Electronic Journal of Probability

Electron. J. Probab. **27** (2022), article no. 41, 1–14.
ISSN: 1083-6489 https://doi.org/10.1214/22-EJP765

Contact processes on general spaces. Models on graphs and on manifolds

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Abstract

The contact process is a particular case of birth-and-death processes on infinite particle configurations. We consider the contact processes on locally compact separable metric spaces. We prove the existence of a one-parameter set of invariant measures in the critical regime under the condition imposed on the associated Markov jump process. This condition means that any pair of independent trajectories of this jump process run away from each other. The general scheme can be applied to the contact process on the lattice in a heterogeneous and random environments as well as to the contact process on graphs and on manifolds.

Keywords: birth and death process; infinite particle configurations; critical regime; correlation functions; hierarchical equations.
MSC2020 subject classifications: 82C22; 82B21; 60K35.
Submitted to EJP on April 3, 2021, final version accepted on March 10, 2022.

1 Introduction

Starting from the pioneer papers of Harris [5], Holley and Liggett [6] the contact process has become one of the most widely used population dynamics model, see also the monographs of Liggett [15, 16]. Notice that the contact model on $\mathbb{Z}^d$ considered in this work is not the same as the basic contact process on $\mathbb{Z}^d$ with the state space $\{0,1\}^{\mathbb{Z}^d}$, see e.g. [16]. While in most of the works the contact processes were considered on the lattice $\mathbb{Z}^d$, much of the interest in the recent years has focused on studying the contact processes in continuous spaces, see e.g. [4, 7, 11]. Contact processes are a particular case of continuous time birth and death processes on infinite particle configurations, and one of the basic problems concerning a contact process is to determine a stationary regime and to prove the existence of stationary measures. In the mathematical literature, the birth and death rates of the contact processes are usually taken to be homogeneous

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(in space); therefore, the corresponding stationary measures on configurations are translation invariant. Since homogeneous models do not quite accurately reflect reality due to heterogeneity in biological or social populations, contact processes in general spaces as well as in heterogeneous and random environments are of great importance for a better understanding of real-world networks.

One of the main features of the contact process is the clustering of the system, i.e. particles are grouped into large clouds of high density, which are located at large distances from each other. It is worth noting that the appearance of a limiting invariant state is only possible in the so-called critical regime, i.e. there is a certain balance between birth and death. As shown in [7], in the case of dimensions $d \geq 3$, there is a continuum of invariant measures parameterized by density values. For small dimensions $d = 1, 2$, the family of invariant measures was constructed under a condition that the dispersal kernel has a heavy tail, see [8]. These invariant measures are described by a simple recurrent relation between their correlation functions and create a concrete (and up to our knowledge, completely new) class of random point fields. For all other regimes, the density of the system tends either to ∞ or to 0 as time grows. The existence of invariant measures in the marked contact model in $\mathbb{R}^d,\ d \geq 3$, with a compact spin space describing dynamics of a population with mutations was proved in [10].

The goal of this work is to construct a family of invariant measures of critical contact processes on general spaces. Our approach is based on the analysis of the infinite system of hierarchical equations for correlation functions, that has been studied earlier for the contact process in $\mathbb{R}^d$, see e.g. [4, 7]. We discuss these constructions and present the main result in Section 3. In Section 2, we formulate assumptions on the model that imply the existence of invariant measures for the contact processes on general state spaces. In particular, our approach can be applied to the contact processes on a hyperbolic (Lobachevsky) space, on a Cayley tree as well as to the contact process on $\mathbb{Z}^d$ in inhomogeneous and random environments, see Section 4. In Section 4 we also present all known results concerning invariant measures of the contact process in $\mathbb{R}^d$, $d \geq 1$. Finally, Section 5 contains the proof of the main results.

2 The model

Let $\mathfrak{X}$ be a locally compact separable metric space, $\mathcal{B}(\mathfrak{X})$ its Borel σ-algebra, and m is a locally finite Borel measure on $\mathcal{B}(\mathfrak{X})$, i.e. m is finite on compact sets. We denote by $\mathcal{M}(\mathfrak{X})$ the space of locally finite Borel measures on $\mathcal{B}(\mathfrak{X})$. The system of all compact sets from $\mathcal{B}(\mathfrak{X})$ is designated by $\mathcal{B}_{\mathrm{b}}(\mathfrak{X})$.

A configuration $\gamma \in \Gamma(\mathfrak{X})$ on $\mathfrak{X}$ is a finite or countably infinite locally finite unordered set of points in $\mathfrak{X}$, and some of them can be multiple, i.e. repetitions are permitted. If the measure m is atomic then any configuration γ can have multiple points. Such situation will be on graphs were the measure m is a counting measure. As the phase space Γ of the continuous contact models, when m is non-atomic (see e.g. [7, 8, 11]), one can take the set of locally finite configurations in $\mathfrak{X}$ with distinct elements:

$$\Gamma_c = \Gamma_c(\mathfrak{X}) := \Big\{ \gamma \subset \mathfrak{X} \ \Big| \ |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathfrak{X}) \Big\}, \tag{2.1}$$

where $|\cdot|$ denotes the number of elements of a set.

In the case of general position, we can identify each $\gamma \in \Gamma$ with an integer-valued measure $\sum_{x\in\gamma} \delta_x \in \mathcal{M}(\mathfrak{X})$, where δ_x is the Dirac measure with unit mass, and the sum is taken considering the multiplicity of elements in the configuration γ. For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathfrak{X})$ we denote by $|\gamma \cap \Lambda|$ the value $\gamma(\Lambda)$ of the measure γ on Λ.

The contact model is a continuous time Markov process on $\Gamma(\mathfrak{X})$ which is a particular case of a general birth-and-death process. The contact model is given by a heuristic generator defined on a proper class of functions $F : \Gamma \to \mathbb{R}$ as follows:

$$(LF)(\gamma) = \sum_{x\in\gamma} [F(\gamma\backslash x) - F(\gamma)] + \int_{\mathfrak{X}} \sum_{x\in\gamma} a(y,x)(F(\gamma \cup y) - F(\gamma))m(dy). \tag{2.2}$$

Notations $\gamma\backslash x$ and $\gamma \cup x$ in (2.2) stand for removing and adding one particle at position $x \in \mathfrak{X}$. The first term in (2.2) corresponds to the death of one particle at position x: each element of the configuration $\gamma \in \Gamma$ dies with the death rate 1. The second term of (2.2) describes the birth of a new particle in a neighborhood dy of the point y with the birth rate density $b(y,\gamma) := \sum_{x\in\gamma} a(y,x)$. Function $a(x,y)$ is called the dispersal kernel.

We assume that $a : \mathfrak{X} \times \mathfrak{X} \to [0,\infty)$ is a non-negative bounded measurable function satisfying the following conditions:

1. *Regularity* condition: there exists a constant $C > 0$, such that

$$\sup_{x\in\mathfrak{X}} \int_{\mathfrak{X}} a(y,x)m(dy) \ < \ C; \tag{2.3}$$

2. *Critical regime* condition:

$$\int_{\mathfrak{X}} a(x,y)m(dy) \ = \ 1 \quad \text{for all } x \in \mathfrak{X}; \tag{2.4}$$

3. *Transience* condition. Let us consider the jump Markov process (random walk in continuum) with generator

$$\mathcal{L}f(x) = \int_{\mathfrak{X}} a(x,y)\big(f(y) - f(x)\big)m(dy). \tag{2.5}$$

   Then we assume that for any two independent copies $X(t)$ and $Y(t)$ of this process starting with $X(0) = x$ and $Y(0) = y$ the following condition holds

$$\sup_{x,y} \int_0^\infty \mathbb{E}_{x,y} a(X(t),Y(t))dt < Q \tag{2.6}$$

   with a constant $Q > 0$. Moreover, we assume that the integral in (2.6) converges uniformly in x, y.

Remark 2.1. Condition (2.4) implies that the average density of configurations is preserved in time. Namely, if we start with a random configuration $\gamma(0)$ such that $\mathbb{E}|\gamma(0) \cap V| = \varrho\, m(V)$ for any compact V with some $\varrho > 0$, then this relation $\mathbb{E}|\gamma(t) \cap V| = \varrho\, m(V)$ will be valid for any $t \geq 0$. It is worth noting that in the case when $a(x,y) = a(y,x)$ or $a(x,y) = a(x-y)$ and $\mathfrak{X} = \mathbb{R}^d$ with the Lebesgue measure $m(dx)$ or $\mathfrak{X} = \mathbb{Z}^d$ with the counting measure m, the critical regime condition (2.4) implies that the branching rate $I(x) = \int a(y,x)m(dy)$ is equal to 1 for any $x \in \gamma$.

Remark 2.2. The sufficient condition for (2.6) together with required uniform convergence reads

$$\int_0^\infty \sup_{x,y} \ \mathbb{E}_x a(X(t),y)dt < Q. \tag{2.7}$$

Proof. Denote by $p(x,dy,t)$ the transition function of the Markov jump process with generator (2.5) at time t. Then we get

$$\sup_{x,y} \int_0^\infty \mathbb{E}_{x,y} a(X(t),Y(t))dt = \sup_{x,y} \int_0^\infty \int_{\mathfrak{X}} \int_{\mathfrak{X}} a(x',y')p(x,dx',t)p(y,dy',t)dt \leq$$

$$\sup_y \int_0^\infty \int_{\mathfrak{X}} \Big(\sup_x \int_{\mathfrak{X}} a(x',y')p(x,dx',t) \Big) p(y,dy',t)dt =$$
$$\sup_y \int_0^\infty \int_{\mathfrak{X}} \Big(\sup_x \mathbb{E}_x a(X(t),y') \Big) p(y,dy',t)dt \le$$
$$\int_0^\infty \sup_y \int_{\mathfrak{X}} \Big(\sup_{y'} \sup_x \mathbb{E}_x a(X(t),y') \Big) p(y,dy',t)dt = \int_0^\infty \sup_{x,y'} \mathbb{E}_x a(X(t),y')dt.$$

Therefore, condition (2.7) implies the uniform convergence in (2.6). □

3 Time evolution of correlation functions. Main results

The study of evolution of the infinite-particle system generated by the operator (2.2) may be realized through the *forward Kolmogorov* (or *Fokker–Planck*) equation with the evolution operator L for probability measures (states) on the configuration space Γ, i.e.

$$\frac{d}{dt}\mu_t(F) = \mu_t(LF), \quad t > 0, \quad \mu_t\big|_{t=0} = \mu_0, \tag{3.1}$$

where

$$\mu(F) := \int_\Gamma F(\gamma)\, d\mu(\gamma)$$

.

Denote by $\mathcal{M}_{fm}(\Gamma)$ the set of all probability measures μ which have finite local moments of all orders, i.e.

$$\int_\Gamma |\gamma \cap \Lambda|^n \; \mu(d\gamma) \; < \; \infty$$

for all $\Lambda \in \mathcal{B}_b(\mathfrak{X})$ and $n \in N$, and let $\mathcal{M}_{\mathrm{corr}}(\Gamma)$ be the subclass of $\mathcal{M}_{fm}(\Gamma)$ consisting of those probability measures on Γ for which correlation functions exist. The terminology originates in statistical mechanics, where the densities of correlation measures with respect to the finite products of the measure m are called correlation functions, see, for instance, [19, Ch. 4] for the case $\mathfrak{X} = \mathbb{R}^d$, or [13, 14], for the general space $\mathfrak{X}$.

The evolution equation for the system of n-point correlation functions corresponding to the continuous contact model in $\mathbb{R}^d$ has been derived previously in [7, 8]. This equation for the general contact model in $\mathfrak{X}$ can be considered in the same way. The equation has the following recurrent forms:

$$\frac{\partial k_t^{(n)}}{\partial t} \; = \; \hat{L}_n^* k_t^{(n)} \; + \; f_t^{(n)}, \quad n \ge 1; \qquad k_t^{(0)} \equiv 1, \tag{3.2}$$

where

$$\begin{aligned} \hat{L}_n^* k^{(n)}(x_1,\ldots,x_n) &= -n\, k^{(n)}(x_1,\ldots,x_n) \\ &+ \sum_{i=1}^n \int_{\mathfrak{X}} a(x_i,y) k^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) m(dy). \end{aligned} \tag{3.3}$$

Here $f_t^{(n)}$ are functions on $\mathfrak{X}^n$ defined for $n \ge 2$ by

$$f_t^{(n)}(x_1,\ldots,x_n) \; = \; \sum_{i=1}^n k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n) \sum_{j \ne i} a(x_i,x_j), \tag{3.4}$$

and $f_t^{(1)} \equiv 0$. The notation $\check{x}_i$ means that the i-th coordinate is excluded.

We consider here the initial data $k_0 = \{k_0^{(n)}\}$ corresponding to the Poisson measure π_ϱ with intensity ϱ:

$$k_0^{(0)} = 1, \quad k_0^{(n)}(x_1,\ldots,x_n) = \varrho^n, \; n \ge 1. \tag{3.5}$$

Let $\mathbb{X}_n = \mathbb{B}(\mathfrak{X}^n)$ be the Banach space of all measurable real-valued bounded functions on $\mathfrak{X}^n$ with the $\sup$-norm. Consider the operator $\hat{L}_n^*$ as an operator on the Banach space $\mathbb{X}_n$ for any $n \geq 1$. Then it is a bounded linear operator in $\mathbb{X}_n$, and the arguments based on the variation of parameters formula yields that

$$k_t^{(n)} \;=\; e^{t\hat{L}_n^*} k_0^{(n)} \;+\; \int_0^t e^{(t-s)\hat{L}_n^*} f_s^{(n)} \, ds, \tag{3.6}$$

where $f_s^{(n)}$ is expressed through $k_s^{(n-1)}$ by (3.4). Thus, the solution to the Cauchy problem (3.2) in $\mathbb{X}_n$ with arbitrary initial values $k_0^{(n)} \in \mathbb{X}_n$ exists and is unique provided $f_t^{(n)}$ is constructed recurrently via the solution to the same Cauchy problem (3.2) for $n-1$.

The goal of this paper is to construct a family of invariant measures on Γ for the critical regime contact processes on general spaces. These measures are described in terms of their correlation functions $\{k^{(n)}\}_{n\geq 0}$ as solutions to the following system:

$$\hat{L}_n^* k^{(n)} + f^{(n)} = 0, \quad n \geq 1, \quad k^{(0)} \equiv 1, \tag{3.7}$$

where $\hat{L}_n^*$, $f^{(n)}$ were defined by (3.3)-(3.4), but with the replacement of $k_t^{(n)}$ by $k^{(n)}$. In the sequel, we say that $k : \Gamma_0 \to \mathbb{R}$ solves the system (3.7) in the Banach spaces $(\mathbb{X}_n)_{n\geq 1}$ if the corresponding $k^{(n)} \in \mathbb{X}_n$, $n \geq 1$ solve (3.7).

The main result of the paper is the following theorem.

Theorem 3.1. *Assume that the contact process satisfies conditions (2.3), (2.4) and (2.6). Then the following assertions hold.*

(i) For any positive constant $\varrho > 0$ there exists a probability measure $\mu^\varrho \in \mathcal{M}_{\mathrm{corr}}(\Gamma)$ on Γ such that its correlation function $k_\varrho : \Gamma_0 \to \mathbb{R}_+$ solves (3.7) in the Banach spaces $(\mathbb{X}_n)_{n\geq 1}$, and the corresponding system $\{k_\varrho^{(n)}\}_{n\geq 1}$ satisfies $k_\varrho^{(1)} \equiv \varrho$. Moreover, there exists a positive constant D such that

$$k_\varrho^{(n)}(x_1, \ldots, x_n) \;\leq D Q^n (n!)^2 \qquad \text{for all} \quad (x_1, \ldots, x_n) \in \mathfrak{X}^n, \tag{3.8}$$

where Q is the same constant as in (2.6).

(ii) Let $\{k_t^{(n)}\}_{n\geq 1}$ be the solution to the Cauchy problem (3.2) with initial value (3.5). Then

$$\|k_t^{(n)} \;-\; k_\varrho^{(n)}\|_{\mathbb{X}_n} \;\to\; 0, \quad t \to \infty, \quad \forall n \geq 1. \tag{3.9}$$

It is worth noting that Theorem 3.1 states the existence of a family of invariant measures $\mu^\varrho \in \mathcal{M}_{\mathrm{corr}}(\Gamma)$ for the contact process in the critical regime. These measures are indexed by a positive parameter $\varrho > 0$. Each measure from this family is uniquely determined by its own system of correlation functions. These correlation functions are constructed as the limit as $t \to \infty$ of solutions of the evolution equation for the correlation functions with initial data corresponding to the Poisson measure with intensity $\varrho > 0$. We do not discuss the uniqueness problem of the invariant measure in this paper.

The main strategy of the proof follows the same line as the proof in the case $\mathfrak{X} = \mathbb{R}^d$, see [7, 8]. However, in the present paper we should modify some of steps of the previous proof for the general models. These modifications include using the condition (2.6) instead of the Fourier transform for homogeneous models in $\mathbb{R}^d$.

Remark 3.2. We can include in our model a possibility to jump. The analogous model in $\mathbb{R}^d$ has been considered earlier in [8, 9]. More precisely, let us consider the following heuristic generator $L \;+\; L_J$, where L was defined by (2.2),

$$L_J F(\gamma) \;=\; \int_{\mathfrak{X}} \sum_{x\in\gamma} J(y,x) \Big(F((\gamma \setminus x) \cup y) - F(\gamma) \Big) \, m(dy). \tag{3.10}$$

Suppose that the total jump rate $\int J(y,x)m(dy)$ is uniformly bounded in x:

$$\sup_x \int_{\mathfrak{X}} J(y,x)m(dy) \ < \ C. \tag{3.11}$$

Then the criticality condition is

$$\int_{\mathfrak{X}} \big(a(x,y)+J(x,y)-J(y,x)\big)\ m(dy)=1 \tag{3.12}$$

("birth"+"immigration"-"emigration"="mortality"). The operator $\mathcal{L}_J$ analogous to (2.5) then takes the form

$$\mathcal{L}_J f(x)=\int_{\mathfrak{X}} \big(a(x,y)+J(x,y)\big)\big(f(y)-f(x)\big)m(dy), \tag{3.13}$$

and "transience" condition (2.6) can be written as

$$\sup_{x,y} \int_0^\infty \mathbb{E}_{x,y} a(\tilde X(t),\tilde Y(t))dt < Q, \tag{3.14}$$

where $a(x,y)$ is the same dispersal kernel as above satisfying (2.3), (2.4), (2.6), while $\tilde X(t)$ and $\tilde Y(t)$ are two independent copies of the Markov process with generator $\mathcal{L}_J$ given by (3.13).

4 Particular models

We start this section with the homogeneous contact model in $\mathbb{R}^d$ that has been studied in [7, 8, 10]. Other examples are new.

1. **The homogeneous contact model in $\mathfrak{X}=\mathbb{R}^d$ generated by dispersal kernel** $a(x-y)$. The homogeneous contact model in $\mathbb{R}^d$ has been studied in papers [7, 8, 10], where we have formulated the condition on $a(x-y)$ guaranteeing the existence of a family of invariant measures of the contact model in the critical regime in any dimension $d\ge 1$. Namely, we assume that $a(\cdot)$ possesses the following properties:

- Boundedness and Normalization

$$a(x)\ge 0; \quad a(x)\in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} a(x)dx=1 \tag{4.1}$$

- Regularity condition

$$\hat a(p) \ := \ \int_{\mathbb{R}^d} e^{-i(p,u)}a(u)du \in L^1(\mathbb{R}^d), \tag{4.2}$$

- Existence of the second moment in dimensions $d\ge 3$

$$\int_{\mathbb{R}^d} |x|^2 a(x)dx<\infty; \tag{4.3}$$

- Heavy tail conditions in dimensions $d=1,\ 2$:

$$a(x) \ \sim \ \frac{1}{|x|^{\alpha+2}} \quad \text{as} \ \ |x|\to\infty,\ 0<\alpha<2, \quad (d=2), \tag{4.4}$$

$$a(x) \ \sim \ \frac{1}{|x|^{\alpha+1}} \quad \text{as} \ \ |x|\to\infty,\ 0<\alpha<1, \quad (d=1). \tag{4.5}$$

Then the statements of Theorem 3.1 are true, see [7, 8]. In these cases, the fulfillment of the transience condition (2.6) is verified using the Fourier transform.

We consider in [10] a marked continuous contact model on $\mathfrak{X} = \mathbb{R}^d \times S, d \geq 3$, where S is a compact metric space. The dispersal kernel $a(x,y)$ were defined as

$$a(x,y) = \alpha(\tau(x) - \tau(y))\, \mathcal{Q}(\sigma(x), \sigma(y)), \tag{4.6}$$

where τ and σ are projections of $\mathfrak{X}$ on $\mathbb{R}^d$ and S respectively, $\alpha(\cdot) \geq 0$ is a function on $\mathbb{R}^d$ satisfying conditions (4.1) – (4.3) (the case $d \geq 3$). We suppose that the function $\mathcal{Q}$ on $S \times S$ is continuous (and so bounded) and strictly positive. Moreover we assume that the corresponding integral operator with kernel $\mathcal{Q}(\cdot,\cdot)$ has the maximal in absolute value eigenvalue equal to 1. Then the statements of Theorem 3.1 are true, see [10].

2. **The symmetric contact model on the hyperbolic (Lobachevsky) plane: $\mathfrak{X} = L$.** Let $\rho(x,y)$ be the hyperbolic distance and $m(dx)$ be the corresponding measure on $\mathfrak{X}$. Consider a continuous time random walk on $\mathfrak{X}$ with the generator (2.5), where $a(x,y)$ depends only on $\rho(x,y)$:

$$a(x,y) = a(\rho(x,y)) = a(y,x), \quad \text{and} \;\; a(x,y) = 0 \;\; \text{if} \;\; \rho(x,y) > h, \tag{4.7}$$

for some h. We assume also that

$$\int_{\mathfrak{X}} a(x,y) m(dy) \; = \; 1 \quad \text{for all } x \in \mathfrak{X}. \tag{4.8}$$

Then (4.8) is the same as the critical regime condition (2.4), and it remains to check the fulfillment of the transience condition (2.6), or equivalently condition (2.7).

Denote by $D(y,h)$ a disc centered at y of radius h:

$$D(y,h) = \{y' \in \mathfrak{X} : \; \rho(y,y') \leq h\},$$

and let $P(x, D(y,h), t)$ be the probability for the Markov jump process $X(t)$ with generator L given by (2.5) starting at $x \in \mathfrak{X}$ to be in $D(y,h)$ at time t:

$$P(x, D(y,h), t) = \Pr\big(X(t) \in D(y,h) |\; X(0) = x\big).$$

Lemma 4.1. *There exist $\varkappa > 0$ and $C(h)$ such that the probability $P(x, D(y,h), t)$ satisfies the following estimate*

$$P(x, D(y,h), t) \leq C(h) e^{-\varkappa t} \quad \textit{for all} \;\; x, \; y \in \mathfrak{X} \quad \textit{and} \;\; t \geq 0. \tag{4.9}$$

Proof. Using the estimates from [20] (in the proof of [20, Lemma 1]) we conclude that there exist $\alpha > 0$ and $\gamma > 0$ such that

$$P(x, D(x, \alpha t), t) \leq C_1(h) e^{-\gamma t} \quad \text{for all} \;\; t \geq 0. \tag{4.10}$$

Fix a large $t > 0$. If $D(y,h) \subset D(x, \alpha t)$, then we get

$$P(x, D(y,h), t) \leq C_1(h) e^{-\gamma t}. \tag{4.11}$$

Let us consider the case when $\rho(x,y) > \alpha t - h$ for a given t, where $\alpha > 0$ is the same constant as in (4.10). In this case, using that the length of the circle of a radius r is exponentially large in r we conclude that “the visible angular size from x”, i.e. the angle $\varphi(y,h)$ of a sector centered at x and resting on disk $D(y,h)$ admits the following upper bound

$$\varphi(y,h) \leq \tilde{C}_2(h) e^{-\alpha t}.$$

Then isotropy condition (4.7) implies that

$$P(x, D(y,h), t) \leq C_2(h) e^{-\alpha t} \quad \text{if} \;\; \rho(x,y) > \alpha t - h. \tag{4.12}$$

Taking

$$\varkappa = \min\{\alpha, \; \gamma\}, \quad C(h) = \max\{C_1(h), \; C_2(h)\},$$

we obtain desired estimate (4.9) from (4.11) – (4.12). □

This Lemma immediately implies the convergence of the integral in (2.7), since

$$\sup_{x,y} \mathbb{E}_x a(X(t),y) \le A \sup_{x,y} P(x, D(y,h), t) \le AC(h) e^{-\varkappa t},$$

where $A = \sup a(x,y)$.

The symmetric contact model on the hyperbolic space ($d \ge 3$) can be considered in the similar way.

Further we will consider several contact models on discrete spaces $\mathfrak{X}$, where m is the counting measure.

3. **The homogeneous contact model on the Cayley tree: $\mathfrak{X} = T_k$.**

Consider the Cayley tree T_k, i.e. infinite regular tree with vertex degree $k \ge 3$. The continuous time symmetric random walk $X(t)$ on T_k is given by the generator L defined in (2.5), where $a(x,y) = a(y,x) = \frac{1}{k}$ for $d(x,y) = 1$ and $a(x,y) = 0$ otherwise. Here $d(x,y)$ is the distance on T_k, which is the same as the distance between two vertices in a graph. The measure m on T_k is the counting measure.

It is easy to see that for the trajectory of random walk $X(t)$ starting at $X(0) = x$ and any vertex $y \in T_k$ the distance $d_{x,y}(t) = d(X(t), y)$ is a random walk $Z_{x,y}(t)$ on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with rates $r(0,1) = 1$ and $r(n, n+1) = \frac{k-1}{k},\ r(n, n-1) = \frac{1}{k},\ n = 1, 2, \ldots$. Thus, the random walk $Z_{x,y}(t)$ for any x and y has a positive drift, and the following lemma holds.

Lemma 4.2. *There exist $\varkappa > 0$ and C such that the transition probability $P(x,y,t) = \Pr(X(t) = y |\, X(0) = x)$ meets the following estimate:*

$$P(x,y,t) \le Ce^{-\varkappa t} \quad \textit{for all} \;\; x,\ y \in T_k \quad \textit{and} \;\; t \ge 0. \tag{4.13}$$

Proof. The proof of this lemma is completely analogous to the proof of Lemma 4.1. As above, it is a combination of two estimates. The first bound follows from the fact that the random walk $Z_{x,y}(t)$ for any x and y has a positive drift. Namely, there exist $\alpha > 0$ and $\gamma > 0$ such that

$$P(x, D(x, \alpha t), t) = \Pr(Z_{x,x}(t) < \alpha t) \le C_1 e^{-\gamma t} \quad \text{for all} \;\; t \ge 0, \tag{4.14}$$

where $D(x,R)$ is the ball centered at x of radius R. The second estimate is valid for $y \in T_k$ with $d(x,y) > \alpha t$:

$$P(x,y,t) \le C_2 e^{-\alpha t} \quad \text{for some} \;\; \alpha > 0. \tag{4.15}$$

Estimate (4.15) follows from the observation that the number of $y \in T_k$ such that $d(x,y) = n$ is exponential in n. Finally, (4.13) follows from (4.14) – (4.15). $\square$

Thus, as above we conclude that all required conditions on $a(x,y)$ are fulfilled.

Let us note that the analogous result holds for any tree T with uniformly bounded vertex degrees $k_i \ge 3,\ i \in T$.

4. **Inhomogeneous contact models generated by inhomogeneous random walks on a lattice: $\mathfrak{X} = \mathbb{Z}^d$.** Let us consider a random walk on $\mathbb{Z}^d$ with transition probabilities $P(x,y) = \Pr(x \to y)$ that differ from those of the homogeneous irreducible symmetric walk $\pi(y-x) = \pi(x-y)$ only locally, i.e. in a finite neighborhood of the origin:

$$P(x,y) = \pi(y-x) + V(y,x), \qquad \sum_y P(x,y) = 1 \;\; \forall\, x \in \mathbb{Z}^d, \tag{4.16}$$

with

$$V(y,x) = 0 \;\; \text{if} \;\; \max\{|y|,\ |x|\} > R, \quad \text{and} \;\; \pi(u) = 0, \;\; \text{if} \;\; |u| > R_1$$

for some $R, R_1 > 0$. Thus we can not write the transition probability $P(x,y) = \Pr(x \to y)$ in the form $P(x-y)$. We also assume that the perturbed random walk is irreducible in $\mathbb{Z}^d$.

In the paper [18], the probability $p(x,y,t) = \Pr(X(t) = y|\, X(0) = x)$ has been written as the sum of the probability $p_0(x,y,t)$ for the homogeneous random walk with transition probabilities $\pi(y-x)$ and the correction term:

$$p(x,y,t) \ = \ p_0(x,y,t) \ + \ \delta_t(y|\, x).$$

The probability $p_0(x,y,t)$ is bounded from above uniformly in x and y by $\frac{C}{t^{d/2}}$, see e.g. [12]. It follows from the results of [18] that the correction term $\delta_t(y|\, x)$ admits the same bound for $d \geq 3$. This yields the following uniform in x and y estimate

$$p(x,y,t) \ \leq \ 1 \ \wedge \frac{C}{t^{d/2}}, \quad t \geq 0. \tag{4.17}$$

Taking into account critical regime condition (2.4) one can conclude that estimate (4.17) also holds for the continuous time random walk on $\mathbb{Z}^d$ with jump intensities $a(x,y) = P(x,y)$. Thus, estimate (4.17) implies that in the case $d \geq 3$ the transience condition (2.6) is fulfilled. Consequently, all statements of Theorem 3.1 are valid for the inhomogeneous contact process on $\mathbb{Z}^d$ with the dispersal kernel given by $a(x,y) = P(x,y)$.

5. **Contact models on $\mathfrak{X} = \mathbb{Z}^d$ generated by random conductance models.** Let $\mathbb{E}_d$ be the set of non-oriented nearest neighbour bonds of the lattice $\mathbb{Z}^d$:

$$\mathbb{E}_d = \{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d, \ x \sim y\}, \quad x \sim y \text{ means } x, y \ \text{ are neighbors},$$

and $\mu_e, \ e \in \mathbb{E}_d$, are taken as nonnegative i.i.d.r.v defined on a probability space $(\Omega, \mathbb{P})$. Moreover, assume that

$$c^{-1} \leq \mu_e \leq c \quad \text{for some} \ \ c \geq 1. \tag{4.18}$$

Thus, $\mu_{xy} = \mu_{yx}, \ x \sim y$, are i.i.d. random variables satisfying (4.18), and $\mu_{xy} = 0$ if $x \not\sim y$. Set

$$\mu_x = \sum_y \mu_{xy}, \quad a(x,y) = \frac{\mu_{xy}}{\mu_x},$$

and consider a continuous time random walk on $\mathbb{Z}^d$ with transition rates $a(x,y)$. The generator of this random walk is given by

$$\mathcal{L}_C f(x) = \mu_x^{-1} \sum_y \mu_{xy}(f(y) - f(x)).$$

In this case, see e.g. [1], [3], the following upper bound holds $\mathbb{P}$-a.s. for all $t \geq 0$

$$\sup_{x,y} \Pr_\omega(X(t) = y|\, X(0) = x) \ \leq \ 1 \ \wedge \frac{C}{t^{d/2}}.$$

Moreover, the same result holds for the simple random walk on the infinite Bernoulli (bond) percolation cluster in $\mathbb{Z}^d$, see e.g. [2, 17].

Thus, in both models the transience condition (2.7) (and (2.6)) guaranteeing the existence of the invariant measure of the corresponding contact model is fulfilled in the case $d \geq 3$.

5 The proof of Theorem 3.1

In the proof of the first part of Theorem 3.1 we use the induction in $n \in \mathbb{N}$. For $n = 1$ in (3.7) we have

$$-k^{(1)}(x) + \int_{\mathfrak{X}} a(x,y) k^{(1)}(y) m(dy) = 0. \tag{5.1}$$

It follows immediately that $k^{(1)}(x) \equiv \varrho$ is an element of $\mathbb{X}_1$ and it solves (5.1). We notice that ϱ can be interpreted as the spatial density of particles.

Now let us turn to the general case. If for any $n > 1$ we succeed to solve equation (3.7) and express $k^{(n)}$ through $f^{(n)}$, then knowing the expression of $f^{(n)}$ through $k^{(n-1)}$ (see (3.4)), we get the solution $\{k^{(n)}\}_{n\geq 1}$ to the full system (3.7) recurrently.

Lemma 5.1. *The operator $e^{t\hat{L}_n^*}$, where $\hat{L}_n^*$ was defined in (3.3), is positive, i.e. it maps non-negative functions to non-negative functions.*

Proof. The operator

$$A^i k^{(n)}(x_1, \dots, x_n) \ := \ \int_{\mathfrak{X}} a(x_i, y) k^{(n)}(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) m(dy).$$

is positive and bounded on $\mathbb{X}_n$ for any $1 \leq i \leq n$. Taking into account

$$e^{t\hat{L}_n^*} \ = \ \otimes_{i=1}^n \, e^{-t} e^{tA^i}, \tag{5.2}$$

we get the desired conclusion. □

Next we will construct a solution to the system (3.7) satisfying (3.8). As follows from (3.4), the function $f^{(n)}$ is the sum of functions of the form

$$f_{i,j}(x_1, \dots, x_n) = k^{(n-1)}(x_1, \dots, \check{x}_i, \dots, x_n) a(x_i, x_j), \quad i \neq j. \tag{5.3}$$

We suppose by induction that

$$k^{(n-1)}(x_1, \dots, x_{n-1}) \ \leq \ K_{n-1}, \quad \text{for all } \ (x_1, \dots, x_{n-1}) \in \mathfrak{X}^{n-1}, \quad n \geq 2,$$

where $K_n = DC^n(n!)^2$, and D, C are some constants. Consequently,

$$f_{i,j}(x_1, \dots, x_n) \ \leq \ K_{n-1} a(x_i, x_j), \quad (x_1, \dots, x_n) \in \mathfrak{X}^n. \tag{5.4}$$

Using the positivity of the operator $e^{t\hat{L}_n^*}$ and (5.4) we have

$$\left(e^{t\hat{L}_n^*} f_{i,j}\right)(x_1, \dots, x_n) \ \leq \ K_{n-1} \ \left(e^{t\hat{L}_n^*} a(\cdot_i, \cdot_j)\right)(x_1, \dots, x_n). \tag{5.5}$$

Set

$$\begin{aligned} \mathcal{L}^i k^{(n)}(x_1, \dots, x_n) \ &= \int_{\mathfrak{X}} a(x_i, y) k^{(n)}(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) m(dy) \\ &\quad - k^{(n)}(x_1, \dots, x_n). \end{aligned} \tag{5.6}$$

An easy observation $e^{t\mathcal{L}^i} \mathbb{1} = \mathbb{1}$, $\forall i = 1, \dots, n$, where $\mathbb{1}(x) \equiv 1$, shows

$$\left(e^{t\hat{L}_n^*} a(\cdot_i, \cdot_j)\right)(x_1, \dots, x_n) \ = \ \left(e^{t(\mathcal{L}^i + \mathcal{L}^j)} a(\cdot_i, \cdot_j)\right)(x_1, \dots, x_n). \tag{5.7}$$

Note that the latter function depends only on variables x_i and x_j.

Notice that $e^{t\hat{L}_n^*} f_{i,j}$ is integrable with respect to t on $\mathbb{R}_+$. Indeed, it follows from (5.5), (5.7), condition (2.6) and the identity

$$e^{t\hat{L}_n^*} a(x,y) = \mathbb{E}_{x,y} a(X(t), Y(t)), \tag{5.8}$$

that
$$v_{i,j}^{(n)} \;=\; \int_0^\infty e^{t\hat{L}_n^*} f_{i,j}\; dt \;\le K_{n-1}\, Q, \tag{5.9}$$
where Q is the same constant as in (2.6).

Our next goal is to show that
$$v^{(n)} = \sum_{i\neq j} v_{i,j}^{(n)} = \int_0^\infty e^{t\hat{L}_n^*} f^{(n)} dt \tag{5.10}$$
is a solution to (3.7) in $\mathbb{X}_n$. It is easily seen from (5.9) and induction procedure that $v^{(n)} \in \mathbb{X}_n$. Since $e^{t\hat{L}_n^*}$ is a strongly continuous semigroup we have
$$e^{t\hat{L}_n^*} f^{(n)} - f^{(n)} = \hat{L}_n^* \int_0^t e^{s\hat{L}_n^*} f^{(n)} ds. \tag{5.11}$$
Rewrite (5.11) as
$$e^{t\hat{L}_n^*} f^{(n)} = f^{(n)} + \hat{L}_n^* \int_0^t e^{s\hat{L}_n^*} f^{(n)} ds. \tag{5.12}$$
Then using condition (2.6), inequality (5.4), Lemma 5.1 and the fact that $\hat{L}_n^*$ is a bounded operator we conclude that the right hand side of (5.12) has a uniform in $x_1, \dots, x_n$ limit as $t \to \infty$, therefore, the left hand side of (5.12), i.e. $e^{t\hat{L}_n^*} f^{(n)}$, also converges in $\mathbb{X}_n$. Moreover the limit is a nonnegative function in $\mathbb{X}_n$. However, if this function is somewhere strictly positive, then we get a contradiction with (5.9). Thus, we conclude that the following limit holds in $\mathbb{X}_n$:
$$e^{t\hat{L}_n^*} f^{(n)} \to 0, \quad t \to \infty. \tag{5.13}$$
A passage to the limit in (5.11) as $t \to \infty$ together with (5.13) shows that $v^{(n)}$ defined in (5.10) is a solution to (3.7) in $\mathbb{X}_n$.

Since the function $f^{(n)}$ is the sum of functions $f_{i,j}$, $i \neq j$ we deduce from (5.9) that $v^{(n)}$ is bounded by $n^2 K_{n-1} Q$. Thus we get the recurrence inequality
$$K_n \;\le\; n^2 K_{n-1} Q, \tag{5.14}$$
and by induction it follows that
$$K_n \;\le\; Q^n\, (n!)^2. \tag{5.15}$$
Thus
$$v^{(n)}(x_1, \dots, x_n) \;\le\; Q^n\, (n!)^2. \tag{5.16}$$

Thus, we have constructed $\{v^{(n)}\}_{n\ge 1}$ satisfying estimate (5.16). Of course, any functions of the form
$$k^{(1)} \equiv \varrho, \quad k^{(n)} = v^{(n)} + A_n = \int_0^\infty e^{t\hat{L}_n^*} f^{(n)}\; dt + A_n, \quad n \ge 2,$$
where A_n are arbitrary constants and $f^{(n)}$ is defined by (3.4) are solution to the system (3.7). Taking $A_n = \varrho^n$ we conclude that
$$k_\varrho^{(1)} \equiv \varrho, \quad k_\varrho^{(n)} = v^{(n)} + \varrho^n = \int_0^\infty e^{t\hat{L}_n^*} f^{(n)} dt + \varrho^n, \quad n \ge 2, \tag{5.17}$$
is the desired solution to (3.7) in the Banach spaces $(\mathbb{X}_n)_{n\ge 1}$. To emphasize the dependence of $f^{(n)}$ on ϱ, we will use notation $f_\varrho^{(n)}$ for $f^{(n)}$. For the solutions $\{k_\varrho^{(n)}\}_{n\ge 1}$ of (5.17) instead of (5.14) we have the recurrence
$$K_n \;\le\; n^2 K_{n-1} Q \;+\; \varrho^n, \tag{5.18}$$

which yields

$$K_n \;\le\; DQ^n (n!)^2. \tag{5.19}$$

To be certain that the constructed system $\{k_\varrho^{(n)}\}_{n\ge 1}$ is a system of correlation functions, i.e., it corresponds to a probability measure μ^ϱ on the configuration space Γ, we will prove below that $\{k_\varrho^{(n)}\}_{n\ge 1}$ can be constructed as the limit when $t\to\infty$ of the system of correlation functions $\{k_t^{(n)}\}_{n\ge 1}$ associated with the solution to the Cauchy problem (3.2) with the initial data (3.5).

By the variation of parameters formula we have

$$k_t^{(n)} \;=\; e^{t\hat{L}_n^*} k_0^{(n)} \;+\; \int_0^t e^{(t-s)\hat{L}_n^*} f_s^{(n)}\; ds, \tag{5.20}$$

where $f_s^{(n)}$ is expressed through $k_s^{(n-1)}$ by (3.4). On the other hand, we proved above the existence of the solution $\{k_\varrho^{(n)}\}_{n\ge 1}$ of the stationary problem:

$$\hat{L}_n^* k_\varrho^{(n)} \;=\; -f_\varrho^{(n)},$$

where

$$f_\varrho^{(n)}(x_1,\dots,x_n) \;=\; \sum_{i,j:\; i\neq j} k_\varrho^{(n-1)}(x_1,\dots,\check{x}_i,\dots,x_n)\; a(x_i,x_j).$$

This solution meets the following equation

$$\left(e^{t\hat{L}_n^*} - \mathbf{1}\right) k_\varrho^{(n)} \;=\; -\int_0^t \frac{d}{ds} e^{(t-s)\hat{L}_n^*} k_\varrho^{(n)} ds \;\;=\; -\int_0^t e^{(t-s)\hat{L}_n^*} f_\varrho^{(n)}\; ds,$$

and therefore

$$k_t^{(n)} - k_\varrho^{(n)} \;=\; e^{t\hat{L}_n^*}(k_0^{(n)} - k_\varrho^{(n)}) \;+\; \int_0^t e^{(t-s)\hat{L}_n^*}(f_s^{(n)} - f_\varrho^{(n)})\; ds. \tag{5.21}$$

We will prove now that both terms in the right-hand side of (5.21) converge to 0 in the norm of $\mathbb{X}_n$ as $t\to\infty$.

Formula (5.17) yields

$$e^{t\hat{L}_n^*}\big(k_0^{(n)} - k_\varrho^{(n)}\big) \;=\; -e^{t\hat{L}_n^*} v^{(n)}, \tag{5.22}$$

where

$$v^{(n)} \;=\; \int_0^\infty e^{s\hat{L}_n^*} f_\varrho^{(n)}\; ds. \tag{5.23}$$

Consequently, the first term in the r.h.s. of (5.21) can be rewritten using (5.22) as follows

$$e^{t\hat{L}_n^*}\; v^{(n)} = \int_0^\infty e^{(t+s)\hat{L}_n^*} f_\varrho^{(n)}\, ds = \int_t^\infty e^{r\hat{L}_n^*} f_\varrho^{(n)}\, dr.$$

Due to the uniform convergence of the integral in (2.6) we conclude that

$$||e^{t\hat{L}_n^*}\; v^{(n)}||_{\mathbb{X}_n} \to 0, \quad t\to\infty. \tag{5.24}$$

The second term in the r.h.s. of (5.21) can be estimated in the same way as in our previous works [8, 10].

Thus we proved the strong convergence (3.9), and the proof of the second part of Theorem 3.1 is completed.

The final step of the proof is to show that the system of correlation functions $\{k_\varrho^{(n)}\}$ corresponds to a probability measure μ^ϱ on the configuration space Γ. For this we have constructed above $\{k_\varrho^{(n)}\}$ as the limit when $t \to \infty$ of the solution $\{k_t^{(n)}\}$ of the Cauchy problem (3.2) with initial data (3.5)

$$k_\varrho^{(n)} \;=\; \lim_{t\to\infty} k_t^{(n)}. \tag{5.25}$$

We will use next the following Proposition summarizing results of two papers [13] and [14] of A. Lenard.

Proposition 5.2. *(see [13], [14]) If the system of correlation functions $\{k^{(n)}\}$ satisfies Lenard positivity and moment growth conditions then there exists a unique probability measure $\mu \in \mathcal{M}_{fm}(\Gamma)$, whose correlation functions are exactly $\{k^{(n)}\}$.*

It is easy to see that this measure is locally absolutely continuous w.r.t. the Poisson measure. For the convenience of the reader we formulate Lenard positivity and moment growth conditions below.

Lenard positivity. $KG \geq 0$ for any $G \in B_{bs}(\Gamma_0)$ implies

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathfrak{X}} \dots \int_{\mathfrak{X}} G^{(n)}(x_1,\dots,x_n) k^{(n)}(x_1,\dots,x_n) m(dx_1)\dots m(dx_n) \geq 0. \tag{5.26}$$

Moment growth. For any compact set $\Lambda \subset \mathfrak{X}$ and $j \geq 0$

$$\sum_{n=0}^{\infty} (m_{n+j}^{\Lambda})^{-\frac{1}{n}} \;=\; \infty, \tag{5.27}$$

where

$$m_n^{\Lambda} = (n!)^{-1} \int_{\Lambda} \dots \int_{\Lambda} k^{(n)}(x_1,\dots,x_n) m(dx_1)\dots m(dx_n).$$

In our case the inequality $\left(m_n^{\Lambda}\right)^{-\frac{1}{n}} \geq \frac{\tilde{C}}{n}$ follows from bound (5.19). Thus condition (5.27) of the uniqueness holds. To obtain the Lenard positivity condition (5.26) we use (5.25). It follows from results of [7] (Proposition 4.4 and Corollary 4.1) that for any $t > 0$ the solution $\{k_t^{(n)}\}$ of the Cauchy problem (3.2) satisfies condition (5.26) of Lenard positivity, see Appendix in [8] for the detailed proof of this statement. Consequently, the limit system of correlation functions $k_\varrho^{(n)}$ also satisfies the Lenard positivity condition (5.26).

Thus Proposition 5.2 implies that for any $\varrho > 0$ there exists a unique probability measure $\mu^\varrho \in \mathcal{M}_{\mathrm{corr}}(\Gamma)$, whose correlation functions are $\{k_\varrho^{(n)}\}$. This completed the proof of Theorem 3.1.

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Acknowledgments. We are grateful to the anonymous Referee for useful remarks and comments.