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Entanglement percolation and spheres in $\mathbb{Z}^d$

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Abstract

We obtain a new lower bound of 0.06576 for the 1-entanglement critical probability (in dimension 3), and prove that the critical point for the existence of a sphere surrounding the origin and intersecting only closed bonds in $\mathbb{Z}^d$ is greater than $\frac{1}{8(d-1)}$, $d \geq 3$. This substantially improves the previous lower bounds and gives the correct order of magnitude for large d.

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1 Introduction

We start this paper by briefly and informally introducing our main theorem.

While for the bond percolation model in $\mathbb{Z}^3$ one generally deals with the existence of open paths, Kantor and Hassold [13] proposed to study an alternative notion called *entanglement* (a notion that comes from physics). In this paper, we follow the definition of 1-entanglement introduced by Grimmett and Holroyd [8], which, informally, asks for the existence of an infinite sequence of finite clusters linked like rings of a chain.

The notion of 1-entanglement is three dimensional by essence. As a natural generalisation, in dimension 3 and higher, Grimmett and Holroyd introduced the concept of sphere intersecting only closed bonds. Recall that a subset of $\mathbb{R}^d$ is a *sphere*, in the sense of [8], if it is homeomorphic to the unit euclidean sphere $S^{d-1} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1^2 + \ldots x_d^2 = 1\}$ and simplicial complex. Denote by $\mathcal{S}$ the event that there exists a sphere intersecting only closed bonds and with the origin in its inside and put

$$p_c^{\mathcal{S}} := \inf\{p \in [0,1] \text{ such that } \mathbb{P}_p(\mathcal{S}) = 0\}$$

for the corresponding critical probability.

In dimension 3 the notion of sphere intersecting only closed bonds coincides with the notion of 1-entanglement. In that case, following [8], we write $p_e^1 := p_c^{\mathcal{S}}$ (see below for an explanation of the index 1 in such a notation).

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Our aim in this article is to improve upon known results on $p_c^{\mathcal{S}}$ for all $d \geq 3$, hence including the three dimensional 1-entanglement notion.

As a first main result, we will prove the following:

Theorem 1.1. *for all $d \geq 3$, it holds*

$$p_c^{\mathcal{S}} \geq \frac{1}{8(d-1)}. \tag{1.1}$$

See Theorem 2.1 below for a more complete statement. The previous (and unique) known lower bound on $p_c^{\mathcal{S}}$ is due to Grimmett and Holroyd [8] and states that $p_c^{\mathcal{S}} \geq \frac{c_d}{d^2}$ with $c_d \approx 1/4$ in the limit $d \to \infty$. In addition to (1.1), notice that, since an infinite cluster prevents the existence of a sphere, $p_c^{\mathcal{S}} \leq p_c$, where p_c is the usual bond percolation critical probability. Together with $p_c \leq c'_d/d$ with $c'_d \approx 1/2$ for d tending to infinity [10], we thus obtain that

$$\frac{1}{8} \leq \liminf_{d\to\infty} dp_c^{\mathcal{S}} \leq \limsup_{d\to\infty} dp_c^{\mathcal{S}} \leq \frac{1}{2}$$

which shows that $1/d$ is the correct behavior of $p_c^{\mathcal{S}}$ for large d. In fact, the lower bound $p_c^{\mathcal{S}} \geq 1/(8(d-1))$ above improves upon known results not only for large d but also for any fixed $d \geq 3$, see Remark 2.2 below.

One of the ideas of Grimmett and Holroyd in their analysis of $p_c^{\mathcal{S}}$ is to construct a certain class of spheres belonging to the event $\mathcal{S}$. Such spheres appear to be star-shaped, which reveals to be too restrictive. Motivated by this observation, our approach will consist in constructing a more refined class of spheres (not necessarily star-shaped) belonging to $\mathcal{S}$.

Specifying to the dimension $d = 3$, the above lower bound on $p_c^{\mathcal{S}} = p_e^1$ leads to $p_e^1 \geq 1/16 = 0.0625$ which already improves upon the best known result $p_e^1 \geq 0,04453$ [8]. In fact, using a more careful analysis on the number of certain paths, by means of large deviations on Markov chains, we will prove the following theorem which constitutes our second main result:

Theorem 1.2. *The 1-entanglement critical probability verifies*

$$p_e^1 \geq 0.06576.$$

The first lower bound on p_e^1 was $p_e^1 \geq 1/15616$ [11], obtained by a nice and tricky construction of spheres. Then Atapour and Madras [2] improved it to $1/597$, by a combinatorial argument. Finally Grimmett and Holroyd proved $p_e^1 \geq 0,04453$. Let us point out that there is still a long way to go in order to obtain a lower bound close to the expected value of p_e^1. Indeed, numerical investigations indicate that $p_c - p_e^1$ should be of order 10^{-7} (and at least $1.8 \cdot 10^{-7}$) [13], while p_c is estimated with simulations to be near 0.248812 [15]. Therefore, one expects p_e^1 to be about $0.24881...$

In the next section we introduce more formally the different notions of interest for us, state a more complete theorem than Theorem 1.1 and add some more comments on the literature.

2 Percolation, spheres, entanglement

We consider the lattice $\mathbb{Z}^d$, whose elements are called *vertices*, and pairs of vertices of euclidean distance one are called *edges*. Two vertices of an edge are said to be *neighbours*. For $p \in (0,1)$, in the *bond percolation model* on $\mathbb{Z}^d$, edges are open with probability p and closed with probability $1-p$, independently one of each other. For a detailed exposition of the percolation model, we refer the reader to [6].

The terms "bond" and "edge" are very similar. However with "bond" the intention is to insist on the topological embedding in $\mathbb{R}^d$ (a bond refers to the continuous segment in

$\mathbb{R}^d$ joining two neighbours of $\mathbb{Z}^d$), whereas an "edge" refers only to a pair of neighbours of $\mathbb{Z}^d$. We will say that a bond is open or closed according to the state of its corresponding edge.

As already mentioned, a *sphere* is a simplicial complex subset of $\mathbb{R}^d$ that is homeomorphic to the unit euclidean sphere $S^{d-1} := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1\}$. A basic example of a sphere is given by the surface of a parallepiped. The complement of a sphere has a unique bounded component, which we call the *inside* of the sphere. Spheres considered in this article will not intersect $\mathbb{Z}^d$. Our goal will be to select a sphere intersecting only closed bonds.

Following [8], we set

$$\mathrm{rad}[A] = \sup\left\{\sum_{i=1}^{d} |x_i|, (x_1, \dots, x_d) \in A\right\}$$

for the *radius* of $A \subset \mathbb{R}^d$ (understood from the origin).

We are now in position to state a more complete version of Theorem 1.1.

Theorem 2.1. *For all dimension $d \geq 3$, it holds*

$$p_c^{\mathcal{S}} \geq \frac{1}{8(d-1)}. \tag{2.1}$$

Moreover, for all $p < \frac{1}{8(d-1)}$ and all $\alpha \in (\sqrt{8p(d-1)}, 1)$, there exist $C > 0$ and $S \in \mathcal{S}$ such that

$$\mathbb{P}_p(\mathrm{rad}[S] \geq r) \leq C\alpha^r, \qquad \forall r > 0. \tag{2.2}$$

In the next remark we compare our result to [8].

Remark 2.2. Let $\sigma(k)$ be the number of self-avoiding paths with length k in $\mathbb{Z}^d$ and let (see *e.g.* [16]) $\mu_d := \lim_{k\to\infty} \sigma(k)^{1/k}$ be the connective constant[1] of $\mathbb{Z}^d$. In [8] the authors proved (among other things) that $p_c^{\mathcal{S}} \geq \mu_d^{-2}$. Since, see for example [14, 16], $\lim_{d\to\infty} \frac{\mu_d}{2d} = 1$, their result reads as $p_c^{\mathcal{S}} \geq 1/(4d^2)$, asymptotically. Furthermore, the exact lower bounds of the connective constant provided in [3], [12] and [5] for $d \leq 6$, and the trivial fact that $\mu_d \geq d$, ensure that (2.1) is actually an improvement on $p_c^{\mathcal{S}} \geq \mu_d^{-2}$ for all dimensions.

Let us briefly explain the notation p_e^1 for the 1-entanglement critical probability. As already mentioned, entanglement is a notion specific to the dimension $d = 3$. For a finite set of bonds there is no uncertainty, at least heuristically, about what we consider entangled or not. But the picture get more complicated for an infinite set of bonds. In [7], the authors define the notion of entanglement systems, which leads to a family having two extremal elements, $\mathcal{E}_0$ and $\mathcal{E}_1$, the latter being the one considered in this article.

Given a set of edges A, denote by $[A]$ the union of its bonds (recall that a bond refers to the continuous segment joining the end points of the corresponding edge). A set of edges A, finite or infinite, is said to be in $\mathcal{E}_1$ if there is no sphere separating $[A]$ into two disconnected parts. As a direct consequence of the definition we observe that a connected sets of edges A (finite or infinite) belong to $\mathcal{E}_1$. See Figure 1 for an example of set in $\mathcal{E}_1$ and of set not in $\mathcal{E}_1$.

We say that there is 1-*entanglement percolation* if there is an infinite set of open edges containing the origin that is an element of $\mathcal{E}_1$. Hence if a sphere with the origin in its inside intersects only closed bonds, there is no 1-entanglement percolation. Note moreover that if there is percolation in the usual sense (*i.e.* an infinite path of open edges starting from the origin), then there is also 1-entanglement percolation.

[1] We follow here the terminology found for exemple in [6] or [16]. Historically, the connective constant was defined by Hammersley [9] as $\lim_{k\to\infty}(\log \sigma(k))/k = \log(\mu_d)$

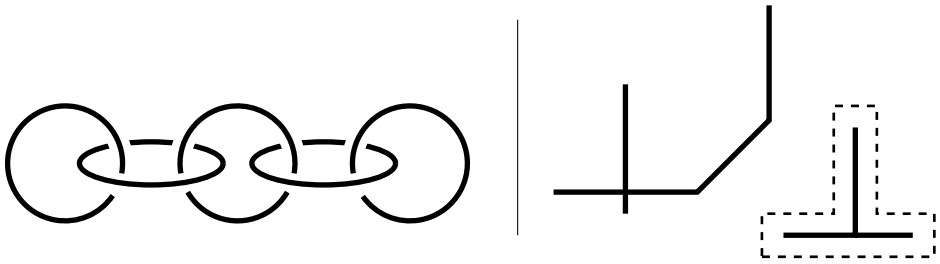

Figure 1: Left: entangled and not connected set. Right: not entangled set.

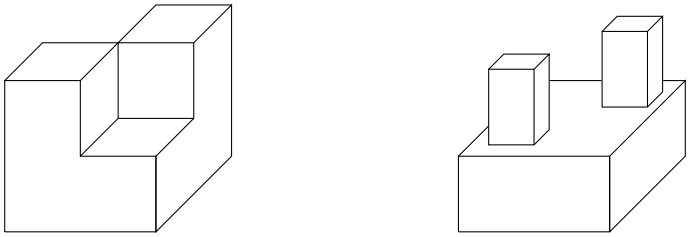

Figure 2: A set of plaquettes that is not a sphere, and how to obtain a sphere.

We end this section with a sketch of our proof.

In order to explain our main ingredient, which is based on an improvement of the ideas from [8], we need first to introduce the notion of plaquette. A *plaquette* is any face of a cube of the form $x + [-\frac{1}{2}, \frac{1}{2}]^d$ with $x \in \mathbb{Z}^d$ (and so is a $(d-1)$-dimensional subet of $\mathbb{R}^d$). A plaquette intersects a unique bond (and is orthogonal to it), and vice versa, so that there is a one to one correspondence between bonds and plaquettes. Based on this correspondence, a plaquette is open/closed according to the state of its corresponding bond.

A simple but key observation is that a sphere of closed plaquettes is necessarily intersecting only closed bonds while the existence of a sphere intersecting only closed bonds does not imply the existence of a sphere of closed plaquettes. To convince the reader, one can consider, in $\mathbb{Z}^3$, a set consisting of the six vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$ and $(1,1,1)$, that is to say four vertices forming a square on the first floor, and two vertices on the second floor, these two being not neighbours. If one considers the set of plaquettes corresponding to the bonds on the outer border of this set, one can see that this is not a sphere due to the intersection of some plaquettes on the second floor. Nevertheless, taking a surface closer to the vertices, one could imagine a sphere intersecting only the bonds of the outer border, as in figure 2. This type of configurations shows that spheres of closed plaquettes are too constrained and therefore potentially not adapted to the study of $p_c^{\mathcal{S}}$.

To ensure the presence of a sphere of plaquettes, Grimmett and Holroyd [8] introduced a notion of good paths. Given the sites $0 = \nu_0, \nu_1, \dots, \nu_k$ of a self-avoiding path, they called it good if, for each i satisfying $\|\nu_{i-1}\|_1 < \|\nu_i\|_1$, the edge $\langle \nu_{i-1}, \nu_i \rangle$ is open (where $\|x\|_1 := \sum_{i=1}^d |x_i|$ is the ℓ^1-norm). In particular a good path can move back (according to the ℓ^1-norm) toward the origin without any constraint (and move away from the origin through open edges).

One of the main ideas of the present article is to modify the notion of good paths, asking for more constraints, therefore leading to a smaller family (of such good paths).

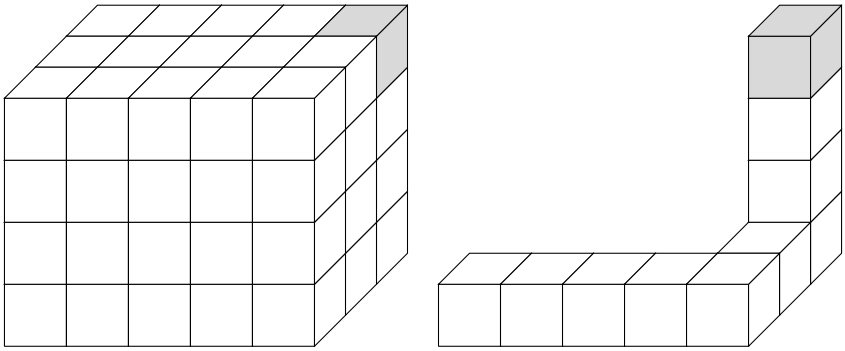

Figure 3: Starting from $(4,2,3)$, the sets of vertices potentially attained with only closed edges: previous definition of good paths and the current one.

Instead of taking the open edges union all the oriented edges pointing "toward" 0, we take the open edges union of the oriented edges pointing toward 0 only along the last non-null coordinate. That is to say, the path is good (in our sense) if for each i, either the edge $\langle \nu_{i-1}, \nu_i \rangle$ is open, or $\nu_i = \nu_{i-1} - e_j(\nu_{i-1})$ where j is the last non-null coordinate of ν_{i-1} and $e_j(v) = \text{sgn}(v_j)e_j$. One can see in figure 3 the difference between the two definitions of good paths. We will show in section 3 how good paths are related to the event $\mathcal{S}$.

As this will become clear after Section 3, in order to obtain the desired lower bound on $p_c^{\mathcal{S}}$ (of Theorem 2.1), one needs to bound the probability of the existence of good paths. In particular, one needs to show that there are not too many good paths of given length. We achieve this, by means of large deviations, in section 4, by carefully controlling the number of closed edges in any good path.

Finally, let us mention that the method developed in section 4 yields to a closed form lower bound on $p_c^{\mathcal{S}}$ in any dimension. This can however be improved numerically, a strategy that we achieve in section 5 and that relies on the study of a Markov chain together with the use of large deviations techniques, proving theorem 1.2. Notice that our procedure, to improve the bound on $p_c^{\mathcal{S}}$, applies to all dimensions as illustrated at the end of the section for $d = 4$ and $d = 5$.

3 Descending sets and spheres

We define in this section a class of finite sets of $\mathbb{Z}^d$, which we will be able to surround by a sphere. Roughly speaking, if $x = (x_1, \dots, x_d)$ is an element of such a set, then the segments $[(x_1, \dots, x_i, 0, \dots, 0), (x_1, \dots, x_{i-1}, 0, \dots, 0)]$ will be contained also in the set.

Definition 3.1. *For x in $\mathbb{R}^d \setminus \{0\}$, write $n(x)$ for the index of the last non-null coordinate of x. For $x, y \in \mathbb{R}^d$, write $y \preceq x$ if the following items hold:*

- $\forall i \in [1,d]$, $x_i y_i \geq 0$
- $\forall i \in [1,d]$, $|y_i| \leq |x_i|$
- $\forall i < n(y)$, $y_i = x_i$

We say that $K \subset \mathbb{Z}^d$ is a descending set if it is a finite set containing 0, with the property that if $x \in K$, then every $y \in \mathbb{Z}^d$ with $y \preceq x$ lies in K.

As an example of the first notion just defined, on the right panel of figure 3, the cubes corrrespond to all the vertices y such that $y \preceq (4,2,3)$.

We now give the adapted version of Proposition 3 of [8]:

Proposition 3.2. *Let $d \geq 2$. Suppose $K \subset \mathbb{Z}^d$ is a descending set. Let $\mathcal{E}$ be the bonds that have one endvertex in K and the other in K^c. Then there exists a sphere (a simplicial complex homeomorph to S^{d-1}) in $\mathbb{R}^d$ that intersects all the bonds of $\mathcal{E}$, and no other one.*

In order to prove this proposition, we shall use a certain homeomorphism on the surface of a hypercube. Consider the hypercube $[-1,1]^d$, and enumerate its $2d$ faces F_i by letting $F_i = [-1,1]^{i-1} \times \{1\} \times [-1,1]^{d-i}$ for $i \in [1,d]$, and $F_i = [-1,1]^{i-d-1} \times \{-1\} \times [-1,1]^{2d-i}$ for $i \in [d+1, 2d]$.

Lemma 3.3. *Let $\tilde{I} \subset \{1,2,\ldots,d-1,d+1,d+2,\ldots,2d-1\}$, $I = \tilde{I} \cup \{2d\}$, and J is the complementary of I in $[1,2d]$. Let $G = \bigcup_{i\in I} F_i$ and $H = \bigcup_{i\in J} F_i$. There exists a homeomorphism between G and H that is the identity on the intersection $G \cap H$.*

The set G contains the face on the bottom (the $2d$-th face), while H contains the face on the top (the d-th face). Using dilations and rotations, we will use this lemma on parallelepipeds and the other directions, not only the last one. The key element is that there are two opposite faces such that G and H contain each one of them.

Proof. We begin with a transformation of the hypercube $B = [-1,1]^d$. For I as in the lemma, $x = (x_1, \ldots, x_d) \in B$, let

$$f_I(x_1,\ldots,x_d) = (g_I^1(x_1,x_d), g_I^2(x_2,x_d), \ldots, g_I^{d-1}(x_{d-1},x_d), x_d),$$

with, for $i \in [1, d-1]$,

$$g_I^i(y,z) = \begin{cases} \frac{1}{4}(z+3)y & \text{if } \left| \begin{array}{l} i \in I \text{ and } y \geq 0 \\ \text{or} \\ i+d \in I \text{ and } y \leq 0 \end{array} \right. \\ \\ \frac{1}{4}(-z+3)y & \text{if } \left| \begin{array}{l} i \notin I \text{ and } y \geq 0 \\ \text{or} \\ i+d \notin I \text{ and } y \leq 0 \end{array} \right. \end{cases}$$

For $i \in [1,2d]$, define the half-space H_I^i by

$$\begin{aligned} H_I^d &= \{x \in \mathbb{R}^d \text{ such that } x_d \leq 1\} \\ H_I^{2d} &= \{x \in \mathbb{R}^d \text{ such that } x_d \geq -1\} \\ &\quad \text{and for } i \text{ different from } d \text{ and } 2d: \\ H_I^i &= \{x \in \mathbb{R}^d \text{ such that } x_i \leq \frac{1}{4}(x_d+3)\} \text{ if } i \leq d, i \in I \\ H_I^i &= \{x \in \mathbb{R}^d \text{ such that } x_{i-d} \geq -\frac{1}{4}(x_d+3)\} \text{ if } i > d, i \in I \\ H_I^i &= \{x \in \mathbb{R}^d \text{ such that } x_i \leq \frac{1}{4}(-x_d+3)\} \text{ if } i \leq d, i \notin I \\ H_I^i &= \{x \in \mathbb{R}^d \text{ such that } x_{i-d} \geq -\frac{1}{4}(-x_d+3)\} \text{ if } i > d, i \notin I \end{aligned}$$

One can show that f_I is a homeomorphism from B to $\tilde{B}_I = \bigcap_{i=1}^{2d} H_I^i$. To define the inverse application of f_I, one would use $\frac{4}{z+3}y$ and $\frac{4}{-z+3}y$ in replacement of the definition of g_I^i.

The set $\tilde{B}_I$ is a convex polyhedron. For $i \in [1,2d]$, we denote by $\tilde{F}_i$ the face of $\tilde{B}_I$ included in H_I^i, face which can be showed to be the image of F_i by f_I. For each $i \in I$, the outer vector of $\tilde{F}_i$ points downwards according to the last coordinate, whereas, for $i \notin I$, the outer vector of $\tilde{F}_i$ points upwards. Now there is a homeomorphism from $\tilde{G} = \bigcup_{i\in I} \tilde{F}_i$ to $\tilde{H} = \bigcup_{i\notin I} \tilde{F}_i$, simply by taking the projection of $\tilde{G}$ onto $\tilde{H}$ along the last dimension. This projection corresponds to the identity on the intersection $\tilde{G} \cap \tilde{H}$. Applying now the inverse of f_I, we obtain a homoemorphism between G and H which is the identity on their intersection, as illustrated on figure 4. □

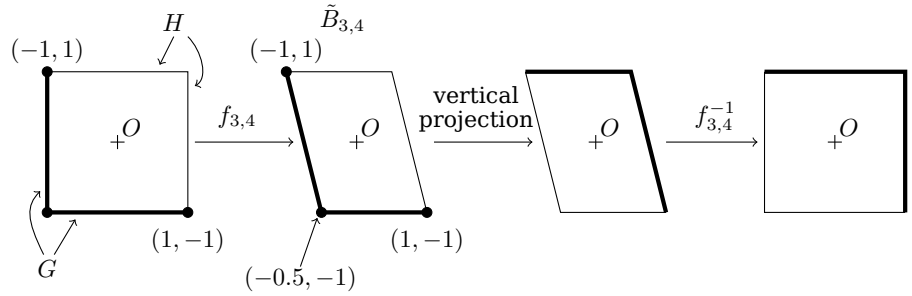

Figure 4: Homeomorphism between two sets of faces.

Proof of Proposition 3.2. For the ease of the exposition, we restrict ourselves to the case where $K \subset \mathbb{Z}^d_+$. Define a sequence $(K_i)_{0\le i\le d}$ by

$$K_i = \{x \in K : x_j = 0 \; \forall j > i\}. \tag{3.1}$$

We will build an increasing sequence of volume $(A_i)_{0\le i\le d}$ such that for each i, $A_i \cap \mathbb{Z}^d = K_i$. In order to achieve this, we will also use for each coordinate i two sequences $(K_{i,n})_{n\le n_i}$ and $(A_{i,n})_{i\le n_i}$ which will be the transitions between i and $i+1$. These sequences will satisfy

$$\begin{aligned} A_{i,n} \cap \mathbb{Z}^d &= K_{i,n} && (3.2)\\ K_{i,0} &= K_{i-1} && (3.3)\\ A_{i,0} &= A_{i-1} && (3.4)\\ K_{i,n_i} &= K_i && (3.5)\\ A_{i,n_i} &= A_i && (3.6) \end{aligned}$$

So we have $K_0 = \{0\}$, and we take

$$A_0 = [-0.4, 0.4]^d.$$

Note that the origin is in the interior of A_0. For $x \in \mathbb{Z}^d$ and $i \in [1, d]$, we shall make use of the boxes

$$B(x,i) = x + [-0.4, 0.4]^{i-1} \times [-0.6, 0.4] \times [-0.4, 0.4]^{d-i}.$$

We will start the following procedure with $i = 1$ and $n = 0$.

Let $K_{i,n}$ and $A_{i,n}$ be fixed. Define

$$Y = \{x \in K_i : x_i = n + 1\}.$$

If Y is not empty, we let

$$K_{i,n+1} = K_{i,n} \cup Y,$$

and

$$A^*_{i,n} = A_{i,n} \cup \{B(x,i) \text{ for } x \in Y\}.$$

For x, y distinct vertices in Y, $B(x,i)$ and $B(y,i)$ do not intersect. Lets take $Y' \subset Y$, $Y' \neq Y$ and $x \in Y \setminus Y'$. The intersection between $B(x,i)$ and $A_{i,n} \cup \{B(y,i) \text{ for } y \in Y'\}$ is simply $B(x,i) \cap A_{i,n}$. Since, by definition of a descending set, $x - e_i$ is in $K_{i,n}$, we have $B(x - e_i, i) \subset A_{i,n}$ and

$$B(x,i) \cap A_{i,n} = x + [-0.4, 0.4]^{i-1} \times \{-0.6\} \times [-0.4, 0.4]^{d-i},$$

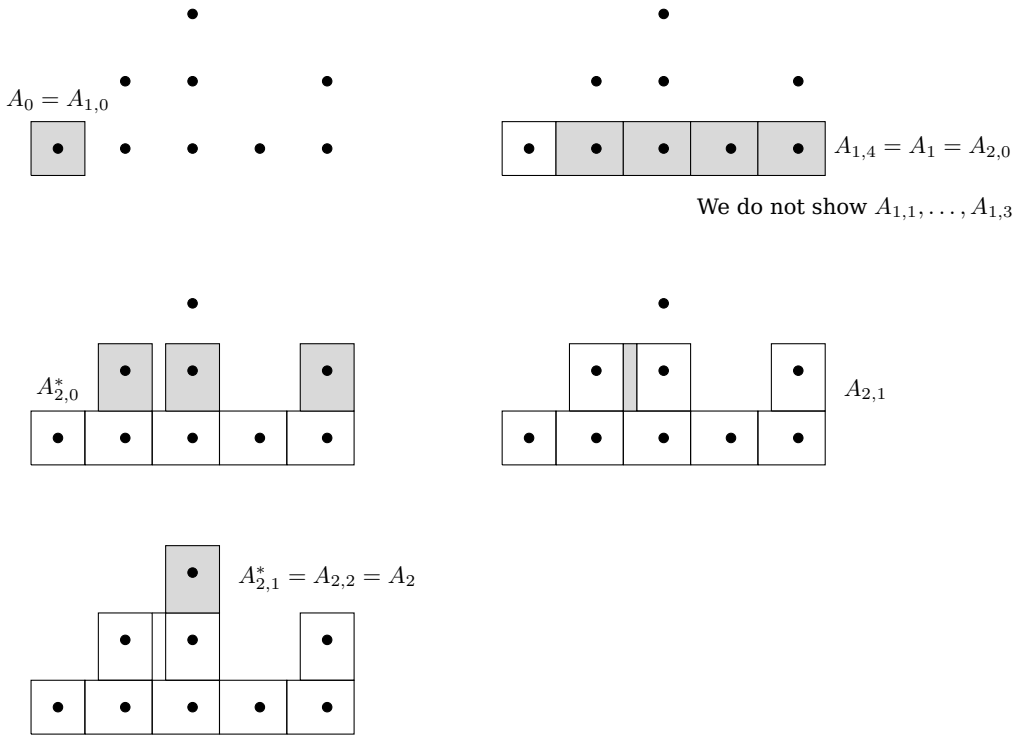

Figure 5: Example for the sequences (A_i), $(A^*_{i,n})$ and $(A_{i,n})$ in dimension two.

which is the $(i+d)$th face of $B(x,i)$. By lemma 3.3, there is a homeomorphism between this face of $B(x,i)$ to the union of its other faces, homeomorphism that is the identity on the intersection of these two sets of faces. So each time we add a box $B(x,i)$ with $x \in Y$, the surfaces of the sets remain homeomorph, and by iteration $\partial A^*_{i,n}$ is homeomorph to $\partial A_{i,n}$. However the set $A^*_{i,n}$ does not fill all our requirements, as its surface intersects the bonds between neighbour vertices of Y. So we have to enhance this set before obtaining $A_{i,n+1}$.

A representation of the set of neighbour vertices in Y is

$$\Gamma = \{(x,k), x \in Y, k \in [1, i-1] \text{ such that } x + e_k \in Y\},$$

where we have used the fact that for all vertex x in Y, x_i is constant (and equals to $n+1$), and $x_j = 0$ for $j > i$. For $(x,k) \in \Gamma$, let

$$\begin{aligned} B(x,k,i) \quad = \quad & x + [-0.4, 0.4]^{k-1} \times [0.4, 0.6] \\ & \times [-0.4, 0.4]^{i-k-1} \times [-0.6, 0.4] \times [-0.4, 0.4]^{d-i}, \end{aligned}$$

and define

$$A_{i,n+1} = A^*_{i,n} \cup \bigcup_{(x,k)\in\Gamma} B(x,k,i).$$

The box $B(x,k,i)$ will serve as a bridge between $B(x,i)$ and $B(x+e_k,i)$. One can see an example of it on the fourth panel of figure 5, at the step $A_{2,1}$. For (x,k), (x',k') two distinct elements of Γ, $B(x,k,i)$ and $B(x',k',i)$ do not intersect. Lets take $\Gamma' \subset \Gamma$, $\Gamma' \neq \Gamma$, $(x,k) \in \Gamma \setminus \Gamma'$, and define

$$A^{*,\Gamma'}_{i,n} = A^*_{i,n} \cup \bigcup_{(x',k')\in\Gamma'} B(x',k',i).$$

Due to the preceding remark,

$$\begin{aligned} A^{*,\Gamma'}_{i,n} \cap B(x,k,i) &= A^{*}_{i,n} \cap B(x,k,i) \\ &= x + \big([-0.4,0.4]^{k-1} \times [0.4,0.6] \\ &\quad \times [-0.4,0.4]^{i-k-1} \times \{-0.6\} \times [-0.4,0.4]^{d-i} \\ &\quad \cup [-0.4,0.4]^{k-1} \times \{0.4\} \times [-0.4,0.4]^{i-k-1} \\ &\quad \times [-0.6,0.4] \times [-0.4,0.4]^{d-i} \\ &\quad \cup [-0.4,0.4]^{k-1} \times \{0.6\} \times [-0.4,0.4]^{i-k-1} \\ &\quad \times [-0.6,0.4] \times [-0.4,0.4]^{d-i}\big) \end{aligned}$$

The intersection was decomposed on $A_{i,n} \cap B(x,k,i)$, $B(x,i) \cap B(x,k,i)$ and $B(x+e_k,i) \cap B(x,k,i)$. We can apply lemma 3.3 again, implying that the surface of $A^{*,\Gamma'}_{i,n} \cup B(x,k,i)$ is homeomorph to $A^{*,\Gamma'}_{i,n}$, and by iteration $A_{i,n+1}$ is homeomorph to $A^{*}_{i,n}$.

If Y is empty, we let $n_i = n$, and by definition of a descending set, we have indeed $K_{i,n_i} = K_i$. If $i < d$, we follow the same instructions, simply incrementing i to $i+1$ and resetting n to 0. If $i = d$, then the algorithm is finished. On figure 5 one can see that for $A^{*}_{i,n}$ we add boxes around the vertices just above $A_{i,n}$, and then we fill the gapes to get $A_{i,n+1}$.

At the end of the previous algorithm, we have that A_d contains $K_d = K$. Since $S_{1,0} = \partial A_0$ is homeomorph to a sphere, by an immediate recurrence $S := \partial A_d$ is homeomorph to a sphere. Let us consider two neighbour vertices x and y in K, take i the smallest integer such that the two vertices are in K_i, and suppose to simplify that the coordinates of x are smaller than the ones of y. There are three cases:

1. If $x_i = 0$, then $x \in K_j$ for a certain $j < i$. In this case, the boxes $B(x,j)$ and $B(y,i)$ are in A_d, and the bond $\langle x,y \rangle$ is contained in the union of these two boxes.
2. If $x_i > 0$ and $x_i < y_i$, then the boxes $B(x,i)$ and $B(y,i)$ are in A_d, and the bond $\langle x,y \rangle$ is contained in the union of these two boxes.
3. If $x_i > 0$ and $x_i = y_i$, then the boxes $B(x,i)$, $B(y,i)$ and $B(x,i,x_i)$ are in A_d, and the bond $\langle x,y \rangle$ is contained in the union of these three boxes.

Hence the surface of A_d does not intersect bonds relying two vertices of K. Since $d_\infty(A_d, K) < 1$, the surface does not intersect bonds between vertices that are both outside K. To conclude, the surface of A_d, which is homeomorph to a sphere, intersects only bonds that have one endvertex in K and the other outside K. □

4 Good paths

This section finishes the proof of theorem 2.1. We give a definition for *good paths* which will generate more paths than just the open paths, and such that, according to the previous section, the set attained from the origin will be enclosed in a sphere intersecting only its outer bonds, these bonds being closed.

Definition 4.1. *A path $(0 = \nu_0, \nu_1, \dots, \nu_k)$ in $\mathbb{Z}^d$ is called a good path if for every i, $1 \le i \le k-1$, either the edge $\langle \nu_{i-1}, \nu_i \rangle$ is open, or $\nu_i \preceq \nu_{i-1}$.*

From this definition and proposition 3.2, we obtain as in [8]:

Lemma 4.2. *Let K be the random set of vertices x such that there exists a good path from 0 to x. If K is finite, there exists a sphere intersecting only closed bonds and containing 0 in its inside.*

Proof. By definition 4.1, all the bonds in $\mathcal{E}$ (as defined in proposition 3.2) are closed, and so this lemma is a consequence of proposition 3.2. □

Proof of Theorem 2.1. Let $r > 0$ be an integer, and $N_p(r)$ the number of good paths that start at 0 and end on $\{x \in \mathbb{Z}^d : \|x\|_1 = r\}$. Then

$$P(\mathrm{rad}[K] \geq r) \leq \mathbb{E}_p(N_p(r)).$$

For any good path π with vertices $0, \nu_1, \ldots, \nu_n = u$ with $\|u\|_1 = r$, we say that π has length n and we let

$$\begin{aligned} A &= \#\{i : \langle \nu_{i-1}, \nu_i \rangle \text{ is not descending, that is } \nu_i \not\preceq \nu_{i-1}\} \\ B &= \#\{i : \langle \nu_{i-1}, \nu_i \rangle \text{ is descending, that is } \nu_i \preceq \nu_{i-1}\}. \end{aligned}$$

As $n - r$ is even, we can let m be the integer such that $n = r + 2m$, and we have the following:

$$\begin{aligned} A + B &= n \\ B &< \frac{n}{2}. \end{aligned}$$

Remark that once we know r, m and B, the values of n and A are determined. Let M be a large even integer to be precised later. We decompose the set of paths as follows:

$$\mathbb{E}(N_p(r)) \leq \sum_{m \geq 0} \sum_{i=0}^{M/2-1} \sum_{\substack{B \geq \frac{i}{M}(r+2m) \\ B < \frac{i+1}{M}(r+2m)}} N(A,B) p^A.$$

Here $N(A,B)$ is the number of self-avoiding paths having (A,B) for characteristics. With the second and the third summation, B runs through the interval $[0, n/2[$. When $B < \frac{i+1}{M}(r+2m)$, we have

$$A > r + 2m - \frac{i+1}{M}(r+2m),$$

and so

$$p^A < p^{(r+2m)(1-\frac{i+1}{M})}. \tag{4.1}$$

Now to provide an upper bound on $N(A,B)$, we simply consider the paths of length $A+B$ that cannot return immediately to the previous vertex (hence $2d-1$ choices after the first) and with B descending steps, that is to say B edges $\langle \nu_{i-1}, \nu_i \rangle$ such that $\nu_i \preceq \nu_{i-1}$. Let G_n be the set of paths of length n that do not return immediately to the previous vertex, and for $\alpha \in [0, 0.5]$, let $G_n(\alpha)$ the subset of G_n of the paths having at least αn descending steps. We have

$$\#G_n = 2d(2d)^{n-1} \leq 2(2d-1)^n. \tag{4.2}$$

Recall that $n = A + B$, so

$$\#G_n(B/n) \geq N(A,B). \tag{4.3}$$

For a path π in G_n with vertices $0, \nu_1, \ldots, \nu_n$, define the variables $(Y_i)_{i=1,\ldots,n}$ by

$$Y_i = \begin{cases} 1 & \text{if } \langle \nu_{i-1}, \nu_i \rangle \text{ is descending} \\ 0 & \text{otherwise.} \end{cases}$$

We will always have $Y_1 = 0$. Let another sequence of variable $(Z_i)_{i=1,\ldots,n}$, independent of the Y_i's, distributed independently according to a Bernoulli of parameter $\frac{1}{2d-1}$. Consider

that π was chosen at random and uniformly in G_n. At each step after the first, the path π has $2d-1$ equally probable possibilities, among which at most one will give a bad step. Hence for each i in $[1,n]$,

$$P(Y_i = 1 \mid Y_1, \dots, Y_{i-1}) \leq \frac{1}{2d-1}.$$

We can use a coupling between (Y_i) and (Z_i) via uniform variables (as one does to compare two binomials) and then apply the Cramer-Chernov large deviations on (Z_i) (see for example [1]). Hence, for $\alpha > 1/(2d-1)$,

$$P\left(\sum_{i=1}^{n} Y_i \geq \alpha n\right) \leq P\left(\sum_{i=1}^{n} Z_i \geq \alpha n\right) \leq \exp -nH, \tag{4.4}$$

with

$$\begin{aligned} H &= \alpha \log \alpha + (1-\alpha)\log(1-\alpha) + \alpha \log(2d-1) - (1-\alpha)\log\left(1 - \frac{1}{2d-1}\right) \\ &\geq -\log(2) + \alpha \log(2d-1) + (1-\alpha)\log\left(\frac{2d-1}{2d-2}\right) \\ &= -\log(2) + \log(2d-1) - (1-\alpha)\log(2d-2) \end{aligned} \tag{4.5}$$

Using the lower bound (4.5) instead of H, inequality (4.4) stands for all $\alpha \in [0, 0.5]$ (and is trivial for $\alpha \leq 1/(2d-1)$ since in that case the lower bound is negative). With (4.2), this gives for all i in $[0, M/2-1]$,

$$\#G_n\left(\frac{i}{M}\right) \leq 2^{n+1}(2d-2)^{n(1-\frac{i}{M})} \tag{4.6}$$

which, with (4.1) and (4.3), implies

$$\begin{aligned} \mathbb{E}(N_p(r)) &\leq \sum_{m\geq 0}\sum_{i=0}^{M/2-1}\left(\frac{1}{M}(r+2m)+1\right)2^{r+2m+1} \\ &\qquad \times (2d-2)^{(r+2m)(1-\frac{i}{M})} p^{(r+2m)(1-\frac{i+1}{M})} \\ &= \sum_{m\geq 0}\sum_{i=0}^{M/2-1}\left(\frac{1}{M}(r+2m)+1\right)2^{r+2m+1} \\ &\qquad \times \left((2d-2)^{\frac{M-i}{M-i-1}} p\right)^{(r+2m)(1-\frac{i+1}{M})} \end{aligned}$$

Now fix the dimension d, and take p such that $p < \frac{1}{8(d-1)}$. Let M be an even integer large enough such that

$$(2d-2)^{1+\frac{2}{M-2}} < \frac{1}{4p}.$$

To simplify calculations, we let $b = (2d-2)^{1+\frac{2}{M-2}} p$. We obtain

$$\begin{aligned} \mathbb{E}(N_p(r)) &\leq \sum_{m\geq 0}\sum_{i=0}^{M/2-1}\left(\frac{1}{M}(r+2m)+1\right)2^{r+2m+1} \\ &\qquad \times b^{(r+2m)(1-\frac{i+1}{M})} \\ &= \sum_{m\geq 0}\left(\frac{1}{M}(r+2m)+1\right)2^{r+2m+1} b^{r+2m} \end{aligned}$$

$$\begin{aligned} & \times \sum_{i=0}^{M/2-1} b^{-(r+2m)\frac{i+1}{M}} \\ \leq & \sum_{m\geq 0}\left(\frac{1}{M}(r+2m)+1\right)2^{r+2m+1}b^{r+2m}b^{-r/2-m}\frac{b}{1-b} \\ = & \frac{b}{1-b}2^{r+1}\sum_{m\geq 0}\left(\frac{1}{M}(r+2m)+1\right)4^m b^{r/2+m} \end{aligned}$$

For r large enough such that $(2d-2)^{1+\frac{2}{r-1}} < \frac{1}{4p}$, we can take $M=r$ when r is even, and $M=r-1$ when r is odd, and we obtain

$$\begin{aligned} \mathbb{E}(N_p(r)) &\leq \frac{b^{\frac{r}{2}+1}}{1-b}2^{r+1}\frac{2}{r-1}\sum_{m\geq 0}(r+2m)4^m b^m \\ &= \frac{4b}{(1-b)(1-4b)}\left(1+\frac{1+4b}{(r-1)(1-4b)}\right)(2\sqrt{b})^r, \end{aligned}$$

which converges exponentially fast towards 0 since we have taken $b<\frac{1}{4}$. This gives the exponential bound (2.2) on the radius of the sphere of theorem 2.1. By the first Borel-Cantelli lemma, the set of vertices attained by good paths from the origin is a.s. finite, and we get the lower bound (2.1) on the critical point p_c^S with the help of lemma 4.2. □

5 Improvement via large deviations on a Markov chain

Theorem 2.1 already gives as a corollary that $p_e^1 \geq 1/16$. We can improve this lower bound by studying more precisely the cardinal of $G_n(\alpha)$ with the help of a Markov chain. Lets first define a chain with three states, W_1, W_2 and W_3. For any site x in $\mathbb{Z}^d \setminus \{0\}$, we recall that its descending edge is the edge $\langle x, x-\mathrm{sgn}(x_{n(x)})e_{n(x)}\rangle$ (as usual $n(x)$ is the index of the last non-null element for x). If $x=0$, there is no descending edge. Furthermore, we call an edge e an ascending edge if $-e$ is the descending edge of $x+e$. If $n(x)\neq d$, there is more than one ascending edge. Actually, all edges (and their opposites) after $e_{n(x)}$ are ascending edges. In particular, if $x=0$, all the edges are ascending. An edge that is neither ascending nor descending is called a neutral edge.

For an infinite immediate self-avoiding walk $(Z_i)_{i\geq 0}$, that is a path that cannot return immediately to its previous site, with $Z_0=0$, consider its ith edge u_i and define $(\tilde{X}_i)_{1\leq i}$ by

- $\tilde{X}_i = W_1$ is u_i is a neutral edge.
- $\tilde{X}_i = W_2$ if u_i is an ascending edge.
- $\tilde{X}_i = W_3$ if u_i is the descending edge.

The sequence $(\tilde{X}_i)$ is not Markovian (one would have to add the current position of the path to get a Markovian couple). Define now a Markov chain, denoted (X_i), also on the three states W_1, W_2 and W_3, and which will be related to $(\tilde{X}_i)$. The initial state X_1 is taken to W_2 (although it is not important), and the transition matrix of (X_i) is taken equal to:

$$\pi = \begin{pmatrix} \frac{2d-3}{2d-1} & \frac{1}{2d-1} & \frac{1}{2d-1} \\ \frac{2d-2}{2d-1} & \frac{1}{2d-1} & 0 \\ \frac{2d-2}{2d-1} & 0 & \frac{1}{2d-1} \end{pmatrix}$$

To get a better understanding of the similarity between these two chains, we describe the general behaviour of $(\tilde{X}_i)$ when the current vertex of the path is not on the hyperplane

$x_d = 0$. Once in state W_1, there are $2d-3$ edges that let $\tilde{X}_{i+1}$ in state W_1, one edge setting $\tilde{X}_{i+1}$ in state W_2 and one edge setting $\tilde{X}_{i+1}$ in state W_3. Once in state W_2, there is one edge, the same as the preceding step, that let $\tilde{X}_{i+1}$ in state W_2, and $2d-2$ setting $\tilde{X}_{i+1}$ in state W_1. Finally, if $\tilde{X}_i$ is in state W_3, there is one edge that let $\tilde{X}_{i+1}$ in state W_3 and $2d-2$ edges setting $\tilde{X}_{i+1}$ in state W_1.

So the sequence $(\tilde{X}_i)$ seems to have the same law as (X_i). Unfortunately this is not the case. When the last edge used by $(\tilde{X}_i)$ is $-e_d$, that the last coordinate of the corresponding vertex is null and the penultimate is strictly positive, there are at least two possibilities for X_{i+1} to be in state W_2, namely e_{d-1} and $-e_d$, and one to be in state W_3, namely $-e_{d-1}$.

Hence the sequences (X_i) and $(\tilde{X}_i)$ are not identical in law, but it is possible to define a coupling between the random path and (X_i) with the property that if $\tilde{X}_i$ is in state W_1, then X_i is in state W_1 or W_3, and if $\tilde{X}_i$ is in state W_3, then X_i is in state W_3. As a consequence, the time spent in the state W_3 is greater or equal for (X_i) than for $(\tilde{X}_i)$.

We use an i.i.d. sequence $(U_i)_{i\geq 2}$ of uniform random variables on $[0,1]$. For $i \geq 2$, we let $a_1(i) = P(X_i = W_1 \mid X_{i-1})$ and $a_2(i) = P(X_i = W_2 \mid X_{i-1})$. These quantities are actually random variables. We recall that we had arbitrarily taken $X_1 = W_2$. Now we apply the following rules:

- If $U_i < a_2(i)$, we set X_i in the state W_2.
- If $U_i \in [a_2(i), a_2(i) + a_1(i)[$, we set X_i in the state W_1.
- Otherwise, we set X_i in the state W_3.

Concerning the random path, always for $i \geq 2$, we let $\tilde{a}_1(i) = P(\tilde{X}_i = W_1 \mid Z_{i-2}, Z_{i-1})$ and $\tilde{a}_2(i) = P(\tilde{X}_i = W_2 \mid Z_{i-2}, Z_{i-1})$. We recall that $Z_0 = 0$ and that we always have $\tilde{X}_1 = W_2$. The path chooses for its first step a random edge taken uniformly among the $2d$ possibilities. For the subsequent steps, the rules are:

- If $U_i < \tilde{a}_2(i)$, the path takes uniformly one of the ascending edges. This implies that $\tilde{X}_i$ is in the state W_2.
- If $U_i \in [\tilde{a}_2(i), \tilde{a}_2(i)+\tilde{a}_1(i)[$, the path takes uniformly one of the neutral edges. Hence $\tilde{X}_i$ is in the state W_1.
- Otherwise the path takes the descending edge, and so $\tilde{X}_i$ is in the state W_3.

In that way we have a coupling between the random path and the Markov chain (X_i). We prove now by recurrence the two following properties: if X_i is in state W_2, so is $\tilde{X}_i$, and if $\tilde{X}_i$ is in state W_3, so is X_i. These properties are true for $i = 1$ since X_1 and $\tilde{X}_1$ are in state W_2. Suppose they are true at step $i-1$. The possible configurations for the couple $(\tilde{X}_{i-1}, X_{i-1})$ are (W_2, W_2), (W_2, W_1), (W_2, W_3), (W_1, W_1), (W_1, W_3) and (W_3, W_3). In all these cases, we have $a_2(i) \leq \tilde{a}_2(i)$ and $a_2(i) + a_1(i) \leq \tilde{a}_2(i) + \tilde{a}_1(i)$. So, according to the coupling described, if X_i is in state W_2, that means $U_i < a_2(i)$, and so $\tilde{X}_i$ is equally in state W_2. If $\tilde{X}_i$ is in state W_3, that means $U_i \geq \tilde{a}_2(i) + \tilde{a}_1(i)$, and so X_i is also in state W_3, and the two properties hold by recurrence. As previously claimed, we obtained a coupling between the path and (X_i), with (X_i) spending more time in W_3 than $(\tilde{X}_i)$.

We now use large deviations techniques on the Markov chain (X_i), as explained in sections 3.1.1 and 3.1.3 of [4]. If a path of length n coupled to the Markov chain $(X_i)_{1\leq i\leq n}$ has at least a proportion of α descending edges, then

$$\sum_{i=1}^{n} \mathbb{1}_{X_i = W_3} \geq \alpha n.$$

This event is controlled by large deviations, the rate function being the infinimum, with respect to the parameters a, b, c and x, of the entropy of distributions of the type

$$q = \begin{pmatrix} a & b & x \\ b & c & 0 \\ x & 0 & \alpha - x \end{pmatrix}$$

with respect to π. The matrix q is taken of this form since when $\pi(i,j)$ is null, $q(i,j)$ must also be null, and for each $j = 1, 2, 3$, the sum of the j-th line must be equal to the sum of the j-th column. As this matrix is a representation of a distribution, we have the constraint $a + 2b + c + x + \alpha = 1$, and all the elements of the matrix are of course positive. The formula of the entropy is

$$H(q,\pi) \quad = \quad \sum_{i=1}^{3}\sum_{j=1}^{3} q(i,j) \log \frac{q(i,j)}{q_1(i)\pi(i,j)},$$

with $q_1(i) = \sum_{j=1}^{3} q(i,j)$. This gives

$$\begin{aligned} H(q,\pi) \quad = \quad & a \log\left(\frac{a(2d-1)}{(2d-3)(a+b+x)}\right) + b\log\left(\frac{b(2d-1)}{a+b+x}\right) \\ & + x\log\left(\frac{x(2d-1)}{a+b+x}\right) \\ & + b\log\left(\frac{b(2d-1)}{(b+c)(2d-2)}\right) + c\log\left(\frac{c(2d-1)}{b+c}\right) \\ & + x\log\left(\frac{x(2d-1)}{\alpha(2d-2)}\right) + (\alpha - x)\log\left(\frac{(\alpha-x)(2d-1)}{\alpha}\right). \end{aligned}$$

We let $\sigma_d(\alpha)$ be the infinimum of these entropies, and denote $\sigma_d = \sigma_d(0.5)$, as this particular value will appear important. Large deviations results on Markov chains imply that

$$G_n(\alpha) \quad \leq \quad 2 \cdot (2d-1)^n \cdot \exp(-n\sigma_d(\alpha)). \tag{5.1}$$

We searched a solution for $\sigma_d(\alpha)$ with the three variables b, c and x, but the derivatives yield a non-linear system of three equations with three variables, that we couldn't solve. It is however possible to get numerically a lower bound for $\sigma_d(\alpha)$.

We finish to explain now the procedure for the dimension 3. In this case, the transition matrix is

$$\pi = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} & 0 \\ \frac{4}{5} & 0 & \frac{1}{5} \end{pmatrix}$$

As an example, consider the value $\alpha = 0.5$. We let $f(x,b,c)$ the function for which we search a lower bound. For a block $[x_1,x_2] \times [b_1,b_2] \times [c_1,c_2]$, we can get a lower bound for f using either

- the monotony of its parts. For example $x\log(x) \geq x_2\log(x_2)$ if $x_2 \leq \exp(-1)$, $x\log(x) \geq x_1\log(x_1)$ if $x_1 \geq \exp(-1)$, and $x\log(x) \geq -\exp(-1)$ in the third case;
- the value of $f(x_1,b_1,c_1)$ and a lower bound of the negative parts of the gradient of f on the block;
- the value of f and its gradient at the point (x_1,b_1,c_1), together with a lower bound of the negative parts of the Hessian of f on the block.

Starting with the block $[0, 0.5] \times [0, 0.25] \times [0, 0.5]$ which covers the set of definition of f, we calculate the best lower bound among the three possibilities just described. If the lower bound is less than 0.24857770256 (a candidate value obtained with gradient search), we split the block in two, cycling over the axes x, b and c, and we reiterate the procedure. With this method, we effectively obtain that $\sigma_3 \geq 0.24857770256$. We note that it is a good approximation, since we have $f(0.24582, 0.035321, 0.005248) = 0.2485777026...$ With (5.1), this yields

$$G_n(0.5) \leq 2 \cdot 5^n \exp(-0.24857770256n) \leq 2 \cdot 3.899546288^n,$$

to compare with $G_n(0.5) \leq 2 \cdot 4^n$ of the previous section. We point out that the second method with a lower bound on the gradient was hardly used by the algorithm. The first method is adapted when we are near the border of the set of definition of f, whereas the third method is adapted when we are near the optimal value.

Now we shall choose a finite strictly increasing sequence $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k = 0.5$, to which we associate

$$L \quad = \quad \max\{5 \exp(-\sigma_3(\alpha_{i-1}))/(2 \cdot 4^{1-\alpha_i}) : i \in [2, k]\}$$

We build the sequence (α_i) in the reverse order. So starting from 0.5, we begin with 1000 elements with a step of 10^{-13}, then sequences of 900 elements with steps ranging from 10^{-12} to 10^{-5}, and finally 81 with a step of 10^{-4}, leading to $\alpha_1 = 0.32$, and we complete with $\alpha_0 = 0$. With this sequence, we are able to verify for each i in $[2, k]$, as in the case $\alpha = 0.5$, that with $L_0 := 0.974886571911$,

$$\sigma_3(\alpha_{i-1}) \geq \log(5/2) + (\alpha_i - 1)\log(4) - \log(L_0), \tag{5.2}$$

implying $L \leq L_0$. As before, this value is a good approximation of the true maximum, since L is bounded from below by $5 \exp(-\sigma_3)/4$, which is greater than 0.97488657191. Note that when i is small, the algorithm needs coarser blocks than when i is near k (that is to say α_i near 0.5), and the maximum for the definition of L corresponds certainly to the index k. A way to optimize the algorithm is then to remark that the partition used for $\alpha = 0.5$ is certainly sufficient for all others α. So instead of considering separately the different α_i, the algorithm seeks a partition sufficient for all the α_i together.

For each $i \geq 2$, inequality (5.2) implies that

$$5 \exp(-\sigma_3(\alpha_{i-1})) \leq L_0 \cdot 2 \cdot 4^{1-\alpha_i},$$

and so with (5.1) and the monotony on α, for all $\alpha \in [\alpha_{i-1}, \alpha_i]$,

$$G_n(\alpha) \leq G_n(\alpha_{i-1}) \leq 2 \cdot (L_0 \cdot 2 \cdot 4^{1-\alpha_i})^n \leq 2 \cdot (L_0 \cdot 2 \cdot 4^{1-\alpha})^n. \tag{5.3}$$

The inequality between the first and the last member is also valid for $\alpha \in [\alpha_0, \alpha_1]$ since $G_n(\alpha) \leq 2 \cdot 5^n$ for all α, and $L_0 \cdot 2 \cdot 4^{1-\alpha_1} > 5$. Now we use (5.3) in replacement of the bound in (4.6), so in each line 2^{r+2m+1} becomes $2 \cdot (2L_0)^{r+2m}$, yielding to

$$p_1^e = p_c^{\mathcal{S}} \geq \frac{1}{16 L_0^2} \geq 0.065761519632,$$

and theorem 1.2 is proved. □

For the other dimensions, we can choose similar sequences of the α_i's to improve the lower bound of theorem 2.1, the general formula for L being

$$L = \max\{(2d-1)\exp(-\sigma_d(\alpha_{i-1}))/(2 \cdot (2d-2)^{1-\alpha_i}) : i \in [2, k]\}.$$

In dimension 4, we were able to obtain $p_c^{\mathcal{S}} \geq 0.04322$, and in dimension 5, $p_c^{\mathcal{S}} \geq 0.03214$, to compare with the respective previous values of $1/24 = 0.041666\ldots$ and $1/32 = 0.03125$.

The preceding gives improved numerically lower bounds on $p_c^{\mathcal{S}}$, but not in a closed form, about which we discuss in the following. It seems plausible that with an infinitely small partition (α_i), particularly near 0.5, the value of L would be given for "$\alpha_{i-1} = \alpha_i = 0.5$", that is

$$L = \frac{(2d-1)\exp(-\sigma_d)}{2\sqrt{2d-2}}.$$

Assuming one could prove this value satisfies $L < 1$, and since we still have

$$p_c^{\mathcal{S}} \geq \frac{1}{8(d-1)L^2},$$

then the following conjecture would follow:

Conjecture 5.1. *For all dimension $d \geq 3$,*

$$p_c^{\mathcal{S}} \geq \frac{\exp(2\sigma_d)}{(2d-1)^2} > \frac{1}{8(d-1)}.$$

The values obtained in dimensions 3 to 5 seem to indicate that the two members on the middle and on the right may be asymptotically equivalent.

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