

# Large deviations for Gibbs ensembles of the classical Toda chain\*

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## Abstract

We prove large deviation principles for the distribution of the empirical measure of the eigenvalues of Lax matrices following the Generalized Gibbs ensembles of the classical Toda chain introduced in [9]. We deduce the almost sure convergence of this empirical measure towards a limit which we describe in terms of the limiting empirical measure of Beta-ensembles. Our results apply to general smooth potentials.

**Keywords:** Toda chain; generalized Gibbs ensemble; random matrices; large deviations; beta ensembles; empirical measure of eigenvalues.

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## 1 Introduction

In a breakthrough paper [9], Herbert Spohn introduced the generalized Gibbs ensembles of the classical Toda chain as invariant measures of the dynamics of the classical Toda lattice. He analyzes them by comparing the Toda Lax matrices for these Generalized Gibbs ensembles with Dumitriu-Edelman tri-diagonal representations of  $\beta$ -ensembles. Thanks to this beautiful comparison, [9] showed that the empirical measure of the eigenvalues of Toda Lax matrices converges towards a probability measure related with the equilibrium measure for  $\beta$ -ensembles. One of the key tools of Herbert Spohn analysis is the use of transfer matrices, which are restricted to polynomial potentials. We refer the interested reader to subsequent developments in [7, 8, 10] and [5] where the transfer matrix approach is used in the similar context of the so-called Ablowitz-Laddik lattice.

The main goal of this article is to generalize some of the results of [9] by using large deviations theory, which allows to consider more general potentials. More precisely, we will show the convergence of the free energy and of the empirical measure of the eigenvalues of Toda Lax matrices following these Generalized Gibbs ensembles. Moreover, we will express the limits in terms of the well known  $\beta$ -ensembles. Indeed, a key tool is again to compare the Toda Lax matrices with Dumitriu-Edelman tri-diagonal representations of  $\beta$ -ensembles. Moreover, we will derive large deviation principles for the empirical measure of the eigenvalues of tri-diagonal matrices with more general coefficients. However, in this generality, the rate functions and the limits will not be explicit as the comparison with  $\beta$ -ensembles is not possible.

More precisely, the Hamiltonian of the Toda chain on sites  $j = 1, \dots, N$  is given by

$$H = \sum_{j=1}^N \left( \frac{1}{2} p_j^2 + e^{-r_j} \right), \quad r_j = q_{j+1} - q_j$$

with the periodic conditions  $q_{N+j} = q_j + cN$  for some real constant  $c$ . The equations of motion are then given by

$$\frac{d}{dt} q_j = p_j, \quad \frac{d}{dt} p_j = e^{-r_{j-1}} - e^{-r_j}. \tag{1}$$

Let  $L_N$  be the Lax matrix given by the  $N \times N$  tri-diagonal matrix with entries

$$(L_N)_{j,j} = p_j \text{ and } (L_N)_{j,j+1} = (L_N)_{j+1,j} = e^{-r_j/2} \tag{2}$$

with periodic boundary conditions  $(L_N)_{1,N} = (L_N)_{N+1,N}$  and  $(L_N)_{N,1} = (L_N)_{N,N+1}$ , then for all integer number  $p$ ,

$$Q_N^p = \text{Tr}(L_N^p)$$

is conserved by the dynamics (1) as well as  $\sum_{i=1}^N r_i$ . It is therefore natural to consider that the finite  $N$  Toda chain is distributed according to the Gibbs measure with density  $e^{-\text{Tr}(W(L_N)) - P \sum r_i}$  with respect to Lebesgue measure. Here,  $P > 0$  controls the pressure of the chain and  $W$  is a potential to be chosen later, which can be a polynomial or a

general measurable function from  $\mathbb{R}$  into  $\mathbb{R}$ . We will assume it goes to infinity faster than  $x^2$ : namely there exists  $c > 0$  and a finite constant  $C$  such that for all  $x \in \mathbb{R}$

$$W(x) \geq cx^2 + C. \tag{3}$$

This assumption is used to compare our distribution to the case where  $W(x) = cx^2$  in which case the entries of the Lax matrix  $L_N$  are independent. We can without loss of generality assume  $c = \frac{1}{2}$  up to rescaling and therefore put

$$W(x) = \frac{1}{2}x^2 + V(x). \tag{4}$$

In the following we will denote

$$d\mathbb{T}_N^{V,P}(p, r) = \frac{1}{\mathbb{Z}_{N,\mathbb{T}}^{V,P}} \exp\{-\text{Tr}(V(L_N)) - \frac{1}{2}\text{Tr}(L_N^2)\} \prod_{i=1}^N e^{-Pr_i} dr_i dp_i \tag{5}$$

where  $\mathbb{Z}_{N,\mathbb{T}}^{V,P}$  is the partition function of the Toda Gibbs measure:

$$\mathbb{Z}_{N,\mathbb{T}}^{V,P} = \int \exp\{-\text{Tr}(V(L_N)) - \frac{1}{2}\text{Tr}(L_N^2)\} \prod_{i=1}^N e^{-Pr_i} dr_i dp_i. \tag{6}$$

We denote in short  $\mathbb{T}_N^P$  for  $\mathbb{T}_N^{0,P}$ . Our goal in this article is to study the empirical measure of the eigenvalues  $\lambda_N \leq \dots \leq \lambda_1$  of  $L_N$  denoted by

$$\hat{\mu}_{L_N} = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}.$$

We shall call  $\hat{\mu}_{L_N}$  the empirical measure of  $L_N$ , or the empirical density of states of the Lax matrix following [9]. Our main result is a large deviations principle for the distribution of  $\hat{\mu}_{L_N}$  under  $d\mathbb{T}_N^{V,P}$ , from which we deduce the almost sure convergence of  $\hat{\mu}_{L_N}$  under  $d\mathbb{T}_N^{V,P}$ .

**Theorem 1.1.** *Let  $P > 0$  and assume that  $V$  is continuous. Assume that either  $V$  is uniformly bounded or there exists  $k \in \mathbb{N}^*$  such that*

$$\lim_{|x| \rightarrow \infty} \frac{V(x)}{x^{2k}} = a, \tag{7}$$

with  $a > 0$  if  $k > 1$  and  $a > -1/2$  if  $k = 1$ . Then:

- (1) *The law of  $\hat{\mu}_{L_N}$  under  $\mathbb{T}_N^{V,P}$  satisfies a large deviation principle in the scale  $N$  with good rate function  $T_P^V$ .*
- (2)  *$T_P^V$  achieves its minimal value at a unique probability measure  $\nu_P^V$ .*
- (3) *As a consequence  $\hat{\mu}_{L_N}$  converges almost surely and in  $L^1$  towards  $\nu_P^V$ .*

$\nu_P^V$  corresponds to the density of states of the Lax matrix in [9]. Moreover, following [9], we can identify the equilibrium measure  $\nu_P^V$  using the equilibrium measure for Coulomb gases in dimension one at temperature of order of the number of particles. More precisely, for a probability measure  $\mu$  on the real line, we define the function  $f_P^V$  by

$$f_P^V(\mu) = \frac{1}{2} \int \left( \frac{1}{2}(x^2 + y^2) + V(x) + V(y) - 2P \ln|x - y| \right) d\mu(x)d\mu(y) + \int \ln \frac{d\mu}{dx} d\mu(x)$$

if  $\mu \ll dx$ , whereas  $f_P^V$  is infinite otherwise.  $f_P^V$  achieves its minimal value at a unique probability measure  $\mu_P^V \ll dx$  which satisfies the non-linear equation

$$\frac{1}{2}x^2 + V(x) - 2P \int \ln|x - y|d\mu_P^V(y) + \ln \frac{d\mu_P^V}{dx} = \lambda_P^V \quad a.s \tag{8}$$

where  $\lambda_P^V$  is a finite constant. We show in section 3 that  $\mu_P^V$  is absolutely continuous with respect to Lebesgue measure and that its depends smoothly on the parameter  $P$ . In Lemma 3.6, we show it is in fact differentiable in  $P$ . We then show that

**Theorem 1.2.** *Let  $P$  be a positive real number. Then, for any bounded continuous function  $f$  on the real line,*

$$\int f(x)d\nu_P^V(x) = \partial_P(P \int f(x)d\mu_P^V(x))$$

This result was already shown in [9] when  $V$  is a polynomial. Our strategy is to prove first a large deviation principle in the case when  $V$  vanishes: then,  $L_N$  has independent entries (modulo the symmetry constraint) under  $\mathbb{T}_N^P$ . We then derive large deviation principles for more general bounded continuous potentials by using Varadhan’s Lemma, see section 2.

Indeed, in the case where  $V$  vanishes, the random variables  $(p_j, r_j)_{1 \leq j \leq N}$  are independent,  $(L_N)_{j,j}$  are standard Gaussian  $N(0, 1)$  variables and  $\sqrt{2}(L_N)_{j,j+1}$  follows a  $\chi_{2P}$  distribution with density with respect to Lebesgue measure given by

$$\chi_{2P}(x) = \frac{2^{1-P} x^{2P-1} e^{-x^2/2}}{\Gamma(P)} \mathbf{1}_{x>0}. \tag{9}$$

The central observation is that we can compare this matrix to the tri-diagonal matrix  $C_N^\beta$  introduced by Dumitriu and Edelman [3]. This is the symmetric matrix with independent (up to symmetry) entries whose diagonal elements are independent standard Gaussian variables, and off diagonal elements so that  $\sqrt{2}C_N^\beta(j, j + 1)$  follow a  $\chi$  distribution with parameter  $\beta(N - j)$ . When  $\beta = 2P/N$ , the matrix is therefore similar to  $L_N$  except that the parameters of the off-diagonal entries vary linearly. The key point is that the law of the eigenvalues of  $C_N^\beta$  is explicit and given by the  $\beta$ -ensemble, see Section 3. This comparison allows to compare the free energy, the rate function and the equilibrium measure of the Toda chain with those of Coulomb gases in section 3. In section 4, we study the case of general potentials. The proof is nearly independent from the quadratic case, but requires additional arguments in particular because the eigenvalues of the Toda matrix are not simple functions of the empirical measure of the entries. Note that the proof given in section 4 also applies to the case where  $V$  is bounded. We nevertheless choose to give a separate proof, dedicated to this case: the computations being simpler, the core of the proof seems more accessible and introduces ideas we re-use in the case where  $V$  is unbounded.

Moreover, our result allows to derive large deviation principles for the empirical measure of the tri-diagonal matrices with independent standard Gaussian entries on the diagonal and independent chi distributed variables with general parameters profile on the off-diagonal. Namely let  $L_N^\sigma$  be a tri-diagonal symmetric matrix with independent Gaussian variables on the diagonal and independent variables  $\sqrt{2}L_N^\sigma(j, j + 1)$  chi distributed with parameter  $\sigma(\frac{i}{N}), 1 \leq i \leq N$ . Let  $\mathbb{T}_N^{V,\sigma}$  be the distribution with density  $e^{-\text{Tr}(V(L_N^\sigma))}/Z$  with respect to the distribution of  $L_N^\sigma$ .

**Theorem 1.3.** *Assume that  $V$  is continuous and satisfies (7). Then, if  $\sigma$  is bounded continuous,*

- (1) the law of  $\hat{\mu}_{L_N^\sigma}$  under  $\mathbb{T}_N^{V,\sigma}$  satisfies a large deviation principle in the scale  $N$  with good rate function  $T_\sigma^V$ ,
- (2)  $T_\sigma^V$  achieves its minimal value at a unique probability measure  $\nu_\sigma^V = \int_0^1 \nu_{\sigma(P)}^V dP$ ,
- (3) As a consequence,  $\hat{\mu}_{L_N^\sigma}$  converges almost surely and in  $L^1$  towards  $\nu_\sigma^V$ .

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## 2 Large deviation principles for tri-diagonal matrices

In this section, we consider a tri-diagonal matrix  $M_N$  with entries

$$(M_N)_{j,j} = a_j \text{ and } (M_N)_{j,j+1} = (M_N)_{j+1,j} = b_j \tag{10}$$

with periodic boundary conditions, the random variables  $(a_i, b_i)_{1 \leq i \leq N}$  being iid, with  $(a_1, b_1)$  with law  $Q_a \otimes Q_b$  on  $\mathbb{R}^2$ . We denote by  $\hat{\mu}_{M_N}$  the empirical measure of the eigenvalues of  $M_N$  and prove the existence of a large deviation principle for the distribution of  $\hat{\mu}_{M_N}$ . In [11, Theorem 4.2], the author proves a large deviation principle for the empirical moments  $\hat{\mu}_{M_N}(x^k)$  by noticing that

$$\hat{\mu}_{M_N}(x^k) = \frac{1}{N} \sum_{i=1}^N f_k(a_j, b_j, |i - j| \leq k)$$

where  $f_k(a_j, b_j, |i - j| \leq k) = (M_N^k)_{ii}$  is an homogeneous polynomial of degree  $k$  in the entries  $a_j, b_j, |i - j| \leq k$ . Noting that  $f_k$  does not depend on  $i$ , one can use the large deviation principle for Markov chains (or  $k$ -dependent large deviation principle), see e.g [2, Theorem 3.1.2 or Section 6.5.2], as well as the contraction principle, to deduce a large deviation principle for the distribution of the empirical moments  $\{\hat{\mu}_{M_N}(x^k), k \leq p\}$ . This could be used to deduce the existence of a large deviation principle for  $\hat{\mu}_{M_N}$  for the weak topology after approximations, but the rate function would not be particularly explicit. We prefer to develop a more straightforward sub-additivity argument and prove separately the existence of a weak large deviation principle and exponential tightness, see e.g [2, Lemma 1.2.18].

### 2.1 Exponential tightness

In this section we assume that

**Assumption 2.1.** *There exists  $\gamma > 0$  such that*

$$D_\gamma := \int e^{\gamma x^2} dQ_a(x) \times \int e^{\gamma y^2} dQ_b(y) < \infty.$$

We equip the set of probability measures on the real line  $\mathcal{P}(\mathbb{R})$  with the weak topology. We then show that

**Lemma 2.2.** *If  $(a_j, b_j)_{1 \leq j \leq N}$  are iid with law  $Q_a \otimes Q_b$  satisfying Assumption 2.1, the sequence  $(\hat{\mu}_{M_N})_{N \geq 0}$  is exponentially tight, namely for each  $L \geq 0$  there exists a compact set  $K_L$  ( $K_L = \{\mu \in \mathcal{P}(\mathbb{R}) : \int x^2 d\mu(x) \leq \frac{2}{\gamma}(L + \ln D_\gamma)\}$  with  $\gamma$  as in Assumption 2.1) such that*

$$\limsup_N \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in K_L^c) < -L. \tag{11}$$

*Proof.* For  $N \geq 1$ , notice that

$$\begin{aligned} \int x^2 d\hat{\mu}_{M_N}(x) &= \frac{1}{N} \text{Tr}(M_N^2) \\ &= \frac{1}{N} \sum_{j=1}^N ((M_N)_{j,j})^2 + \frac{1}{N} \sum_{j=1}^N \left( \sqrt{2} (M_N)_{j,j+1} \right)^2. \end{aligned} \tag{12}$$

As a consequence, Tchebychev's inequality implies that, for any  $\gamma > 0$ ,

$$\mathbb{P} \left( \int x^2 d\hat{\mu}_{M_N}(x) > K \right) \leq e^{-\frac{1}{2}\gamma NK} \mathbb{E}[e^{\frac{1}{2}N\gamma \int x^2 d\hat{\mu}_{M_N}(x)}] \leq e^{-\frac{1}{2}\gamma NK} D_\gamma^N.$$

The conclusion follows by taking  $K = \frac{2}{\gamma}(L + \ln D_\gamma)$ . □

### 2.2 Weak large deviation principle

We next establish a weak large deviation principle, based on the general ideas developed in [2, Lemma 6.1.7]. To this end, we use the following distance on  $\mathcal{P}(\mathbb{R})$ :

$$d(\mu, \nu) = \sup_{\|f\|_{\text{BV}} \leq 1, |f|_{\text{Lip}} \leq 1} \left\{ \left| \int_{\mathbb{R}} f(x) d\mu(x) - \int_{\mathbb{R}} f(x) d\nu(x) \right| \right\}, \tag{13}$$

where  $\|f\|_{\text{BV}}$  is the total variation norm of  $f$  given by

$$\|f\|_{\text{BV}} = \sup \sum_{k \in \mathbb{N}} |f(x_{k+1}) - f(x_k)|,$$

where the supremum holds over all increasing sequences  $(x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ .  $\|f\|_L$  is the Lipschitz norm of  $f$ . If  $f$  is continuously differentiable and we put without loss of generality  $f(0) = 0$ ,  $\|f\|_{\text{BV}} = \int_{-\infty}^{+\infty} |f'(y)| dy$  and  $\|f\|_L = \|f'\|_\infty$ . The distance  $d$  is smaller than the Wasserstein distance where one takes the supremum over all functions whose  $L^\infty$  and Lipschitz norms are bounded by one, and is easily seen to be as well compatible with the weak topology. Then, we shall prove that if  $B_\mu(\delta) = \{\nu \in \mathcal{P}(\mathbb{R}) : d(\mu, \nu) < \delta\}$  denotes the open ball with radius  $\delta$  centered at  $\mu$ , we have:

**Lemma 2.3.** *For any  $\mu$  in  $\mathcal{P}(\mathbb{R})$ , there exists a limit*

$$\lim_{\delta \rightarrow 0} \liminf_N \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in B_\mu(\delta)) = \lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in B_\mu(\delta)). \tag{14}$$

We denote this limit by  $-J_M(\mu)$ .

*Proof.* The advantage of the distance  $d$  is the following control: For any symmetric  $N \times N$  matrices  $A$  and  $B$  with empirical measures of eigenvalues  $\hat{\mu}_A$  and  $\hat{\mu}_B$ , we have:

$$d(\hat{\mu}_A, \hat{\mu}_B) \leq \min \left\{ \frac{\text{rank}(A - B)}{N}, \frac{1}{N} \sum_{i,j} |A(i, j) - B(i, j)| \right\}. \tag{15}$$

Indeed, for any function  $f$  with bounded variation we have thanks to Weyl interlacing property, see e.g. [6, (1.17)],

$$\left| \int f d\hat{\mu}_A - \int f d\hat{\mu}_B \right| \leq \frac{1}{N} \text{rank}(A - B). \tag{16}$$

Moreover, one can check that, if  $f$  is continuously differentiable, we have

$$\begin{aligned} \int f d\hat{\mu}_A - \int f d\hat{\mu}_B &= \int_0^1 \frac{1}{N} \text{Tr}((A - B)f'(\alpha A + (1 - \alpha)B)) d\alpha \\ &= \int_0^1 \left( \frac{1}{N} \sum_{i,j=1}^N (A - B)_{ij} f'(\alpha A + (1 - \alpha)B)_{ji} \right) d\alpha \end{aligned}$$

which implies since for all indices  $i, j$ ,  $|f'(\alpha A + (1 - \alpha)B)_{ji}| \leq \|f'\|_\infty$  that

$$\left| \int f d\hat{\mu}_A - \int f d\hat{\mu}_B \right| \leq \|f'\|_\infty \frac{1}{N} \sum_{i,j=1}^N |(A - B)_{ij}|. \tag{17}$$

Since continuously differentiable functions with bounded  $L^\infty$  norm are dense in Lipschitz functions, we deduce (15) from (16) and (17). We are now ready to prove Lemma 2.3. To this end, we shall approximate our matrix  $M_N$  by a diagonal block matrix with independent blocks. Let  $q \geq 1$ . For  $N \geq 1$  we decompose  $N = k_N q + r_N$  with  $r_N \in \{0, \dots, q - 1\}$  and set  $M_N = M_N^q + R_N^q$ , where  $M_N^q$  is the diagonal block matrix

$$M_N^q = \begin{bmatrix} M_q^1 & & & \\ & \ddots & & \\ & & M_q^{k_N} & \\ & & & B \end{bmatrix}. \tag{18}$$

Here, for all  $i \in \{1, \dots, k_N\}$ ,  $M_q^i$  has the same distribution than  $M_q$  and  $B$  the same distribution than  $M_{r_N}$ . The matrices  $M_q^i$ ,  $1 \leq i \leq k_N$ , are independent, and are independent from  $B$ .  $R_N^q$  is the self-adjoint matrix with null entries except  $R_N^q(1, N) = R_N^q(N, 1) = b_N$ ,  $R_N^q(k_N q + 1, N) = R_N^q(N, k_N q + 1) = -b_N$ , and those given, for  $k \in \{1, \dots, k_N\}$ , by  $R_N^q(kq + 1, kq) = R_N^q(kq, kq + 1) = b_{kq}$ ,  $R_N^q((k - 1)q + 1, kq) = R_N^q(kq, (k - 1)q + 1) = -b_{kq}$ . Therefore  $\text{rank}(R_N^q) \leq 2k_N + 2 \leq 4k_N$ . By (15), we deduce that

$$d(\hat{\mu}_{M_N}, \hat{\mu}_{M_N^q}) \leq \frac{4}{q}. \tag{19}$$

Moreover, we can write  $\hat{\mu}_{M_N^q}$  as the sum

$$\hat{\mu}_{M_N^q} = \sum_{i=1}^{k_N} \frac{q}{N} \hat{\mu}_{M_q^i} + \frac{r_N}{N} \hat{\mu}_B.$$

Therefore, for any  $\mu \in \mathcal{P}(\mathbb{R})$  and  $\delta > 0$ , we have

$$\begin{aligned} \mathbb{P} \left( \hat{\mu}_{M_q^1} \in B_\mu(\delta) \right)^{k_N} \mathbb{P} \left( \hat{\mu}_{M_{r_N}} \in B_\mu(\delta) \right) &= \mathbb{P} \left( \forall i \in \{1, \dots, k_N\}, \hat{\mu}_{M_q^i} \in B_\mu(\delta), \hat{\mu}_B \in B_\mu(\delta) \right) \\ &\leq \mathbb{P} \left( \hat{\mu}_{M_N^q} \in B_\mu(\delta) \right) \\ &\leq \mathbb{P} \left( \hat{\mu}_{M_N} \in B_\mu \left( \delta + \frac{4}{q} \right) \right), \end{aligned}$$

where we used the convexity of balls and (19). As a consequence,

$$u_N(\delta) := -\ln \mathbb{P}(\hat{\mu}_{M_N} \in B_\mu(\delta))$$

satisfies

$$u_N(\delta + 4/q) \leq k_N u_q(\delta) + u_{r_N}(\delta).$$

It is easy (and classical) to deduce the convergence of  $u_N(\delta)/N$  when  $N$  goes to infinity, and then  $\delta$  goes to zero. Indeed let  $\delta > 0$  be given and choose  $q$  large enough so that  $\frac{4}{q} < \delta$ . Then, since  $\delta \rightarrow u_N(\delta)$  is decreasing and non-negative, we have:

$$\frac{u_N(2\delta)}{N} \leq \frac{u_N(\delta + 4/q)}{N} \leq \frac{u_q(\delta)}{q} + \frac{u_{r_N}(\delta)}{N}. \tag{20}$$

Since  $\frac{u_{r_N}(\delta)}{N} \leq \frac{\max_{1 \leq i \leq q-1} u_i(\delta)}{N}$  goes to zero when  $N \rightarrow \infty$ , we conclude that

$$\limsup_N \frac{u_N(2\delta)}{N} \leq \frac{u_q(\delta)}{q}.$$

Since this is true for all  $q$  large enough, we get

$$\limsup_N \frac{u_N(2\delta)}{N} \leq \liminf_N \frac{u_N(\delta)}{N}.$$

Since the left and right hand sides decrease as  $\delta$  goes to zero, we conclude that

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} -\frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in B_\mu(\delta)) \leq \lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} -\frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in B_\mu(\delta)),$$

and the conclusion follows. □

### 2.3 Full large deviation principle

As a consequence of Lemmas 2.2 and 2.3, we have by [2, Theorem 1.2.18] the following large deviation principle.

**Theorem 2.4.** *Under Assumption 2.1, the law of  $\hat{\mu}_M$  satisfies a large deviation principle in the scale  $N$  with a good rate function  $J_M$ . Moreover,  $J_M$  is convex. In other words,*

- $J_M : \mathcal{P}(\mathbb{R}) \rightarrow [0, +\infty]$  has compact level sets  $\{\mu : J_M(\mu) \leq L\}$  for all  $L \geq 0$ . Moreover,  $J_M$  is convex.
- For any closed set  $F \subset \mathcal{P}(\mathbb{R})$ ,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in F) \leq -\inf_F J_M,$$

whereas for any open set  $O \subset \mathcal{P}(\mathbb{R})$

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{P}(\hat{\mu}_{M_N} \in O) \geq -\inf_O J_M.$$

*Proof.*  $J_M$  exists and is defined by Lemma 2.3. The lower semi-continuity of  $J_M$  follows from [2, Theorem 4.1.11]. We then deduce that the level sets of  $J_M$  are compact by the exponential tightness, see [2, Lemma 1.2.18 (b)].

In the spirit of [2, Lemma 4.1.21], we show that  $J_M$  is convex. Let  $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{R})$ . Since  $\hat{\mu}_{M_{2N}}$  can be decomposed as the independent sum of  $\hat{\mu}_{M_N}$  divided by 2 plus an error term of smaller than  $4/N$  by (16), we have for all  $\delta_1, \delta_2 > 0$

$$\mathbb{P}(d(\hat{\mu}_{M_N}, \mu_1) < \delta_1) \mathbb{P}(d(\hat{\mu}_{M_N}, \mu_2) < \delta_2) \leq \mathbb{P}\left(d(\hat{\mu}_{M_{2N}}, \frac{\mu_1 + \mu_2}{2}) < \delta_3\right). \tag{21}$$

for any  $\delta_3 \geq \frac{1}{2}(\delta_1 + \delta_2) + \frac{4}{N}$ . Taking the logarithm, dividing by  $2N$  and letting  $N$  go to infinity,  $\delta_1, \delta_2$  and then  $\delta_3$  to zero, we conclude that

$$J_M\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{2}\left(J_M(\mu_1) + J_M(\mu_2)\right), \tag{22}$$

from which we deduce the convexity of  $J_M$  as in [2, Lemma 4.1.21].

The second point, namely that a weak large deviation principle and exponential tightness implies a full large deviation principle, is classical, see [2, Lemma 1.2.18]. □



**2.4 Large deviation principle for the Toda-Chain with quadratic potential**

Recall that we denoted by  $Q_a$  and  $Q_b$  respectively the laws of the  $a_i$ 's and  $b_i$ 's, see (10). In the case of the Toda chain with Gaussian potential, that is  $V = 0$ , with entries following  $\mathbb{T}_N^P$ , we take  $Q_a$  to be the standard Gaussian law and  $Q_b$  to be the chi distribution  $\sqrt{2}^{-1} \chi_{2P}$  given in (9). We let  $L_N(P)$  be the tridiagonal matrix whose entries follow  $\mathbb{T}_N^P$ . These entries clearly satisfy Assumption 2.1 and therefore we have

**Corollary 2.5.** *For any  $P > 0$ , the law of  $\hat{\mu}_{L_N(P)}$  satisfies a large deviation principle in the scale  $N$  with good, convex, rate function denoted by  $T_P$ .*

For further use, we show that

**Lemma 2.6.** *For each  $\mu \in \mathcal{P}(\mathbb{R})$ , the map  $P \in (0, +\infty) \mapsto T_P(\mu)$  is lower semi-continuous.*

*Proof.* Let  $P, h$  be positive real numbers. We first couple the matrices  $(L_N(P), L_N(P + h))_N$ , where  $L_N(u)$  follows  $\mathbb{T}_N^u$  for  $u = P$  and  $u = P + h$ , in such a way that there exists a finite constant  $c$  so that

$$\mathbb{P} \left( d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(P+h)}) > \delta \right) \leq e^{N(c - \sqrt{-\ln(h)}\delta/2)}. \tag{23}$$

This coupling is done as follows:

- The diagonal coefficients are the same set of standard independent Gaussian variables
- The coefficient below and above the diagonal  $X_u^i$ , follow a  $\sqrt{2}^{-1} \chi_{2u}$  for  $u = P, u = h$  and  $P + h$ . By definition of the  $\chi$  distribution we can construct these variables so that almost surely

$$X_{P+h}^i = \sqrt{(X_P^i)^2 + (X_h^i)^2}. \tag{24}$$

This coupling allows by (15) to write

$$d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(P+h)}) \leq \frac{2}{N} \sum_{i=1}^N |X_{P+h}^i - X_P^i| = \frac{2}{N} \sum_{i=1}^N (X_{P+h}^i - X_P^i) \leq \frac{2}{N} \sum_{i=1}^N X_h^i,$$

where we ultimately used that, for all  $i \in \{1, \dots, N\}$ ,  $X_{P+h}^i \leq X_h^i + X_P^i$  because  $X_h^i X_P^i$  is non-negative and (24) holds. Equation (23) follows by Tchebychev inequality since  $\mathbb{E}[\exp\{\sqrt{\ln h^{-1}} X_h^i\}]$  is finite, see (40). (23) implies that  $(\hat{\mu}_{L_N(P+h)})_{N \geq 0}$  is an exponential approximation of  $(\hat{\mu}_{L_N(P)})_{N \geq 0}$  when  $h$  goes to zero. By [2, Theorem 4.2.16], we deduce that for any  $\mu \in \mathcal{P}(\mathbb{R})$ , we have

$$T_P(\mu) = \lim_{\delta \rightarrow 0} \liminf_{h \rightarrow 0} \inf_{B_\mu(\delta)} T_{P+h}.$$

By monotonicity of the right hand side and the lower semi-continuity of  $T_{P+h}$  we deduce that, see [2, (4.1.2)],

$$\lim_{\delta \rightarrow 0} \inf_{B_\mu(\delta)} T_{P+h} = T_{P+h}(\mu),$$

and therefore

$$T_P(\mu) = \lim_{\delta \rightarrow 0} \liminf_{h \rightarrow 0} \inf_{B_\mu(\delta)} T_{P+h} \leq \liminf_{h \rightarrow 0} T_{P+h}(\mu),$$

and so  $P \mapsto T_P(\mu)$  is lower semi-continuous. □

We shall also use later that Corollary 2.5 gives a large deviation principle for the empirical measure of the Toda chain with general bounded continuous potential.

**Corollary 2.7.** *Let  $V$  be a bounded continuous function on the real line and  $P$  be a positive real number. Let  $L_N(P)$  be the tridiagonal matrix whose entries follow  $\mathbb{T}_N^{V,P}$ . Then:*

- The law of  $\hat{\mu}_{L_N(P)}$  satisfies a large deviation principle in the scale  $N$  with convex good rate function given, for any  $\mu \in \mathcal{P}(\mathbb{R})$ ,

$$T_P^V(\mu) = T_P(\mu) + \int V d\mu - \inf_{\nu} \{T_P(\nu) + \int V d\nu\}.$$

- The set  $M_P^V$  where  $T_P^V$  achieves its minimum value is a compact convex subset of  $\mathcal{P}(\mathbb{R})$ . It is continuous in the sense that for any  $\varepsilon > 0$ , there exists  $\delta_\varepsilon > 0$  such that for all  $\delta < \delta_\varepsilon$ , any  $P, Q > 0$  such that for  $|P - Q| \leq \delta$ ,

$$M_P^V \subset (M_Q^V)^\varepsilon$$

where  $A^\varepsilon = \{\mu : d(\mu, A) \leq \varepsilon\}$ .

*Proof.* The first point is a direct consequence of Varadhan’s lemma since when  $V$  is bounded continuous,  $\mu \rightarrow \int V(x)d\mu(x)$  is also continuous. We hence need only to prove the second point, that is the continuity of  $P \in (0, +\infty) \mapsto M_P^V$ . Note that since  $T_P^V$  is a good rate function,  $M_P^V$  is compact for all positive real number  $P$ . We let  $\mathbb{T}_N$  be the coupling of  $L_N(P)$  and  $L_N(Q)$  introduced in Lemma 2.6. By definition, for  $R = P$  and  $Q$ ,  $B$  a measurable subset of  $\mathcal{P}(\mathbb{R})$ , we have

$$\mathbb{T}_N^{V,R}(\hat{\mu}_{L_N} \in B) = \frac{1}{Z_{N,\mathbb{T}}^{V,R}} \int \mathbf{1}_{\{\hat{\mu}_{L_N(R)} \in B\}} e^{-N \int V(x)d\hat{\mu}_{L_N(R)}(x)} d\mathbb{T}_N,$$

where we used the notation

$$Z_{N,\mathbb{T}}^{V,R} = \int e^{-N \int V(x)d\hat{\mu}_{L_N(R)}(x)} d\mathbb{T}_N^R.$$

Therefore, since  $((M_Q^V)^\varepsilon)^c$  is open, we can use the large deviation principle for the empirical measure of  $L_N(P)$ , Corollary 2.5, to state that for any  $\kappa > 0$

$$\begin{aligned} - \inf_{((M_Q^V)^\varepsilon)^c} T_P^V &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int_{\{d(\hat{\mu}_{L_N(P)}, M_Q^V) > \varepsilon\}} e^{-N \int V(x)d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N \\ &\leq \max\left\{ \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int_{\{d(\hat{\mu}_{L_N(P)}, M_Q^V) > \varepsilon\} \cap \{d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)}) \leq \kappa\}} e^{-N \int V(x)d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N, \right. \\ &\quad \left. 2\|V\|_\infty + c - \sqrt{-\ln|P - Q|} \kappa/2 \right\} \end{aligned} \tag{25}$$

where we used (23) and  $Z_{N,\mathbb{T}}^{V,P} \geq e^{-N\|V\|_\infty}$ . We next remark that by Lemma 2.2, there exists a positive constant  $c$  and a finite constant  $C$  such that uniformly on  $P$  in a compact set, if we denote by  $K_L = \{\int x^2 d\mu(x) \leq L\}$ ,

$$\mathbb{T}_N^P(\hat{\mu}_{L_N} \in K_L^c) \leq e^{-(cL+C)N}.$$

Hence, fixing some  $L > 0$ , (25) implies

$$\begin{aligned} - \inf_{((M_Q^V)^\varepsilon)^c} T_P^V &\leq \max\left\{ 2\|V\|_\infty + c - \sqrt{-\ln|P - Q|} \kappa/2, 2\|V\|_\infty - cL - C, \right. \\ &\quad \limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \\ &\quad \left. \times \int \mathbf{1}_{d(\hat{\mu}_{L_N(P)}, M_Q^V) > \varepsilon} \mathbf{1}_{d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)}) \leq \kappa} \mathbf{1}_{\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)} \in K_L} e^{-N \int V(x)d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N \right\}. \end{aligned} \tag{26}$$

We next notice that  $\int V(d\mu - d\nu)$  is bounded by some  $\varepsilon_V^L(\kappa)$  going to zero as  $\kappa$  does uniformly on  $\{d(\mu, \nu) \leq \kappa\}$  and  $\mu, \nu$  in the compact set  $K_L$ . Indeed, this is obvious if

$V$  has bounded variation and Lipschitz norms, with  $\varepsilon_V(\kappa) = \max\{\|V\|_{\text{BV}}, \|V\|_L\}\kappa$ . If  $V$  is bounded continuous, we let  $\eta > 0$  and choose  $M = \sqrt{2\eta^{-1}L}$  so that  $\mu([-M, M]^c) + \nu([-M, M]^c) \leq \eta$  for  $\mu, \nu \in K_L$ , and then  $V_\eta$  with finite bounded variation and Lipschitz norm so that

$$\sup_{x \in [-M, M]} |V(x) - V_\eta(x)| \leq \eta.$$

We then check that

$$\left| \int V(d\mu - d\nu) \right| \leq \|V\|_\infty \eta + 2\eta + \left| \int V_\eta(d\mu - d\nu) \right| \leq (\|V\|_\infty + 2)\eta + \max\{\|V_\eta\|_{\text{BV}}, \|V_\eta\|_L\}\kappa.$$

We finally choose  $\eta = \eta(\kappa)$  going to zero slowly enough with  $\kappa$  so that the above right hand side goes to zero. Hence, we can bound the third term in the right hand side of (26) to find that

$$\begin{aligned} & \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int \mathbf{1}_{d(\hat{\mu}_{L_N(P)}, M_Q^V) > \varepsilon} \mathbf{1}_{d(\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)}) \leq \kappa} \mathbf{1}_{\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)} \in K_L} e^{-N \int V(x) d\hat{\mu}_{L_N(P)}(x)} d\mathbb{T}_N \\ & \leq e^{N\varepsilon_V^L(\kappa)} \frac{Z_{N,\mathbb{T}}^{V,Q}}{Z_{N,\mathbb{T}}^{V,P}} \frac{1}{Z_{N,\mathbb{T}}^{V,Q}} \int \mathbf{1}_{\{d(\hat{\mu}_{L_N(Q)}, M_Q^V) \geq \varepsilon - \kappa\}} e^{-N \int V(x) d\hat{\mu}_{L_N(Q)}(x)} d\mathbb{T}_N. \end{aligned}$$

Similarly, we find that

$$\begin{aligned} Z_{N,\mathbb{T}}^{V,Q} & \leq \int e^{-N \int V d\hat{\mu}_{L_N(Q)}} \mathbf{1}_{\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)} \in K_L} \mathbf{1}_{\{d((\hat{\mu}_{L_N(P)}, \hat{\mu}_{L_N(Q)}) \leq \kappa)\}} d\mathbb{T}_N \\ & \quad + e^{(\|V\|_\infty + c - \sqrt{-\ln|P-Q|}\kappa/2)N} + 2e^{(\|V\|_\infty - cL - C)N} \\ & \leq Z_{N,\mathbb{T}}^{V,P} (e^{N\varepsilon_V^L(\kappa)} + e^{(2\|V\|_\infty + c - \sqrt{-\ln|P-Q|}\kappa/2)N} + 2e^{(2\|V\|_\infty - cL - C)N}) \end{aligned}$$

where we used that the partition function is bounded from below by  $e^{-\|V\|_\infty N}$ . Moreover the previous large deviation principle implies if  $\kappa \leq \varepsilon/2$  that

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \ln \frac{1}{Z_{N,\mathbb{T}}^{V,Q}} \int_{\{d(\hat{\mu}_{L_N(Q)}, M_Q^V) \geq \varepsilon - \kappa\}} e^{-N \int V(x) d\hat{\mu}_{L_N(Q)}(x)} d\mathbb{T}_N \leq - \inf_{d(\mu, M_Q^V) \geq \varepsilon/2} \{T_Q^V\}.$$

Hence, we find that if  $L$  is big enough,  $P - Q$  small enough so that  $\varepsilon_V^L(\kappa) > \max\{2\|V\|_\infty + c - \sqrt{-\ln|P-Q|}\kappa, 2\|V\|_\infty - cL - C\}$ , (26) yields

$$- \inf_{((M_Q^V)^\varepsilon)^c} T_P^V \leq 2\varepsilon_V^L(\kappa) - \inf_{d(\mu, M_Q^V) \geq \varepsilon/2} \{T_Q^V\}$$

We then conclude that the right hand side is negative for such choices of parameters if  $\kappa$  is small enough and therefore  $\inf_{((M_Q^V)^\varepsilon)^c} T_P^V > 0$  so that  $((M_Q^V)^\varepsilon)^c \subset (M_P^V)^c$  which yields the result.  $\square$

### 3 $\beta$ -ensembles

#### 3.1 Large deviation principles for $\beta$ -ensembles

In this section we consider the  $\beta$ -ensembles and collect already known results about their large deviation principles. We then relate these large deviation principles with the previous ones thanks to Dumitriu-Edelman tri-diagonal representation, as pioneered in [9]. Coulomb gases on the real line are given by the following  $\beta$ -ensembles distribution:

$$d\mathbb{P}_{N,C}^{V,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,C}^{V,\beta}} \prod_{i < j} |x_i - x_j|^\beta e^{-\sum_{i=1}^N (\frac{1}{2}x_i^2 + V(x_i))} dx_1 \dots dx_N. \quad (27)$$

$V$  will be a continuous potential. When  $V = 0$  and  $\beta = 1$ , it is well known [1, Section 2.5.2] that  $d\mathbb{P}_N^{0,1}$  is the law of the eigenvalues of the Gaussian orthogonal ensemble of random matrices with standard Gaussian entries. Hereafter, we keep the potential to be under the form of a quadratic potential plus a general potential only to have simpler notations later on. In this article, we are however interested in the scaling where  $\beta = \frac{2P}{N}$ . The large deviation principles for the empirical measure  $\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$  have been derived in [4] and yields the following result.

**Theorem 3.1.** [4] *Let  $P_N$  be a sequence of positive real numbers converging towards  $P > 0$ . Let  $W(x) = \frac{1}{2}x^2 + V(x)$  be a continuous function such that for some  $P' > P + 1$  there exists a finite constant  $C_V$  such that for all  $x$*

$$W(x) \geq P' \ln(|x|^2 + 1) + C_V \tag{28}$$

Then the law of  $\hat{\mu}_N$  under  $\mathbb{P}_N^{V, \frac{2P_N}{N}}$  satisfies a large deviation principle in the scale  $N$  and with good rate function  $I_P^V(\mu) = f_P^V(\mu) - \inf f_P^V$  where

$$f_P^V(\mu) = \frac{1}{2} \int (W(x) + W(y) - 2P \ln|x - y|) d\mu(x) d\mu(y) + \int \ln \frac{d\mu}{dx} d\mu(x)$$

if  $\mu \ll dx$  and  $\ln \frac{d\mu}{dx}$  is  $\mu$ -integrable, whereas  $f_P^V$  is infinite otherwise.

In fact, neglecting the singularity of the logarithm, this result would be a direct consequence of Sanov's theorem and Varadhan's lemma. Dealing with this singularity requires extra-care, a difficulty which was addressed in [4]. Indeed, [4, Theorem 1.1] can be applied, as was kindly shown to us by David García-Zelada. For  $\frac{1}{2P'} < \alpha < 1 - \frac{P}{P'}$ , we can rewrite

$$d\mathbb{P}_N^{V, \frac{2P_N}{N}}(x_1, \dots, x_N) = \frac{1}{Z_N^{V, P_N}} e^{-2P_N N H_N(x_1, \dots, x_N)} d\pi(x_1) \dots d\pi(x_N),$$

where, if  $\gamma(N) = (1 - N^{-1})\frac{1}{2P'} + \frac{\alpha-1}{2P_N}$ , we set

$$H_N(x_1, \dots, x_N) = \frac{1}{N^2} \sum_{1 \leq i < j \leq N} \left( \frac{W(x_i)}{2P'} + \frac{W(x_j)}{2P'} - \ln|x_i - x_j| \right) - \frac{\gamma(N)}{N} \sum_{i=1}^N W(x_i)$$

and  $\pi$  is the probability measure given by

$$d\pi(x) = \frac{1}{Z} e^{-\alpha W(x)} dx.$$

The sequence  $(H_N)_{N \geq 0}$  is (up to considering  $N$  large enough) uniformly bounded from below by (28). Moreover, letting  $\gamma(\infty) = \frac{1}{2P'} + \frac{\alpha-1}{2P}$ , we set for  $\mu \in \mathcal{P}(\mathbb{R})$ ,

$$H(\mu) := \frac{1}{2} \int \left( \frac{W(x)}{2P'} + \frac{W(y)}{2P'} - \ln|x - y| \right) d\mu(x) d\mu(y) - \gamma(\infty) \int W(x) d\mu(x),$$

we find [4, Lemma 2.1] that the couple  $(\{H_N\}_{N \geq 0}, H)$  fulfills the assumptions of [4, Theorem 1.1]. Thus the law of  $\hat{\mu}_N$  satisfies a large deviation principle at speed  $N$  with rate function  $I_P^V = f_P^V - \inf f_P^V$ , where

$$f_P^V(\mu) = \begin{cases} 2PH_V(\mu) + \int \ln \frac{d\mu}{d\pi} d\mu & \text{if } \mu \ll \pi \text{ and } \ln \frac{d\mu}{d\pi} \text{ is } \mu\text{-integrable} \\ +\infty & \text{otherwise.} \end{cases}$$

It is not hard to see that

**Lemma 3.2.** *For any continuously differentiable function  $W$ , any  $P' > P + 1$  such that (28) holds,*

- $\mu \mapsto I_P^V(\mu)$  is strictly convex,
- $I_P^V$  achieves its minimal value at a unique probability measure  $\mu_P^V(dx) \ll dx$  which satisfies the non-linear equation

$$W(x) - 2P \int \ln|x - y| d\mu_P^V(y) + \ln \frac{d\mu_P^V}{dx} = \lambda_P^V \quad a.s \tag{29}$$

where  $\lambda_P^V$  is a finite constant. Furthermore the support of  $\mu_P^V$  is the whole real line and the density of  $\frac{d\mu_P^V}{dx}$  is bounded from above by  $C_P(|x| + 1)^{2(P-P')}$  where  $C_P$  is a constant which is uniformly bounded on compact subsets of  $(0, P' - 1)$ .

- Let  $D$  be the distance on  $\mathcal{P}(\mathbb{R})$  given by

$$\begin{aligned} D(\mu, \mu') &= \left( - \int \ln|x - y| d(\mu - \mu')(x) d(\mu - \mu')(y) \right)^{1/2} \\ &= \left( \int_0^\infty \frac{1}{t} \left| \int e^{itx} d(\mu - \mu')(x) \right|^2 dt \right)^{1/2} \end{aligned} \tag{30}$$

Then  $P \mapsto \mu_P^V$  is locally 1/2-Hölder for the distance  $D$ : For any  $\delta > 0$  such that  $[P - \delta, P + \delta] \subset (0, P' - 1)$ , there exists a constant  $D > 0$  such that for all  $P - \delta \leq R \leq P + \delta$ , we have

$$D(\mu_P^V, \mu_R^V) \leq D\sqrt{|P - R|}.$$

We will see later that in fact  $P : (0, P' - 1) \rightarrow \mu_P^V$  is differentiable, see Lemma 3.6. Observe that if  $f$  is in  $L^2$  with derivative in  $L^2$ , we can set  $\|f\|_{\frac{1}{2}} = (\int_0^\infty t|\hat{f}_t|^2 dt)^{1/2}$ . Then, for any measure  $\nu$  with zero mass,

$$\int f(x) d\nu(x) = \int_{-\infty}^\infty \hat{f}_t \hat{\nu}_t dt = \int_{-\infty}^\infty \sqrt{t} \hat{f}_t \frac{1}{\sqrt{t}} \hat{\nu}_t dt$$

so that by Cauchy-Schwartz inequality, we get,

$$\left| \int f(x) d\nu(x) \right|^2 \leq \int_{-\infty}^\infty |t \hat{f}_t|^2 dt \int_{-\infty}^\infty \frac{1}{|t|} |\hat{\nu}_t|^2 dt = 4\|f\|_{1/2}^2 D(\nu, 0)^2 \tag{31}$$

In particular, the last point in the theorem shows that for any  $f$  with finite  $\|f\|_{1/2}$ ,  $P \rightarrow \int f d\mu_P^V$  is Hölder 1/2.

*Proof.* For  $P'' > 1$ , we denote by  $\lambda_{P''}$  the probability measure on the real line given by  $\lambda_{P''}(dx) := Z_{P''}^{-1}(|x^2| + 1)^{-P''/2} dx$  and rewrite  $f_P^V$  (up to a constant  $\ln Z_{P''}$ ) as

$$f_P^V(\mu) = \frac{1}{2} \int (\bar{W}(x) + \bar{W}(y) - 2P \ln|x - y|) d\mu(x) d\mu(y) + \int \ln \frac{d\mu(x)}{d\lambda_{P''}(x)} d\mu(x)$$

where  $\bar{W}(y) := W(y) - \frac{1}{2}P'' \ln(|y|^2 + 1)$ . Because  $\lambda_{P''}$  is a probability measure so that, for every probability measure  $\mu$ ,

$$\int \ln \frac{d\mu}{d\lambda_{P''}}(x) d\mu(x) \geq 0$$

by Jensen's inequality since  $x \mapsto x \ln x$  is convex.

The first point of the lemma is clear as  $\mu \mapsto \int (\bar{W}(x) + \bar{W}(y) - 2P \ln|x - y|) d\mu(x) d\mu(y)$  is strictly convex [1, Lemma 2.6.2] whereas the relative entropy  $\mu \mapsto \int \ln \frac{d\mu}{d\lambda_{P''}}(y) d\mu(y)$  is well known to be convex. Since  $f_P^V$  is a good rate function, it achieves its minimal value

at a unique probability measure  $\mu_P^V$ . Writing that for any measure  $\nu$  with mass zero such that  $\mu_P^V + \varepsilon\nu$  is a probability measure for small enough  $\varepsilon$ ,  $I_P^V(\mu_P^V + \varepsilon\nu) \geq I_P^V(\mu_P^V)$ , we get that (29) holds  $\mu_P^V$  almost surely and that the left hand side in (29) is greater or equal than the right hand side outside of the support of  $\mu_P^V$ . Since the left hand side equals  $-\infty$  when the density vanishes, we conclude that the support is the whole real line. We finally show the boundedness of the density. Note that (29) implies that

$$\frac{d\mu_P^V}{dx}(x) = e^{\lambda_P^V} e^{-W(x)+2P \int \ln|x-y|d\mu_P^V(y)} \tag{32}$$

We get from (28), and the fact that  $\ln|x-y| \leq \frac{1}{2} \ln(|x|^2 + 1) + \frac{1}{2} \ln(|y|^2 + 1)$  the bound

$$-W(x) + 2P \int \ln|x-y|d\mu_P^V(y) \leq -(P' - P) \ln(|x|^2 + 1) + C_V + P \int \ln(|x|^2 + 1)d\mu_P^V.$$

We thus only need to bound  $\int \ln(|x|^2 + 1)d\mu_P^V$  and  $\lambda_P^V$  from above. We first notice that  $P \mapsto \inf f_P^V$  is concave since it is the limit of the free energy  $-N^{-1} \ln Z_N^{V, \frac{2P}{N}}$ . This is enough to guarantee that this quantity is uniformly bounded on compact sets (as it is at any given point). We denote by  $C$  such a bound for a fixed compact set. As in [1, Lemma 2.6.2 (b)], since the relative entropy is non-negative we find that

$$\int (\bar{W}(x) - P \ln(|x|^2 + 1))d\mu_P^V(x) \leq f_P^V(\mu_P^V) \leq C.$$

This implies by our hypothesis (28) that

$$(P' - P'' - P) \int \ln(|x|^2 + 1)d\mu_P^V(x) \leq C - C_V$$

and therefore plugging this estimate in the infimum of  $f_P^V$  gives if  $P' - P - P'' > 0$  (which is always possible as we assumed  $P' - P > 1$ )

$$\int W(x)d\mu_P^V(x) \geq C + \frac{C - C_V}{2(P' - P - P'')}$$

Moreover, again because the relative entropy is non-negative,

$$\begin{aligned} -P\Sigma(\mu_P^V) &:= -P \int \ln|x-y|d\mu_P^V(x)d\mu_P^V(y) \\ &\leq C - \int \bar{W}(x)d\mu_P^V(x) \leq C - 2(P' - P'') \int \ln(|x|^2 + 1)d\mu_P^V(x) - C_V \end{aligned}$$

is uniformly bounded. Finally, from (29) we have after integration under  $\mu_P^V$

$$\lambda_P^V = \inf f_P^V - P \int \ln|x-y|d\mu_P^V(x)d\mu_P^V(y) \tag{33}$$

is thus uniformly bounded from above. This completes the proof of the upper bound of the density:  $\frac{d\mu_P^V}{dx}$  is bounded by  $C_P(|x| + 1)^{2(P-P')}$  where  $C_P$  is uniformly bounded on compacts so that  $P' - P - 1 \geq \varepsilon > 0$  for some fixed  $\varepsilon$ .

We next study the regularity of the equilibrium measure  $\mu_P^V$  in the parameter  $P$ . Let  $\delta > 0$  be such that  $[P - \delta, P + \delta] \subset (0, P' - 1)$ , and let  $P - \delta \leq R \leq P + \delta$ . If  $\Delta\mu = \mu_P^V - \mu_R^V$ ,

since  $\mu_P^V$  minimizes  $f_P^V$ , we have

$$\begin{aligned} 0 &\geq f_P^V(\mu_P^V) - f_P^V(\mu_R^V) \\ &= \int W(x)d\Delta\mu(x) - 2P \int \ln|x-y|d\mu_R^V(x)d\Delta\mu(y) - P \int \ln|x-y|d\Delta\mu(x)d\Delta\mu(y) \\ &\quad + \int \ln \frac{d\mu_P^V}{dx} d\mu_P^V - \int \ln \frac{d\mu_R^V}{dx} d\mu_R^V \\ &= \int (2R \int \ln|x-y|d\mu_R^V(y) - \ln \frac{d\mu_R^V}{dx})(x)d\Delta\mu(x) - 2P \int \ln|x-y|d\mu_R^V(x)d\Delta\mu(y) \\ &\quad - P \int \ln|x-y|d\Delta\mu(x)d\Delta\mu(y) + \int \ln \frac{d\mu_P^V}{dx} d\mu_P^V - \int \ln \frac{d\mu_R^V}{dx} d\mu_R^V \\ &= 2(R-P) \int \ln|x-y|d\mu_R^V(x)d\Delta\mu(y) - P \int \int \ln|x-y|d\Delta\mu(x)d\Delta\mu(y) \\ &\quad + \int \ln \frac{d\mu_P^V}{d\mu_R^V} d\mu_P^V \end{aligned}$$

where in the second line we used (29) and the fact that  $\Delta\mu(1) = 0$ . By using the Fourier transform of the logarithm, the centering of  $\Delta\mu$  and the definition (30) we deduce

$$\int \ln \frac{d\mu_P^V}{d\mu_R^V} d\mu_P^V + PD(\mu_P^V, \mu_R^V)^2 \leq 2(P-R) \int \int \ln|x-y|d\mu_R^V(x)d\Delta\mu(y). \tag{34}$$

We can assume without loss of generality that  $R < P$ . We now show that the integral of the right hand side is bounded independently of  $R \in [P - \delta, P]$ . We have  $\frac{d\mu_R^V}{dx} \leq \frac{C_R}{(1+|x|)^{2(P'-R)}}$ , where  $R \mapsto C_R$  is bounded on any compact of  $(0, P' - 1)$ , and in particular on  $[P - \delta, P + \delta]$ . Thus there exists  $C > 0$  such that  $\frac{d\mu_R^V}{dx} \leq \frac{C}{(1+|x|)^2}$ , and the same bound holds for  $\mu_P^V$ . Using that for any  $x, y$  with  $x \neq y$  we have  $\ln(|x-y|) \leq \ln(1+|x|) + \ln(1+|y|)$  and the previous bound on the density of  $\mu_R^V$ , we conclude that  $\int \int \ln|x-y|d\mu_R^V(x)d\Delta\mu(y)$  is uniformly bounded in  $R \in [P - \delta, P + \delta]$ . Since  $\int \ln \frac{d\mu_P^V}{d\mu_R^V} d\mu_P^V \geq 0$  by Jensen's inequality equation (34) gives the existence of a finite constant  $D$  such that

$$D(\mu_P^V, \mu_R^V) \leq D\sqrt{|P-R|}. \tag{35}$$

### 3.2 Relation with the large deviation principle for Toda matrices with quadratic potential

When  $V = 0$ , for any  $\beta > 0$ , Dumitriu and Edelman [3, Theorem 2.12] have shown that  $\mathbb{P}_N^{0,\beta}$  is the law of the eigenvalues of a  $N \times N$  tri-diagonal matrix  $C_N^\beta$  such that  $\left( (C_N^\beta)_{j,j} \right)_{1 \leq j \leq N}$  are independent standard normal variables, independent from the off diagonal entries  $(C_N^\beta)_{j,j+1} = (C_N^\beta)_{j+1,j}$  which are independent and such that  $\sqrt{2}C_N^\beta(j, j + 1)$  follows a  $\chi_{(N-j)\beta}$  distribution. As in the case of the Toda measure we hereafter identify  $\mathbb{P}_N^{0,\beta}$  with  $\mathbb{P}_N^\beta$ . We are now going to give an alternate large deviation principle for the empirical measure under  $\mathbb{P}_N^{2P/N}$  based on this representation, this will allow to relate the rate function  $I_P = I_P^0$  of the Coulomb Gas in terms of the large deviation rate function  $T_s, s \leq P$  for Toda matrices.

**Lemma 3.3.** *The law of the empirical measure  $\hat{\mu}_N$  under  $\mathbb{P}_N^{2P/N}$  satisfies a large deviation principle in the scale  $N$  and with good rate function*

$$I_P(\mu) = \lim_{\delta \rightarrow 0} \liminf_{M \rightarrow \infty} \inf_{\substack{\nu_{P/M}, \dots, \nu_{P/M} \text{ s.t.} \\ \frac{1}{M} \sum_i \nu_{iP/M} \in B_\mu(\delta)}} \left\{ \frac{1}{M} \sum_{i=1}^M T_{iP/M}(\nu_{iP/M}) \right\}. \tag{35}$$

Observe for later purpose that we must have  $I_P = I_P^0$  where  $I_P^V$  is defined just above Lemma 3.2.

*Proof.* We shall proceed by exponential approximation. We write  $N = k_N M + r_N$ ,  $0 \leq r_N \leq M - 1$ , and consider the matrices

$$S_N^M = \begin{pmatrix} L_{k_N}^1 & & & \\ & \ddots & & \\ & & L_{k_N}^M & \\ & & & 0 \end{pmatrix},$$

with  $(L_{k_N}^i)_{1 \leq i \leq M}$  a family of independent square matrices with size  $k_N$  distributed according to  $\mathbb{T}_{k_N}^{(P \frac{N - ik_N}{N})}$ , and a block with null entries of size  $r_N \times r_N$ . We shall prove that they provide good exponential approximation for the matrix  $C_N^{\frac{2P}{N}}$  following the distribution  $\mathbb{P}_N^{2P/N}$ , see [2, Definition 4.2.14]. More precisely, we show that for any positive real number  $\delta$ :

$$\lim_{M \rightarrow +\infty} \limsup_N \frac{1}{N} \ln \mathbb{P}(d(\hat{\mu}_{C_N^{\frac{2P}{N}}}, \hat{\mu}_{S_N^M}) > \delta) = -\infty. \tag{36}$$

The lemma is then a direct application of [2, Theorem 4.2.16 and Exercise 4.2.7]. We first approximate  $S_N^M$  by the following matrix

$$U_N^M = \begin{pmatrix} \boxed{C_1} & & & & \\ & * & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \\ & & * & & \boxed{C_M} & & \\ & & & & & * & \\ & & & & & & \boxed{R_N^M} \end{pmatrix},$$

where the symbols  $*$  denote entries following the law of a matrix distributed according to  $\mathbb{P}_N^{2P/N}$ :

$$U_N(ik_N, ik_N + 1) = U_N(ik_N + 1, ik_N) \sim \frac{1}{\sqrt{2}} \chi_{2P \frac{N - ik_N}{N}}, \quad 1 \leq i \leq M;$$

$R_N^M$  has same distribution as the  $r_N \times r_N$ -bottom-right corner of a  $\mathbb{P}_N^{2P/N}$ -distributed matrix.  $C_i$  has the same coefficients as  $L_{k_N}^i$  except for the top-right and bottom-left corner entries which are put to zero:

$$C_i = \begin{pmatrix} g_{(i-1)k_N+1} & \ddots & & 0 \\ \ddots & \ddots & & \frac{1}{\sqrt{2}} c_j^i \\ & \frac{1}{\sqrt{2}} c_j^i & \ddots & \ddots \\ 0 & & \ddots & g_{ik_N} \end{pmatrix}.$$





Thus for all  $\delta > 0$ , for any integer numbers  $N$  such that  $\frac{3}{k_N} \leq \delta/2$  (i.e for  $N$  larger than some  $N_0$  depending on  $M$ ) and for any non-negative function  $A : M \mapsto A(M)$

$$\begin{aligned} \mathbb{P} \left( d(\hat{\mu}_{S_N^M}, \hat{\mu}_{C_N^{2P/N}}) > \delta \right) &\leq \mathbb{P} \left( \sum_{i=1}^{k_N M} \chi_{2P/M} > \frac{k_N M \delta}{2\sqrt{2}} \right) \\ &\leq e^{-A(M)k_N M \delta / (2\sqrt{2})} \mathbb{E} \left[ e^{A(M)\chi_{2P/M}} \right]^{k_N M}. \end{aligned}$$

It is not hard to see that with  $A(M) = \sqrt{\ln(M)}$ , there exists a finite constant  $K$  such that

$$\sup_{M \geq 0} \mathbb{E} \int e^{A(M)x} d\chi_{1/M}(x) \leq K \tag{40}$$

insuring that

$$\frac{1}{N} \ln \mathbb{P}(d(\hat{\mu}_{C_N^{2P/N}}, \hat{\mu}_{S_N^M}) > \delta) \leq -A(M) \frac{\delta}{2\sqrt{2}} + K,$$

which yields the result. □

We shall use the previous lemma to study the case with a non trivial potential. Indeed, as a direct consequence of Lemma 3.3 and Varadhan’s lemma, we deduce the following Theorem.

**Theorem 3.4.** *For any continuous function  $V$  such that*

$$\limsup_{|x| \rightarrow \infty} \frac{|V(x)|}{x^2} = 0, \tag{41}$$

*the law of the empirical measure  $\hat{\mu}_N$  under  $\mathbb{P}_N^{V, 2P/N}$  satisfies a large deviation principle in the scale  $N$  and with good rate function  $I_P^V(\mu) = f_P^V(\mu) - \inf f_P^V$  where*

$$f_P^V(\mu) = \lim_{\delta \rightarrow 0} \liminf_M \inf_{\substack{\nu_{P/M}, \dots, \nu_{P/M} \text{ s.t.} \\ \frac{1}{M} \sum_i \nu_{iP/M} \in B_\mu(\delta)}} \left\{ \frac{1}{M} \sum_{i=1}^M (T_{iP/M}(\nu_{iP/M}) + \int V d\nu_{iP/M}) \right\}. \tag{42}$$

**Remark 3.5.** Varadhan’s lemma gives the result for bounded continuous function  $V$ . However, we can approximate  $V$  by  $V(x)(1 + \varepsilon x^2)^{-1}$  with overwhelming probability thanks to Lemma 2.2, which allows to conclude for any potential  $V$  satisfying (41)

We shall use this relation to give a better description of the rate function  $T_P$ . In fact we first consider the free energy

$$F_{\mathbb{T}}^{V,P} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_{N,\mathbb{T}}^{V,P}, \quad F_C^{V,P} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_{N,C}^{V,P} = - \inf f_P^V.$$

**Lemma 3.6.** *For any continuous function  $V$  satisfying (41),*

- $P \mapsto F_C^{V,P} = - \inf f_P^V$  is continuously differentiable on  $(0, +\infty)$ . Moreover, for any  $P > 0$

$$F_{\mathbb{T}}^{V,P} = \partial_P (P F_C^{V,P})$$

- For any bounded continuous function  $f$ , the map  $P \in (0, +\infty) \mapsto P \mu_P^V(f)$  is continuously differentiable. Moreover, there exists a unique minimizer  $\nu_P^V$  of  $\mu \mapsto T_P(\mu) + \int V d\mu(x)$ , which satisfies, for any bounded continuous function  $f$ ,

$$\nu_P^V(f) = \partial_P (P \mu_P^V(f)).$$

Therefore, we have

$$\nu_P^V = \partial_P (P \mu_P^V). \tag{43}$$

- For any probability measure  $\mu$ ,

$$T_P(\mu) = - \inf_{V \in C_b^0} \left\{ \int_{\mathbb{R}} V d\mu + F_{\mathbb{T}}^{V,P} \right\}. \tag{44}$$

*Proof.* First notice that, for any probability measure  $\mu$ , Lemma 3.3 implies that

$$\begin{aligned} f_P^V(\mu) &= I_P(\mu) + \int_{\mathbb{R}} V d\mu \geq \liminf_M \inf_{\nu} \frac{1}{M} \sum_{i=1}^M \left\{ T_{iP/M}(\nu) + \int_{\mathbb{R}} V d\nu \right\} \\ &= \int_0^1 \inf_{\nu} \left\{ T_{sP}(\nu) + \int_{\mathbb{R}} V d\nu \right\} ds = - \int_0^1 F_{\mathbb{T}}^{V,sP} ds. \end{aligned} \tag{45}$$

In the equality between the lim inf and the integral, we used the fact that  $s \in (0, 1) \mapsto F_{\mathbb{T}}^{V,sP}$  is convex and therefore continuous. We claim that this lower bound is achieved. For  $s \in [0, 1]$ , let  $\nu_{sP}^*$  be a minimizer of  $\mu \mapsto T_{sP}(\mu) + \int V d\mu$ . By Corollary 2.7, we can choose  $\nu_{sP}^*$  such that  $s \mapsto \nu_{sP}^*$  is continuous. Hence,  $\mu_P^* := \int_0^1 \nu_{sP}^* ds$  makes sense and is a probability measure on  $\mathbb{R}$ . We claim it minimizes  $f_P^V$ . Indeed, by Lemma 3.3, we have

$$\begin{aligned} f_P^V(\mu_P^*) &= \lim_{\delta \rightarrow 0} \liminf_M \inf_{\frac{1}{M} \sum_{i=1}^M \nu_{iP/M} \in B_{\mu_P^*}(\delta)} \left\{ \frac{1}{M} \sum_{i=1}^M T_{iP/M}(\nu_{iP/M}) + \int_{\mathbb{R}} V d\nu_{iP/M} \right\} \\ &\leq \liminf_M \frac{1}{M} \sum_{i=1}^M \left\{ T_{iP/M}(\nu_{iP/M}^*) + \int_{\mathbb{R}} V d\nu_{iP/M}^* \right\} \\ &= \liminf_M \frac{1}{M} \sum_{i=1}^M \inf_{\nu} \left\{ T_{iP/M}(\nu) + \int_{\mathbb{R}} V d\nu \right\} \\ &= \int_0^1 \inf_{\nu} \left\{ T_{sP}(\nu) + \int_{\mathbb{R}} V d\nu \right\} ds = - \int_0^1 F_{\mathbb{T}}^{V,Ps} ds. \end{aligned}$$

With (45), we deduce that the above inequality is an equality and that  $f_P^V$  achieves its minimal value at  $\mu_P^*$ . By Lemma 3.2, this minimizer is unique and therefore  $\mu_P^* = \mu_P^V$  for any choices of paths  $\nu^*$  and any positive real number  $P$ . Hence, we find that

$$-F_C^{V,P} = \inf f_P^V = I_P(\mu_P^V) + \int_{\mathbb{R}} V d\mu_P^V = - \int_0^1 F_{\mathbb{T}}^{V,Ps} ds.$$

By a change of variable we deduce

$$PF_C^{V,P} = \int_0^P F_{\mathbb{T}}^{V,s} ds.$$

Since  $s \mapsto F_{\mathbb{T}}^{V,s}$  is convex, it is continuous. This shows that  $P \mapsto PF_C^{V,P}$  is continuously differentiable, and that for all  $P > 0$ ,

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P}).$$

Moreover, we have seen that for any choice of continuous minimizing path  $\nu^*$  of  $\mu \mapsto T(\mu) + \int V d\mu$  and any positive real number  $P$ ,

$$\mu_P^V = \int_0^1 \nu_{sP}^* ds = \frac{1}{P} \int_0^P \nu_s^* ds.$$

Integrating the last equality against  $f$  bounded continuous we have

$$\mu_P^V(f) = \frac{1}{P} \int_0^P \nu_s^*(f) ds.$$

By continuity of  $s \mapsto \nu_s^*(f)$ , we deduce that  $P \mapsto \mu_P^V(f)$  is continuously differentiable and that

$$\nu_P^*(f) = \partial_P(P\mu_P^V(f)).$$

But Corollary 3.2 implies that any probability which minimizes  $T_P^V$  can be seen as the endpoint of a continuous path  $s \in (0, P] \mapsto \nu_s^*$  where each  $\nu_s^*$  minimizes  $T_s^V$ . By the latter, such a measure is then equal to  $\partial_P(P\mu_P^V(f))$ , showing the uniqueness of the minimizer  $\nu_P^V$  of  $T_P^V$  and the equality

$$\nu_P^V = \partial_P(P\mu_P^V).$$

The last point of the Lemma is a direct consequence of [2, Theorem 4.5.10] since  $T_P^V$  is convex for all bounded continuous function  $V$ .  $\square$

By Lemma 3.2,  $\nu_P^V$  is a probability measure which satisfies almost surely

$$d\nu_P^V(x) = (C_P^V + 2P \int \ln|x - y|d\nu_P^V(y))d\mu_P^V(x)$$

with  $C_P^V$  a constant such that

$$C_P^V + 2P \int \ln|x - y|d\nu_P^V(y)d\mu_P^V(x) = 1$$

Furthermore we must have  $C_P^V + 2P \int \ln|x - y|d\nu_P^V(y) \geq 0$  for all  $x$ .

## 4 Large deviations for Toda Gibbs measure with general potentials

We now consider the measures  $\mathbb{T}_N^{V,P}$  given by (5), with potential given by  $W : x \in \mathbb{R} \mapsto ax^{2k} + U(x)$ ,  $k \geq 2$ , with  $U(x)/x^{2k}$  going to zero at infinity. We show that under these laws, the law of the empirical measures  $(\hat{\mu}_{L_N})_{N \geq 1}$  still fulfills a large deviation principle, by extending the subadditivity argument previously used. We then identify the rate function as before. By Varadhan’s Lemma, it is enough to consider the case where  $U(x) = \frac{1}{2}x^2$  (we detail this in Section 5). We hereafter continue to use the notation (5) with now  $V(x) = ax^{2k}$ .

### 4.1 Exponential tightness

In this section we prove that if  $W(x) = ax^{2k} + \frac{1}{2}x^2$ , i.e  $V(x) = ax^{2k}$  with  $k \geq 2$  and  $a > 0$ , then the law of the empirical measure of the eigenvalues is exponentially tight under  $\mathbb{T}_N^{V,P}$ . More precisely, we let  $\mathcal{K}_L = \{\mu \in \mathcal{P}(\mathbb{R}) \mid \int V(x)d\mu(x) \leq L\}$  which is a compact of  $\mathcal{P}(\mathbb{R})$ . Then we shall prove

**Lemma 4.1.** *There exists a finite constant  $c_W$  such that*

$$\mathbb{T}_N^{V,P}(\hat{\mu}_N \in \mathcal{K}_L^c) \leq e^{-(L-c_W)N}.$$

*Proof.* We first bound from below the free energy by Jensen’s inequality

$$Z_{N,\mathbb{T}}^{V,P} = \int_{\mathbb{R}^{2N}} e^{-N \int_{\mathbb{R}} V d\hat{\mu}_N} d\mathbb{T}_N^P \geq \exp\{-N \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}} V d\hat{\mu}_N d\mathbb{T}_N^P\} \geq \exp\{-c_V N\}. \tag{46}$$

From here we deduce exponential tightness for  $(\hat{\mu}_N)_N$  under  $\mathbb{T}_N^{V,P}$ : for  $L > 0$ ,

$$\begin{aligned} \mathbb{T}_N^{V,P} \left( \int_{\mathbb{R}} V d\hat{\mu}_N \geq L \right) &= \frac{1}{Z_{N,\mathbb{T}}^{V,P}} \int_{\mathbb{R}^{2N}} \mathbf{1}_{\{\int_{\mathbb{R}} V d\hat{\mu}_N \geq L\}} e^{-N \int_{\mathbb{R}} V d\hat{\mu}_N} d\mathbb{T}_N^P \\ &\leq e^{N(c_V - L)}. \end{aligned} \tag{47}$$

$\square$

For later purpose we prove the following result showing that the off diagonal entries  $b_i = e^{-r_i/2}$ ,  $1 \leq i \leq N$  of the Lax matrix  $L_N$  do not become too small:

**Lemma 4.2.** For any  $P > 0$

$$\limsup_L \limsup_N \frac{1}{N} \ln \mathbb{T}_N^{V,P} \left( \frac{1}{N} \sum_{i=1}^N \ln b_i \leq -L \right) = -\infty.$$

*Proof.* Since  $V$  is bounded from below and we have bounded from below the partition function (46), it enough to prove this estimate when  $V = 0$ . But, in this case the entries are independent and so we only need to prove it for independent chi distributed variables. But then, for any  $0 < \delta < P$ , with  $Z_{N,\mathbb{T}}^P = Z_{N,\mathbb{T}}^{0,P}$  the partition function in (6), we find

$$\mathbb{T}_N^P \left( \frac{1}{N} \sum_{i=1}^N \ln b_i \leq -L \right) \leq e^{-\delta LN} \frac{Z_{N,\mathbb{T}}^{P-\delta/2}}{Z_{N,\mathbb{T}}^P} = e^{-\delta LN} \left( \frac{\Gamma(P-\delta/2)}{2^{\delta/2} \Gamma(P)} \right)^N$$

from which the result follows by taking for instance  $\delta = P/2$ . □

### 4.2 Weak LDP

In this section, we prove that  $\hat{\mu}_{L_N}$  satisfies a weak large deviation principle, namely Lemma 2.3. In this more general setup, we follow again a subadditivity argument, which is however more sophisticated since the entries of  $L_N$  are not independent anymore. We will restrict ourselves to the case where  $V(x) = ax^{2k}$ ,  $a > 0$ , the case of a more general potential with the same asymptotic behavior being again a consequence of Varadhan’s Lemma. We first show that the large deviation principles is the same if we remove the entries (equal to  $b_N$ ) in the corners  $(N, 1)$  and  $(1, N)$  in the Toda matrix. Namely, let  $\tilde{L}_N$  be the tridiagonal matrix with entries equal to those of  $L_N$  except for the entries  $(1, N)$  and  $(N, 1)$  which vanish and consider the following modification of  $\mathbb{T}_N^{V,P}$  given by

$$d\tilde{\mathbb{T}}_N^{V,P} = \frac{1}{\tilde{Z}_N^{V,P}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P. \tag{48}$$

**Lemma 4.3.** For any probability measure  $\mu$ , we have

$$\lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P = \lim_{\delta \rightarrow 0} \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{\tilde{L}_N}, \mu) < \delta} e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P$$

Moreover,

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P = \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int e^{-\text{Tr}V(\tilde{L}_N)} d\tilde{\mathbb{T}}_N^P.$$

The same results hold if we replace all the liminf by limsup.

*Proof.* To simplify the notations we take  $a = 1$  in the proof. First notice that  $V(L_N) - V(\tilde{L}_N)$  is an homogeneous polynomial of degree  $2k$  in  $L_N$  and  $\Delta L_N = L_N - \tilde{L}_N$ , with degree at least one in the latter. Observe that  $\Delta L_N$  only depends on  $b_N$ . Therefore, there exists a finite constant  $C_k$  such that on  $B_N^{K,M} := \{b_N \leq K\} \cap \{\frac{1}{N} \text{Tr}(L_N^{2k}) \leq M\}$  (or  $\tilde{B}_N^{M,K} := \{b_N \leq K\} \cap \{\frac{1}{N} \text{Tr}(\tilde{L}_N^{2k}) \leq M\}$ ), Hölder’s inequality implies

$$\begin{aligned} \left| \frac{1}{N} \text{Tr} \left( V(L_N) - V(\tilde{L}_N) \right) \right| &\leq C_k \sum_{l=1}^{2k} \left( \frac{1}{N} \text{Tr} \left( (\Delta L_N)^{2k} \right) \right)^{l/2k} \left( \frac{1}{N} \text{Tr} \left( L_N^{2k} \right) \right)^{\frac{2k-l}{2k}} \\ &\leq C(M, K) N^{-\frac{1}{2k}} \end{aligned}$$

where  $C(M, K)$  is a finite constant depending only on  $M, K, k$ . Note that in the above right hand side  $\text{Tr}(L_N^{2k})$  can be replaced by  $\text{Tr}(\tilde{L}_N^{2k})$  as they play a symmetric role. Moreover, by (16),  $d(\hat{\mu}_{L_N}, \hat{\mu}_{\tilde{L}_N}) \leq 2/N$  since  $\Delta L_N$  has rank at most two. We fix a probability measure  $\mu$  and first prove that

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P \geq \liminf_{N \rightarrow \infty} \frac{1}{N} \ln \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P. \tag{49}$$

We can assume without loss of generality that the right hand side does not equal  $-\infty$ . Then, we have by the previous remark

$$\begin{aligned} \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P &\geq e^{-C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\tilde{B}_N^{M,K} \cap \{d(\hat{\mu}_{L_N}, \mu) < \delta - \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\geq C' e^{-C(M,K)N^{\frac{2k-1}{2k}}} \\ &\quad \times \int 1_{\{\text{Tr}V(\tilde{L}_N) \leq NM\} \cap \{d(\hat{\mu}_{L_N}, \mu) < \delta - \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\geq C' e^{-C(M,K)N^{\frac{2k-1}{2k}}} \\ &\quad \times \left\{ \int 1_{\{d(\hat{\mu}_{L_N}, \mu) < \delta - \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P - e^{-NM} \right\} \end{aligned}$$

where in the second line we integrated over  $b_N \leq K$  and in the last line we used that

$$\int 1_{\{\text{Tr}V(\tilde{L}_N) \geq NM\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \leq e^{-NM}.$$

We next choose  $M$  so that this term is smaller than the first term (which we assumed bounded below by  $e^{-NC}$  for some finite  $C$ ). We deduce that (49) holds. To prove the converse inequality, we notice that there exists one  $b_i$  bounded by  $K$  with probability greater than  $1 - e^{-a(K)N}$  under  $\mathbb{T}_N^P$ , with  $a(K) = -\ln P(b \geq K) > 0$  which goes to  $+\infty$  when  $K$  does. By symmetry with respect to the order of the indices, we may assume it is  $b_N$ . Therefore, because  $V$  is bounded below by some finite constant  $C$ , setting  $a'(K) = a(K) - C$ , and using Lemma 4.1, we find

$$\begin{aligned} \int 1_{d(\hat{\mu}_{L_N}, \mu) < \delta} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P &\leq e^{-Na'(K)} + N \int 1_{\{b_N \leq K\} \cap \{d(\hat{\mu}_{L_N}, \mu) < \delta\}} e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P \\ &\leq e^{-Na'(K)} + Ne^{-N(M-c_V)} \\ &\quad + Ne^{C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\tilde{B}_N^{M,K} \cap \{d(\hat{\mu}_{L_N}, \mu) < \delta + \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \\ &\leq e^{-Na'(K)} + Ne^{-N(M-c_V)} + Ne^{C(M,K)N^{\frac{2k-1}{2k}}} \int 1_{\{d(\hat{\mu}_{L_N}, \mu) < \delta + \frac{2}{N}\}} e^{-\text{Tr}V(\tilde{L}_N)} d\mathbb{T}_N^P \end{aligned}$$

which gives the converse bound, letting  $N$  going to infinity, provided  $K$  and  $M$  are large enough. The same arguments also hold when there is no indicator function, giving the same estimates for the free energy.  $\square$

**Lemma 4.4.** *Let  $V(x) = ax^{2k}$  and  $P > 0$ . For any  $\mu$  in  $\mathcal{P}(\mathbb{R})$ , there exists a limit*

$$\lim_{\delta \rightarrow 0} \liminf_N \frac{1}{N} \ln \mathbb{T}_N^{V,P}(\hat{\mu}_{L_N} \in B_\mu(\delta)) = \lim_{\delta \rightarrow 0} \limsup_N \frac{1}{N} \ln \mathbb{T}_N^{V,P}(\hat{\mu}_{L_N} \in B_\mu(\delta)). \tag{50}$$

We denote this limit by  $-T_P^V(\mu)$ . Then,  $\mu \mapsto T_P^V(\mu)$  is convex.

*Proof.* We use the notations of Lemma 2.3. Let  $q \geq 1$  be fixed. For  $N \geq 1$  we write  $N = k_N q + r_N$ ,  $0 \leq r_N \leq q - 1$ , and define  $L_N^q$  by removing the off diagonal entries

$b_{\ell q} = L_N(\ell q, \ell q + 1), L_N(\ell q + 1, \ell q), 1 \leq \ell \leq k_N$ , as well as the entries  $L_N(1, N), L_N(N, 1)$ , from  $L_N$ . We set  $R_N^q = L_N - L_N^q$ . Let  $Z_N^V = Z_{N,T}^{V,P}$  denote in short the partition function for the Toda Gibbs measure with potential  $V$  and set

$$Z_{N,q}^V = \mathbb{E}_{\mathbb{T}_N^P} \left[ e^{-\text{Tr}V(L_N^q)} \right] = \int e^{-\text{Tr}V(L_N^q)} d\mathbb{T}_N^P.$$

We first show that there is some constant  $C_k$  (independent of  $N$ ) such that for all  $N \geq 1$ ,

$$\frac{1}{N} \ln \frac{Z_{N,q}^V}{Z_N^V} \geq -\frac{C_k}{q^{1/2k}}. \tag{51}$$

By Jensen’s inequality we have

$$\frac{1}{N} \ln \frac{Z_{N,q}^V}{Z_N^V} = \frac{1}{N} \ln \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ e^{\text{Tr}(V(L_N) - V(L_N^q))} \right] \geq \frac{1}{N} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ \text{Tr}(V(L_N) - V(L_N^q)) \right]. \tag{52}$$

As in the proof of Lemma 4.1, we bound the right hand side by first noticing that  $V(L_N) - V(L_N^q)$  is an homogeneous polynomial of degree  $2k$  in  $L_N$  and  $L_N - L_N^q$ , with degree at least one in the latter. Therefore, Hölder’s inequality implies that there exists a finite constant  $C$  depending only on  $k$  such that

$$\begin{aligned} & \left| \frac{1}{N} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ \text{Tr}(V(L_N) - V(L_N^q)) \right] \right| \\ & \leq C \sum_{l=1}^{2k} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ \frac{1}{N} \text{Tr} \left( (L_N - L_N^q)^{2k} \right) \right]^{l/2k} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ \frac{1}{N} \text{Tr}(L_N^{2k}) \right]^{\frac{2k-l}{2k}} \end{aligned}$$

Now,  $R_N^q = L_N - L_N^q$  has non zero entries only at the sites  $(i, i + 1)$  and  $(i + 1, i)$ ,  $i \in J = \{\ell q, 1 \leq \ell \leq k_N\}$ , as well as  $(N, 1)$  and  $(1, N)$ . We can assume without loss of generality that  $q > 2k$  so that  $\text{Tr}(R_N^q)^{2k}$  simply depends on the  $2k$ th power of the its non-vanishing entries. Thus, there exists a finite constant  $C_k$  which only depends on  $k$  such that

$$\text{Tr} \left( (R_N^q)^{2k} \right) \leq C_k \sum_{i \in J} L_N(i, i + 1)^{2k} + C_k L_N(N, 1)^{2k}.$$

Next notice that

$$L_N(i, i + 1)^2 \leq L_N(i, i)^2 + L_N(i, i + 1)^2 + L_N(i, i - 1)^2 = L_N^2(i, i).$$

Moreover, diagonalizing  $L_N = \sum \lambda_j v_j v_j^T$ , we find by Hölder’s inequality (since  $\sum v_j(i)^2 = 1$  for all  $i \in \{1, \dots, N\}$ ) that

$$L_N^2(i, i)^k = \left( \sum \lambda_j^2 v_j(i)^2 \right)^k \leq \sum \lambda_j^{2k} v_j(i)^2 = L_N^{2k}(i, i).$$

Thus,

$$L_N(i, i + 1)^{2k} \leq L_N^2(i, i)^k \leq L_N^{2k}(i, i).$$

Because  $L_N$  has periodic boundary conditions, the distribution of the entries of  $L_N$  are invariant under the shift  $\theta : i \rightarrow i + 1$ , so that under  $\mathbb{T}_N^{V,P}$ ,  $L_N(i, i + 1)$  has the same law than  $L_N(i + 1, i + 2)$ , and  $L_N(i, i)$  has the same law than  $L_N(i + 1, i + 1)$ . As a consequence, we have

$$\mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ \frac{1}{N} \text{Tr} \left( (L_N - L_N^q)^{2k} \right) \right] \leq \frac{1}{N} C_k \sum_{i \in J} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ L_N^{2k}(i, i) \right] = C_k \frac{k_N}{N} \mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ \frac{1}{N} \text{Tr}(L_N^{2k}) \right].$$

But (47) implies that  $\mathbb{E}_{\mathbb{T}_N^{V,P}} \left[ \frac{1}{N} \text{Tr}(L_N^{2k}) \right]$  is bounded by some finite constant independent of  $N$ . We therefore deduce (51) from (52).

We next prove the subadditivity property. Let  $\delta > 0$  and  $L > 0$  be given. Let  $\mathcal{K}_L = \{\hat{\mu}_{L_N}(V) \leq L\}$ . As in equation (19), we have for  $q$  big enough,

$$\mathbb{T}_N^{V,P}(\{\hat{\mu}_{L_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L) \geq \frac{Z_{N,q}^V}{Z_N^V} \frac{1}{Z_{N,q}^V} \int_{\mathcal{K}_L \cap K_A} \mathbf{1}_{\hat{\mu}_{L_N^q} \in B_\mu(\delta-4/q)} e^{-\text{Tr}(V(L_N))} d\mathbb{T}_N^P, \quad (53)$$

where we set  $K_A = K_{A,N} = \cap_{i \in J} \{b_i^{2k} \leq A\} \cap \{b_N^{2k} \leq A\}$ . As before, noticing that  $V(L_N) - V(L_N^q)$  is a polynomial in  $L_N^q$  and  $L_N - L_N^q$ , we find a finite constant  $C$  such that, on  $\mathcal{K}_L \cap K_A$ , for  $N$  large enough,

$$\frac{1}{N} |\text{Tr}(V(L_N) - V(L_N^q))| \leq C \left( \frac{k_N}{N} C_k A \right)^{1/2k} L^{\frac{2k-1}{2k}}.$$

Therefore if we set  $\mathcal{K}_L^q = \{\hat{\mu}_{L_N^q}(V) \leq L\}$ , we deduce that  $K_A \cap \mathcal{K}_L$  contains  $K_A \cap \mathcal{K}_{L-\varepsilon(q)}^q$  for some  $\varepsilon(q)$  going to zero as  $q$  goes to infinity. We deduce from (51) and (53) that there exists a finite constant  $C$  independent of  $q$  (but dependent on  $L$  and  $k$ ) such that

$$\mathbb{T}_N^{V,P}(\{\hat{\mu}_{L_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L) \geq \frac{e^{-NCq^{-1/2k}}}{Z_{N,q}^V} \int_{K_A \cap \mathcal{K}_{L-\varepsilon(q)}^q} \mathbf{1}_{\hat{\mu}_{L_N^q} \in B_\mu(\delta-4/q)} e^{-\text{Tr}(V(L_N^q))} d\mathbb{T}_N^P, \quad (54)$$

Since  $L_N^q$  is independent of the entries  $b_i, i \in J$  and therefore of  $K_A$ , we see that we can integrate the indicator function of  $K_A$  yielding a contribution  $C_A^{k_N}$  for some positive constant  $C_A$  depending only on  $A$ . We observe as well that  $L_N^q$  is a block diagonal matrix  $\text{diag}(L_q^1, \dots, L_q^{k_N}, B)$  where  $L_q^i, 1 \leq i \leq k_N$ , are independent and independent from  $B$ ,  $L_q^i$  following  $\tilde{\mathbb{T}}_q^P$  defined in (48) and  $B$  following  $\tilde{\mathbb{T}}_{r_N}^P$ . Finally, we notice that  $\mathcal{K}_{L-\varepsilon(q)}^q$  contains  $\cap_{1 \leq i \leq k_N} \{\frac{1}{q} \text{Tr}((L_q^i)^{2k}) \leq L - \varepsilon(q)\} \cap \{\frac{1}{N-k_N q} \text{Tr}(B^{2k}) \leq L - \varepsilon(q)\}$  since the trace of  $(L_N^q)^{2k}$  is a linear combination of the latter traces. Thus by independence of the matrices  $L_q^1, \dots, L_q^{k_N}$  under  $\frac{1}{Z_{N,q}^V} e^{-\text{Tr}V(L_N^q)} d\mathbb{T}_N^P$  and convexity of balls, we deduce by taking the logarithm that if we set  $u_N(\delta, L) = -\ln \mathbb{T}_N^{V,P}(\{\hat{\mu}_{M_N} \in B_\mu(\delta)\} \cap \mathcal{K}_L)$  and  $v_N(\delta, L) = -\ln \tilde{\mathbb{T}}_N^{V,P}(\{\hat{\mu}_{\tilde{L}_N} \in B_\mu(\delta)\} \cap \{\text{Tr}(\tilde{L}_N)^{2k} \leq LN\})$ , then we have

$$u_N(\delta + 4/q, L + \varepsilon(q)) \leq N(Cq^{-1/2k} + \ln(C_A)/q) + k_N v_q(\delta, L) + v_{r_N}(\delta, L). \quad (55)$$

We conclude as in Lemma 2.3 that

$$\limsup_N \frac{u_N(\delta + 4/q, L + \varepsilon(q))}{N} \leq \frac{v_q(\delta, L)}{q} + Cq^{-1/2k} + \frac{\ln(C_A)}{q}. \quad (56)$$

We then notice that for all  $N, \delta, u_N(\delta, L) \geq u_N(\delta, \infty)$  and  $v_N(\delta, L) \leq v_N(\delta, \infty) + \ln 2$  for  $L$  large enough by Lemma 2.2 (for  $\tilde{L}_N$ ). If therefore we choose a subsequence  $q$  going to infinity along which the liminf is taken, we deduce by Lemma 4.3 that

$$\limsup_N \frac{u_N(2\delta, \infty)}{N} \leq \liminf_{q \rightarrow \infty} \frac{v_q(\delta, \infty)}{q} = \liminf_{q \rightarrow \infty} \frac{u_q(\delta, \infty)}{q}$$

If there is no such subsequence then both sides go to infinity and there is nothing to say. Otherwise we conclude as in Lemma 2.3.

We see that we can adapt in the same fashion the proof of Theorem 2.4 (which stands for quadratic  $V$ ) to our setting and get that  $\mu \mapsto T_P^V(\mu)$  is convex, which concludes the proof.  $\square$



**4.3 Convergence of the free energy and large deviation principle**

In the case where  $V(x) = ax^{2k}, a > 0$ , Lemmas 4.1 and 4.4 of the previous two sections showed that a large deviation principle holds for the empirical measure of the eigenvalues of  $L_N$  under  $\mathbb{T}_N^{V,P}$  with good, convex rate function which, using [2, Theorem 4.5.10], can be represented as

$$T_P^V(\mu) = - \inf_{W \in C_b^0} \left\{ \int W d\mu + F_{\mathbb{T}}^{V+W,P} - F_{\mathbb{T}}^{V,P} \right\} \tag{57}$$

where

$$F_{\mathbb{T}}^{V,P} = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \int e^{-\text{Tr}V(L_N)} d\mathbb{T}_N^P.$$

To identify  $T_P^V$  and its minimizer, our goal is to show that

**Lemma 4.5.** *For  $a > 0$  and  $V(x) = ax^{2k} + U(x)$  with  $U \in C_b^0(\mathbb{R})$ , for every  $P > 0$ , we have*

$$\int_0^1 F_{\mathbb{T}}^{V,sP} ds = F_C^{V,P}. \tag{58}$$

As a consequence, the unique minimizer of  $T_P^V$  is given by  $\nu_P^V = \partial_P(P\mu_P^V)$  with  $\mu_P^V$  the equilibrium measure for the  $\beta$ -ensemble with parameter  $\beta = 2P/N$ .

*Proof.* We first prove (58). Clearly, for all bounded continuous functions  $U, U'$ , uniformly in  $P$ ,

$$|F_{\mathbb{T}}^{ax^{2k}+U,P} - F_{\mathbb{T}}^{ax^{2k}+U',P}| \leq \|U - U'\|_{\infty} \text{ and } |F_C^{ax^{2k}+U,P} - F_C^{ax^{2k}+U',P}| \leq \|U - U'\|_{\infty}.$$

Therefore it is enough to prove (58) for  $U \in C_b^1(\mathbb{R})$  by density. We prove that for  $U \in C_b^1(\mathbb{R})$ ,

$$F_{\mathbb{T}}^{V,P} = \partial_P(PF_C^{V,P}). \tag{59}$$

Let us consider the tridiagonal matrix  $C_P^N$  of the Coulomb model with distribution  $\mathbb{P}_N^{\frac{2P}{N}}$ . We decompose, for  $\varepsilon > 0$ , this matrix as

$$C_P^N = \begin{pmatrix} M_P^{[\varepsilon N]} & R_N \\ R_N^T & C_{P_N^{\varepsilon}}^{N_{\varepsilon}} \end{pmatrix}$$

where  $M_P^{[\varepsilon N]}$  is a  $[N\varepsilon] \times [N\varepsilon]$  tri-diagonal symmetric matrix with standard independent Gaussian variables on the diagonal and chi distributed variables above the diagonal with parameters  $2\frac{i}{N}P, N - [\varepsilon N] \leq i \leq N - 1$ ,  $C_{P_N^{\varepsilon}}^{N_{\varepsilon}}$  is a  $N_{\varepsilon} = N - [\varepsilon N]$  square tridiagonal Coulomb matrix with parameter  $2P_N^{\varepsilon}/N$  with  $P_N^{\varepsilon} = N_{\varepsilon}N^{-1}P = (1 - [\varepsilon N]/N)P$ , and  $R_N$  has only one non-zero entry  $r$  at position  $([\varepsilon N], [\varepsilon N] + 1)$ . Our first goal is to show that, with  $V(x) = ax^{2k} + U(x)$ , we have

$$\lim_{N \rightarrow \infty} \frac{1}{\varepsilon N} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] = \frac{1}{\varepsilon} (F_C^{V,P} - F_C^{V,P-\varepsilon}) + F_C^{V,P-\varepsilon}. \tag{60}$$

We will then complete the argument by showing that

$$\lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} \frac{1}{\varepsilon N} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] = F_{\mathbb{T}}^{V,P} \tag{61}$$

We next turn to the proof of (60). Let us denote

$$\tilde{C}_P^N = \begin{pmatrix} M_P^{[\varepsilon N]} & 0 \\ 0 & C_{P_N^{\varepsilon}}^{N_{\varepsilon}} \end{pmatrix}.$$

We now show that

$$\text{Tr}((C_P^N)^{2k}) \geq \text{Tr}((\tilde{C}_P^N)^{2k}). \tag{62}$$

Indeed, by Klein’s lemma [1, Lemma 4.4.12],  $B \mapsto \text{Tr}(B^{2k})$  is convex on the set of symmetric matrices. Moreover  $\nabla \text{Tr}(B^{2k}) = (2kB^{2k-1})_{ij}$ . As a consequence, for any symmetric matrices  $A, B$

$$\text{Tr}((A + B)^{2k}) - \text{Tr}(B^{2k}) \geq \text{Tr}(2kB^{2k-1}A).$$

We apply the above inequality with  $A = C_P^N - \tilde{C}_P^N$  and  $B = \tilde{C}_P^N$  and notice that the entry  $[\varepsilon N], [\varepsilon N] + 1$  of  $(\tilde{C}_P^N)^{2k-1}$  vanishes so that  $\text{Tr}((\tilde{C}_P^N)^{2k-1}(C_P^N - \tilde{C}_P^N)) = 0$ , proving (62).

Moreover, if  $U$  is  $C_b^1$ ,

$$|\text{Tr}(U(C_P^N)) - \text{Tr}(U(\tilde{C}_P^N))| \leq \int_0^1 |\text{Tr}(U'(\alpha C_P^N + (1-\alpha)\tilde{C}_P^N)(C_P^N - \tilde{C}_P^N))| d\alpha \leq \|U'\|_\infty |r| \tag{63}$$

Consequently, using the independence of  $r$  and  $\tilde{C}_P^N$  and the fact that  $C_U = \mathbb{E}[e^{+\|U'\|_\infty|r|}]$  is finite since  $r$  has sub-Gaussian distribution, we deduce from (62) that

$$\mathbb{E}[e^{-\text{Tr}(V(C_P^N))}] \leq \mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N)) + \|U'\|_\infty|r|}] \leq C_U \mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N))}]. \tag{64}$$

As a consequence

$$\mathbb{E}[e^{-\text{Tr}(V(C_P^N))}] \leq C_U \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \mathbb{E}[e^{-\text{Tr}(V(C_{P_N^\varepsilon}^N))}]$$

which gives the desired lower bound:

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \geq F_C^{P,V} - (1-\varepsilon)F_C^{P(1-\varepsilon),V} \tag{65}$$

where we used that Theorem 3.1 is valid for  $P_N^\varepsilon \rightarrow (1-\varepsilon)P$ .

To get the complementary lower bound we restrict ourselves to

$$\{|r| \leq \frac{1}{N}\} \cap \{\frac{1}{N} \text{Tr}((\tilde{C}_P^N)^{2k}) \leq M\}$$

Because of (63) and applying Hölder’s inequality as in the proof of Lemma 4.3, we see that on this set  $\text{Tr}(V(C_P^N)) - \text{Tr}(V(\tilde{C}_P^N))$  goes to zero uniformly for all  $M$ . On the other hand the probability of the set  $\{|r| \leq \frac{1}{N}\}$  is of order  $1/N$ . Again by independence we deduce that

$$\begin{aligned} \mathbb{E}[e^{-\text{Tr}(V(C_P^N))}] &\geq e^{o(1)} \mathbb{E}[1_{\{|r| \leq \frac{1}{N}\} \cap \{\frac{1}{N} \text{Tr}((\tilde{C}_P^N)^{2k}) \leq M\}} e^{-\text{Tr}(V(\tilde{C}_P^N))}] \\ &\geq e^{o(1)} \left( \mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N))}] - \mathbb{E}[1_{\{\frac{1}{N} \text{Tr}((\tilde{C}_P^N)^{2k}) \geq M\}} e^{-\text{Tr}(V(\tilde{C}_P^N))}] \right). \end{aligned} \tag{66}$$

But we can show exactly as in the proof of Lemma 4.1 that for  $M$  large enough

$$\limsup_{N \rightarrow \infty} \frac{\mathbb{E}[1_{\{\text{Tr}((\tilde{C}_P^N)^{2k}) \geq MN\}} e^{-\text{Tr}(V(\tilde{C}_P^N))}]}{\mathbb{E}[e^{-\text{Tr}(V(\tilde{C}_P^N))}]} \leq \frac{1}{2},$$

yielding the desired lower bound and therefore (60).

To prove (61), we proceed by approximation. We notice that if we denote by  $D_T^\varepsilon$  the density of the distribution of  $M_P^{[\varepsilon N]}$  with respect to the distribution of a Toda matrix  $\tilde{L}_{[\varepsilon N]}$  with parameter  $P$  to which we removed the extreme entries at  $(1, [\varepsilon N])$  and  $([\varepsilon N], 1)$ , then we get

$$D_T^\varepsilon = \prod_{i=1}^{N\varepsilon} b_i^{-2P(\frac{i}{N})}.$$

Therefore

$$\begin{aligned} \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] &\geq e^{-\varepsilon^2 NM} \mathbb{E}[e^{-\text{Tr}V(\tilde{L}_{[\varepsilon N]})} \mathbf{1}_{-2P \sum_{i=1}^{\varepsilon N} \frac{i}{N} \ln b_i \geq -\varepsilon^2 NM}] \\ &= e^{-\varepsilon^2 NM} \mathbb{E}[e^{-\text{Tr}V(\tilde{L}_{[\varepsilon N]})}] (1 - \tilde{\mathbb{T}}_{[N\varepsilon]}^{V,P}(-2P \sum_{i=1}^{\varepsilon N} \frac{i}{N} \ln b_i \leq -\varepsilon^2 NM)) \end{aligned}$$

On the other hand

$$\{2P \sum_{i=1}^{\varepsilon N} \frac{i}{N} \ln b_i \geq \varepsilon^2 NM\} \subset \{P \frac{1}{N\varepsilon} \sum_{i=1}^{\varepsilon N} b_i^2 \geq M\} \subset \{\frac{1}{N\varepsilon} \text{Tr}((\tilde{L}_{[N\varepsilon]})^2) \geq M/P\}$$

has exponentially small probability under  $\tilde{\mathbb{T}}_{[N\varepsilon]}^{V,P}$  for  $\varepsilon$  enough. This shows, using Lemma 4.3, that there exists a finite constant  $M$  such that

$$\liminf_{N \rightarrow \infty} \frac{1}{N\varepsilon} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \geq F_{\mathbb{T}}^{V,P} + M\varepsilon$$

Similarly, we can see that the density  $\tilde{D}_T^\varepsilon = \prod_{i=1}^{N\varepsilon} b_i^{2P(\frac{i}{N}-\varepsilon)}$  of the law a Toda matrix  $\tilde{L}_{[\varepsilon N]}$  with respect to  $M_P^{[\varepsilon N]}$  is bounded below by  $-\varepsilon^2 NM$  on  $\{\sum_{i=1}^{\varepsilon N} (\varepsilon - \frac{i}{N}) \ln b_i \leq \varepsilon^2 NM\}$  so that we get similarly a finite constant  $M'$  such that

$$\limsup_{N \rightarrow \infty} \frac{1}{N\varepsilon} \ln \mathbb{E}[e^{-\text{Tr}V(M_P^{[\varepsilon N]})}] \leq F_{\mathbb{T}}^{V,P(1-\varepsilon)} + M'\varepsilon \tag{67}$$

We hence conclude by the continuity of  $\varepsilon \rightarrow F_{\mathbb{T}}^{V,P(1-\varepsilon)}$  (which is due to its convexity) Equality (59) follows then from (67).

We finally show that (58) implies that  $T_P^V$  achieves its minimum value at  $\partial_P(P\mu_P^V)$ . Indeed, by (57), for any bounded continuous  $U$ , any probability measure  $\nu$ , we have

$$T_P^V(\nu) \geq - \left( \int U d\nu + F_{\mathbb{T}}^{V+U,P} - F_{\mathbb{T}}^{V,P} \right)$$

We integrate this inequality at  $\nu = \nu_{sP}$  a measurable probability measure valued process such that  $\mu = \int_0^1 \nu_{sP} ds$  to deduce from (58) that

$$\int_0^1 T_P^V(\nu_{sP}) ds \geq - \left( \int U d\mu + F_C^{V+U,P} - F_C^{V,P} \right).$$

We finally optimize over  $U$  to conclude that

$$\int_0^1 T_P^V(\nu_{sP}) ds \geq - \inf_U \left( \int U d\mu + F_C^{V+U,P} - F_C^{V,P} \right) = I_P^V(\mu).$$

Since  $I_P^V$  vanishes only at  $\mu_P^V$  we deduce that any measurable minimizing path  $(\nu_{sP})_{0 \leq s \leq 1}$  must satisfy  $\int_0^1 \nu_{sP} ds = \mu_P^V$ . If we can consider a continuous  $s \mapsto \nu_{sP}$ , we conclude that  $\partial_P(P\mu_P^V)$  makes sense and that it is equal to  $\nu_P$ . We therefore now show that such a path can be chosen to be continuous. But we can follow arguments similar to those of Corollary 2.7 to show that the set  $M_P^V$  where  $T_P^V$  achieves its minimum value is a compact convex subset of  $\mathcal{P}(\mathbb{R})$  and is continuous in the sense that for any  $\varepsilon > 0$ , there exists  $\delta_\varepsilon > 0$  such that for all  $\delta < \delta_\varepsilon$ , any  $P, Q > 0$  such that for  $|P - Q| \leq \delta$

$$M_Q^V \subset (M_P^V)^\varepsilon.$$

Indeed, even if we do not have the coupling of Corollary 2.7, we easily see that the density of  $\mathbb{T}_N^{V,Q}$  with respect to  $\mathbb{T}_N^{V,P}$  is bounded by  $e^{MN|P-Q|}$  with probability greater

than  $1 - e^{-c(M)N}$  with  $c(M)$  going to infinity when  $M$  goes to infinity. Indeed, the density equals  $(P - Q) \sum \ln b_i$  from which the remark follows from Lemma 4.2. This implies that

$$- \inf_{((M_P^V)^\varepsilon)^c} T_Q^V \leq \max\{M|Q - P| - \inf_{((M_P^V)^\varepsilon)^c} T_P^V, -c(M)N\}$$

which implies that for any  $\varepsilon > 0$ , for  $M$  large enough and  $|Q - P|$  small enough  $\inf_{((M_P^V)^\varepsilon)^c} T_Q^V > 0$ , from which the continuity follows.  $\square$

## 5 Proof of Theorem 1.1 and 1.3

Lemma 4.4 combined with the exponential tightness of Lemma 4.1 proves a large deviation principle for the potential  $V(x) = ax^{2k}$ . If now we consider the case where  $V(x)/x^{2k}$  goes to  $a > 0$  at infinity, we can always write  $V(x) = ax^{2k} + U(x)$  where  $U(x)/x^{2k}$  goes to zero at infinity. We have seen by Lemma 4.1 that under  $\mathbb{T}_N^{P,V}$ , the event  $\{\frac{1}{N} \text{Tr}(L_N^{2k}) > M\}$  has exponentially small probability. Let for  $\varepsilon > 0$ ,  $V_\varepsilon(x) = ax^{2k} + (1 + \varepsilon x^{2k})^{-1}U(x)$ . Then, the large deviation principle for the distribution of  $\hat{\mu}_{L_N}$  under  $\mathbb{T}_N^{V_\varepsilon, P}$  follows from Varadhan's lemma. Moreover, on  $\{\text{Tr}(L_N^{2k}) \leq MN\}$ , if  $|U(x)| \leq \delta x^{2k}$  on  $|x| \geq L$ ,

$$\begin{aligned} \left| \frac{1}{N} \text{Tr}V(L_N) - \frac{1}{N} \text{Tr}V_\varepsilon(L_N) \right| &\leq \frac{\varepsilon L^{2k}}{1 + \varepsilon L^{2k}} \max_{|x| \leq L} |U(x)| + \delta \varepsilon \frac{1}{N} \text{Tr}\left(\frac{L_N^{4k}}{1 + \varepsilon L_N^{2k}}\right) \\ &\leq \frac{\varepsilon L^{2k}}{1 + \varepsilon L^{2k}} \max_{|x| \leq L} |U(x)| + M\delta \end{aligned}$$

which is as small as wished if  $M$  is fixed,  $L$  taken large so that  $\delta$  is small, provided  $\varepsilon$  is taken small enough. This shows that we can approximate  $\mathbb{T}_N^{V,P}$  by  $\mathbb{T}_N^{V_\varepsilon, P}$  in the exponential scale from which the result follows.

The proof of Theorem 1.3 follows the same arguments than those developed in the last section: we approximate the general variance profile by a stepwise constant profile, remove a negligible number of off diagonal entries and then use the large deviation principle for the Toda matrices. We leave the details to the reader.

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