \leq 2$

$$\frac{1}{2} \int \int \frac{\chi(x)\chi(y)}{xy} [\mathcal{J}_\alpha(Nx)]_i [\mathcal{J}_\alpha(Ny)]_j \epsilon(x - y) dx dy = O(N^{-1} \log(N)). \tag{6.44}$$

This integral is antisymmetric under the interchange of i and j . Thus it suffices to prove the case of $(i, j) = (1, 2)$. By employing Lemma 6.5 we may rewrite (6.44) in the case $(i, j) = (1, 2)$ as the sum of three integrals $I + II + III$

$$\begin{aligned} I &= \int x^{-1} [\mathcal{J}_\alpha(Nx)]_1 I_2(Nx) \chi(x) \chi(x) dx, \\ II &= \int I_1(Nx) I_2(Nx) \chi(x) \chi'(x) dx, \\ III &= \frac{1}{2} \int \int I_1(Nx) I_2(Ny) \chi'(x) \chi'(y) \epsilon(x - y) dx dy. \end{aligned}$$

As χ' vanishes in a neighborhood of 0, and $I_i(x) = O(|x|^{-1})$, we see that $II, III = O(N^{-2})$. To deal with I , we first rewrite it as

$$I = \int x^{-1} [\mathcal{J}_\alpha(x)]_1 I_2(x) \chi(x/N)^2 dx.$$

As $[\mathcal{J}_\alpha(x)]_1 = O(1)$, we note that as $x^{-1} [\mathcal{J}_\alpha(x)]_1 I_2(x) = O((1 + |x|)^{-2})$, we see that $x^{-1} [\mathcal{J}_\alpha(x)]_1 I_2(x)$ is integrable. In addition, we see that there is C such that

$$\left| \int x^{-1} [\mathcal{J}_\alpha(x)]_1 I_2(x) (1 - \chi(x/N)^2) dx \right| \leq C \int_{|x| \geq \delta N} x^{-2} dx = 2C\delta^{-1} N^{-1}.$$

Combining these, we see that

$$I = \int x^{-1} [\mathcal{J}_\alpha(x)]_1 I_2(x) dx + O(N^{-1}) = 2\sqrt{\pi} \int_0^\infty x^{-1/2} J_{\alpha+1/2}(x) I_2(x) dx + O(N^{-1}).$$

We write

$$I' = \frac{1}{\sqrt{\pi}} \int_0^\infty x^{-1/2} J_{\alpha+1/2}(x) I_2(x) dx = \int_0^\infty \int_x^\infty x^{-1/2} J_{\alpha+1/2}(x) y^{-1/2} J_{\alpha-1/2}(y) dy dx,$$

so we now need only show that $I' = 0$. Making the change of variables $y = ax$, this integral becomes

$$\int_0^\infty \int_1^\infty a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) da dx. \tag{6.45}$$

We note that by equation 11.4.41 of [1], we have that

$$\int_0^\infty a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) dx = 0; \quad a > 1.$$

This shows that (6.45) vanishes when the integrals are taken in the opposite order. As the integrals are not absolutely convergent though, we must proceed with care. Applying the above bounds on I_2 and $J_{\alpha+1/2}$, we have that

$$\int_0^L \int_L^\infty a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) dadx = \frac{1}{\sqrt{\pi}} \int_0^L x^{-1/2} J_{\alpha+1/2}(x) I_2(Lx) dx$$

is of order $O(L^{-1} \log(L))$. Thus we have that

$$\begin{aligned} I' &= \lim_{L \rightarrow \infty} \int_0^L \int_1^\infty a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) dadx \\ &= \lim_{L \rightarrow \infty} \int_0^L \int_1^L a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) dadx. \end{aligned}$$

By employing (6.6) and the double-angle formula, we may compute that

$$\begin{aligned} &\int_1^L \int_L^\infty a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) dx da \\ &= -\frac{2}{\pi} \int_1^L \int_L^\infty \left(\frac{\sin(x + \alpha \frac{\pi}{2}) \cos(ax + \alpha \frac{\pi}{2})}{xa} + O(x^{-2} a^{-1}) \right) dx da \\ &= -\frac{2}{\pi} \int_1^L \frac{1}{2a} \left[\int_L^\infty (\sin((1+a)x + \alpha\pi)x^{-1} - \sin((a-1)x)x^{-1}) dx \right] da + O(L^{-1} \log(L)). \end{aligned}$$

We note that there is C such that for $y > 0$, we have that $\int_y^\infty \sin(x)x^{-1} dx \leq Cy^{-1}$. Employing this, we see that

$$\left| \int_1^L \frac{1}{a} \int_L^\infty \sin((1+a)x + \alpha\pi)x^{-1} dx da \right| \leq C^2 \int_1^L \frac{1}{aL} da = O(L^{-1} \log(L)).$$

We also note that there is C such that for $y > 0$

$$\left| \int_y^\infty \sin(x)x^{-1} dx \right| \leq C.$$

From this, we see that

$$\begin{aligned} &\left| \int_1^L \frac{1}{a} \int_L^\infty \sin((a-1)x)x^{-1} dx da \right| \\ &\leq \left| \int_1^{1+L^{-1}} \frac{1}{a} \int_{L(a-1)}^\infty \sin(x)x^{-1} dx da \right| + \left| \int_{1+L^{-1}}^L \frac{1}{a} \int_{L(a-1)}^\infty \sin(x)x^{-1} dx da \right| \\ &\leq C \left| \int_{1+L^{-1}}^L \frac{1}{La(a-1)} dx da \right| + C \left| \int_{1+L^{-1}}^L \frac{1}{a} da \right|. \end{aligned} \tag{6.46}$$

The first integral on the right-most side of (6.46) is of order $O(L^{-1})$ and the second is of order $O(L^{-1} \log(L))$. Altogether we see that

$$\lim_{L \rightarrow \infty} \int_1^L a^{-1/2} \int_L^\infty J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) dx da = 0.$$

Thus finally

$$\begin{aligned} & \lim_{L \rightarrow \infty} \int_0^L \int_1^L a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) da dx \\ &= \int_0^\infty \int_1^\infty a^{-1/2} J_{\alpha+1/2}(x) J_{\alpha-1/2}(ax) da dx = 0. \end{aligned}$$

This shows that $I' = 0$ which completes the proof of (6.41). □

7 Proof of Proposition 2.13

In this section, we will provide a proof of Proposition 2.13. The method of proof will be almost identical in structure to the proof of Proposition 2.9 given in the last section. The main differences that occur are that we must now choose N -dependent partitions of unity to isolate the regions of each specified asymptotics and that now the error asymptotics of R_N are of order $O(N^{\gamma-1})$ in the regions around $\{\lambda_1, \lambda_2\}$. We will now, for the remainder of this section, assume that we are in the case of Proposition 2.13. That is, we will fix $1 > \gamma > 0$ and $\epsilon > 0$, and only consider choices of (possibly N -dependent) $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$ such that $(\lambda_1 - \lambda_2) > N^{-\gamma}$.

We will first focus on integral results in the region $(\lambda_2 - \delta, \lambda_1 + \delta)$. Let us fix $\epsilon, \delta_0 > 0$ as in Proposition 4.17, and $\delta < \min(\epsilon/4, 1/4, \delta_0/2)$. Let us denote $\delta_N = \delta(\lambda_1 - \lambda_2)$. We will fix a symmetric smooth function χ , such that $\chi(x) = 1$ for $|x| < 1/8$ and $\chi(x) = 0$ for $|x| > 1/4$. We define $\chi_{N,i}(x) = \chi((x - \lambda_i)\delta_N^{-1})$. We additionally choose another smooth compactly-supported function $\bar{\chi}$, supported on $(\lambda_2 - \delta/2, \lambda_1 + \delta/2)$, such that $\bar{\chi}(x) = 1$ for $x \in (\lambda_2 - \delta/4, \lambda_1 + \delta/4)$. We define $\chi_{N,0}(x) = \bar{\chi}(x) - \chi_{N,1}(x) - \chi_{N,2}(x)$. A key property of these functions is the following: for $k \in \mathbb{N}$ and $i \in \{0, 1, 2\}$, there is C such that

$$\int |\chi_{N,i}^{(k)}(x)| dx \leq C(\delta_N^{-k+1} + \delta_{ik}\delta_{k0}) \leq C^2(N^{\gamma(k-1)} + \delta_{ik}\delta_{k0}), \quad \sup_x |\chi_{N,i}^{(k)}(x)| \leq C\delta_N^{-k} \leq C^2N^{\gamma k}. \tag{7.1}$$

In addition, we will define $\chi_{N,i,j}(x) = \chi_{N,i}(x)/(x - \lambda_j)$. An important subtlety that arises in the current context is that for $i \neq j$, the function $\chi_{N,i,j}(x)$ is no longer $O(1)$. On the other hand, we instead observe that for $i \neq j$, and $k \in \mathbb{N}$, there is $C > 0$, such that

$$\begin{aligned} \int |\chi_{N,i,j}^{(k)}(x)| dx &\leq C(\delta_N^{-k} + \log(\delta_N)\delta_{ik}\delta_{k0}) \leq C^2(N^{\gamma k} + \gamma \log(N)\delta_{ik}\delta_{k0}), \\ \sup_x |\chi_{N,i,j}^{(k)}(x)| &\leq C\delta_N^{-k-1} \leq C^2N^{\gamma(k+1)}. \end{aligned} \tag{7.2}$$

The key property we will use for $\chi_{N,i}$ and $\chi_{N,i,j}$ will be (7.1) and (7.2), respectively, the majority of proofs of the results below are identical for both functions, with slightly worse bounds for the latter. We begin our analysis with the following modification of Lemma 6.1.

Lemma 7.1. *Let ϕ be a smooth function. Then for each fixed $k \in \mathbb{Z}$, $i = 1, 2$, and choice of $c \in \mathbb{N}$, we have that*

$$\int \phi(x)\chi_{N,0}(x)p_{N-k}(x)w_N(x)^{1/2} dx = O(N^{-c}), \tag{7.3}$$

$$\int \phi(x)\chi_{N,0,i}(x)p_{N-k}(x)w_N(x)^{1/2} dx = O(N^{-c}). \tag{7.4}$$

Proof. We may assume that $c > 1$. We first establish (7.3). For notational clarity, we

assume again that $k = 0$. We define $f_{N,l}$ as in Lemma 6.1. By (6.3) we see that for $c \in \mathbb{N}$,

$$\begin{aligned} & \left| \int \phi(x) \chi_{N,0}(x) p_N(x) w_N(x)^{1/2} dx \right| \\ & \leq \frac{1}{\pi^{1/2}} \sum_{l=1,2} \frac{1}{N^c} \int \left| \frac{d^c}{dy^c} ((s^{-1})'(y) \chi_{N,0}(s^{-1}(y)) f_{N,l}(s^{-1}(y))) \right| dy. \end{aligned}$$

We observe that $f_{N,l}^{(c)}(x) = O(1)$ for $c \in \mathbb{N}$. By the product formula, we may inductively write

$$\frac{d^c}{dy^c} ((s^{-1})'(y) \chi_{N,0}(s^{-1}(y)) f_{N,l}(s^{-1}(y))) = \sum_{k=0}^c \chi_{N,0}^{(k)}(s^{-1}(y)) a_{k,l,N}(y),$$

where $a_{k,l,N}(y)$ are smooth functions with $a_{k,l,N}(y) = O(1)$ uniformly. Thus we have that there is C such that

$$\begin{aligned} \int \left| \frac{d^c}{dy^c} ((s^{-1})'(y) \chi_{N,0}(s^{-1}(y)) f_{N,l}(s^{-1}(y))) \right| dy & \leq \sum_{k=0}^c \int |\chi_{N,0}^{(k)}(y) a_{k,l,N}(s(y)) s'(y)| dy \\ & \leq C \sum_{k=0}^c \int |\chi_{N,0}^{(k)}(y)| dy \leq C^2 N^{c\gamma}, \end{aligned}$$

where in the second inequality, we have used (7.1). In particular, we see that

$$\int \phi(x) \chi_{N,0}(x) p_N(x) w_N(x)^{1/2} dx = O(N^{c(\gamma-1)}).$$

As $c \in \mathbb{N}$ was arbitrary, we see that taking $c \mapsto \lceil c/(1-\gamma) \rceil$, we obtain (7.3).

To show (7.4), the above proof works, replacing (7.1) for (7.2) to show that the left-hand side of (7.4) is $O(N^{(c+1)\gamma-c})$. Adjusting c again gives (7.4). \square

We now state the modification of Lemma 6.2 that we shall use.

Lemma 7.2. *Let ϕ and φ be smooth functions. Then for fixed choice of n and m and $i, j \in \{1, 2\}$, we have that*

$$\begin{aligned} & \frac{1}{2} \int \int \phi(x) \varphi(y) \chi_{N,0}(x) \chi_{N,0}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ & = \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x) \varphi(x) \chi_{N,0}(x)^2}{(1-x^2)} \sin((m-n) \arccos(x)) dx + O(N^{\gamma-2}), \end{aligned} \tag{7.5}$$

$$\begin{aligned} & \int \int \phi(x) \varphi(y) \chi_{N,0,i}(x) \chi_{N,0,j}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ & = O(N^{-1}), \end{aligned} \tag{7.6}$$

$$\begin{aligned} & \int \int \phi(x) \varphi(y) \chi_{N,0,i}(x) \chi_{N,0,j}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ & = O(N^{-\min(1, 2(1-\gamma))}). \end{aligned} \tag{7.7}$$

Proof. We begin with the proof of (7.5). As in the proof of Lemma 6.2, we write

$$\begin{aligned} & \frac{1}{2} \int \int \phi(x) \varphi(y) \chi_{N,0}(x) \chi_{N,0}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ & = \frac{1}{2\pi} \sum_{j,k=1,2} \int \int \chi_{N,0}(x) \chi_{N,0}(y) f_{N,j,k}(x, y) e^{-(-1)^j N i s(x) - (-1)^k N i s(y)} \epsilon(x-y) dx dy, \end{aligned}$$

where here $f_{N,j,k}$ is defined as in the proof of Lemma 6.2. We denote

$$\bar{f}_{N,j,k}(x, y) = (s^{-1})'(x) (s^{-1})'(y) f_{N,j,k}(s^{-1}(x), s^{-1}(y)).$$

We have that for $c \in \mathbb{N}$ that $\bar{f}_{N,j,k}^{(c)}(x) = O(1)$ uniformly in x . One may apply Lemma 6.3 as in the proof of Lemma 6.2 to write

$$\begin{aligned} & \frac{1}{2} \sum_{j,k=1,2} \int \int \chi_{N,0}(x)\chi_{N,0}(y)f_{N,j,k}(x,y)e^{-(-1)^j Nis(x)-(-1)^k Nis(y)}\epsilon(x-y)dx dy \\ &= -\frac{i}{N} \int \frac{f_{N,1,2}(x,x)\chi_{N,0}(x)^2}{|s'(x)|} dx + \frac{i}{N} \int \frac{f_{N,2,1}(x,x)\chi_{N,0}(x)^2}{|s'(x)|} dx + E_N, \end{aligned} \tag{7.8}$$

where E_N is bounded by

$$\begin{aligned} & \sum_{j,k=1,2} \frac{4}{N^2} \left(\int \int \|\nabla(\chi_{N,0}(s^{-1}(x))\chi_{N,0}(s^{-1}(y))\bar{f}_{N,j,k}(x,y))\| dx dy + \right. \\ & \left. 4 \int \left| \frac{d}{dx}(\chi_{N,0}(s^{-1}(x))\chi_{N,0}(s^{-1}(x))\bar{f}_{N,j,k}(x,x)) \right| dx \right). \end{aligned} \tag{7.9}$$

As in the proof of Lemma 7.1, and recalling that $\|(x,y)\| \leq |x| + |y|$, we see that there is C such that (7.9) is bounded by

$$\begin{aligned} & CN^{-2} \left(\int |\chi'_{N,0}(x)| dx \int |\chi_{N,0}(y)| dy + \int |\chi_{N,0}(x)| dx \int |\chi_{N,0}(y)| dy \right. \\ & \left. + \int |\chi'_{N,0}(x)||\chi_{N,0}(x)| dx + \int |\chi_{N,0}(x)|^2 dx \right). \end{aligned}$$

We further note that

$$\begin{aligned} & \int |\chi'_{N,0}(x)||\chi_{N,0}(x)| dx + \int |\chi_{N,0}(x)|^2 dx \\ & \leq \left(\sup_x |\chi_{N,0}(x)| \right) \left(\int |\chi'_{N,0}(x)| dx + \int |\chi_{N,0}(x)| dx \right). \\ & \int |\chi'_{N,0}(x)||\chi_{N,0}(x)| dx + \int |\chi_{N,0}(x)|^2 dx \\ & \leq \left(\sup_x |\chi_{N,0}(x)| \right) \left(\int |\chi'_{N,0}(x)| dx + \int |\chi_{N,0}(x)| dx \right). \end{aligned}$$

All of these terms are $O(1)$ by (7.1). Thus we see that we have that $E_N = O(N^{-2})$. As we have that $R_N(x), \bar{R}_N(x) = O(N^{\gamma-1})$ we have that

$$\begin{aligned} f_{N,j,k}(x,y) &= \phi(x)\varphi(y)[T_\infty(x)]_{1j}[T_\infty(y)]_{1k}e^{(-1)^j i(\text{narccos}(x)+\pi(I(x<\lambda_1)+I(x<\lambda_2)))\alpha} \\ & \times e^{(-1)^k i(\text{marccos}(y)+\pi(I(y<\lambda_1)+I(y<\lambda_2)))\alpha} + O(N^{\gamma-1}). \end{aligned} \tag{7.10}$$

The remainder of the proof of (7.5) consists of inserting (7.10) into (7.8) and simplifying the resulting expression identically to as in the proof of Lemma 6.2.

To establish (7.6), the entire proof works identically, except E_N is now bounded by

$$\begin{aligned} & CN^{-2} \left[\int |\chi'_{N,0,i}(x)| dx \int |\chi_{N,0}(x)| dx + \int |\chi'_{N,0}(x)| dx \int |\chi_{N,0,i}(x)| dx \right. \\ & \left. + \int |\chi_{N,0}(x)| dx \int |\chi_{N,0,i}(x)| dx + \left(\sup_x |\chi'_{N,0,i}(x)| \right) \int |\chi_{N,0}(x)| dx \right. \\ & \left. \left(\sup_x |\chi_{N,0}(x)| \right) \left(\int |\chi'_{N,0,i}(x)| dx + \int |\chi_{N,0,i}(x)| dx \right) \right], \end{aligned}$$

for some $C > 0$. Employing (7.1) and (7.2), and maintaining the order of the terms for clarity, we see that for sufficiently large C , this expression is less than

$$C^2 N^{-2} \left[N^\gamma + \log(N) + \log(N) + N^\gamma + (N^\gamma + 1) \right].$$

In particular, we see that $E_N = O(N^{\gamma-2})$, which establishes (7.6).

In the case of (7.7), we have that E_N is now bounded by

$$\begin{aligned}
 & CN^{-2} \left[\int |\chi'_{N,0,i}(x)| dx \int |\chi_{N,0,j}(x)| dx + \int |\chi'_{N,0,j}(x)| dx \int |\chi_{N,0,i}(x)| dx \right. \\
 & \quad + \int |\chi_{N,0,j}(x)| dx \int |\chi_{N,0,i}(x)| dx + \left(\sup_x |\chi_{N,0,j}(x)| \right) \int |\chi'_{N,0,i}(x)| dx \\
 & \quad \left. + \left(\sup_x |\chi_{N,0,i}(x)| \right) \left(\int |\chi'_{N,0,i}(x)| dx + \int |\chi_{N,0,i}(x)| dx \right) \right]
 \end{aligned}$$

for some large C . Again employing (7.2), and possibly enlarging C , we see this is further less than

$$CN^{-2} [N^\gamma \log(N) + N^\gamma \log(N) + \log(N)^2 + N^{2\gamma} + \log(N)(N^\gamma + \log(N))].$$

From this we see that $E_N = O(N^{2\gamma-2})$, which establishes (7.7). □

We now will focus on the modifications that need to be made in the region around λ_i . We note that the preparatory Lemma 6.4 still holds as stated.

Lemma 7.3. *Let ϕ and φ be smooth functions. Then for any $i, j \in \{0, 1, 2\}$, and fixed n and m we have that*

$$\int \phi(x) \chi_{N,i}(x) p_{N-n}(x) w_N(x)^{1/2} dx = O(N^{-1}), \tag{7.11}$$

$$\begin{aligned}
 & \frac{1}{2} \int \int \phi(x) \varphi(y) \chi_{N,i}(x) \chi_{N,j}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\
 & = \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x) \varphi(x) \chi_{N,i}(x) \chi_{N,j}(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx + O(N^{\gamma-2}). \tag{7.12}
 \end{aligned}$$

In addition, with $r, t = 1, 2$, such that $i \neq r$ and $j \neq t$, we have that

$$\begin{aligned}
 & \int \phi(x) \chi_{N,i,r}(x) p_{N-n}(x) w_N(x)^{1/2} dx \\
 & = O(N^{-\min(1, 2(1-\gamma))}), \tag{7.13}
 \end{aligned}$$

$$\begin{aligned}
 & \int \int \phi(x) \varphi(y) \chi_{N,i,r}(x) \chi_{N,j}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\
 & = O(N^{-1}), \tag{7.14}
 \end{aligned}$$

$$\begin{aligned}
 & \int \int \phi(x) \varphi(y) \chi_{N,i,r}(x) \chi_{N,j,t}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\
 & = O(N^{-\min(1, 2(1-\gamma))}). \tag{7.15}
 \end{aligned}$$

Proof. We begin with the proof of (7.11). We have already established the case of $i = 0$, so we assume that $i > 0$, and denote $\lambda_i = \lambda$. We observe that proceeding identically to Lemma 6.6 we have that there is C such that

$$\begin{aligned}
 & \left| \int_{\lambda}^{\lambda \pm \delta} \phi(x) p_{N-n}(x) \chi_{N,i}(x) w_N(x)^{1/2} dx \right| \\
 & \leq N^{-1} \pi^{1/2} (f_{N,n,\lambda}^{-1})'(0) (D(\alpha + 1/2, 1/2) |g_{N,1}(\lambda)| + D(\alpha - 1/2, 1/2) |g_{N,2}(\lambda)|) \\
 & \quad + \frac{C}{N^2} \sum_{k=1,2} \sum_{l=0}^2 \left| \int_0^{\pm\infty} \frac{d^l}{dx^l} ((f_{N,n,\lambda}^{-1})'(x) g_{N,k}(f_{N,n,\lambda}^{-1}(x)) \chi_{N,i}(f_{N,n,\lambda}^{-1}(x))) \right| dx, \tag{7.16}
 \end{aligned}$$

where $g_{N,k}$ is defined as in the proof of Lemma 6.6. The first term on the right-hand side is clearly $O(N^{-1})$. Additionally, we may show, as before, that there is C , such that the second term of the right-hand side of (7.16) may be bounded by

$$\frac{C^2}{N^2} \sum_{l=0}^2 \int |\chi_{N,i}^{(l)}(x)| dx = O(N^{\gamma-2}).$$

Together these bounds establish (7.11). The same argument may be used to establish (7.13).

To show (7.12), we first observe that the case of $i = j = 0$ was established in Lemma 7.2. In the case that $i > 0$ and $j = 0$, let us write $\hat{\chi}(x) = \chi(4x)$ and $\hat{\chi}_{N,i}(x) = \hat{\chi}((x - \lambda_i)\delta_N^{-1})$, so that $\text{supp}(\hat{\chi}_{N,i}) \cap \text{supp}(\chi_{N,0}) = \emptyset$. Thus we see that

$$\begin{aligned} & \int \int \phi(x)\varphi(y)\hat{\chi}_{N,i}(x)\chi_{N,0}(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx dy \\ &= \int \phi(x)\hat{\chi}_{N,i}(x)p_{N-n}(x)w_N(x)^{1/2} dx \int \varphi(y)\chi_{N,0}(y)p_{N-m}(y)\epsilon(\lambda_i - y)w_N(y)^{1/2} dy. \end{aligned} \tag{7.17}$$

We note that as $\hat{\chi}_{N,i}$ clearly satisfies the bounds (7.1) and has a support contained within that of $\chi_{N,i}$, the above proof of (7.11) with $\chi_{N,i}$ replaced by $\hat{\chi}_{N,i}$ shows that

$$\int \phi(x)\hat{\chi}_{N,i}(x)p_{N-n}(x)w_N(x)^{1/2} dx = O(N^{-1}).$$

We also observe that while Lemma 7.1 does not formally apply to the second integral on the right-hand side of (7.17), as the function $\phi(x)\epsilon(\lambda_i - x)$ is N -dependant though λ_i , this function (as well as its derivatives of up to any finite order) are still uniformly bounded on the support of $\chi_{N,0}$, so that the proof of Lemma 7.1 with $\phi(x)$ replaced by $\phi(x)\epsilon(\lambda_i - x)$ shows that for any $c > 0$ we also have that

$$\int \varphi(y)\chi_{N,0}(y)p_{N-m}(y)\epsilon(\lambda_i - y)w_N(y)^{1/2} dy = O(N^{-c}).$$

Together these results show that

$$\int \int \phi(x)\varphi(y)\hat{\chi}_{N,i}(x)\chi_{N,0}(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx dy = O(N^{-2}). \tag{7.18}$$

We also observe that $\text{supp}(\chi_{N,i} - \hat{\chi}_{N,i}) \subseteq (\lambda_i - \delta/2, \lambda_i - \delta/32) \cup (\lambda_i + \delta/32, \lambda_i + \delta/2)$, and that $\chi_{N,i} - \hat{\chi}_{N,i}$ satisfies the bounds (7.1). From this, we see that the proof of Lemma 7.2 with $\chi_{N,0}$ replaced by $\chi_{N,i} - \hat{\chi}_{N,i}$ establishes that

$$\begin{aligned} & \frac{1}{2} \int \int \phi(x)\varphi(y)(\chi_{N,i}(x) - \hat{\chi}_{N,i}(x))\chi_{N,0}(y) \\ & \quad \times p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2} dx dy \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x)\varphi(x)(\chi_{N,i}(x) - \hat{\chi}_{N,i}(x))\chi_{N,0}(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx + O(N^{\gamma-2}) \\ &= \frac{1}{2\pi N} \int_{-1}^1 \frac{\phi(x)\varphi(x)\chi_{N,i}(x)\chi_{N,0}(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx + O(N^{\gamma-2}) \end{aligned} \tag{7.19}$$

where in the last step we have used that $\hat{\chi}_{N,i}(x)\chi_{N,0}(x) = 0$. Combining (7.18) with (7.19) establishes (7.12) in this case. The case of $i = 0$ and $j > 0$ of (7.12) follows symmetrically, so we now assume that $i, j > 0$.

We note that if additionally $i \neq j$, then (7.12) follows from (7.11) by employing the disjointness of the support of $\chi_{N,i}$ and $\chi_{N,j}$ (6.1) to rewrite this as a product of single integrals. Thus we assume that $i = j$ and write, as before, $\lambda_i = \lambda$. As before, we additionally note that by the proof of (7.11), we have that

$$\begin{aligned} & \int \int \phi(x)\varphi(y)\chi_{N,i}(x)\chi_{N,i}(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ &= \int_{\lambda}^{\infty} \int_{\lambda}^{\infty} \phi(x)\varphi(y)\chi_{N,i}(x)\chi_{N,i}(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \\ & \quad + \int_{-\infty}^{\lambda} \int_{-\infty}^{\lambda} \phi(x)\varphi(y)\chi_{N,i}(x)\chi_{N,i}(y)p_{N-n}(x)p_{N-m}(y)\epsilon(x-y)w_N(x)^{1/2}w_N(y)^{1/2}dxdy \end{aligned} \tag{7.20}$$

up to a term of order $O(N^{\gamma-2})$. We will compute the asymptotics of the first integral on the right-hand side, the other being identical. Proceeding as in Lemma 6.6 we may rewrite this integral as the sum $I + II + III$ with

$$\begin{aligned} I &= \frac{1}{\pi} \sum_{k,l=1,2} \int_{\lambda}^{\infty} [\mathcal{J}_{\alpha}(g_{N,n}(x))]_k \left(\int_0^{g_{N,m}(x)} [\mathcal{J}_{\alpha}(z)]_l dz \right) \frac{h_{N,k}(x)l_{N,l}(x)\chi_{N,i}(x)^2}{g'_{N,m}(x)} dx, \\ II &= \frac{1}{\pi} \sum_{k,l=1,2} \int_{\lambda}^{\infty} \left(\int_0^{g_{N,n}(x)} [\mathcal{J}_{\alpha}(z)]_k dz \right) \left(\int_0^{g_{N,m}(x)} [\mathcal{J}_{\alpha}(z)]_l dz \right) \\ & \quad \times \frac{\chi_{N,i}(x)h_{N,k}(x)}{g'_{N,n}(x)} \frac{d}{dx} \left(\frac{\chi_{N,i}(x)l_{N,l}(x)}{g'_{N,m}(x)} \right) dx, \\ III &= \frac{1}{2\pi} \sum_{k,l=1,2} \int_{\lambda}^{\infty} \int_{\lambda}^{\infty} \left(\int_0^{g_{N,n}(x)} [\mathcal{J}_{\alpha}(z)]_k dz \right) \left(\int_0^{g_{N,m}(y)} [\mathcal{J}_{\alpha}(z)]_l dz \right) \\ & \quad \times \frac{d}{dx} \left(\frac{\chi_{N,i}(x)h_{N,k}(x)}{g'_{N,n}(x)} \right) \frac{d}{dy} \left(\frac{\chi_{N,i}(y)l_{N,l}(y)}{g'_{N,m}(y)} \right) \epsilon(x-y)dxdy, \end{aligned}$$

where $l_{N,k}, h_{N,k}$ are as in the proof of Lemma 6.6. We note that for any $c \in \mathbb{N}$, we have that $l_{N,k}^{(c)}(x), h_{N,k}^{(c)}(x) = O(1)$, which when supplemented with the asymptotics of (6.17) establish that there is C such that

$$|III| \leq \frac{C}{N^2} \int \int (|\chi'_{N,i}(x)| + |\chi_{N,i}(x)|) (|\chi_{N,i}(y)| + |\chi'_{N,i}(y)|) dxdy,$$

which shows that $III = O(N^{-2})$. Similarly

$$|II| \leq \frac{C}{N^2} \left(\sup_x |\chi_{N,i}(x)| \right) \left(\int |\chi'_{N,i}(x)| dx + \int |\chi_{N,i}(x)| dx \right),$$

so that $II = O(N^{-2})$. As before, we may write I as the sum of two terms

$$\begin{aligned} I' &= - \sum_{k,l=1,2} \int_{\lambda}^{\infty} \sqrt{g_{N,n}(x)} J_{\alpha-(-1)^k 1/2}(g_{N,n}(x)) I_{\alpha-(-1)^l 1/2, 1/2}(g_{N,m}(x)) \\ & \quad \times \frac{h_{N,k}(x)l_{N,l}(x)\chi_{N,i}(x)^2}{g'_{N,m}(x)} dx, \\ II' &= \sum_{k,l=1,2} D(\alpha - (-1)^l 1/2, 1/2) \int_{\lambda}^{\infty} J_{\alpha-(-1)^k 1/2}(g_{N,n}(x)) \frac{h_{N,k}(x)l_{N,l}(x)\chi_{N,i}(x)^2}{g'_{N,m}(x)} dx. \end{aligned}$$

Employing Lemma 6.7, we see that there is C such that

$$\begin{aligned} |II'| &\leq \frac{C}{N^2} + \frac{C}{N^3} \sum_{l=0}^2 \int_0^\infty \left| \frac{d^l}{dx^l} (\chi_{N,i}(x)^2) \right| dx \\ &\leq C^2 \left(N^{-2} + N^{-3} \left(\sup_x |\chi_{N,i}(x)| \right) \sum_{l=0}^2 \int_0^\infty |\chi_{N,i}^{(l)}(x)| dx \right. \\ &\quad \left. + N^{-3} \left(\sup_x |\chi'_{N,i}(x)| \right) \int_0^\infty |\chi'_{N,i}(x)| dx \right) \\ &\leq C^3 (N^{-2} + N^{\gamma-3}). \end{aligned}$$

Employing the asymptotics (6.16) and (4.29), as in the proof of Lemma 6.6, we see that

$$I' = \frac{1}{N\pi} \sum_{k,l=1,2} \int_\lambda^\infty [\mathcal{I}_\alpha(g_{N,n}(x))]_k [\mathcal{I}_{\alpha-1}(g_{N,m}(x))]_l \frac{h_{N,k}(x) l_{N,l}(x) \chi_{N,i}(x)^2}{f'_{N,m,\lambda}(x)} dx + O(N^{-2}).$$

Now using the fact that $R_N(x), \bar{R}_N(x) = O(N^{\gamma-1})$ for $x \in (\lambda - \delta_N, \lambda + \delta_N)$, we derive that

$$\begin{aligned} I' &= \frac{1}{\pi} \sum_{k,l=1,2} \int_\lambda^\infty [\mathcal{I}_\alpha(g_{N,n}(x))]_k [\mathcal{I}_{\alpha-1}(g_{N,m}(x))]_l \phi(x) \varphi(x) \chi_{N,i}(x)^2 \\ &\quad \times \frac{[T_{N,n,\lambda}(x)]_{1k} [T_{N,m,\lambda}(x)]_{1l}}{f'_{N,m,\lambda}(x)} dx + O(N^{\gamma-2}). \end{aligned}$$

The remainder of the proof of (7.12) now proceeds identically to that of Lemma 6.6.

To show (7.14) one proceeds identically to the case of (7.12) above, with the only differences being that now the error in (7.20) is given by $O(N^{-1-\min(1,2(1-\gamma))})$, $II, III = O(N^{\gamma-2})$ and $II' = O(N^{-\min(3-2\gamma,2)})$. The case of (7.15) is similar as well, with the error in (7.19) now being given by $O(N^{2(\gamma-1)})$, the error in (7.20) now being given by $O(N^{-2\min(1,2(1-\gamma))})$, and with the new error bounds $II, III = O(N^{2(\gamma-1)})$ and $II' = O(N^{-\min(3(1-\gamma),2)})$ \square

Now we will now discuss the remaining asymptotics which we need outside of $(\lambda_2 - \delta, \lambda_1 + \delta)$. As we will not need N -dependant partitions in this region, and as the bound on the error term given in Proposition 4.16 is still of order $O(N^{-1})$, we see that the proofs given for Lemmas 6.1, 6.2, and 6.10, require no modifications on their respective regions. Explicitly, together they imply the following result.

Lemma 7.4. *Let ϕ and φ be smooth functions of subexponential growth, both vanishing on $(\lambda_2 - \delta, \lambda_1 + \delta)$. Then we have that for any fixed n and m that*

$$\begin{aligned} &\int \phi(x) p_{N-k}(x) w_N(x)^{1/2} dx = O(N^{-1/2}). \\ &\int J_N^{-1}(\varphi p_{N-m})(x) \phi(x) p_{N-n}(x) w_N(x) dx \\ &= \frac{1}{2\pi N} \left[\int_{-1}^1 \frac{\phi(x) \varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x)) dx \right] \\ &\quad + \frac{1}{4N} [(-1)^{N-m} \phi(1) \varphi(-1) - (-1)^{N-n} \phi(-1) \varphi(1)] + O(N^{-7/6}). \end{aligned}$$

Combining this Lemma with the above results, we achieve the following.

Lemma 7.5. *Let ϕ and φ be smooth functions of subexponential growth. Then we have that for any fixed n and m that*

$$\begin{aligned} & \int J_N^{-1}(\varphi p_{N-m})(x)\phi(x)p_{N-n}(x)w_N(x)dx \\ &= \frac{1}{2\pi N} \left[\int_{-1}^1 \frac{\phi(x)\varphi(x)}{(1-x^2)} \sin((m-n)\arccos(x))dx \right] \\ & \quad + \frac{1}{4N} [(-1)^{N-m}\phi(1)\varphi(-1) - (-1)^{N-n}\phi(-1)\varphi(1)] + O(N^{-\min(2-\gamma, 7/6)}). \end{aligned}$$

Proceeding as in the prior section, we see that this result implies that when N is even that

$$\int J_N^{-1}(p_N)(x)p_{N-1}(x)w_N(x)dx = O(N^{-\min(2-\gamma, 7/6)}),$$

which is the first result required by Proposition 2.13.

We will now begin discussing the required integrals involving ℓ_i and q_i . A technicality present in this case is that the term c_i defined in Proposition 4.14 is now N -dependant and diverging as $|\lambda_1 - \lambda_2| \rightarrow 0$. To avoid confusion, we will use the notation $c_{i,N}$ for the remainder of the section to emphasize this N -dependence. We also observe that for $i \in \{1, 2\}$, both $c_{i,N}|\lambda_1 - \lambda_2|^\alpha$ and $c_{i,N}^{-1}|\lambda_1 - \lambda_2|^{-\alpha}$ are uniformly bounded for $\lambda_1, \lambda_2 \in (-1 + \epsilon, 1 - \epsilon)$.

Lemma 7.6. *For ϕ a smooth function supported on $(-1 + \delta, 1 - \delta)$, and any choice of $i \in 1, 2$ and $j \in \{1, 2, 3\}$, we have that*

$$\begin{aligned} \int \phi(x)\chi_{N,j}(x)\ell_i(x)w_N(x)^{1/2}dx &= c_{i,N}^{-1}O(N^{\gamma-1}), \\ \int \phi(x)\chi_{N,j}(x)q_i(x)w_N(x)^{1/2}dx &= c_{i,N}O(1). \end{aligned}$$

Proof. In sight of Proposition 4.17, and proceeding as in the proof of Lemma 6.13, we are reduced to showing that

$$\int \phi(x)\chi_{N,j,i}(x) \begin{bmatrix} \pi^{1/2}p_N(x) \\ -i\pi^{1/2}p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2}dx = T_{N,\lambda_i}(\lambda_i) \begin{bmatrix} O(1) \\ 0 \end{bmatrix} + O(N^{\gamma-1}). \quad (7.21)$$

We will actually show that the error is of order $O(N^{-1})$, but as the error terms in Proposition 4.17 are of order $O(N^{\gamma-1})$, this does not improve the result. Now in the case that $i \neq j$, (7.21) follows from (7.13), and thus we will focus on the case of $i = j$, and denote $\lambda = \lambda_i$ and $\alpha = \alpha_i$ as before. Applying Proposition 4.17 we may write

$$\begin{aligned} \int \phi(x)\chi_{N,i,i}(x) \begin{bmatrix} \pi^{1/2}p_N(x) \\ -i\pi^{1/2}p_{N-1}(x) \end{bmatrix} w_N(x)^{1/2}dx &= \int \frac{\xi_N(x)G_{N,2}(x)}{x} \sqrt{\pi|Nx|} J_{\alpha-1/2}(N|x|)dx \\ & \quad + \int \frac{\xi_N(x)G_{N,1}(x)}{x} \epsilon(x) \sqrt{\pi|Nx|} J_{\alpha+1/2}(N|x|)dx \end{aligned}$$

where $G_{N,i}$ is defined as in the proof of Lemma 6.13, and $\xi_N(x) = \chi_{N,i,i}(f_\lambda^{-1}(x))x$. If we write the first integral on the right-hand side as I and the second as II , then by Lemma 6.7 we have that

$$I = N^{-1}2\pi^{1/2}D(\alpha + 1/2, -1/2)G_{N,1}(0)\xi_N(0) + E_N,$$

where there is C such that

$$|E_N| \leq \frac{C}{N^2} \sum_{l=0}^2 \int |\chi_{N,i}^{(l)}(x)|dx \leq C^2 N^{\gamma-2}.$$

Thus $I = O(N^{-1})$. Proceeding similarly, one obtains that $II = O(N^{-1})$. □

As in (6.39), we see from the asymptotics provided in Proposition 4.17 that for $i \in \{1, 2\}$, there are constants $C_{k,N}$, for $k \in \{1, 2, 3, 4\}$, such that

$$c_{i,N} \ell_i(x) = C_{1,N} \frac{p_N(x)}{x - \lambda_i} + C_{2,N} \frac{p_{N-1}(x)}{x - \lambda_i}, \tag{7.22}$$

$$c_{i,N}^{-1} q_i(x) = C_{3,N} \frac{p_N(x)}{x - \lambda_i} + C_{4,N} \frac{p_{N-1}(x)}{x - \lambda_i}, \tag{7.23}$$

and such that $C_{k,N} = O(1)$ for $k \in \{1, 2, 3, 4\}$.

We recall the functions $\hat{\chi}_{N,i}$ introduced in the proof of Lemma 7.3, which are similar to $\chi_{N,i}$, but defined so that $\text{supp}(\hat{\chi}_{N,i}) \cap \text{supp}(\chi_{N,0}) = \emptyset$. Proceeding as in the previous section, we see that

$$\begin{aligned} \int J_N^{-1} \ell_i(x) p_{N-1}(x) w_N(x) dx &= \int (J_N^{-1} \chi_{N,l} \ell_i)(x) \hat{\chi}_{N,l}(x) p_{N-1}(x) w_N(x) dx \\ &\quad + c_{l,N}^{-1} O(N^{-\min(1, (1-\gamma)-1/2)}), \\ \int J_N^{-1} p_N(x) q_k(x) w_N(x) dx &= \int (J_N^{-1} \chi_{N,l} p_N)(x) \hat{\chi}_{N,k}(x) q_k(x) w_N(x) dx + c_{k,N} O(N^{-1/2}), \\ \int J_N^{-1} \ell_i(x) q_k(x) w_N(x) dx &= \delta_{k,l} \int J_N^{-1} (\chi_{N,l} \ell_i)(x) \hat{\chi}_{N,k}(x) q_k(x) w_N(x) dx \\ &\quad + c_{k,N} c_{l,N}^{-1} O(N^{-\min(1/2, 2(1-\gamma))}). \end{aligned}$$

We observe that $c_{k,N} c_{l,N}^{-1} = O(1)$. By these formulas, and (7.22-7.23), we see that to establish the remaining cases of Proposition 2.13, it suffices to show the following.

Lemma 7.7. *Let ϕ and φ be smooth. Then for any choice of $i = 1, 2$, and any n and m , we have that*

$$\begin{aligned} &\int \int \phi(x) \varphi(y) \chi_{N,i,i}(x) \chi_{N,i,i}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ &= O(N^{-1} \log(N)). \end{aligned} \tag{7.24}$$

Moreover, if $|n - m| \leq 1$, then additionally

$$\begin{aligned} &\int \int \phi(x) \varphi(y) \chi_{N,i,i}(x) \chi_{N,i,i}(y) p_{N-n}(x) p_{N-m}(y) \epsilon(x-y) w_N(x)^{1/2} w_N(y)^{1/2} dx dy \\ &= O(N^{\gamma-1} \log(N)). \end{aligned} \tag{7.25}$$

Proof. We begin with the proof of (7.24). We may proceed identically to the proof of (6.41) to write the left-hand side of (7.24) as the sum $I + II$ where

$$\begin{aligned} I &= \frac{1}{\pi} \sum_{k,l=1,2} \int [\mathcal{J}_\alpha(g_{N,m}(x))]_l I_k(g_{N,n}(x)) \frac{h_{N,k}(x) l_{N,l}(x) g_{N,n}(x) \chi_{N,i,i}(x)^2}{g'_{N,n}(x)(x-\lambda)} dx, \\ II &= -\frac{1}{2\pi} \sum_{k,l=1,2} \int \int [\mathcal{J}_\alpha(g_{N,m}(x))]_l I_k(g_{N,n}(y)) \chi_{N,i,i}(y) l_{N,l}(y) \\ &\quad \times \frac{d}{dx} \left(\frac{\chi_{N,i,i}(x) h_{N,k}(x) g_{N,n}(x)}{g'_{N,n}(x)(x-\lambda)} \right) \epsilon(x-y) dx dy, \end{aligned}$$

with notation as in the proof of Lemma 6.14. As $|\chi_{N,j}(x)|$ is bounded, and as the integrand of I remains of order $O(\min(N|x|^{-1}, 1))$, we see that $I = O(N^{-1} \log(N))$, as before. In

addition, we see there is C such that

$$\begin{aligned} |II| &\leq \frac{1}{2\pi} \sum_{k,l=1,2} \int |\chi_{N,i}(y) I_k(g_{N,n}(y)) l_{N,l}(y)| dy \\ &\quad \times \int \left| [\mathcal{J}_\alpha(g_{N,m}(x))]_l \frac{d}{dx} \left(\frac{\chi_{N,i}(x) h_{N,k}(x) g_{N,n}(x)}{g'_{N,n}(x)(x-\lambda)} \right) \right| dx \\ &\leq \frac{C}{N} \int \min(|y|^{-1}, N) dy \int (|\chi_{N,i}(x)| + |\chi'_{N,i}(x)|) dx = O(N^{-1} \log(N)). \end{aligned}$$

This completes the proof of (7.24).

We now discuss the proof of (7.25). The proof of (7.25) may also be reduced, by modifying the proof of (6.42) as above, to showing that for $k, l = 1, 2$,

$$\int \int \frac{\chi_{N,i}(x-\lambda)\chi_{N,i}(y-\lambda)}{xy} [\mathcal{J}_\alpha(Nx)]_k [\mathcal{J}_\alpha(Ny)]_l \epsilon(x-y) dx dy = O(N^{\gamma-1} \log(N)). \quad (7.26)$$

On the other hand, we may rescale the left-hand side of (7.26) to become

$$\int \int \frac{\chi(x)\chi(y)}{xy} [\mathcal{J}_\alpha(N\delta_N x)]_k [\mathcal{J}_\alpha(N\delta_N y)]_l \epsilon(x-y) dx dy.$$

This coincides with a rescaled case of (6.44), which we have already shown to be of order

$$O(\delta_N^{-1} N^{-1} \log(\delta_N^{-1} N^{-1})),$$

and thus is further of order $O(N^{\gamma-1} \log(N))$. This completes the proof of (7.26). \square

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