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Approximation of a degenerate semilinear PDE with a nonlinear Neumann boundary condition*

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Abstract

We consider a system of semilinear partial differential equations (PDEs) with a nonlinearity depending on both the solution and its gradient. The Neumann boundary condition depends on the solution in a nonlinear manner. The uniform ellipticity is not required for the diffusion coefficient. We show that this problem admits a viscosity solution which can be approximated by a penalization. The Lipschitz condition is required for the coefficients of the diffusion part. The nonlinear part as well as the Neumann condition are Lipschitz. Moreover, the nonlinear part is monotone in the solution variable. Note that the existence of a viscosity solution to this problem has been established in [13] then completed in [15]. In the present paper, we construct a sequence of penalized systems of decoupled forward backward stochastic differential equations (FBSDEs) then we directly show its strong convergence. This allows us to deal with the case where the nonlinearity depends on both the solution and its gradient. Our work extends, in particular, the result of [4] and, in some sense, those of [1, 3]. In contrast to works [1, 3, 4], we do not pass by the weak compactness of the laws of the stochastic system associated to our problem.

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1 Introduction

Let D be a regular convex, open and bounded subset of $\mathbb{R}^d$. We can construct a function $\rho \in \mathcal{C}^1(\mathbb{R}^d)$ such that $\rho = 0$ in $\bar{D}$, $\rho > 0$ in $\mathbb{R}^d \setminus \bar{D}$ and $\rho(x) = (d(x, \bar{D}))^2$ in a neighborhood of $\bar{D}$. On the other hand, since the domain D is smooth (say $\mathcal{C}^3$), it is possible to consider an extension $l \in \mathcal{C}_b^2(\mathbb{R}^d)$ of the function $d(\cdot\,, \partial D)$ defined on the restriction to D of a neighborhood of ∂D such that D and ∂D are characterized by

$$D = \{x \in \mathbb{R}^d \,:\, l(x) > 0\} \quad \text{and} \quad \partial D = \{x \in \mathbb{R}^d \,:\, l(x) = 0\}\,,$$

and for every $x \in \partial D$, $\nabla l(x)$ coincides with the unit normal pointing toward the interior of D (see for example [9, Remark 3.1]). In particular we may and do choose ρ and l such that

$$\langle \nabla l(x)\,,\, \delta(x)\rangle \le 0\,, \quad \text{for all } x \in \mathbb{R}^d\,, \tag{1.1}$$

where $\delta(x) := \nabla\rho(x)$ and is called the penalization term. We have in particular

$$\frac{1}{2}\delta(x) = x - \pi_{\bar{D}}(x),$$

where $\pi_{\bar{D}}$ is the projection operator on $\bar{D}$. Consider the second-order differential operator

$$\mathcal{L} = \frac{1}{2}\sum_{i,j}(\sigma\sigma^*(.))_{ij}\,\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(.)\frac{\partial}{\partial x_i}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d'}$ are given measurable coefficients satisfying suitable assumptions.

Our first aim is to establish the existence of a viscosity solution via penalization to the following system of partial differential equations with nonlinear Neumann boundary condition, defined for $1 \le i \le m$, $0 \le t \le T$, $x \in D$.

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t,x) + \mathcal{L}u_i(t,x) + f_i(t,x,u(t,x),(\nabla u_i \sigma)(t,x)) = 0, \\ u(T,x) = g(x)\,, \quad x \in D, \\ \quad \dfrac{\partial u}{\partial n}(t,x) + h(t,x,u(t,x)) = 0\,, \;\; (t,x) \in [0,T) \times \partial D\,. \end{cases} \tag{1.2}$$

To this end, we consider a sequence (u^n) of viscosity solutions of the following semi-linear partial differential equations ($1 \le i \le m$, $0 \le t \le T$, $x \in \mathbb{R}^d$, $n \in \mathbb{N}$).

$$\begin{cases} \dfrac{\partial u_i^n}{\partial t}(t,x) + \mathcal{L}\, u_i^n(t,x) + f_i(t,x,u^n(t,x),(\nabla u_i^n \sigma)(t,x)) \\ \qquad\qquad - n\,\langle \delta(x), \nabla u_i^n(t,x)\rangle - n\langle \delta(x), \nabla l(x)\rangle h_i(t,x,u^n(t,x)) = 0\,, \\ u^n(T,x) = g(x)\,. \end{cases} \tag{1.3}$$

then we show that for each n, equation (1.3) has a viscosity solution u^n which converges to a function u, and u is a viscosity solution to (1.2). Our method is probabilistic.

The authors of [1, 3, 4] considered the case where f does not depends on ∇u. Using the connection between backward stochastic differential equations (BSDEs) and partial differential equations (PDEs), the convergence of u^n to u has been established in [4] for bounded and uniformly Lipschitz coefficients b and σ. The authors of [1] extended the result of [4] to the case where b and σ are bounded continuous. The case where b, σ and f are bounded measurable is considered in [3] in the framework of L^p-viscosity solution. The techniques developed in the previous works rely on tightness properties of the associated sequence of BSDEs in the Jakubowski S-topology. The main drawback of this method is that it does not allow to deal with the nonlinearity f depending on ∇u. Here, our method is direct and does not pass by weak compactness properties. Usually, when the nonlinearity f depends on the gradient of the solution, PDE techniques are used to control the gradient ∇u in order to get the convergence of the associated BSDE. And generally, a uniform ellipticity of the diffusion is required to get a good control of the gradient ∇u, see for instance [2, 5, 6] where this method is used in homogenization of nonlinear PDEs. Our approach is completely different: We use a purely probabilistic method, which allows us to deal with (possibly) degenerate PDEs. The convergence of the penalized BSDE is provided only by the convergence of the penalized forward SDE. Our proof essentially uses [14, Proposition 6.80, Annex C]. The latter has been already used by the authors of [15] in order to prove the continuity of the solution of a system of SDE-BSDE in its initial data (t,x). By bringing essential modifications in the idea developed in [15], we prove the convergence of our sequence of penalized BSDEs.

The paper is organized as follows: Section 2 contains some facts about reflected stochastic differential equations (SDEs) and generalized BSDEs. This mainly consists in approximation, existence, uniqueness results and a priori estimates of the solutions. Section 3 is devoted to the penalization of the nonlinear Neumann PDE.

2 Preliminaries and formulation of the problem

Throughout the paper, for a fixed $T>0$, $(W_t; t\in[0,T])$ is a d'-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ and for every $t\in[0,T]$, $\mathcal{F}_s^t$ is the σ-algebra $\sigma(W_r;\, t\le r\le s)\vee\mathcal{N}$ if $s\ge t$ and $\mathcal{F}_s^t=\mathcal{N}$ if $s\le t$, where $\mathcal{N}$ is the collection of $\mathbb{P}$-zero sets of $\mathcal{F}$. For $q\ge 0$, we denote by $\mathcal{S}_d^q[0,T]$ the space of continuous progressively measurable stochastic processes $X:\Omega\times[0,T]\to\mathbb{R}^d$, such that for $q>0$ we have

$$\mathbb{E}\sup_{t\in[0,T]}|X_t|^q<+\infty.$$

For $q\ge 0$, we denote by $\mathcal{M}_d^q(0,T)$ the space of progressively measurable stochastic processes $X:\Omega\times[0,T]\to\mathbb{R}^d$ such that:

$$\mathbb{E}\left[\left(\int_0^T|X_t|^2dt\right)^{\frac{q}{2}}\right]<+\infty \quad\text{if}\quad q>0;\quad \mathit{and}\quad \int_0^T|X_t|^2dt<+\infty\quad \mathbb{P}-a.s.\quad\text{if}\quad q=0.$$

2.1 Penalization for reflected stochastic differential equation

Let $(t,x)\in[0,T]\times\bar{D}$. The reflected SDE under consideration is

$$\begin{cases} X_s^{t,x}=x+\displaystyle\int_t^s b(X_r^{t,x})\,dr+\int_t^s\sigma(X_r^{t,x})\,dW_r+K_s^{t,x},\\ K_s^{t,x}=\displaystyle\int_t^s\nabla l(X_r^{t,x})d|K^{t,x}|_{[t,r]},\\ |K^{t,x}|_{[t,s]}=\displaystyle\int_t^s 1_{\{X_r^{t,x}\in\partial D\}}d|K^{t,x}|_{[t,r]},\quad s\in[t,T],\end{cases}\tag{2.1}$$

where the notation $|K^{t,x}|_{[t,s]}$ stands for the total variation of $K^{t,x}$ on the interval $[t,s]$, we denote this continuous increasing process by $k_s^{t,x}$. In particular we have

$$k_s^{t,x}=\int_t^s\langle\nabla l(X_r^{t,x}),dK_r^{t,x}\rangle.\tag{2.2}$$

Several authors have studied the problem of the existence of solutions of the reflected diffusion and its approximation by solutions of equations with penalization terms, we refer for example to [9, 7, 18, 19, 20]. We consider the following sequence of penalized SDEs associated with our reflected diffusion $X^{t,x}$

$$X_s^{t,x,n} = x + \int_t^s \left[b(X_r^{t,x,n}) - n\delta(X_r^{t,x,n})\right] dr + \int_t^s \sigma(X_r^{t,x,n}) dW_r, \quad s \in [t,T]. \tag{2.3}$$

For $s \in [t,T]$, we put

$$K_s^{t,x,n} := \int_t^s -n\delta(X_r^{t,x,n}) dr \quad \text{and} \quad k_s^{t,x,n} := \int_t^s \langle \nabla l(X_r^{t,x,n}), dK_r^{t,x,n} \rangle. \tag{2.4}$$

We introduce the following assumption

$(A.1)$: There exist positive constants C and μ such that for every $(x,y) \in \mathbb{R}^d$:

(i) $|b(x)| + \|\sigma(x)\| \le C(1+|x|)$,

(ii) $|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\| \le \mu|x-y|$.

Remark. It's obvious that assumption $(A.1)$(ii) implies $(A.1)$(i).

It is known that under assumption $(A.1)$ equation (2.3) admits, for any fixed $n \in \mathbb{N}$, a unique strong solution, and we have for every $q \ge 1$:

$$\sup_{n\ge 0} \mathbb{E} \sup_{s\in[t,T]} |X_s^{t,x,n}|^{2q} + \sup_{n\ge 0} \mathbb{E} \sup_{s\in[t,T]} |K_s^{t,x,n}|^{2q} + \sup_{n\ge 0} \mathbb{E}|K^{t,x,n}|_{[t,T]}^q < +\infty. \tag{2.5}$$

The proof of the previous estimates can be found e.g. in [1, Lemma 3.1].

The first assertion of the following theorem is proved in [20], while the second one follows from [18].

Theorem 2.1. *Under assumption* (A.1), we have

(i) the system (2.1) admits a unique solution,

(ii) for every $1 \le q < \infty$ and $0 < T < \infty$,

$$\mathbb{E}\left[\sup_{t\le s\le T} |X_s^{t,x,n} - X_s^{t,x}|^q\right] \longrightarrow 0, \quad as\ n \to \infty,$$

the limit is uniform in $(t,x) \in [0,T] \times \bar{D}$.

We extend the processes $(X^{t,x}, K^{t,x})$ and $(X^{t,x,n}, K^{t,x,n})$ to $[0,t)$ by putting

$$X_s^{t,x} = X_s^{t,x,n} := x, \quad K_s^{t,x} = K_s^{t,x,n} := 0, \quad \text{for } s \in [0,t).$$

As a consequence of Theorem 2.1, we have the following convergence, which is established in [4, Lemma 2.2].

Lemma 2.2. *Under assumptions of Theorem 2.1, we have, for any $q \ge 1$:*

(i) $\displaystyle \lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le s\le T} \left|K_s^{t,x,n} - K_s^{t,x}\right|^q\right] = 0;$

(ii) $\displaystyle \lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le s\le T} \left|\int_0^s \varphi(X_r^n)\, dK_r^{t,x,n} - \int_0^s \varphi(X_r)\, dK_r^{t,x}\right|^q\right] = 0$ *for every* $\varphi \in \mathcal{C}_b^1(\mathbb{R}^d)$.

Remark 2.3. (i) Using Lemma 2.2 and the representations (2.2), (2.4), it holds that for any $q \ge 1$ and any $(t,x) \in [0,T] \times \bar{D}$

$$\lim_{n\to+\infty} \mathbb{E} \sup_{s\in[0,T]} |k_s^{t,x,n} - k_s^{t,x}|^q = 0. \tag{2.6}$$

(ii) From [17, Corollary 2.5], it follows that for each $q \ge 1$ and each $(t,x) \in [0,T] \times \bar{D}$

$$\mathbb{E} \sup_{s\in[0,T]} |X_s^{t,x}|^{2q} + \mathbb{E} \sup_{s\in[0,T]} |K_s^{t,x}|^{2q} + \mathbb{E}|K^{t,x}|_{[0,T]}^q < +\infty. \tag{2.7}$$

2.2 Backward inequality

We state a lemma which is a version of [14, Proposition 6.80, Annex C]. This lemma is essential in our proofs. We give its proof for convenience.

Lemma 2.4. *Let* $(Y, Z) \in \mathcal{S}_m^0[0,T] \times \mathcal{M}^0_{m\times d'}(0,T)$ *satisfying*

$$Y_t = Y_T + \int_t^T d\mathcal{K}_r - \int_t^T Z_r dW_r, \quad 0 \le t \le T,\ \mathbb{P} - a.s.,$$

where $\mathcal{K} \in \mathcal{S}_m^0$ *and, for almost all* $\omega \in \Omega$, $\mathcal{K}_\cdot(\omega) \in BV([0,T];\mathbb{R}^m)$ *(the space of bounded variation processes).*

Assume be given

- *a non-decreasing stochastic process* L *with* $L_0 = 0$*,*
- *a stochastic process* R *whose sample paths are* $\mathbb{P}$*-a.s. in the space* $BV([0,T],\mathbb{R})$*, with* $R_0 = 0$*,*
- *a continuous stochastic process* V *with trajectories* $\mathbb{P}$*-a.s. in the space* $BV([0,T],\mathbb{R})$*, and such that* $V_0 = 0$*,*

$$\mathbb{E}\left(\sup_{s\in[0,T]}\int_s^T e^{2V_r} dR_r\right) < \infty,$$

and

(i) $\langle Y_r, d\mathcal{K}_r\rangle \le \frac{\alpha}{2}\|Z_r\|^2 dr + |Y_r|^2 dV_r + |Y_r| dL_r + dR_r$ *as measures on* $[0,T]$*, with* $\alpha \in \mathbb{R}$*,*

(ii) $\mathbb{E}\sup_{r\in[0,T]} e^{2V_r}|Y_r|^2 < +\infty$.

We have the following conclusion: if $\alpha < 1$*, then there exist positive constants* C_1*,* C_2 *and* C_3*, depending only on* α*, such that*

$$\begin{aligned}
&\mathbb{E}\left(\sup_{r\in[0,T]}|e^{V_r}Y_r|^2\right) + \mathbb{E}\left(\int_0^T e^{2V_r}\|Z_r\|^2 dr\right)\\
&\quad\le C_1\ \mathbb{E}|e^{V_T}Y_T|^2 + C_2\mathbb{E}\left(\int_0^T e^{V_r} dL_r\right)^2 + C_3\mathbb{E}\sup_{s\in[0,T]}\int_s^T e^{2V_r} dR_r.
\end{aligned}$$

Proof. By Itô's formula, we have

$$\begin{aligned}
&|e^{V_t}Y_t|^2 = |e^{V_T}Y_T|^2 - 2\int_t^T e^{2V_s}|Y_s|^2 dV_s + 2\int_t^T e^{2V_s}\langle Y_s, d\mathcal{K}_s\rangle\\
&-\int_t^T e^{2V_s}\|Z_s\|^2 ds - 2\int_t^T \langle e^{V_s}Y_s, e^{V_s}Z_s dW_s\rangle\\
&= |e^{V_T}Y_T|^2 + 2\int_t^T e^{2V_s}\left(\langle Y_s, d\mathcal{K}_s\rangle - |Y_s|^2 dV_s\right) - \int_t^T e^{2V_s}\|Z_s\|^2 ds\\
&-2\int_t^T \langle e^{V_s}Y_s, e^{V_s}Z_s dW_s\rangle.
\end{aligned}$$

Using (i) of Lemma 2.4, we get

$$\begin{aligned}
&|e^{V_t}Y_t|^2 + (1-\alpha)\int_t^T e^{2V_s}\|Z_s\|^2 ds \le\\
&\qquad |e^{V_T}Y_T|^2 + 2\int_t^T e^{2V_s}\left(dR_s + |Y_s| dL_s\right) - 2\int_t^T \langle e^{V_s}Y_s, e^{V_s}Z_s dW_s\rangle.
\end{aligned}$$

We consider the following sequence of stopping times

$$T_n := T \wedge \inf\{s \geq t : \sup_{r\in[t,s]} |e^{V_r}Y_r - e^{V_t}Y_t| + \int_t^s e^{2V_r}\|Z_r\|^2 dr + \int_t^s e^{V_r} dL_r \geq n\}.$$

For fixed $n \geq 1$, we consider the local martingale $N_s := 2\int_0^s 1_{[t,T_n]}(r)\langle e^{V_r}Y_r, e^{V_r}Z_r dW_r\rangle$, $s \in [0,\ T]$. Next, we show that N is a true martingale. We obviously have

$$\mathbb{E}\left(\langle N\rangle_T\right)^{\frac{1}{2}} \leq 2\mathbb{E}\left(\int_t^{T_n} e^{4V_r}|Y_r|^2\|Z_r\|^2 dr\right)^{\frac{1}{2}}.$$

Thanks to the definition of the stopping time T_n together with the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we get the following estimate.

$$\begin{aligned}\mathbb{E}\left(\langle N\rangle_T\right)^{\frac{1}{2}} &\leq 2\sqrt{2}\mathbb{E}\left(\left[\sup_{r\in[t,T_n]} |e^{V_r}Y_r - e^{V_t}Y_t|^2 + |e^{V_t}Y_t|^2\right]\int_t^{T_n} e^{2V_r}\|Z_r\|^2 dr\right)^{\frac{1}{2}} \\ &\leq 2\sqrt{2}\mathbb{E}\left(|e^{V_t}Y_t| + n\right)\sqrt{n} < +\infty.\end{aligned}$$

Then, by the Burkholder–Davis–Gundy inequality we can deduce that the process $\{N_s; s \in [0,T]\}$ is a true martingale.

By the foregoing, we have

$$\begin{aligned}|e^{V_t}Y_t|^2 + (1-\alpha)\int_t^{T_n} e^{2V_s}\|Z_s\|^2 ds \leq |e^{V_{T_n}}Y_{T_n}|^2 + 2\int_t^{T_n} e^{2V_s}\left(dR_s + |Y_s|dL_s\right) \\ -(N_{T_n} - N_t).\end{aligned}$$

Taking expectation, it follows that

$$\begin{aligned}\mathbb{E}\left(|e^{V_t}Y_t|^2 + (1-\alpha)\int_t^{T_n} e^{2V_s}\|Z_s\|^2 ds\right) \leq \\ \mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2 + 2\int_t^{T_n} e^{2V_s}\left(dR_s + |Y_s|dL_s\right)\right).\end{aligned} \tag{2.8}$$

On the other hand, since $\alpha < 1$ we deduce

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T_n]} |e^{V_r}Y_r|^2 \leq \mathbb{E}\Bigg(|e^{V_{T_n}}Y_{T_n}|^2 + 2\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r \\ +2\int_0^{T_n} e^{2V_s}|Y_s|dL_s + 2\sup_{r\in[0,T_n]}|N_r|\Bigg).\end{aligned}$$

The Burkholder-Davis-Gundy inequality shows that

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T_n]} |e^{V_r}Y_r|^2 &\leq \mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2 + 2\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r + 2\int_0^{T_n} e^{2V_s}|Y_s|dL_s\right) \\ &\quad + 2C_{BDG}\mathbb{E}\left(\int_0^{T_n} e^{4V_r}|Y_r|^2\|Z_r\|^2 dr\right)^{\frac{1}{2}} \\ &\leq \mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2 + 2\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r + 2\int_0^{T_n} e^{2V_s}|Y_s|dL_s\right) \\ &\quad + 2C_{BDG}\mathbb{E}\left(\sup_{r\in[0,T_n]} e^{2V_r}|Y_r|^2\int_0^{T_n} e^{2V_r}\|Z_r\|^2 dr\right)^{\frac{1}{2}},\end{aligned}$$

where C_{BDG} denotes the Burkholder-Davis-Gundy constant. Applying the elementary inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for the last term in the right-hand side, we get

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2 \leq \mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2 + 2\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r + 2\int_0^{T_n} e^{2V_s}|Y_s|dL_s\right)\\ + \mathbb{E}\left(\frac{1}{2}\sup_{r\in[0,T_n]} e^{2V_r}|Y_r|^2 + 2C_{BDG}^2\int_0^{T_n} e^{2V_r}\|Z_r\|^2dr\right).\end{aligned}$$

We pass the term $\frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,T_n]} e^{2V_r}|Y_r|^2\right)$ to the left-side of the inequality to get

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2 \leq 2\mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2 + 2\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r + 2\int_0^{T_n} e^{2V_s}|Y_s|dL_s\right)\\ + 4C_{BDG}^2\mathbb{E}\left(\int_0^{T_n} e^{2V_r}\|Z_r\|^2dr\right).\end{aligned}$$

From inequality (2.8), we get

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2 \leq \left(2+\frac{4C_{BDG}^2}{1-\alpha}\right)\mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2\right)\\ + \left(4+\frac{8C_{BDG}^2}{1-\alpha}\right)\mathbb{E}\left(\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r\right)\\ + \left(2+\frac{4C_{BDG}^2}{1-\alpha}\right)\mathbb{E}\left(\int_0^{T_n} e^{2V_s}|Y_s|dL_s\right),\end{aligned} \tag{2.9}$$

Again by estimate (2.8), we have

$$\begin{aligned}\mathbb{E}\left(\int_0^{T_n} e^{2V_s}\|Z_s\|^2ds\right) \leq \frac{1}{1-\alpha}\mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2\right) + \frac{2}{1-\alpha}\mathbb{E}\left(\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r\right)\\ + \frac{2}{1-\alpha}\mathbb{E}\left(\int_0^{T_n} e^{2V_s}|Y_s|dL_s\right)\end{aligned} \tag{2.10}$$

Combining the inequalities (2.9) and (2.10) we find

$$\begin{aligned}\mathbb{E}\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2 + \frac{1}{2}\mathbb{E}\int_0^{T_n} e^{2V_r}\|Z_r\|^2dr \leq \mu\,\mathbb{E}\left(|e^{V_{T_n}}Y_{T_n}|^2\right)\\ +2\mu\,\mathbb{E}\left(\sup_{s\in[0,T_n]}\int_s^{T_n} e^{2V_r}dR_r\right) + 2\mu\,\mathbb{E}\left(\int_0^{T_n} e^{2V_s}|Y_s|dL_s\right),\end{aligned}$$

where

$$\mu = 2 + \frac{4C_{BDG}^2}{1-\alpha} + \frac{1}{2(1-\alpha)}.$$

It follows that

$$\mathbb{E}\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2+\frac{1}{2}\mathbb{E}\int_0^{T_n}e^{2V_r}\|Z_r\|^2dr\leq$$
$$\mathbb{E}\left(\mu\,|e^{V_{T_n}}Y_{T_n}|^2+2\mu\sup_{s\in[0,T_n]}\int_s^{T_n}e^{2V_r}dR_r\right)+2\mu\mathbb{E}\left(\sup_{r\in[0,T_n]}e^{V_r}|Y_r|\int_0^{T_n}e^{V_r}dL_r\right)$$
$$\mathbb{E}\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2+\frac{1}{2}\mathbb{E}\int_0^{T_n}e^{2V_r}\|Z_r\|^2dr\leq$$
$$\mathbb{E}\left(\mu|e^{V_{T_n}}Y_{T_n}|^2+2\mu\sup_{s\in[0,T_n]}\int_s^{T_n}e^{2V_r}dR_r\right)+\frac{1}{2}\mathbb{E}\sup_{r\in[0,T_n]}e^{2V_r}|Y_r|^2$$
$$+2\mu^2\mathbb{E}\left(\int_0^{T_n}e^{V_r}dL_r\right)^2$$

Hence,

$$\mathbb{E}\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2+\mathbb{E}\int_0^{T_n}e^{2V_r}\|Z_r\|^2dr\leq$$
$$\mathbb{E}\left(2\mu|e^{V_{T_n}}Y_{T_n}|^2+4\mu\sup_{s\in[0,T_n]}\int_s^{T_n}e^{2V_r}dR_r\right)+4\mu^2\mathbb{E}\left(\int_0^{T_n}e^{V_r}dL_r\right)^2.$$

Put

$$C_1:=2\mu,\qquad C_2:=4\mu\qquad\text{and}\qquad C_3:=4\mu^2.$$

We then have

$$\mathbb{E}\left(\sup_{r\in[0,T_n]}|e^{V_r}Y_r|^2+\int_0^{T_n}e^{2V_r}\|Z_r\|^2dr\right)\leq$$
$$C_1\mathbb{E}|e^{V_{T_n}}Y_{T_n}|^2+C_2\mathbb{E}\sup_{s\in[0,T_n]}\int_s^{T_n}e^{2V_r}dR_r+C_3\mathbb{E}\left(\int_0^{T}e^{V_r}dL_r\right)^2.$$

Letting n tends to $+\infty$, T_n increases a.s. to T, since $\mathbb{E}\sup_{r\in[0,T]}e^{2V_r}|Y_r|^2<+\infty$ and Z is in the space $\mathcal{M}^0_{m\times d'}(0,T)$. We conclude by using the Beppo-Levi theorem for the left-hand side term of the previous inequality, and the Lebesgue dominated convergence theorem for the right-hand side term. □

2.3 The BSDEs associated to the nonlinear Neumann problem

We introduce the generalized BSDEs which we have to use. Let $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times d'}\to\mathbb{R}^m$, $h:[0,T]\times\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}^m$ and $g:\mathbb{R}^d\to\mathbb{R}^m$ be continuous functions satisfying the following assumptions:

$(A.2)$**:** There exist C, l_f positive constants and $\beta<0$, $\mu_f\in\mathbb{R}$ such that for every $t\in[0,T]$ and every $(x,x',y,y',z,z')\in\left(\mathbb{R}^d\right)^2\times(\mathbb{R}^m)^2\times(\mathbb{R}^{m\times d'})^2$ we have:

(i) $\langle y-y',f(t,x,y,z)-f(t,x,y',z)\rangle\leq\mu_f\,|y-y'|^2$,

(ii) $|f(t,x,y,z)-f(t,x,y,z')|\leq l_f\|z-z'\|$,

(iii) $|f(t,x,y,0)|\leq C\,(1+|y|)$,

(iv) $\langle y-y',h(t,x,y)-h(t,x,y')\rangle\leq\beta|y-y'|^2$,

(v) $|h(t,x,y)|\leq C\,(1+|y|)$,

(vi) $|g(x)| \leq C(1+|x|)$.

For every $t \in [0,T]$ and $s \in [t,T]$, consider the following generalized BSDEs

$$\begin{aligned} Y_s^{t,x,n} = g(X_T^{t,x,n}) + \int_s^T f(r, X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n})\, dr \\ + \int_s^T h(r, X_r^{t,x,n}, Y_r^{t,x,n}) dk_r^{t,x,n} - \int_s^T Z_r^{t,x,n}\, dW_r, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\, dr + \int_s^T h(r, X_r^{t,x}, Y_r^{t,x}) dk_r^{t,x} \\ - \int_s^T Z_r^{t,x}\, dW_r. \end{aligned} \tag{2.12}$$

According to [12, 13], assumption $(A.2)$ ensures the existence of unique solutions to equations (2.11) and (2.12). The solutions of equations (2.11) and (2.12) will be respectively denoted by $(Y_s^{t,x,n}, Z_s^{t,x,n})_{s\in[t,T]}$ and $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$.

Lemma 2.5. *Under assumptions* (A.1)(i) and (A.2), it holds that for any $t \in [0,T]$,

$$\sup_{n\geq 1} \mathbb{E}\left(\sup_{r\in[t,T]} |Y_r^{t,x,n}|^2 + \int_t^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} + \int_t^T \|Z_r^{t,x,n}\|^2 dr\right) < +\infty \tag{2.13}$$

and

$$\mathbb{E}(\sup_{r\in[t,T]} |Y_r^{t,x}|)^q + \mathbb{E}\left(\int_t^T \|Z_r^{t,x}\|^2 dr\right)^{\frac{q}{2}} < +\infty, \qquad \forall\, q > 1. \tag{2.14}$$

Proof. We prove the first assertion. Itô's formula gives

$$\begin{aligned} &|Y_s^{t,x,n}|^2 + \int_s^T \|Z_r^{t,x,n}\|^2 dr = |g(X_T^{t,x,n})|^2 \\ &\quad +2\int_s^T \langle Y_r^{t,x,n}, f(r, X_r^{t,x,n}, Y_r^{t,x,n}, Z_r^{t,x,n})\rangle dr \\ &\quad +2\int_s^T \langle Y_r^{t,x,n}, h(r, X_r^{t,x,n}, Y_r^{t,x,n})\rangle dk_r^{t,x,n} - 2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r\rangle. \end{aligned}$$

Using assumptions (A.2)(i) and (A.2)(vi), we find

$$\begin{aligned} &|Y_s^{t,x,n}|^2 + \int_s^T \|Z_r^{t,x,n}\|^2 dr \leq 2C^2(1+|X_T^{t,x,n}|^2) + 2\mu_f \int_s^T |Y_r^{t,x,n}|^2 dr \\ &+2\int_s^T \langle Y_r^{t,x,n}, f(r, X_r^{t,x,n}, 0, Z_r^{t,x,n}) - f(r, X_r^{t,x,n}, 0, 0)\rangle dr \\ &+2\int_s^T \langle Y_r^{t,x,n}, f(r, X_r^{t,x,n}, 0, 0)\rangle dr \\ &+2\int_s^T \langle Y_r^{t,x,n}, h(r, X_r^{t,x,n}, Y_r^{t,x,n}) - h(r, X_r^{t,x,n}, 0)\rangle dk_r^{t,x,n} \\ &+2\int_s^T \langle Y_r^{t,x,n}, h(r, X_r^{t,x,n}, 0)\rangle dk_r^{t,x,n} - 2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r\rangle \end{aligned}$$

We employ hypothesis (A.2)(iii), (A.2)(iv) and (A.2)(v) to get

$$\begin{aligned}
&|Y_s^{t,x,n}|^2 + \int_s^T \|Z_r^{t,x,n}\|^2 dr \leq \\
&2C^2(1+|X_T^{t,x,n}|^2) + 2\mu_f \int_s^T |Y_r^{t,x,n}|^2 dr + 2l_f \int_s^T |Y_r^{t,x,n}||Z_r^{t,x,n}| dr \\
&+2C \int_s^T |Y_r^{t,x,n}| dr + 2\beta \int_s^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} \\
&+2C \int_s^T |Y_r^{t,x,n}| dk_r^{t,x,n} - 2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r \rangle.
\end{aligned}$$

We apply the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for the terms $2l_f|Y_r^{t,x,n}||Z_r^{t,x,n}|$ and $2C|Y_r^{t,x,n}|$ to obtain

$$\begin{aligned}
&|Y_s^{t,x,n}|^2 + \int_s^T \|Z_r^{t,x,n}\|^2 dr \leq \\
&2C^2(1+|X_T^{t,x,n}|^2) + 2\mu_f \int_s^T |Y_r^{t,x,n}|^2 dr + 2l_f^2 \int_s^T |Y_r^{t,x,n}|^2 dr + \frac{1}{2}\int_s^T |Z_r^{t,x,n}|^2 dr \\
&+C^2 T + \int_s^T |Y_r^{t,x,n}|^2 dr + 2\beta \int_s^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} + 2C \int_s^T |Y_r^{t,x,n}| dk_r^{t,x,n} \\
&\qquad -2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r \rangle,
\end{aligned}$$

we arrange the terms and get

$$\begin{aligned}
&|Y_s^{t,x,n}|^2 + \frac{1}{2}\int_s^T \|Z_r^{t,x,n}\|^2 dr \leq 2C^2 + C^2 T + 2C^2|X_T^{t,x,n}|^2 \\
&+(1+2\mu_f+2l_f^2)\int_s^T |Y_r^{t,x,n}|^2 dr + 2\int_s^T \left(\beta|Y_r^{t,x,n}|^2 + C|Y_r^{t,x,n}|\right) dk_r^{t,x,n} \\
&\qquad -2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r \rangle.
\end{aligned}$$

Since $-\beta > 0$, we use the inequality $2ab \leq (-\beta)a^2 + \frac{b^2}{(-\beta)}$, with $a = |Y_r^{t,x,n}|$ and $b = C$, we have

$$\begin{aligned}
&|Y_s^{t,x,n}|^2 + \frac{1}{2}\int_s^T \|Z_r^{t,x,n}\|^2 dr \leq \\
&\qquad 2C^2 + C^2 T + 2C^2|X_T^{t,x,n}|^2 + (1+2\mu_f+2l_f^2)\int_s^T |Y_r^{t,x,n}|^2 dr \\
&\qquad -\frac{C^2}{\beta}k_T^{t,x,n} + \beta\int_s^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} - 2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r \rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&|Y_s^{t,x,n}|^2 + |\beta|\int_s^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} + \frac{1}{2}\int_s^T \|Z_r^{t,x,n}\|^2 dr \leq \\
&2C^2 + C^2 T + 2C^2|X_T^{t,x,n}|^2 + \frac{C^2}{|\beta|}k_T^{t,x,n} + (1+2\mu_f+2l_f^2)\int_s^T |Y_r^{t,x,n}|^2 dr \\
&\qquad -2\int_s^T \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r \rangle.
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality and the a priori estimate of solution of BSDE, the process $(\int_0^\cdot \langle Y_r^{t,x,n}, Z_r^{t,x,n} dW_r \rangle)$ is a uniformly integrable martingale. We take expectation in the previous inequality to show that

$$\mathbb{E}\left(|Y_s^{t,x,n}|^2 + |\beta| \int_s^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} + \frac{1}{2}\int_s^T \|Z_r^{t,x,n}\|^2 dr\right) \leq$$
$$2C^2 + C^2T + 2C^2\mathbb{E}|X_T^{t,x,n}|^2 + \frac{C^2}{|\beta|}\mathbb{E}k_T^{t,x,n} + (1+2\mu_f + 2l_f^2)\mathbb{E}\int_s^T |Y_r^{t,x,n}|^2 dr.$$

Using estimate (2.5) and Gronwall's inequality, we obtain

$$\sup_{n\geq 1}\sup_{s\in[t,T]} \mathbb{E}\left(|Y_s^{t,x,n}|^2 + \int_s^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} + \int_s^T \|Z_r^{t,x,n}\|^2 dr\right) < +\infty.$$

The Burkholder-Davis-Gundy inequality shows that

$$\sup_{n\geq 1}\mathbb{E}\left(\sup_{s\in[t,T]} |Y_s^{t,x,n}|^2 + \int_t^T |Y_r^{t,x,n}|^2 dk_r^{t,x,n} + \int_t^T \|Z_r^{t,x,n}\|^2 dr\right) < +\infty.$$

Inequality (2.13) is proved. Using [10, Proposition A.2], we prove inequality (2.14). □

We extend the processes $(Y^{t,x,n}, Z^{t,x,n})$ and $(Y^{t,x}, Z^{t,x})$ to $[0,t)$ as follows

$$Y_s^{t,x,n} := Y_t^{t,x,n}, \quad Y_s^{t,x} := Y_t^{t,x} \quad \text{and} \quad Z_s^{t,x,n} = Z_s^{t,x} := 0, \quad s \in [0,t). \tag{2.15}$$

3 Penalization of the nonlinear Neumann PDE

We divide this section into two parts. The first one concerns the convergence of the solution of the BSDE (2.11). The second one is an application of our convergence to the nonlinear Neumann boundary problem.

3.1 Convergence of the penalized BSDE

For $(t,x) \in [0,T]\times\bar{D}$, let $(Y_s^{t,x,n}, Z_s^{t,x,n})_{s\in[0,T]}$ and $(Y_s^{t,x}, Z_s^{t,x})_{s\in[0,T]}$ be respectively, the solutions of BSDEs (2.11) and (2.12). Our first main result is

Theorem 3.1. *Let assumptions* (A.1) and (A.2) hold. Then, we have the following convergence

$$\mathbb{E}\left(\sup_{r\in[0,T]} |Y_r^{t,x,n} - Y_r^{t,x}|^2 + \int_0^T \|Z_r^{t,x,n} - Z_r^{t,x}\|^2 dr\right) \to 0, \quad \text{as } n \to +\infty.$$

Proof. We adapt the proof of [15, Theorem 3.1] to our situation by bringing some modifications. From now on, we omit the superscripts (t,x), and C will denote a nonnegative constant, which may vary from one line to another, but does not depend on n. We shall apply Lemma 2.4 to the following BSDE

$$Y_s^n - Y_s \quad = \quad g(X_T^n) - g(X_T) + \int_s^T d\mathcal{K}_r^n - \int_s^T (Z_r^n - Z_r)dW_r$$

where

$$d\mathcal{K}_r^n \quad := \quad [f(r,X_r^n,Y_r^n,Z_r^n) - f(r,X_r,Y_r,Z_r)]\,dr + h(r,X_r^n,Y_r^n)dk_r^n - h(r,X_r,Y_r)dk_r.$$

Using (A.2)(i)-(A.2)(ii), we get for every $0 \le t \le s \le T$,

$$
\begin{aligned}
&\int_s^T \langle Y_r^n - Y_r, f(r, X_r^n, Y_r^n, Z_r^n) - f(r, X_r, Y_r, Z_r)\rangle dr \\
&= \int_s^T \langle Y_r^n - Y_r, f(r, X_r^n, Y_r^n, Z_r^n) - f(r, X_r^n, Y_r, Z_r^n)\rangle dr \\
&\quad + \int_s^T \langle Y_r^n - Y_r, f(r, X_r^n, Y_r, Z_r^n) - f(r, X_r^n, Y_r, Z_r)\rangle dr \\
&\quad + \int_s^T \langle Y_r^n - Y_r, f(r, X_r^n, Y_r, Z_r) - f(r, X_r, Y_r, Z_r)\rangle dr \\
&\le \int_s^T \mu_f |Y_r^n - Y_r|^2 dr + l_f |Y_r^n - Y_r|\, \|Z_r^n - Z_r\| dr \\
&\quad + \int_s^T |Y_r^n - Y_r||f(r, X_r^n, Y_r, Z_r) - f(r, X_r, Y_r, Z_r)| dr \\
&\le \int_s^T (l_f^2 + \mu_f)|Y_r^n - Y_r|^2 dr + \frac{1}{4}\|Z_r^n - Z_r\|^2 dr \\
&\quad + \int_s^T |Y_r^n - Y_r||f(r, X_r^n, Y_r, Z_r) - f(r, X_r, Y_r, Z_r)| dr.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_s^T \langle Y_r^n - Y_r, h(r, X_r^n, Y_r^n) dk_r^n - h(r, X_r, Y_r) dk_r\rangle \\
&= \int_s^T \langle Y_r^n - Y_r, h(r, X_r^n, Y_r^n) - h(r, X_r^n, Y_r)\rangle dk_r^n \\
&\quad + \int_s^T \langle Y_r^n - Y_r, h(r, X_r^n, Y_r) - h(r, X_r, Y_r)\rangle dk_r^n \\
&\quad + \int_s^T \langle Y_r^n - Y_r, h(r, X_r, Y_r)(dk_r^n - dk_r)\rangle
\end{aligned}
$$

recall that by (1.1) the process k^n is increasing. Thanks to assumption (A.2)(iv), we obtain

$$
\begin{aligned}
&\int_s^T \langle Y_r^n - Y_r, h(r, X_r^n, Y_r^n) dk_r^n - h(r, X_r, Y_r) dk_r\rangle \\
&\le \beta \int_s^T |Y_r - Y_r^n|^2 dk_r^n + \int_s^T |Y_r^n - Y_r||h(r, X_r^n, Y_r) - h(r, X_r, Y_r)| dk_r^n \\
&\quad + \int_s^T \langle Y_r^n - Y_r, h(r, X_r, Y_r)\rangle (dk_r^n - dk_r) \\
&\le \int_s^T |Y_r^n - Y_r||h(r, X_r^n, Y_r) - h(r, X_r, Y_r)| dk_r^n + \int_s^T \langle Y_r^n - Y_r, h(r, X_r, Y_r)\rangle (dk_r^n - dk_r),
\end{aligned}
$$

where the last inequality follows from the fact that $\beta < 0$.

By the foregoing, it holds that, for $\lambda = (l_f^2 + \mu_f) \vee l_f^2$,

$$
\begin{aligned}
\langle Y_r^n - Y_r, d\mathcal{K}_r^n\rangle \quad \le \quad & \frac{1}{4}\|Z_r^n - Z_r\|^2 dr + \lambda |Y_r^n - Y_r|^2 dr \\
& + |Y_r^n - Y_r||h(r, X_r^n, Y_r) - h(r, X_r, Y_r)| dk_r^n \\
& + |Y_r^n - Y_r||f(r, X_r^n, Y_r, Z_r) - f(r, X_r, Y_r, Z_r)| dr \\
& + \langle Y_r^n - Y_r, h(r, X_r, Y_r)\rangle (dk_r^n - dk_r) \\
= \quad & \frac{1}{4}\|Z_r^n - Z_r\|^2 dr + \lambda |Y_r^n - Y_r|^2 dr + |Y_r^n - Y_r| dL_r^n + dR_r^n,
\end{aligned}
$$

where L^n and R^n are defined by

$$\begin{aligned} dL_r^n &:= |f(r,X_r^n,Y_r,Z_r)-f(r,X_r,Y_r,Z_r)|dr \\ &\quad +|h(r,X_r^n,Y_r)-h(r,X_r,Y_r)|dk_r^n \qquad (3.1)\\ dR_r^n &:= \langle Y_r^n-Y_r, h(r,X_r,Y_r)\rangle(dk_r^n-dk_r). \qquad (3.2)\end{aligned}$$

Therefore, applying Lemma 2.4 with $V_r=\lambda r$, $L_r=L_r^n$, $R_r=R_r^n$ and $\alpha=\frac{1}{2}$, there exist positive constants C_1, C_2 and C_3 such that

$$\begin{aligned} &\mathbb{E}\left(\sup_{r\in[0,T]} e^{2\lambda r}|Y_r^n-Y_r|^2\right)+\mathbb{E}\left(\int_0^T e^{2\lambda r}\|Z_r^n-Z_r\|^2dr\right) \qquad (3.3)\\ &\le C_1\,\mathbb{E}e^{2\lambda T}|g(X_T^n)-g(X_T)|^2+C_2\mathbb{E}\left(\int_0^T e^{\lambda r}dL_r^n\right)^2+C_3\mathbb{E}\sup_{s\in[0,T]}\int_s^T e^{2\lambda r}dR_r^n.\end{aligned}$$

We shall give several auxiliary assertions ensuring that the right-hand side term of the previous inequality converges to zero as n goes to $+\infty$.

Lemma 3.2. *Under assumptions* (A.1) and (A.2)(vi), the following convergence holds

$$\lim_{n\to\infty}\mathbb{E}\left(e^{2\lambda T}|g(X_T^n)-g(X_T)|^2\right)=0.$$

Proof. Taking into account the convergence of X_T^n to X_T, the continuity of g, assumption (A.2)(vi) together with the estimate (2.5), the result follows by using the uniform integrability of the sequence X_T^n. □

Lemma 3.3. *Let L^n be the processes given by equation (3.1). Assume that* (A.1) and (A.2) are satisfied. Then,

$$\lim_{n\to\infty}\mathbb{E}\left(\int_0^T e^{\lambda r}dL_r^n\right)^2=0.$$

Proof. Using the inequality $(x+y)^2\le 2(x^2+y^2)$ we obtain

$$\begin{aligned}\mathbb{E}\left(\int_0^T e^{\lambda r}dL_r^n\right)^2 &\le 2\mathbb{E}\left(\int_0^T e^{\lambda r}|f(r,X_r^n,Y_r,Z_r)-f(r,X_r,Y_r,Z_r)|dr\right)^2\\ &\quad +2\mathbb{E}\left(\int_0^T e^{\lambda r}|h(r,X_r^n,Y_r)-h(r,X_r,Y_r)|dk_r^n\right)^2\\ &:= I_1^n+I_2^n.\end{aligned}$$

We shall show that I_1^n and I_2^n tend to zero as n tends to ∞. Hölder's inequality leads to

$$\begin{aligned} I_1^n &= 2\,\mathbb{E}\left(\int_0^T e^{\lambda r}|f(r,X_r^n,Y_r,Z_r)-f(r,X_r,Y_r,Z_r)|dr\right)^2\\ &\le 2Te^{2\lambda T}\mathbb{E}\left(\int_0^T|f(r,X_r^n,Y_r,Z_r)-f(r,X_r,Y_r,Z_r)|^2dr\right). \qquad (3.4)\end{aligned}$$

Again by the convergence of X^n to X in each L^q with respect to the uniform norm and the continuity of f we deduce that the sequence $|f(r,X_r^n,Y_r,Z_r)-f(r,X_r,Y_r,Z_r)|^2$ converges to zero in probability, for every $r\in[0,T]$. Since by assumptions (A.2)(ii) and (A.2)(iii) on f we have

$$|f(r,X_r^n,Y_r,Z_r)-f(r,X_r,Y_r,Z_r)|^2\le C(1+|Y_r|^2+\|Z_r\|^2),\quad r\in[0,T],$$

by uniform integrability argument it follows that

$$\mathbb{E}|f(r,X_r^n,Y_r,Z_r)-f(r,X_r,Y_r,Z_r)|^2\to 0,\quad \text{as}\quad n\to+\infty,\quad r\in[0,T].$$

Using the Lebesgue dominated convergence theorem, thanks to (2.14), we get $\lim_{n\to+\infty} I_1^n=0$. Concerning I_2^n, Hölder's inequality yields

$$\begin{aligned} I_2^n &= 2\mathbb{E}\left(\int_0^T e^{\lambda r}|h(r,X_r^n,Y_r)-h(r,X_r,Y_r)|dk_r^n\right)^2 \\ &\leq 2\,e^{2\lambda T}\left(\mathbb{E}\sup_{r\in[0,T]}|h(r,X_r^n,Y_r)-h(r,X_r,Y_r)|^4\right)^{\frac{1}{2}}\left(\sup_{n\geq 1}\mathbb{E}\left(k_T^n\right)^4\right)^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

On the other hand, by the linear growth assumption on h, we have for each $q>1$

$$\mathbb{E}\sup_{r\in[0,T]}|h(r,X_r^n,Y_r)-h(r,X_r,Y_r)|^{4q}\leq C(1+\mathbb{E}\sup_{r\in[0,T]}|Y_r|^{4q}).$$

It follows from estimates (2.14) that the sequence of random variables $\sup_{r\in[0,T]}|h(r,X_r^n,Y_r)-h(r,X_r,Y_r)|^4$ is uniformly integrable. Since X^n converges to X in each L^q for the uniform norm, we deduce that the sequence $\sup_{r\in[0,T]}|h(r,X_r^n,Y_r)-h(r,X_r,Y_r)|^4$ converges to zero in probability as n goes to $+\infty$. This combined with estimate (2.5) ensures that $\lim_{n\to+\infty} I_2^n=0$. Lemma 3.3 is proved. □

We will show an estimate for the solution Y^n that will be used to control the term $\mathbb{E}\sup_{s\in[0,T]}\int_s^T e^{2\lambda r}dR_r^n$. To this end, let $N\in\mathbb{N},\ N>T$ and the partition of $[0,T]$, $r_i=\frac{iT}{N}$ $i=0,...,N$. For $r\in[0,T]$, we denote $\langle\langle r\rangle\rangle:=\max\{r_i;\,r_i\leq r\}$. Given a continuous stochastic process $(H_r)_{r\in[0,T]}$, we define

$$H_r^N:=\sum_{i=0}^{N-1}H_{r_i}1_{[r_i,r_{i+1})}(r)+H_T1_{\{T\}}(r)=H_{\langle\langle r\rangle\rangle}.$$

Lemma 3.4. *Assume* (A.1) and (A.2) *hold. Then, for any $q\in]1,\ 2[$, there exists a positive constant C depending on T, q and independent of N, such that:*

$$\limsup_{n\to\infty}\mathbb{E}\left(\int_0^T|Y_r^n-Y_r^{n,N}|^q(dk_r^n+dk_r)\right)\leq\frac{C}{N^{q/2}}+C\left[\mathbb{E}\max_{i=1,\dots,N}\left(k_{r_i}-k_{r_{i-1}}\right)^{\frac{2q}{2-q}}\right]^{\frac{2-q}{4}}.$$

Proof. Throughout the proof, C is a generic constant which is independent of n and N. We write BSDE (2.11) between $\langle\langle s\rangle\rangle$ and s

$$\begin{aligned} Y_s^{n,N} &= Y_s^n+\int_{\langle\langle s\rangle\rangle}^s f(r,X_r^n,Y_r^n,Z_r^n)dr \\ &\quad +\int_{\langle\langle s\rangle\rangle}^s h(r,X_r^n,Y_r^n)dk_r^n-\int_{\langle\langle s\rangle\rangle}^s Z_r^n dW_r. \end{aligned}$$

Hölder's inequality gives

$$\begin{aligned} |Y_s^{n,N}-Y_s^n|^q &\leq \frac{C}{N^{q/2}}\left[\int_{\langle\langle s\rangle\rangle}^s|f(r,X_r^n,Y_r^n,Z_r^n)|^2dr\right]^{q/2} \\ &\quad +C\left(k_s^n-k_{\langle\langle s\rangle\rangle}^n\right)^{q/2}\left[\int_{\langle\langle s\rangle\rangle}^s|h(r,X_r^n,Y_r^n)|^2dk_r^n\right]^{q/2}+C\left|\int_{\langle\langle s\rangle\rangle}^s Z_r^n dW_r\right|^q. \end{aligned}$$

It follows that

$$\mathbb{E}\left(\int_0^T |Y_r^n - Y_r^{n,N}|^q (dk_r^n + dk_r)\right) \leq J_1^{n,N} + J_2^{n,N} + J_3^{n,N}$$

where

$$J_1^{n,N} := \frac{C}{N^{q/2}} \mathbb{E} \int_0^T \left[\int_{\langle\langle s\rangle\rangle}^s |f(r, X_r^n, Y_r^n, Z_r^n)|^2 dr\right]^{q/2} (dk_s^n + dk_s),$$

$$J_2^{n,N} := C\,\mathbb{E} \int_0^T (k_s^n - k_{\langle\langle s\rangle\rangle}^n)^{q/2} \left(\int_{\langle\langle s\rangle\rangle}^s |h(r, X_r^n, Y_r^n)|^2 dk_r^n\right)^{q/2} (dk_s^n + dk_s),$$

$$J_3^{n,N} := C\,\mathbb{E} \int_0^T \left|\int_{\langle\langle s\rangle\rangle}^s Z_r^n dW_r\right|^q (dk_s^n + dk_s).$$

We shall estimate $J_1^{n,N}$, $J_2^{n,N}$ and $J_3^{n,N}$. We use Hölder's inequality to obtain

$$\begin{aligned} J_1^{n,N} &= \frac{C}{N^{q/2}} \mathbb{E} \int_0^T \left[\int_{\langle\langle s\rangle\rangle}^s |f(r, X_r^n, Y_r^n, Z_r^n)|^2 dr\right]^{q/2} (dk_s^n + dk_s) \\ &\leq \frac{C}{N^{q/2}} \mathbb{E} \left((k_T^n + k_T) \left[\int_0^T |f(r, X_r^n, Y_r^n, Z_r^n)|^2 dr\right]^{q/2}\right) \\ &\leq \frac{C}{N^{q/2}} \left(\mathbb{E}(k_T^n + k_T)^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \left(\mathbb{E} \int_0^T |f(r, X_r^n, Y_r^n, Z_r^n)|^2 dr\right)^{q/2}. \end{aligned}$$

By the linear growth of f in its third variable and Lipschitz continuity with respect to the fourth argument, we get

$$\begin{aligned} J_1^{n,N} &\leq \frac{C}{N^{q/2}} \left(\mathbb{E}(k_T^n + k_T)^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \left(1 + \mathbb{E} \sup_{r\in[0,T]} |Y_r^n|^2 + \mathbb{E} \int_0^T \|Z_r^n\|^2 dr\right)^{q/2} \\ &\leq \frac{C}{N^{q/2}} \end{aligned}$$

where in the last line we have used inequalities (2.5), (2.7) and (2.13).

Concerning $J_2^{n,N}$, we use Hölder's inequality, the monotonicity of k^n and k and the linear growth condition on h, to obtain

$$\begin{aligned} J_2^{n,N} &= C\,\mathbb{E} \int_0^T (k_s^n - k_{\langle\langle s\rangle\rangle}^n)^{q/2} \left(\int_{\langle\langle s\rangle\rangle}^s |h(r, X_r^n, Y_r^n)|^2 dk_r^n\right)^{q/2} (dk_s^n + dk_s) \\ &\leq C\,\mathbb{E} \left(\int_0^T |h(r, X_r^n, Y_r^n)|^2 dk_r^n\right)^{q/2} \sum_{i=1}^N \int_{r_{i-1}}^{r_i} (k_s^n - k_{\langle\langle s\rangle\rangle}^n)^{q/2} (dk_s^n + dk_s) \\ &\leq C \left(\mathbb{E} \int_0^T |h(r, X_r^n, Y_r^n)|^2 dk_r^n\right)^{q/2} \\ &\qquad \times \left[\mathbb{E}\left(\sum_{i=1}^N \int_{r_{i-1}}^{r_i} (k_s^n - k_{\langle\langle s\rangle\rangle}^n)^{q/2} (dk_s^n + dk_s)\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}} \end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\mathbb{E}\int_0^T |h(r,X_r^n,Y_r^n)|^2 dk_r^n\right)^{q/2}\\
&\qquad\times\left[\mathbb{E}\left(\sum_{i=1}^N (k_{r_i}^n-k_{r_{i-1}}^n)^{q/2}(k_{r_i}^n+k_{r_i}-k_{r_{i-1}}^n-k_{r_{i-1}})\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\\
&\leq C\left(\mathbb{E}\int_0^T 1+|Y_r^n|^2 dk_r^n\right)^{q/2}\\
&\qquad\times\left[\mathbb{E}\left(\sum_{i=1}^N (k_{r_i}^n-k_{r_{i-1}}^n)^{q/2}(k_{r_i}^n+k_{r_i}-k_{r_{i-1}}^n-k_{r_{i-1}})\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\\
&\leq C\left(\mathbb{E}k_T^n+\mathbb{E}\int_0^T |Y_r^n|^2 dk_r^n\right)^{q/2}\\
&\qquad\left[\mathbb{E}\left(\sum_{i=1}^N (k_{r_i}^n-k_{r_{i-1}}^n)^{q/2}(k_{r_i}^n+k_{r_i}-k_{r_{i-1}}^n-k_{r_{i-1}})\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}.
\end{aligned}
$$

Hence, by inequalities (2.5) and (2.13), we see that

$$
J_2^{n,N} \quad\leq\quad C\left[\mathbb{E}\left(\sum_{i=1}^N (k_{r_i}^n-k_{r_{i-1}}^n)^{q/2}(k_{r_i}^n+k_{r_i}-k_{r_{i-1}}^n-k_{r_{i-1}})\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}.
$$

Taking into account the convergence of k^n to k (Remark 2.3 (i)), estimates (2.5) and (2.7), we pass to the limit as n goes $+\infty$ then we use the Lebesgue dominated convergence theorem to get

$$
\begin{aligned}
\limsup_{n\to\infty} J_2^{n,N} &\leq C\left[\mathbb{E}\left(\sum_{i=1}^N (k_{r_i}-k_{r_{i-1}})^{q/2}(k_{r_i}-k_{r_{i-1}})\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\\
&\leq C\left[\mathbb{E}\left(\max_{i=1,\dots,N}(k_{r_i}-k_{r_{i-1}})^{q/2}k_T\right)^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\\
&\leq C\left[\mathbb{E}\max_{i=1,\dots,N}(k_{r_i}-k_{r_{i-1}})^{\frac{q}{2-q}}k_T^{\frac{2}{2-q}}\right]^{\frac{2-q}{2}}\\
&\leq C\left[\mathbb{E}\max_{i=1,\dots,N}(k_{r_i}-k_{r_{i-1}})^{\frac{2q}{2-q}}\right]^{\frac{2-q}{4}}\left[\mathbb{E}k_T^{\frac{4}{2-q}}\right]^{\frac{2-q}{4}}.
\end{aligned}
$$

For $J_3^{n,N}$, we have

$$
\begin{aligned}
J_3^{n,N} &= C\,\mathbb{E}\int_0^T \left|\int_{\langle\langle s\rangle\rangle}^s Z_r^n dW_r\right|^q (dk_s^n+dk_s)\\
&= C\,\mathbb{E}\sum_{i=1}^N\int_{r_{i-1}}^{r_i} \left|\int_{\langle\langle s\rangle\rangle}^s Z_r^n dW_r\right|^q (dk_s^n+dk_s)\\
&\leq C\sum_{i=1}^N\mathbb{E}\sup_{s\in[r_{i-1},r_i]}\left|\int_{\langle\langle s\rangle\rangle}^s Z_r^n dW_r\right|^q (k_{r_i}^n-k_{r_{i-1}}^n+k_{r_i}-k_{r_{i-1}})
\end{aligned}
$$

$$\leq \quad C\sum_{i=1}^{N}\left(\mathbb{E}\sup_{s\in[r_{i-1},r_i]}\left|\int_{r_{i-1}}^{s}Z_r^n dW_r\right|^2\right)^{\frac{q}{2}}\left(\mathbb{E}(k_{r_i}^n-k_{r_{i-1}}^n+k_{r_i}-k_{r_{i-1}})^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}.$$

Using the Burkholder-Davis-Gundy inequality, we obtain

$$J_3^{n,N} \quad\leq\quad C\sum_{i=1}^{N}\left(\mathbb{E}\int_{r_{i-1}}^{r_i}\|Z_r^n\|^2\,dr\right)^{\frac{q}{2}}\left(\mathbb{E}(k_{r_i}^n-k_{r_{i-1}}^n+k_{r_i}-k_{r_{i-1}})^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}.$$

Again by Hölder's inequality, we find

$$\begin{aligned}
J_3^{n,N} \quad&\leq\quad C\left(\sum_{i=1}^{N}\mathbb{E}\int_{r_{i-1}}^{r_i}\|Z_r^n\|^2\,dr\right)^{\frac{q}{2}}\left(\sum_{i=1}^{N}\mathbb{E}(k_{r_i}^n-k_{r_{i-1}}^n+k_{r_i}-k_{r_{i-1}})^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\\
&\leq\quad C\left(\mathbb{E}\int_0^T\|Z_r^n\|^2\,dr\right)^{\frac{q}{2}}\left(\sum_{i=1}^{N}\mathbb{E}(k_{r_i}^n-k_{r_{i-1}}^n+k_{r_i}-k_{r_{i-1}})^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}.
\end{aligned}$$

Keeping in mind inequality (2.13) and the convergence of k^n to k, we pass to the limit as $n\to+\infty$, to obtain

$$\begin{aligned}
\lim_{n\to\infty}J_3^{n,N} \quad&\leq\quad C\left(\sum_{i=1}^{N}\mathbb{E}(k_{r_i}-k_{r_{i-1}})^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\\
&\leq\quad C\left(\mathbb{E}\max_{i=1,N}(k_{r_i}-k_{r_{i-1}})^{\frac{q}{2-q}}\sum_{i=1}^{N}(k_{r_i}-k_{r_{i-1}})\right)^{\frac{2-q}{2}}\\
&\leq\quad C\left(\mathbb{E}\max_{i=1,N}(k_{r_i}-k_{r_{i-1}})^{\frac{q}{2-q}}k_T\right)^{\frac{2-q}{2}}\\
&\leq\quad C\left(\mathbb{E}k_T^2\right)^{\frac{2-q}{4}}\left(\mathbb{E}\max_{i=1,\ldots,N}(k_{r_i}-k_{r_{i-1}})^{\frac{2q}{2-q}}\right)^{\frac{2-q}{4}}\\
&\leq\quad C\left(\mathbb{E}\max_{i=1,\ldots,N}(k_{r_i}-k_{r_{i-1}})^{\frac{2q}{2-q}}\right)^{\frac{2-q}{4}}.
\end{aligned}$$

This completes the proof of Lemma 3.4. ☐

Lemma 3.5. *Let R^n be the process defined by (3.2). Under assumptions* (A.1) *and* (A.2), *the following inequality holds*

$$\limsup_{n\to\infty}\mathbb{E}\sup_{s\in[0,T]}\int_s^T e^{2\lambda r}dR_r^n\leq 0.$$

Proof. Set $h_r=h(r,X_r,Y_r)$ and $|h|_\infty=\sup_{r\in[0,T]}|h_r|$, then

$$\begin{aligned}
\langle Y_r^n-Y_r,h_r\rangle \quad&=\quad \langle Y_r^{n,N}-Y_r^N,h_r-h_r^N\rangle+\langle Y_r^N-Y_r,h_r\rangle\\
&\quad +\langle Y_r^{n,N}-Y_r^N,h_r^N\rangle+\langle Y_r^n-Y_r^{n,N},h_r\rangle.
\end{aligned}$$

We have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]}\int_s^T e^{2\lambda r}dR_r^n\right)= \\
&\mathbb{E}\left(\sup_{s\in[0,T]}\int_s^T e^{2\lambda r}\langle Y_r^n-Y_r,h(r,X_r,Y_r)(dk_r^n-dk_r)\rangle\right) \\
&\leq \mathbb{E}\left[\left((|Y^n|_\infty+|Y|_\infty)|h-h^N|_\infty+|Y^N-Y|_\infty|h|_\infty\right)e^{2\lambda T}(k_T^n+k_T)\right] \\
&\quad+\mathbb{E}\left(\sup_{s\in[0,T]}\sum_{i=1,\ s\leq r_{i-1}}^{N}\langle Y_{r_{i-1}}^n-Y_{r_{i-1}},h_{r_{i-1}}\rangle\int_{s\vee r_{i-1}}^{r_i}e^{2\lambda r}d(k_r^n-k_r)\right) \\
&\quad+\mathbb{E}\left(e^{2\lambda T}|h|_\infty\int_0^T|Y_r^{n,N}-Y_r^n|(dk_r^n+dk_r)\right).
\end{aligned}$$

We explain briefly how to estimate the second term in the right-hand side of the previous inequality. Using an integration by parts we find

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]}\sum_{i=1,\ s\leq r_{i-1}}^{N}\langle Y_{r_{i-1}}^n-Y_{r_{i-1}},h_{r_{i-1}}\rangle\int_{s\vee r_{i-1}}^{r_i}e^{2\lambda r}d(k_r^n-k_r)\right)= \\
&\mathbb{E}\left(\sup_{s\in[0,T]}\sum_{i=1,\ s\leq r_{i-1}}^{N}\langle Y_{r_{i-1}}^n-Y_{r_{i-1}},h_{r_{i-1}}\rangle\right. \\
&\left.\times\left(e^{2\lambda r_i}(k_{r_i}^n-k_{r_i})-e^{2\lambda s\vee r_{i-1}}(k_{s\vee r_{i-1}}^n-k_{s\vee r_{i-1}})-\int_{s\vee r_{i-1}}^{r_i}2\lambda(k_r^n-k_r)e^{2\lambda r}dr\right)\right) \\
&\leq\mathbb{E}\left(\sup_{s\in[0,T]}\sum_{i=1,\ s\leq r_{i-1}}^{N}\left(|Y^n|_\infty+|Y|_\infty\right)|h|_\infty(2e^{2\lambda T}+2\lambda Te^{2\lambda T})\sup_{s\in[0,T]}|k_s^n-k_s|\right) \\
&\leq 2Ne^{2\lambda T}(1+\lambda T)\mathbb{E}\left(\left(|Y^n|_\infty+|Y|_\infty\right)|h|_\infty\sup_{s\in[0,T]}|k_s^n-k_s|\right)
\end{aligned}$$

We comeback now to our main estimate. Let $1<q<2$. Using Hölder's inequality repeatedly we obtain

$$\begin{aligned}
&\mathbb{E}\left(\sup_{s\in[0,T]}\int_s^T e^{2\lambda r}dR_r^n\right)\leq \\
&\left[\mathbb{E}\left[e^{2\lambda T}(|Y^n|_\infty+|Y|_\infty)\right]^2\right]^{\frac12}\left[\mathbb{E}(k_T^n+k_T)^4\right]^{\frac14}\left[\mathbb{E}|h-h^N|_\infty^4\right]^{\frac14} \\
&+\left[\mathbb{E}\left[e^{2\lambda T}(k_T^n+k_T)|h|_\infty\right]^2\right]^{\frac12}\left[\mathbb{E}|Y^N-Y|_\infty^2\right]^{\frac12} \\
&+2Ne^{2\lambda T}(1+\lambda T)\left[\mathbb{E}\left[e^{2\lambda T}(|Y^n|_\infty+|Y|_\infty)\right]^2\right]^{\frac12}\left[\mathbb{E}|h|_\infty^4\right]^{\frac14}\left(\mathbb{E}\left[\sup_{s\in[0,T]}|k_s^n-k_s|^4\right]\right)^{\frac14} \\
&+e^{2\lambda T}\left[\mathbb{E}|h|_\infty^{\frac{2q}{q-1}}\right]^{\frac{q-1}{2q}}\left[\mathbb{E}\left(k_T^n+k_T\right)^2\right]^{\frac{q-1}{2q}}\left[\mathbb{E}\int_0^T|Y_r^{n,N}-Y_r^n|^q(dk_r^n+dk_r)\right]^{\frac1q}. \qquad (3.6)
\end{aligned}$$

The linear growth hypothesis on h combined with estimate (2.14) show that for every $p\geq 1$,

$$\mathbb{E}|h|_\infty^p=\mathbb{E}\sup_{r\in[0,T]}|h(r,X_r,Y_r)|^p\leq C(1+\mathbb{E}\sup_{r\in[0,T]}|Y_r|^p)<+\infty.$$

On the other hand, we use inequalities (2.5), (2.7), (2.13) and (2.14) along with Lemma 3.4 then we pass to the limit as n goes to $+\infty$ in inequality (3.6) to get, for all $N \in \mathbb{N}^*$,

$$\begin{aligned}\limsup_{n\to+\infty} \mathbb{E}\left(\sup_{s\in[0,T]}\int_s^T e^{2\lambda r} dR_r^n\right) \quad &\leq \quad C\left(\mathbb{E}|h-h^N|_\infty^4\right)^{1/4} + C\left(\mathbb{E}|Y^N - Y|_\infty^2\right)^{1/2} \\ &\quad + \left[\frac{C}{N^{q/2}} + C\left[\mathbb{E}\max_{i=1,N}\left(k_{r_i} - k_{r_{i-1}}\right)^{\frac{2q}{2-q}}\right]^{\frac{2-q}{4}}\right]^{1/q}.\end{aligned}$$

Since the integrands are uniformly integrable, hence passing to the limit as $N \to +\infty$, we get the result. Lemma 3.5 is proved. □

Now, combining inequality (3.3) with Lemmas 3.2, 3.3 and 3.5, we complete the proof of Theorem 3.1. □

3.2 Convergence of the penalized PDE

This subsection is devoted to an application of our convergence of the BSDE. Namely, we will establish the convergence of a viscosity solution of the following systems

$$\begin{cases}\dfrac{\partial u_i^n}{\partial t}(t,x) + \mathcal{L}\, u_i^n(t,x) + f_i(t,x,u^n(t,x),(\nabla u_i^n\sigma)(t,x)) \\ \qquad\qquad - \, n\langle\delta(x),\nabla u_i^n(t,x)\rangle - n\langle\delta(x),\nabla l(x)\rangle h_i(t,x,u^n(t,x)) = 0\,, \\ u^n(T,x) = g(x)\,, 1\leq i\leq m, 0\leq t\leq T, x\in\mathbb{R}^d, n\in\mathbb{N}\end{cases} \tag{3.7}$$

to a viscosity solution of a system of the form

$$\begin{cases}\dfrac{\partial u_i}{\partial t}(t,x) + \mathcal{L}u_i(t,x) + f_i(t,x,u(t,x),(\nabla u_i\sigma)(t,x)) = 0\,, \\ 1\leq i\leq m\,,\ (t,x)\in[0,T)\times D\,, \\ u(T,x) = g(x)\,,\ \ x\in D\,, \\ \dfrac{\partial u}{\partial n}(t,x) + h(t,x,u(t,x)) = 0\,,\ \forall (t,x)\in[0,T)\times\partial D\,.\end{cases} \tag{3.8}$$

Since we consider viscosity solutions, we introduce the following condition

(A.3): f_i, the i-th coordinate of f, depends only on the i-th row of the matrix z.

For the sake of completeness, we recall the definition of the viscosity solution of system (3.8).

Definition 3.6. *(i)* $u \in \mathcal{C}([0,T]\times\bar{D},\mathbb{R}^m)$ *is called a viscosity sub solution of system* (3.8) *if* $u_i(T,x)\leq g_i(x)$, $x\in\bar{D}$, $1\leq i\leq m$ *and, for any* $1\leq i\leq m$, $\varphi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$, *and* $(t,x)\in(0,T]\times\bar{D}$ *at which* $u_i-\varphi$ *has a local maximum, one has*

$$-\frac{\partial\varphi}{\partial t}(t,x) - \mathcal{L}\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \leq 0, \quad \textit{if} \ \ x\in D,$$

$$\begin{aligned}\min\Bigg(&-\frac{\partial\varphi}{\partial t}(t,x) - \mathcal{L}\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)), \\ &-\frac{\partial\varphi}{\partial n}(t,x) - h_i(t,x,u(t,x))\Bigg) \leq 0, \quad \textit{if}\, x\in\partial D.\end{aligned}$$

(ii) $u\in\mathcal{C}([0,T]\times\bar{D},\mathbb{R}^d)$ *is called a viscosity super-solution of (3.8) if* $u_i(T,x)\geq g_i(x)$, $x\in\bar{D}$, $1\leq i\leq m$ *and, for any* $1\leq i\leq m$, $\varphi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$, *and* $(t,x)\in(0,T]\times\bar{D}$ *at which* $u_i-\varphi$ *has a local minimum, one has*

$$-\frac{\partial\varphi}{\partial t}(t,x) - \mathcal{L}\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \geq 0, \quad \textit{if} \ \ x\in D,$$

$$\max\left(-\frac{\partial\varphi}{\partial t}(t,x)-\mathcal{L}\varphi(t,x)-f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)),\right.$$
$$\left.-\frac{\partial\varphi}{\partial n}(t,x)-h_i(t,x,u(t,x))\right)\geq 0,\quad \textit{if } x\in\partial D.$$

(iii) $u \in \mathcal{C}([0,T]\times\bar{D},\mathbb{R}^m)$ *is called a viscosity solution of system (3.8) if it is both a viscosity sub and super-solution.*

We recall the continuity of the map $(t,x)\mapsto Y_t^{t,x}$ where $Y^{t,x}$ is the solution of BSDE (2.12). This continuity has been proved in [15].

Proposition 3.7 ([15])**.** *Under assumptions* (A.1) and (A.2), the mapping $(t,x)\to Y_t^{t,x}$ is continuous.

Our second main result is

Theorem 3.8. *Assume* (A.1), (A.2) and (A.3). There exist a sequence of continuous functions $u^n : [0,T]\times\mathbb{R}^d\to\mathbb{R}^m$ and a function $u : [0,T]\times\bar{D}\to\mathbb{R}^m$ such that: u^n is a viscosity solution to system (3.7), u is a viscosity solution to system (3.8) and the following convergence holds for every $(t,x)\in[0,T]\times\bar{D}$

$$\lim_{n\to+\infty} u^n(t,x) = u(t,x).$$

Proof. We set,

$$u^n(t,x) := Y_t^{t,x,n} \quad \text{and} \quad u(t,x) := Y_t^{t,x}. \tag{3.9}$$

It follows from Theorem 3.2 of [12] that u^n is a viscosity solution of PDEs (3.7). Thanks to [15, 13], u is a viscosity solution of PDEs (3.8). Further, we have for each $(t,x)\in[0,T]\times\bar{D}$

$$|u^n(t,x)-u(t,x)|^2 = |Y_t^{t,x,n}-Y_t^{t,x}|^2 \leq \mathbb{E}\sup_{s\in[0,T]}|Y_s^{t,x,n}-Y_s^{t,x}|^2.$$

Thanks to Theorem 3.1, we have $\lim_{n\to\infty}\mathbb{E}\sup_{s\in[0,T]}|Y_s^{t,x,n}-Y_s^{t,x}|^2=0$. It follows that,

$$\lim_{n\to+\infty} u^n(t,x) = u(t,x).$$

The theorem is proved. □

Remark 3.9. When u is the unique viscosity solution of system (3.8), then it is constructible by penalization. See [14, Theorem 5.43, p 423] and [15, Theorem 5.1] for cases where the viscosity solution of system (3.8) is unique.

References

[1] Bahlali, K., Maticiuc, L. and Zalinescu, A. : Penalization method for a nonlinear Neumann PDE via weak solution of reflected SDEs, *Elctron. J. Probab. 18(2013), no. 102, 1–19* MR3145049

[2] Bahlali, K., Elouaflin, A. and Pardoux, E. : Averaging for BSDEs with null recurrent fast component. Application to homogenization in a non periodic media. Stochastic Process. Appl. 127 (2017), no. 4, 1321–1353. MR3619273

[3] Bahlali, K., Boufoussi, B. and Mouchtabih, S. : Penalization for a PDE with a nonlinear Neumann boundary condition and measurable coefficients. Stochastics and Dynamics, 2021. https://doi.org/10.1142/S0219493721500532 MR4380129

[4] Boufoussi, B. and Casteren, J.V. : An approximation result for a nonlinear Neumann boundary value problem via BSDEs, *Stochastic Process. Appl. 114 (2004), 331–350.* MR2101248

[5] Delarue, F. : Equations différentielles stochastiques progressives-rétrogrades. Application à l'homogenéisation des EDP quasi-linéaires. *Thèse de Doctorat,* (2002), Aix Marseille Université. Formerly Université de Provence, Aix-Marseille I.

[6] Delarue, F. : Auxiliary SDEs for homogenization of quasilinear PDEs with periodic coefficients. Ann. Probab. 32 (2004), no. 3B, 2305–2361. MR2078542

[7] Laukajtys, W. and Slominski, L. : Penalization methods for reflecting stochastic differential equations with jumps, *Stoch. Stoch. Rep. 75 no. 5 (2003), 275–293* MR2017780

[8] Lions, P.L., Menaldi, J.L. and Sznitman, A.S. : Construction de processus de diffusion réfléchis par pénalisation du domaine, *C. R. Acad. Sc. Paris, t. 229, Série I (1981) 459–462*. MR0614669

[9] Lions, P.L. and Sznitman, A.S. : Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. XXXVII (1984) 511–537. MR0745330

[10] Maticiuc, L. and Rascanu, A. : Backward stochastic variational inequalities on random interval, Bernoulli 21(2) (2015), pp. 1166–1199. MR3338660

[11] Menaldi, J.L. : Stochastic Variational Inequality for Reflected Diffusion, *Indiana University Mathematical Journal, Vol.32, No 5 (1983)*. MR0711864

[12] Pardoux, E. : BSDEs, weak convergence and homogenization of semilinear PDEs, Nonlinear Analysis, Differential Equations and Control (Montreal, QC, 1998), Kluwer Academic Publishers, Dordrecht (1999), 503–549. MR1695013

[13] Pardoux, E. and Zhang, S. : Generalized BSDEs and nonlinear Neumann boundary value problems, *Prob. Theory and Rel. Fields, 110 (1998) 535–558*. MR1626963

[14] Pardoux, E. and Rascanu, A. : Stochastic differential equations, Backward SDEs, Partial differential equations, Stoch. Model. Appl. Probab. 69 (2014). Springer MR3308895

[15] E. Pardoux, E. and Rascanu, A. : Continuity of the Feynman-Kac formula for a generalized parabolic equation, Stochastics (89) 5, 726–752. 2017 http://dx.doi.org/10.1080/17442508.2016.1276911 MR3640792

[16] A. Rozkosz, A. and Slominski, L. : On stability and existence of solutions of SDEs with reflection at the boundary, *Stoc. Proc. Appl. 68, (1997), 285–302*. MR1454837

[17] Slominski, L. : Eulers approximations of solutions of SDEs with reflecting boundary, Stochastic Process. Appl. 94 (2001) 317–337. MR1840835

[18] Slominski, L. : Weak and strong approximations of reflected diffusions via penalization methods, Stochastic Process. Appl. 123 (2013) 752–763. MR3005004

[19] Stroock, D.W. and Varadhan, S.R.S. : Diffusion Processes with boundary conditions,*Comm. Pure Appl. Math. 24 (1971), 147–225*. MR0277037

[20] Tanaka, H. : Stochastic differential equations with reflecting boundary condition in convex regions, Hiroshima Math. J. 9 (1979) 163–177. MR0529332

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