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A driven tagged particle in asymmetric exclusion processes*

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Abstract

We consider the asymmetric exclusion process with a driven tagged particle on $\mathbb{Z}$ which has different jump rates from other particles. When the non-tagged particles have non-nearest-neighbor jump rates , we show that the tagged particle can have a speed which has a different sign from the mean derived from its jump rates. We also show the existence of some non-trivial invariant measures for the environment process viewed from the tagged particle. Our arguments are based on coupling, martingale methods, and analyzing currents through fixed bonds.

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1 Introduction

The exclusion process on the lattice $\mathbb{Z}^d$ with a driven tagged particle can be formally described as: a collection of red particles and a tagged green particle performing continuous-time random walks on the lattice $\mathbb{Z}^d$ with respect to the exclusion rule, i.e. at most one particle is at each site and jumps are suppressed if the target site is already occupied. Red particles have independent exponential clocks with rates $\lambda = \sum_z p(z)$. When a clock rings, the particle at site x jumps to a vacant site $x+z$ with probability $\frac{p(z)}{\lambda}$; the jump is suppressed if the site $x+z$ is occupied. The green tagged particle follows similar rules, but it has different jump rates $q(.)$. In particular, for both types of particles, the jump rates $p(.)$ and $q(.)$ are independent of where the particles are. We would like to study the long-time behavior of the displacement D_t of the tagged particle.

The behavior of the tagged particle is mostly studied when $p(.) = q(.)$. Limit theorems for the displacement D_t were obtained by works [1, 21, 10, 12, 26, 24]. The environment process ξ_t viewed from the tagged particle turns out to be a convenient tool to study: ξ_t denotes the sites occupied by the red particles relative to the tagged particle. There is a

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class of invariant and ergodic measures for ξ_t: Bernoulli measures μ_ρ with parameter ρ ($0 \leq \rho \leq 1$). As a consequence, the speed of the tagged particle can be computed explicitly as $(1-\rho)\sum_z z \cdot q(z)$. For details on the exclusion process, the tagged particle process, and their invariant measures when $p(.) = q(.)$, see Chapter III.4 [15]. The fluctuation of D_t in equilibrium is known to be subdiffusive when $d = 1$ and $p(.) = q(.)$ are nearest-neighbor symmetric [1], and diffusive in most other finite-range cases [10, 12, 26, 24]. A powerful method, developed by Kipnis and Varadhan [12], is to study the additive functionals of reversible Markov processes. It is also extended to asymmetric models [26, 24]. The only open cases are when non-mean-zero $p(.) = q(.)$ is non-nearest-neighbor in dimension $d = 1$, and when $p(.) = q(.)$ is non-mean-zero in dimension $d = 2$.

The case where $d = 1$, $p(.) = q(.)$ are nearest-neighbor is special. Particles are trapped, and orders are preserved. The gaps between particles follow a zero-range process [10]. The displacement D_t can be considered jointly with the current through the bond between 0 and 1 in either the zero-range process[10], or the exclusion process [23]. On the other hand, when jump rates $p(.)$ are symmetric, we can use the stirring system to construct the symmetric exclusion process, see Chapter VIII.4 [18]. With these considerations, one can study the density fields and apply hydrodynamic limit results to analyze the displacement D_t. Some related works are [1, 9, 23, 5, 7].

However, when jump rates $p(.), q(.)$ are different and non-nearest-neighbor, the asymptotic behavior of the displacement D_t is less understood. A primary difficulty in providing rigorous proofs is the lack of explicit knowledge of invariant measures for the environment process, which is essential in the analysis in [1, 12, 26, 24]. The difference between $p(.)$ and $q(.)$ introduces asymmetry making explicit computations of invariant measures seemingly impossible. In dimension $d \leq 2$, it is unclear whether there are multiple invariant measures for different values of density ρ, except for two trivial ones, Bernoulli measures μ_0 and μ_1. In $d \geq 3$, Loulakis's result [19] provides a partial answer when $p(.)$ is symmetric. Also, in dimension $d = 1$, the orders of gaps between particles are no longer preserved due to the non-nearest-neighbor assumption on $p(.)$. Meanwhile, we should notice two special cases in general one-dimensional asymmetric models without a tagged particle: in asymmetric exclusion process, there are stationary blocking measures [4, 6]; in the totally asymmetric simple K-exclusion process, invariant measures are also unknown [22]. In the former model, blocking measures are nontranslation invariant measures concentrated on configurations that are completely occupied by particles after some point to the positive infinity and completely empty before some point to the negative infinity. The existence of blocking measures in principle implies the displacement D_t of a tagged particle may grow sub-linearly because their neighbor particles often block particle jumps. For the K-exclusion process, Seppäläinen [22] managed to show the hydrodynamic limit of the system even though the invariant measures are unknown. His arguments are based on coupling the process with a growth model.

Alternatively, when $p(.)$ is different from $q(.)$, we can view the tagged particle in the exclusion process as a special particle driven by an external force, and consider our model as a perturbed system of the case when $p(.)$ equals $q(.)$. One approach is to verify the Einstein relation, which connects mobility and diffusivity. The mobility describes the speed of the tagged particle in the perturbed system, while the diffusivity describes the variance of the tagged particle in the unperturbed system. For exclusion processes, the Einstein relation is verified in some symmetric and reversible scenarios [14, 19, 20]. In dimension $d = 1$, when $p(.)$ is symmetric and $p(.), q(.)$ are nearest-neighbor, Landim, Olla and Volchan, by studying the dynamics of gaps, [20] showed that the displacement D_t grows as $\sqrt{t}$, and there is an Einstein relation for D_t. They further conjectured that D_t

grows linearly in t when the mean $\sum_z z \cdot q(z)$ is positive, and $p(.)$ is non-nearest-neighbor in $d = 1$ or general in $d \geq 2$. This conjecture is partially verified when $d \geq 3$ and $p(.)$ is symmetric [19], and it remains open for most of the other cases. When $q(.)$ is close to $p(.)$, one can show the displacement D_t grows linearly in t with a corresponding Einstein relation [19]. However, the speed of the tagged particle is unknown because there is no explicit formula for the invariant measures. For a mixing dynamical environment with a positive spectral gap, Komorowski and Olla [14] obtained a full expansion of invariant measures, and showed the explicit speed and the corresponding Einstein relation.

Another approach is to study the currents through a fixed bond in the one-dimensional asymmetric exclusion process (AEP) with coupling arguments. The current describes the average number of particles across a site, and it is a natural object to study especially when $\sum_z z \cdot p(z)$ is non-zero. Liggett [16, 17] computed the currents and limiting measure in AEP explicitly for a class of general initial measures by couplings. For a more general class of asymmetric conservative particle systems with a blockage, one can show a hydrodynamic limit result with a coupling argument different from Liggett's [2]. In these systems, the current across the blockage is a key quantity in the hydrodynamic limit because it describes the densities near the blockage. Although this second type of couplings is different from Liggett's [16, 17], it is available in the one-dimensional AEP case. When jump rates $p(.)$ satisfy certain monotonicity conditions, Ferrari, Lebowitz, and Speer [6] showed a coupling of two AEPs and applied this coupling to prove the existence of blocking measures. In the case of our model, when a driven tagged particle is present, we can also consider the current across a particular site, the (moving) tagged particle, and obtain estimates of currents by coupling different AEPs with a driven tagged particle.

This article will consider the case where $d = 1$ and $p(.)$ is non-nearest-neighbor and asymmetric with a positive mean $\sum z \cdot p(z) > 0$. The main tools are the couplings and martingale arguments. There are two types of couplings similar to those in [6, 16, 17]. These two types of couplings allow us to compare currents in different processes and obtain estimates of currents. With martingale arguments, we can relate estimates of currents to estimates of the displacement D_t and some invariant measures. In the end, we will show that the displacement D_t grows linearly in t in three scenarios (Theorems 2.1, 2.2, 2.3). These results suggest behavior of the tagged particle depends on jump rates $p(.), q(.)$ and the initial measure in a nontrivial way. By characterizing some nontrivial invariant measure, we will show that the tagged particle can have a positive speed in AEP even when it has a negative drift, $\sum_z z \cdot q(z) < 0$ (Theorem 2.3). We will make some mild assumptions in the next section.

2 Notation and results

In this section, we first introduce the problem and describe the environment process viewed from the tagged particle; next we describe the assumptions and introduce some notation; and lastly we state the main results and provide an outline of the proofs.

A configuration $\xi(.)$ on $\mathbb{Z} \setminus \{0\}$ indicates which sites are occupied relative to the tagged particle: $\xi(x) = 1$ if site x is occupied, and $\xi(x) = 0$ otherwise. The collection of all configurations $\mathbb{X} = \{0, 1\}^{\mathbb{Z} \setminus \{0\}}$ forms a state space for the environment process ξ_t.

Local functions on $\mathbb{X}$ are functions of the form $g(\xi(x_1), \ldots, \xi(x_n))$ for some finite integer n, such that $g : \{0, 1\}^n \to \mathbb{R}$. We will use $\boldsymbol{C}$ to denote the space of local functions on $\mathbb{Z} \setminus \{0\}$ and $\mathbf{M}_1$ to denote the space of probability measures on $\mathbb{X}$. Examples of local functions are:

$$\xi_A(\xi) = \prod_{x \in A} \xi(x), \text{ A is a finite subset of } \mathbb{Z} \setminus \{0\}. \tag{2.1}$$

When $A = \{x\}$ for some integer x, we abuse the notation and write it as ξ_x. We will always use subscript to stress that ξ_x is a local function. We also hope this will not cause confusion when ξ_t is the configuration at time t, as we will see in a moment.

The environment process ξ_t with respect to the simple exclusion process is a Feller process. Starting from any initial configuration in $\mathbb{X}$, ξ_t is described by its generator $L = L^{ex} + L^{sh}$. The action of L on any local function f is given by:

$$\begin{aligned} Lf(\xi) =& (L^{ex} + L^{sh})f(\xi) \\ =& \sum_{x,y\neq 0} p(y-x)\xi(x)\left(1-\xi(y)\right)\left(f(\xi^{x,y}) - f(\xi)\right) \\ &+ \sum_{z\neq 0} q(z)\left(1-\xi(z)\right)\left(f(\theta_z\xi) - f(\xi)\right) \end{aligned} \tag{2.2}$$

where $\xi^{x,y}$ represents the configuration after exchanging particles at sites x and y of ξ,

$$\xi^{x,y}(z) = \begin{cases} \xi(z) & \text{if } z \neq x, y \\ \xi(y) & \text{if } z = x \\ \xi(x) & \text{if } z = y \end{cases}, \tag{2.3}$$

and $\theta_z\xi$ represents the configuration shifted by $-z$ unit due to the jump of the tagged particle to an empty site z,

$$(\theta_z\xi)(x) = \begin{cases} \xi(x+z) & \text{if } x \neq -z \\ \xi(z) & \text{if } x = -z \end{cases}. \tag{2.4}$$

The generator L^{ex} corresponds to the motion of red particles, while the generator L^{sh} corresponds to the motion of the tagged particle.

Denote by $\mathbb{P}^{\eta,q}$ the probability measure on the space of càdlàg paths on $\mathbb{X}$ starting from a deterministic configuration $\xi_0 = \eta$, and let $\mathbb{P}^{\nu_0,q} = \int \mathbb{P}^{\eta,q}\, d\nu_0(\eta)$ when the initial configuration ξ_0 is distributed according to some measure ν_0 on $\mathbb{X}$. We also denote by $\mathbb{E}^{\nu_0,q}$ the expectation with respect to $\mathbb{P}^{\nu_0,q}$. A special initial measure is the step measure $\mu_{1,0}$, which concentrates on the configuration ξ, with $\xi(x) = 1$, for $x < 0$, and $\xi(x) = 0$, for $x > 0$. Also, we use $\mathbb{P}^{\nu_0,0}$ and $\mathbb{E}^{\nu_0,0}$ in the case when $q(.)$ is a zero function.

Lastly, we will denote by D_t the displacement of the green tagged particle up to time t. Initially, $D_0 = 0$ a.s. When $q(.)$ is nearest-neighbor, we can represent D_t as the difference of numbers of right and left jumps, see (3.9) in section 3. The main problem is to investigate the long time behavior of D_t when $q(.)$ is different from $p(.)$.

To illustrate the result, we consider the case where red particles have positive drifts

$$w = \sum_z z \cdot p(z) > 0 \tag{2.5}$$

while the tagged particle has jump rates $q(.)$. We want $p(.)$ to satisfy the following assumptions:

- A1 (Radially Decreasing and Range 2) $p(-1) \geq p(-2)$, $p(1) \geq p(2)$, and $p(k) = 0$ for all $|k| > 2$.
- A2 (Positive Mean) $p(2) = p(-2) > 0$, and $p(1) > p(-1)$.

It turns out that these assumptions can be generalized and we can get similar results. We will mention them and give outlines of their proofs after the proofs of the main results. See Remark 7.1 and Remark 8.4 at the ends of sections 7, 8. For the more general cases, we assume $p(.)$ satisfies

A'1 (Radially Decreasing) $p(x)$ is increasing on $(-\infty, -1]$ and decreasing on $[1, \infty)$,

A'2 (Positive Mean) $p(k) \geq p(-k)$ for all $k > 0$, and $p(k) > p(-k)$ for some k,

A'3 (Finite-range) there is an $R > 0$ such that $p(x) = 0$ for $|x| > R$.

Our main results are the ballistic behavior of a driven tagged particle in asymmetric exclusion processes under different assumptions. The first result is the most natural one. When the initial measure is the step measure $\mu_{1,0}$ and the tagged particle has only pure left jump rates, it has a ballistic behavior towards left, i.e. $\frac{D_t}{t}$ has a strictly negative asymptotic upper bound.

Theorem 2.1. *(Ballistic Behavior of a Tagged Particle in AEP with Only Left Jumps)*
Consider the AEP with a driven tagged particle. Let the jump rates $p(.)$ for the red particles satisfy A1 and A2, and the jump rates $q(.)$ be supported on negative axis with $q(-1) > 0$. Then, starting from the step initial measure $\mu_{1,0}$, there exists a negative constant c such that

$$\limsup_{t\to\infty} \frac{D_t}{t} \leq c < 0, \ \mathbb{P}^{\mu_{1,0},q} - a.s.$$

When the tagged particle can jump in both directions, we can also obtain ballistic behavior. In the case when $p(.) = q(.)$, and the initial measure is the Bernoulli measure μ_ρ, for some $0 < \rho < 1$, the tagged particle has a speed $(1-\rho)\sum_z z \cdot p(z)$, see [18]. Now, if we change the jump rate $q(.)$ such that the drift $\sum_z z \cdot q(z)$ is greater than $\sum_z z \cdot p(z)$, we expect its mean displacement to have the same asymptotic lower bound, $(1-\rho)\sum_z z \cdot p(z)$. The second result confirms that under some conditions on $p(.)$ and $q(.)$, the displacement D_t has an asymptotic lower bound $(1-\rho)\sum_z z \cdot p(z)$. For the second result, we make the following assumptions on jump rates $p(.), q(.)$.

A"1 (Supports) $p(.)$ has a support on $-2, -1, 1$; $q(.)$ has a support on $-1, 1, 2$,

A"2 (Radially decreasing) $p(-1) \geq p(-2)$, $q(1) \geq q(2) > 0$,

A"3 (Dominance and Positive) $q(1) \geq p(1), q(-1) \leq p(-1)$, $w = \sum_z z \cdot p(z) > 0$.

Theorem 2.2. *(Ballistic Behavior of a Fast Tagged Particle in AEP)*
Consider the AEP with a driven tagged particle. Let the jump rates $p(.), q(.)$ satisfy assumptions A"1, A"2, and A"3. Then, starting from a Bernoulli product measure μ_ρ with $\rho \in (0, 1)$ (on $\{0, 1\}^{\mathbb{Z}\setminus\{0\}}$), we have

$$\liminf_{t\to\infty} \frac{D_t}{t} \geq (1-\rho)\sum_z z \cdot p(z), \ \mathbb{P}^{\mu_\rho,q} - a.s.$$

The assumptions on jump rates imply that we can couple two continuous-time random walks with jump rates $p(.), q(.)$ such that the walk with $q(.)$ always stays on the right of the walk with $p(.)$. The supports of jump rates $p(.), q(.)$ imply red particles do not jump to the right of the tagged particle, and the tagged particle does not jump to the left of red particles.See Remark 8.2 in section 8 below for some discussion on these assumptions.

The final result is that a slow tagged particle in AEP can follow the general behavior of red particles even if it has jump rates $q(.)$ with a negative mean $\sum_z z \cdot q(z) < 0$. By slow, we mean that the size of $q(.)$, $\sum_z q(z)$, is sufficiently small relative to $w = \sum_z z \cdot p(z)$.

Theorem 2.3. *(Ballistic Behavior of a Slow Tagged Particle in AEP)*
Consider the AEP with a driven tagged particle. Let the jump rates for the red particles satisfy assumptions A1 and A2. Then, there exist nearest-neighbor jump rates $q(.)$ for the tagged particle and an ergodic invariant measure ν_e for the environment process viewed from the tagged particle such that, under $\mathbb{P}^{\nu_e,q}$, we have

a. the tagged particle has a negative drift: $-q(-1) + q(1) < 0$*,*

b. the tagged particle has a positive speed under $\mathbb{P}^{\nu_e,q}$*, that is,*

$$\lim_{t\to\infty} \frac{D_t}{t} = m > 0,\ \mathbb{P}^{\nu_e,q} - a.s.$$

In these results, we have shown ballistic behavior of the tagged particle in AEP. With arguments to be introduced in section 3, the ballistic behavior (with estimates) implies the existence of some non-trivial invariant measures, measures other than μ_0 or μ_1, for the environment process viewed from the tagged particle. In the driven tagged particle problem, the invariant measure is in general impossible to compute due to the break of symmetry. The Bernoulli product measure is no longer invariant. In principle, there could be multiple invariant measures, which makes the behavior of the tagged particle hard to predict.

To get these three results, we use similar ideas. The proofs of Theorem 2.1 and Theorem 2.3 are similar, and they are in section 7. The proof of Theorem 2.2 is in section 8. We will mainly discuss the approach to Theorem 2.3. It consists of three parts.

We first start from any initial measure ν_0 and obtain a candidate $\bar{\nu}$ for the invariant measure in Theorem 2.3 and some estimates of the displacement D_t. Let N_t be the number of red particles which initially start from the left of the tagged particle and move to the right of the tagged particle by time t. By standard martingale arguments and an algebraic identity, we can see that, up to an error of $q(1) - q(-1)$, a multiple of $\mathbb{E}^{\nu_0,q}\left[\frac{N_t}{t}\right]$ is a lower bound for $\mathbb{E}^{\nu_0,q}\left[\frac{D_t}{t}\right]$. This is done in section 3. On the other hand, we will show that the speed of the tagged particle is $\mathbb{E}^{\nu_0,q}\left[\frac{D_t}{t}\right]$ if ν_0 is ergodic. This is done in section 7.

Next, we want to prove a positive lower bound for $\mathbb{E}^{\nu_0,q}\left[\frac{N_t}{t}\right]$ for some ν_0, and we use two steps. The first step is to obtain an estimate for $\mathbb{E}^{\nu_0,q}[N_t] - \mathbb{E}^{\nu_0,0}[N_t]$, which allows us to consider the case where the tagged particle does not move. This estimate indicates that the case when the tagged particle is moving slowly can be viewed as a pertubation of the case when the tagged particle is fixed. This estimate requires a coupling result, which is the main subject in section 4. The existence of coupling requires mainly assumptions A'1 and A'3, and we will show it in Appendix A.

The second step is to prove a positive current $\mathbb{E}^{\nu_0,0}\left[\frac{N_t}{t}\right]$ for some initial measure ν_0. When the tagged particle does not move, the environment process evolves as the AEP with a blockage at site 0. A blockage is simply a site which particles are not allowed to jump to. We consider the case where ν_0 is the step measure $\mu_{1,0}$ and prove that the current is strictly positive by contradiction. The idea is to consider the limiting measure of an invariant measure under translations $\{\tau_x\bar{\nu}\}$ in the Cesàro sense. We will get an estimate for this limiting measure by comparing this process with another process called asymmetric exclusion process (AEP) on the half-line even though with creation and annihilation. The analysis of the latter process requires a second coupling argument, and follows results and ideas of Liggett, [16, 17]. The second step is done in section 5 and section 6.

We end this section with some remarks on the coupling result to be introduced in section 4 and the current through a fixed bond.

Remark 2.4. 1. Ferrari, Lebowitz, and Speer considered a coupling in [6]. This is the same as the couplings in section 4. We give an alternative construction in Appendix A. See Lemma 4.2 in [6] and Theorem A.4 in Appendix A. The main improvement in this article is the couplings of two environment processes when the tagged particle has jump rates different from $p(.)$.

2. If $p(2) > p(-1) + 2p(-2)$, we can obtain a positive lower bound for $\frac{1}{t}\mathbb{E}^{\nu_0,0}[N_t]$ using couplings in section 4. However, the proof does not work with the general assumption A1 so we will use arguments in section 5 and section 6 instead.

3. The step initial measure $\mu_{1,0}$ gives us the maximal value for $\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}^{\nu_0,0}[N_t]$. By Theorem 2.2 from [2] and the couplings in section 4, we can get positive lower bounds for $\frac{1}{t}\mathbb{E}^{\nu_0,0}[N_t]$ for more general initial measures ν_0, such as Bernoulli measures μ_ρ for ρ close to 0 or 1. This is indeed a hydrodynamic limit result for the AEP with a blockage.

4. The current through a fixed bond in the AEP with a blockage is of independent interest. The current in the usual AEP starting from a step initial measure $\mu_{1,0}$ is computed explicitly by Liggett [16, 17] as $\frac{1}{4}\sum z \cdot p(z)$. However, in the case where there is a local perturbation, the size of current is open. Whether the value of the current in the perturbed system is strictly smaller than $\frac{1}{4}\sum z \cdot p(z)$ is not well understood, and it is known as the "Slow Bond Problem". Recently, there is a progress in the nearest-neighbor case by Basu, Sidoravicius and Sly [3]. In the current article, we will show the lower bound in the perturbed case is strictly positive in some non-nearest-neighbor cases (Theorem 5.5).

3 Invariant measure and the lower bound for the displacement of a tagged particle

In this section, we will assume that $q(.)$ is nearest-neighbor and $p(2) = p(-2) > 0$, and assumptions A2, A1 are in force. This simplifies the computation, for some generalization, see Remark 7.1. We construct a candidate invariant measure by using the empirical measures. We also relate the displacement D_t of the tagged particle to the current through bond $(-1, 1)$. Most results in this section are shown by standard martingale arguments.

We start with a tightness result on $\mathbf{M}_1$ with weak topology. Since $\mathbb{X} = \{0,1\}^{\mathbb{Z}^d\setminus\{0\}}$ is equipped with the product topology, it is compact. By Prokhorov's Theorem, $\mathbf{M}_1$ is precompact with the weak topology.

Define the (random) empirical measure μ_t for process ξ_t and its mean ν_t by their actions on local functions:

$$\langle \mu_t, f\rangle := \frac{1}{t}\int_0^t f(\xi_s)\, ds, \tag{3.1}$$

and

$$\langle \nu_t, f\rangle := \frac{1}{t}\mathbb{E}^{\nu_0,q}[\int_0^t f(\xi_s)\, ds], \tag{3.2}$$

for all f in $\boldsymbol{C}$ and $t > 0$. We also have continuity at $t = 0$, $\nu_0 = \lim_{t\downarrow 0}\nu_t$. By precompactness of $\mathbf{M}_1$, we can obtain a measure $\bar{\nu}$ as the weak limit of a subsequence ν_{T_n}. It is an invariant measure by Theorem B7 [15].

Let $\mathcal{F}_t := \sigma(\xi_s : s \le t)$ and let N_t be the net number of the red particles moving from the left of the tagged particle to the right of the tagged particle up to time t (or the integrated current through bond (-1,1)). Since the tagged particle has only nearest-neighbor jumps, the jumps of the tagged particle do not change the value of N_t and N_t is the difference of two numbers:

$$N_t := R_t - L_t = \sum_{s\le t}\chi_{\left\{\xi_s=\xi_{s_-}^{-1,1},\xi_s(1)=1,\xi_s(-1)=0\right\}} - \sum_{s\le t}\chi_{\left\{\xi_s=\xi_{s_-}^{-1,1},\xi_s(1)=0,\xi_s(-1)=1\right\}} \tag{3.3}$$

Under $\mathbb{P}^{\xi,q}$, R_t has (varying) jump rates $\lambda_1(\xi_t) = p(2)(1-\xi_t(1))\xi_t(-1)$, and L_t has (varying) jump rates $\lambda_2(\xi_t) = p(-2)(1-\xi_t(-1))\xi_t(1)$. By using $\mathbb{P}^{\nu_0,q}$-martingales, and uniform integrability, we can obtain the following result:

Lemma 3.1. *Let $p(.)$ satisfy assumptions A2 and A1. For a sequence of $T_n \uparrow \infty$, $\bar{\nu} = \lim_{n\to\infty} \nu_{T_n}$ exists, and $\bar{\nu}$ is an invariant measure for the environment process ξ_t. We have*

$$\langle \bar{\nu}, C_{-1,1} \rangle = \lim_{n\to\infty} \mathbb{E}^{\nu_0,q}\left[\frac{N_{T_n}}{T_n}\right], \tag{3.4}$$

where $C_{-1,1} = p(2)\xi_{-1}(1-\xi_1)\rangle - p(-2)\xi_1(1-\xi_{-1})$. Furthermore, if there is a $C_0 > 0$, such that

$$\liminf_{t\to\infty} \mathbb{E}^{\nu_0,q}\left[\frac{N_t}{t}\right] \geq C_0, \tag{3.5}$$

we also have

$$\langle \bar{\nu}, \xi_{-1} - \xi_1 \rangle \geq \frac{C_0}{p(2)} > 0. \tag{3.6}$$

Proof. We write two $\mathbb{P}^{\nu_0,q}$–martingales

$$M_t = R_t - \int_0^t \lambda_1(\xi_s)\, ds \tag{3.7}$$

$$\tilde{M}_t = L_t - \int_0^t \lambda_2(\xi_s)\, ds. \tag{3.8}$$

These two martingales are generalizations of the classical martingale for a Poisson process n_t with a rate λ, $n_t - \lambda t$. Combining them, we can get a $\mathbb{P}^{\nu_0,q}-$ martingale,

$$N_t - \int_0^t C_{-1,1}(\xi_s)\, ds.$$

For more details, see Chapter 6.2 [13]. Taking expectation with respect to $\mathbb{P}^{\nu_0,q}$, we obtain

$$\langle \nu_{T_n}, p(2)\xi_{-1}(1-\xi_1)) - p(-2)\xi_1(1-\xi_{-1}) \rangle = \frac{1}{T_n}\mathbb{E}^{\nu_0,q}[N_{T_n}]$$

Passing through the weak limit, we get the equation (3.4). As L_t and R_t are both dominated by a Poisson Process with rate 1, $\{\frac{M_t}{t}\}_{t>1}$ and $\{\frac{\tilde{M}_t}{t}\}_{t>1}$ are uniformly integrable. Using $p(2) = p(-2)$, we get (3.6) from (3.4),(3.5). □

We can also write the displacement of the tagged particle D_t as the difference of two numbers, r_t and l_t, the numbers of right jumps and left jumps of the tagged particle:

$$D_t := r_t - l_t = \sum_{s\leq t} \chi_{\left\{\xi_s = \theta_1 \xi_{s-}\right\}} - \sum_{s\leq t} \chi_{\left\{\xi_s = \theta_{-1}\xi_{s-}\right\}}. \tag{3.9}$$

With a similar argument, we see the displacement D_t has a lower bound which is a multiple of C_0, up to an error (the difference of $q(-1)$ and $q(1)$):

Lemma 3.2. *Let jump rates $p(.)$ satisfy assumptions A2 and A1, and $q(.)$ be nearest-neighbor. There is a sequence of $T_n \uparrow \infty$ such that $\bar{\nu} = \lim_{n\to\infty} \nu_{T_n}$ exists and it is invariant, and D_t has an estimate:*

$$q(1)\langle \bar{\nu}, 1-\xi_1 \rangle - q(-1)\langle \bar{\nu}, 1-\xi_{-1} \rangle = \liminf_{t\to\infty} \mathbb{E}^{\nu_0,q}\left[\frac{D_t}{t}\right]. \tag{3.10}$$

Furthermore, if (3.5) *holds with $C_0 > 0$, then*

$$\liminf_{t\to\infty} \mathbb{E}^{\nu_0,q}\left[\frac{D_t}{t}\right] \geq \frac{q(1)}{p(2)} C_0 - (q(-1) - q(1)). \tag{3.11}$$

Proof. It is almost the same as that of Lemma 3.1. We notice that, $l_t - q(-1)\int_0^t(1 - \xi_s(-1))\,ds$ and $r_t - q(1)\int_0^t(1-\xi_s(1))\,ds$ are $\mathbb{P}^{\nu_0,q}$- martingales, and that the left hand side of (3.10) can be rewritten as

$$q(1)\,\langle\bar{\nu},\xi_{-1}-\xi_1\rangle - (q(-1)-q(1))\,\langle\bar{\nu},1-\xi_{-1}\rangle\,. \qquad \square$$

We use $\liminf$ in (3.10) to emphasize that the initial measure ν_0 is arbitrary, and D_t may not satisfy a law of large numbers. From the estimate (3.11) in Lemma 3.2, we can get a positive mean for the displacement when the tagged particle has almost symmetric jump rates, i.e. when $q(-1)-q(1)$ is small, and C_0 is positive. For the next three sections, we will show how to get a positive C_0 with (3.5) in Lemmas 3.1 and 3.2 for some ν_0.

4 An error estimate and couplings of particles on $\mathbb{Z}$

The main result of this section is Theorem 4.4, which gives an estimate of the error $\mathbb{E}^{\nu_0,q}[N_t] - \mathbb{E}^{\nu_0,0}[N_t]$, where $q(.)$ is nearest-neighbor (for extensions, see Remark 4.5). This estimate allows us to consider the problem with a fixed tagged particle instead of a moving tagged particle. The proof relies on couplings of two auxiliary processes, which is the main tool in this section. The couplings are similar to those in [2, 6]. Under the couplings of two auxiliary processes, we will have one auxiliary process which moves "faster" than the other process. We will order particles in increasing order, and compare the positions of particles in two processes in pairs. Typically, the "faster" process has particles with larger coordinates relative to their paired particles in the "slower" process. By coupling jumps of particles, we can preserve the relative orders of paired particles in both processes for all time $t \geq 0$. Next, we introduce some notions, and show the proof of Theorem 4.4 at the end of this section.

4.1 Auxiliary processes

We can view the environment process ξ_t of the asymmetric exclusion process with a tagged particle in another way. We can label all red particles according to the initial configuration in an ascending order, and track their relative positions with respect to the tagged particle.

Starting from an initial configuration ξ with infinitely many particles on both sides of zero, we label particles with their initial positions as $\vec{X}_0 = (X_i)_{i\in\mathbb{Z}} \in (\mathbb{Z}\setminus\{0\})^{\mathbb{Z}} = \tilde{\mathbb{X}}$. In particular, $\vec{X}_0$ satisfies

$$\cdots < X_{-2} < X_{-1} < X_0 < X_1 < X_2 < \dots \tag{4.1}$$

and

$$\xi(x) = 1 \Leftrightarrow X_i = x, \text{for some } i.$$

To extend to the case when there are finitely many particles to the right or the left of zero, it is also convenient for us to add particles at $+\infty$ and $-\infty$, and therefore, we would enlarge the state space to $\hat{\mathbb{X}} = (\mathbb{Z}\setminus\{0\}\bigcup\{-\infty,\infty\})^{\mathbb{Z}}$. For example, for the step measure $\mu_{1,0}$, we can label particles as:

$$\cdots < X_{-2} = -3 < X_{-1} = -2 < X_0 = -1 < X_1 = \infty \leq X_2 = \infty \leq \dots$$

Also, there is no particular rule for the choice of X_0 with respect to the tagged particle.

For each initial configuration $\vec{X}_0$ satisfying (4.1), there is a Markov process $\vec{X}_t$ with generator $\tilde{L}$ corresponding to the process ξ_t with initial configuration ξ. In particular, we will introduce many re-labelings to keep $\vec{X}_t$ satisfying (4.1) for any $t > 0$. There are two types of jumps for the auxiliary process, corresponding to jumps (2.3) and (2.4). The first occurs when the i-th red particle jumps to an empty target site $X_i + z$; the second occurs when the tagged particle jumps to an empty target site z. Due to nearest-neighbor jump rates $q(.)$, a jump of the tagged particle does not result in change of labels, while a jump of a red particle requires re-labelings of particles between the particle and its target site so that (4.1) holds. Note that there are multiple $\vec{X}_0$ corresponding to ξ, so to the process ξ_t there correspond multiple processes $\vec{X}_t$.

Let $T_{i,z}\vec{X}$ and $\Theta_z\vec{X}$ represent the configurations after these two jumps respectively. See (4.2), (4.4), (4.6) below for their expressions. We can see two examples for these two types of jumps in Figure 1 and Figure 2.

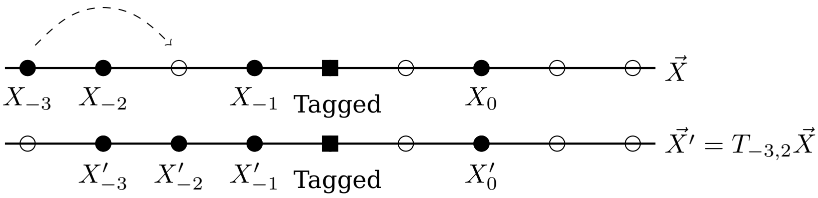

Figure 1: Red Particle X_{-3} Jumps 2 Units

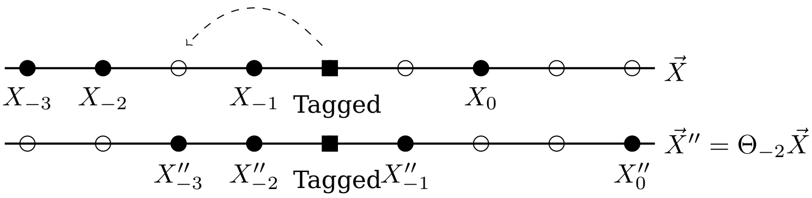

Figure 2: Tagged Particle Jumps -2 Units

For any $z \neq 0$, we have $\Theta_z\vec{X}$ as,

$$(\Theta_z\vec{X})_j = X_j - z. \tag{4.2}$$

For $z > 0$, we denote the index of the right-most particle to the left of site $X_i + z$ by $I_{i,z}(\vec{X})$,

$$I_{i,z}(\vec{X}) = \max\{k : X_k \leq X_i + z\}. \tag{4.3}$$

When a positive jump is possible for the i-th particle, we have the new configuration described by

$$(T_{i,z}\vec{X})_j = \begin{cases} X_j & \text{if } j < i \text{ or } j > I_{i,z}(\vec{X}) \\ X_{j+1} & \text{if } i \leq j < I_{i,z}(\vec{X}) \\ X_i + z & \text{if } j = I_{i,z}(\vec{X}) \end{cases}. \tag{4.4}$$

The conditions for these two types of jumps to occur are $A_{i,z} = \{X_i + z \notin \vec{X} \cup \{0\}\}$ and $B_z = \{z \notin \vec{X} \bigcup\{0\}\}$, respectively. Here we also think of $\vec{X}$ as a subset of $\mathbb{Z} \setminus \{0\}$ (instead of $\mathbb{Z} \setminus \{0\} \cup \{-\infty, \infty\}$).

For negative jumps $z < 0$, we can think of the dynamics by reversing the lattice $\mathbb{Z}$. That is, with a change of variable, $\vec{Y} = \{Y_i\}_{i\in\mathbb{Z}} = R(\vec{X})$, we have

$$\Big(R(\vec{X})\Big)_i = Y_i = -X_{-i} \tag{4.5}$$

$$(T_{i,z}\vec{X}) = R(T_{-i,-z}(R(\vec{X})) \tag{4.6}$$

$$I_{i,z}(\vec{X}) = -I_{-i,-z}(R(\vec{X})) = \min\{k : X_k \geq X_i + z\} \tag{4.7}$$

For $z = 0$, we take $T_{i,0}$ as the identity map and $I_{i,0}(\vec{X}) = i$.

Therefore, we can write down the generator $\tilde{L}$ for the auxiliary process $\vec{X}_t$ by its action on local functions $F : \hat{\mathbb{X}} \longrightarrow \mathbb{R}$ (i.e. $F(\vec{X})$ depends on a finite set $\{X_i\}$) as:

$$\begin{aligned}\tilde{L}F(\vec{X}) =&(\tilde{L}^{ex} + \tilde{L}^{sh})F(\vec{X})\\ =&\sum_{i,z} p(X_i, X_i + z)\mathbb{1}_{A_{i,z}}(\vec{X}) \left[F(T_{i,z}\vec{X}) - F(\vec{X})\right]\\ &+\sum_{y} q(y)\mathbb{1}_{B_y}(\vec{X}) \left[F(\Theta_y\vec{X}) - F(\vec{X})\right].\end{aligned} \tag{4.8}$$

The transition rates are $p(x, y) = p(y - x)$ if $x, y \neq 0, \pm\infty$, and $p(x, y) = 0$ otherwise.

4.2 Shifts of labels

In the environment process, a jump of the tagged particle influences coordinates of all red particles (and does not change labels of particles), while jumps of red particles influence only finitely many coordinates (and change labels of particles). In order to couple jumps of the tagged particle with jumps of red particles and preserve the order of the two processes, we use shifts of labels to offset the global effect on coordinates from jumps of the tagged particle.

For couplings, we also consider two other versions of auxiliary processes with shifts of labels, which correspond to the same process ξ_t. Let $S_z\vec{X}$ represents the configuration after shifting labels by z,

$$(S_z\vec{X})_j = X_{j+z}. \tag{4.9}$$

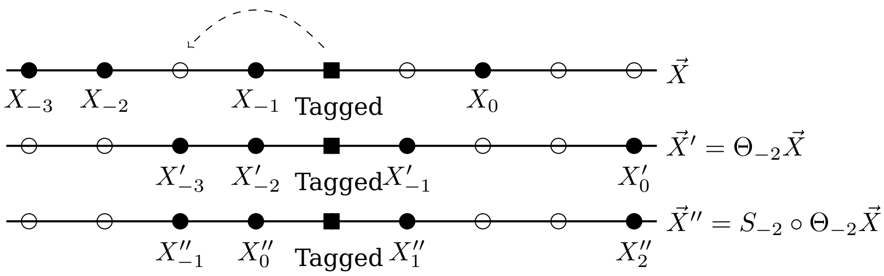

Figure 3: Tagged Particle Jumps -2 Units with Labels Shifted

In addition to shifting configurations when a tagged particle jumps, we can also shift labels after shifting the configurations. See Figure 3. We obtain the first version by adding a shift of labels by z after the tagged particle has a left jump with z units, that is,

$$\begin{aligned}\tilde{L}_LF(\vec{X}) =&(\tilde{L}^{ex} + \tilde{L}_L^{sh,q-} + \tilde{L}^{sh,q+})F(\vec{X})\\ =&\sum_{i,z} p(X_i, X_i + z)\mathbb{1}_{A_{i,z}}(\vec{X}) \left[F(T_{i,z}\vec{X}) - F(\vec{X})\right]\\ &+\sum_{y<0} q(y)\mathbb{1}_{B_y}(\vec{X}) \left[F(S_y \circ \Theta_y\vec{X}) - F(\vec{X})\right]\\ &+\sum_{y>0} q(y)\mathbb{1}_{B_y}(\vec{X}) \left[F(\Theta_y\vec{X}) - F(\vec{X})\right]\end{aligned} \tag{4.10}$$

Similarly, we can have the second version by shifting labels after the tagged particle takes a right jump. See Figure 4 below for an example when the tagged particle has a right jump with size 1.

$$\begin{aligned}\tilde{L}_R F(\vec{X}) =&(\tilde{L}^{ex} + \tilde{L}^{sh,q_-} + \tilde{L}_R^{sh,q_+})F(\vec{X}) \\ =&\sum_{i,z} p(X_i, X_i+z)\mathbb{1}_{A_{i,z}}(\vec{X}) \left[F(T_{i,z}\vec{X}) - F(\vec{X})\right] \\ &+\sum_{y<0} q(y)\mathbb{1}_{B_y}(\vec{X}) \left[F(\Theta_y \vec{X}) - F(\vec{X})\right] \\ &+\sum_{y>0} q(y)\mathbb{1}_{B_y}(\vec{X}) \left[F(S_y \circ \Theta_y \vec{X}) - F(\vec{X})\right] \end{aligned} \tag{4.11}$$

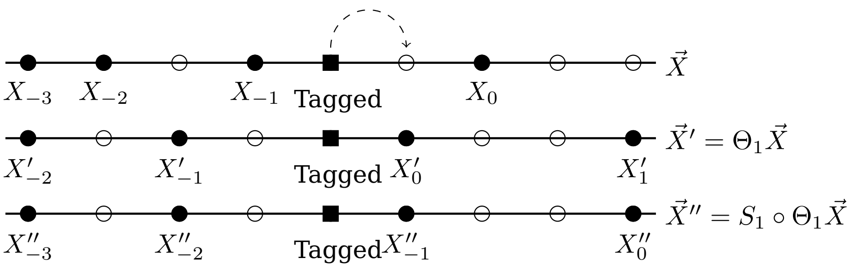

Figure 4: Tagged Particle Jumps 1 Unit with Labels Shifted

We will use $\vec{X}_t = (\vec{X}_0, G, p, q)$ to denote the auxiliary process with $\vec{X}_0$ as the initial configuration, and generator G. In particular, G is one of the forms (4.8),(4.11), and (4.10) with p, q as parameters. And we use $\mathbb{P}^{(\vec{X}_0, G, p, q)}$ or $\mathbb{P}^{\vec{X}_t}$ to denote the corresponding probability measure on the space of càdlàg paths on $\hat{\mathbb{X}}$. $\vec{X}_0$ can also be random.

4.3 Couplings of auxiliary processes and error estimates

There is a natural partial order on the set $\hat{\mathbb{X}}$:

$$\vec{X} \geq \vec{Y} \Leftrightarrow X_i \geq Y_i, \text{ for all } i. \tag{4.12}$$

With this partial order,we can define that two auxiliary processes $\vec{X}_t = (\vec{X}_0, G, p, q)$ and $\vec{Y}_t = (\vec{Y}_0, G', p', q')$ are coupled by stochastic ordering.

Definition 4.1. *We denote $\vec{X}_t \succeq \vec{Y}_t$, if two auxiliary processes $\vec{X}_t$ and $\vec{Y}_t$ can be coupled: that is, there exists a joint process $\vec{Z}_t = (\vec{W}_t, \vec{V}_t)$, with a joint generator Ω on the space of local functions $F : \tilde{\mathbb{X}} \times \tilde{\mathbb{X}} \mapsto \mathbb{R}$, such that*

1. $\vec{W}_t \geq \vec{V}_t, \mathbb{P}^{\vec{Z}_t} - a.s.$
2. *$\vec{Z}_t$ has marginals as $\vec{X}_t$ and $\vec{Y}_t$. That is, for any local functions $F_1(\vec{X}, \vec{Y}) = H_1(\vec{X})$, and $F_2(\vec{X}, \vec{Y}) = H_2(\vec{Y})$, we have,*

$$\begin{gathered}\Omega F_1(\vec{X}, \vec{Y}) = G H_1(\vec{X}) \\ \Omega F_2(\vec{X}, \vec{Y}) = G' H_2(\vec{Y}) \\ \vec{W}_0 \stackrel{d}{=} \vec{X}_0,\ \vec{V}_0 \stackrel{d}{=} \vec{Y}_0.\end{gathered}$$

Our main step towards Theorem 4.4 is the existence of couplings of auxiliary processes. The construction of the couplings is done in Appendix A.

Theorem 4.2. *Let $p(.)$ satisfy assumption A1 and two initial configurations satisfy $\vec{X}_0 \geq \vec{Y}_0$. For any $q(.)$, we can couple below two pairs of auxiliary processes:*

$$(\vec{X}_0, \tilde{L}_R, p, q) \succeq (\vec{Y}_0, \tilde{L}, p, 0) \tag{4.13}$$

$$(\vec{X}_0, \tilde{L}, p, 0) \succeq (\vec{Y}_0, \tilde{L}_L, p, q). \tag{4.14}$$

Proof. See Theorem A.4 in Appendix A. □

Remark 4.3. The couplings (4.13), (4.14) are valid for more general jump rates $p(.)$, see Theorem A.4. We can also extend Theorems 2.2, 2.1, and 2.3 when couplings 4.13 and 4.14 are valid for more general jump rates $p(.), q(.)$. For details, see Remarks 7.1 and 8.4.

Above two couplings provide a lower bound and an upper bound of the error $\mathbb{E}^{\nu_0,q}[N_t] - \mathbb{E}^{\nu_0,0}[N_t]$ respectively, and we can estimate the error by the number of jumps of the tagged particle.

Theorem 4.4. *Let $p(.)$ satisfy assumption A1, and the tagged particle take nearest-neighbor jumps, with rates $q(-1), q(1)$. For any (deterministic) initial configuration ξ, and any $t \geq 0$,*

$$\left|\mathbb{E}^{\xi,q}[N_t] - \mathbb{E}^{\xi,0}[N_t]\right| \leq t \cdot (q(1) + q(-1)). \tag{4.15}$$

Proof. To get (4.15), we will use couplings from Theorem 4.2 to obtain two inequalities,

$$\mathbb{E}^{\xi,q}[N_t] - \mathbb{E}^{\xi,0}[N_t] \geq -\mathbb{E}^{\xi,q}[r_t], \tag{4.16}$$

$$\mathbb{E}^{\xi,q}[N_t] - \mathbb{E}^{\xi,0}[N_t] \leq \mathbb{E}^{\xi,q}[l_t]. \tag{4.17}$$

With the fact that $l_t - q(-1)\int_0^t (1 - \xi_s(-1))\, ds$ and $r_t - q(1)\int_0^t (1 - \xi_s(1))\, ds$ are $\mathbb{P}^{\nu_0,q}$-martingales, we derive (4.15) from (4.16) and (4.17).

To get (4.16) and (4.17), we can consider the following. For any non-zero configuration ξ in $\mathbb{X}$, we can label the particles as $\vec{X}_0 = \{X_i\}_{i\in\mathbb{Z}}$,

$$\cdots \leq X_{-2} \leq X_{-1} \leq X_0 < 0 < X_1 \leq X_2 \leq \dots$$

and equality occurs if both sides are ∞ or $-\infty$. By Theorem 4.2, from the same initial configuration ξ, we have two couplings with $\vec{X}_0 = \vec{Y}_0 = \vec{Z}_0$,

$$\begin{aligned} \vec{X}_t = (\vec{X}_0, \tilde{L}_R, p, q) \succeq (\vec{X}_0, \tilde{L}, p, 0) = \vec{Y}_t \\ \vec{Y}_t = (\vec{X}_0, \tilde{L}, p, 0) \succeq (\vec{X}_0, \tilde{L}_L, p, q) = \vec{Z}_t. \end{aligned} \tag{4.18}$$

Consider a function $F : \hat{X} \to \mathbb{Z}$, $F(\vec{X}) = \max\{i : X_i \leq -1\}$. It is decreasing in $\vec{X}$, that is, if $\vec{X} \geq \vec{Y}$

$$F(\vec{X}) \leq F(\vec{Y}) \tag{4.19}$$

Therefore, we get, under two joint distributions (one for the coupling $\vec{X}_t \succeq \vec{Y}_t$, and the other for the coupling $\vec{Y}_t \succeq \vec{Z}_t$) and $\vec{X}_0 = \vec{Y}_0 = \vec{Z}_0$,

$$F(\vec{X}_0) - F(\vec{X}_t) \geq F(\vec{Y}_0) - F(\vec{Y}_t), a.s., \tag{4.20}$$

$$F(\vec{Y}_0) - F(\vec{Y}_t) \geq F(\vec{Z}_0) - F(\vec{Z}_t), a.s.. \tag{4.21}$$

Notice that the derivation of (4.20) and (4.21) does not require $q(.)$ to be nearest-neighbor.

On the other hand, when $q(.)$ is nearest-neighbor, jumps of the tagged particle do not move particles between positive and negative axes, but they may shift labels. See Figure

4. For $\vec{X}_t$, we can use a decomposition similar to those in (3.3) and (3.9), and see that the change in the label of the right-most particle on the negative axis by time t comes from three sources: jumps of red particles through bond $(-1,1)$ ($N_{\vec{X}}(t)$), right jumps of the tagged particle ($r_{\vec{X}}(t)$) and left jumps of the tagged particle ($l_{\vec{X}}(t)$). In particular, each first type of jump contributes 1 to the change, each second type of jump contributes 1 to the change, and each third type of jump contributes 0 to the change. Therefore, we obtain

$$F(\vec{X}_0) - F(\vec{X}_t) = N_{\vec{X}}(t) + r_{\vec{X}}(t), \tag{4.22}$$

where $N_{\vec{X}}(t), r_{\vec{X}}(t)$ are the same as N_t, r_t for the corresponding environment process ξ_t. Similarly, we obtain two identities for processes $\vec{Y}_t, \vec{Z}_t$,

$$F(\vec{Y}_0) - F(\vec{Y}_t) = N_{\vec{Y}}(t), \tag{4.23}$$

$$F(\vec{Z}_0) - F(\vec{Z}_t) = N_{\vec{Z}}(t) - l_{\vec{Z}}(t), \tag{4.24}$$

where $l_{\vec{Z}}(t)$ is the same as l_t for process $\vec{Z}(t)$.

Taking expectations on (4.20) rewritten in terms of (4.22) and (4.23), we get

$$\mathbb{E}^{\xi,q}[N_t] + \mathbb{E}^{\xi,q}[r_t] \geq \mathbb{E}^{\xi,0}[N_t],$$

which implies (4.16); taking expectations of (4.21), rewritten in terms of (4.23) and (4.24), we get (4.17). □

Remark 4.5. We can obtain further results with similar proofs of Theorem 4.4. We will assume that $p(\cdot)$ satisfies assumption A1 so that couplings in Theorem 4.2 are possible. We mention these results without giving detailed proofs.

1. When $q(.)$ is non-nearest-neighbor and finite-range, we can find an estimate similar to (4.15): there is a $C_{R'} > 0$ depending on the range R' of $q(.)$ such that,

$$\left|\mathbb{E}^{\nu_0,q}[N_t] - \mathbb{E}^{\nu_0,0}[N_t]\right| \leq C_{R'} \sum_z q(z) \cdot t. \tag{4.25}$$

We outline the proof of (4.25): we can first obtain couplings (4.18) by Theorem 4.2. Then, we can apply couplings (4.18) to the decreasing function $F(\vec{X}) = \max\{i : X_i \leq -1\}$ and get (4.20) and (4.21). Each side of (4.20) is the same as the integrated current N_t through the bond $(-1,1)$ for the corresponding environment process ξ_t up to a term corresponding to the shift of labels. For example, we can take the auxiliary process $\vec{X}_t = (\vec{X}_0, \tilde{L}_R, p, q)$. Every jump changing the value of the integrated current N_t also changes the value $F(\vec{X}_t)$ by the same amount, except for right jumps of the tagged particle. A right jump of size z decreases $F(\vec{X}_t)$ by an additional amount z, so we can get (4.22)

$$F(\vec{X}_0) - F(\vec{X}_t) = N_{\vec{X}}(t) + r_{\vec{X}}(t),$$

by interpreting $N_{\vec{X}}(t)$ as the integrated current N_t through the bond $(-1,1)$ for its corresponding environment process, and $r_{\vec{X}}(t)$ as the sum of right jump sizes of the tagged particle. Similarly, we can derive (4.23) and (4.24) with new interpretations. Therefore, we can get (4.25) by taking expectations and the fact that

$$\max\left\{\mathbb{E}\left[r_{\vec{X}}(t)\right], \mathbb{E}\left[l_{\vec{X}}(t)\right]\right\} \leq R \sum_z q(z) \cdot t.$$

2. From the coupling, we can use Kingman Subadditive Ergodic Theorem to show the convergence of $\frac{N_t}{t}$ when the initial measure is the step measure $\mu_{1,0}$, and the tagged particle does not move, $q=0$:

$$\lim_{t\to\infty}\frac{N_t}{t}=\lim_{t\to\infty}\frac{1}{t}\mathbb{E}^{\mu_{1,0},0}\left[N_t\right],\quad \mathbb{P}^{\mu_{1,0},0}-a.s. \tag{4.26}$$

See Remark 2.4 and Lemma 4.8 in [2], or see (7.6) in the proof of Theorem 2.1.

5 Current in AEP with a blockage

In this section, we will show the current in AEP with a blockage at the origin has a positive lower bound (Theorem 5.5).The existence of a positive lower bound helps us to show that the tagged particle has a positive speed under $\mathbb{P}^{\nu_e,q}$, for some small $q(.)$ and some ergodic measure ν_e for the environment process.

5.1 Currents and densities in equilibrium

In sections 5, 6, we make the following assumptions on $p(.,.)$. Let $p(.,.)$ be jump rates for a continuous-time random walk on $\mathbb{Z}$ with the following conditions:

1. $p(.,.)$ is translation invariant: $p(x,y)=p(y-x)$.
2. $p(x,x+k)=p(k)\geq p(-k)=p(x+k,x)$ for all $k>0$, and a strict inequality holds for some k.
3. $p(.,.)$ has a finite jump range $R>1$: $p(k)=0,|k|>R$. Assume further $p(R)>0$.

Notice that the assumptions A1, A2 are sufficient for the above assumptions, but not necessary. Also, we don't need A1 or A'1, which is the main condition for the existence of couplings in section 4; instead, the second assumption above is the main condition for this section. It enables us to construct an increasing sequence G_i, which will be important in the proof of Lemma 5.3.

We will consider a process, the AEP on lattice $\mathbb{Z}$ with a blockage at the origin, i.e., the AEP with a tagged particle when $q=0$, and quantities $C_{x,y}$ that are currents through bond (x,y).

The AEP on lattice $\mathbb{Z}$ with a blockage at the origin has a generator L defined by its action on a local function f,

$$Lf(\eta)=\sum_{x,y\neq 0}p(x,y)\eta(x)\left(1-\eta(y)\right)\left(f(\eta^{x,y})-f(\eta)\right), \tag{5.1}$$

which is the same as (2.2) when $q=0$. Assume the initial configuration is the step measure $\mu_{1,0}$ for the rest of this section. Recall that $C_{-1,1}$ was defined in Lemma 3.1. In general, for any $x<y$, we can define the current $C_{x,y}$ through bond (x,y) as:

$$C_{x,y}=\sum_{\substack{i\leq x,y\leq j,\\ i,j\neq 0}}\left(p(i,j)\eta_i(1-\eta_j)-p(j,i)\eta_j(1-\eta_i)\right). \tag{5.2}$$

Theorem 5.5 is the main result for the next two sections. Before its statement and proof, we shall see three lemmas on invariant measures with respect to L, and currents $C_{x,y}$. The first two lemmas are direct consequences of translation invariance and finite range of $p(.,.)$ and they are standard. In the third lemma, we will need the second condition on $p(.,.)$. The first lemma says the mean of current $C_{x,x+1}$ is constant in x with respect to an invariant measure.

Lemma 5.1. *For an invariant measure $\bar{\nu}$ with respect to the generator L defined in* (5.1)*, we have, for any $x \neq -1, 0$,*

$$\langle \bar{\nu}, C_{x,x+1} \rangle = \langle \bar{\nu}, C_{-1,1} \rangle. \tag{5.3}$$

Proof. The change of density at site x is due to the difference between currents through bonds $(x-1,x)$ and $(x,x+1)$. Computing $L\eta_x$ for $x \neq -1, 0, 1$, we get

$$\begin{aligned} L\eta_x &= C_{x-1,x} - C_{x,x+1}, \\ L\eta_{-1} &= C_{-2,-1} - C_{-1,1}, \\ L\eta_1 &= C_{-1,1} - C_{1,2}. \end{aligned}$$

We show the first one, and the next two are similar: for any $x \neq -1, 0, 1$, we have

$$\begin{aligned} L\eta_x &= \sum_{i,j \neq 0} p(i,j)\eta_i \left(1-\eta_j\right) \left(\eta_x^{i,j} - \eta_x\right) \\ &= \sum_{i \neq 0,x} p(i,x)\eta_i \left(1-\eta_x\right) - \sum_{j \neq 0,x} p(x,j)\eta_x \left(1-\eta_j\right) \\ &= \sum_{i \neq 0,x} \left(p(i,x)\eta_i \left(1-\eta_x\right) - p(x,i)\eta_x \left(1-\eta_i\right)\right). \end{aligned}$$

On the other hand, we can check pairs (i,j) contributing to the difference $C_{x-1,x} - C_{x,x+1}$

$$\begin{aligned} C_{x-1,x} - C_{x,x+1} &= \sum_{\substack{i \leq x-1, x \leq j, \\ i,j \neq 0}} (p(i,j)\eta_i(1-\eta_j) - p(j,i)\eta_j(1-\eta_i)) \\ &\quad - \sum_{\substack{i \leq x, x+1 \leq j, \\ i,j \neq 0}} (p(i,j)\eta_i(1-\eta_j) - p(j,i)\eta_j(1-\eta_i)) \\ &= \sum_{\substack{i \leq x-1, x = j, \\ i,j \neq 0}} (p(i,j)\eta_i(1-\eta_j) - p(j,i)\eta_j(1-\eta_i)) \\ &\quad - \sum_{\substack{i = x, x+1 \leq j, \\ i,j \neq 0}} (p(i,j)\eta_i(1-\eta_j) - p(j,i)\eta_j(1-\eta_i)) \\ &= \sum_{\substack{i \leq x-1, \\ i \neq 0}} (p(i,x)\eta_i(1-\eta_x) - p(x,i)\eta_x(1-\eta_i)) \\ &\quad + \sum_{\substack{x+1 \leq i, \\ i \neq 0}} (p(i,x)\eta_i(1-\eta_x) - p(x,i)\eta_x(1-\eta_i)) = L\eta_x, \end{aligned}$$

where interchanging i and j in the third last line results in a change of sign.

Taking expectation with respect to the invariant measure $\bar{\nu}$, we get (5.3). □

Consider translation operators τ_i on the state space $\mathbb{X}' = \{0,1\}^{\mathbb{Z}}$, for $i, j \in \mathbb{Z}$,

$$(\tau_i \eta)(j) = \eta(j+i).$$

We define translations on local functions f and on measures ν by

$$\tau_i f(\eta) = f(\tau_i \eta), \tag{5.4}$$

$$\langle \tau_i \nu, f \rangle = \langle \nu, \tau_i f \rangle \tag{5.5}$$

In particular, we have that $\tau_i \eta_j = \eta_{i+j}, \langle \tau_i \nu, \eta_j \rangle = \langle \nu, \eta_{i+j} \rangle$.

The second lemma says that any weak limit ν^* of the Cesàro means of $\bar{\nu}$ under translation is a mixture of Bernoulli measures μ_ρ, $0 \leq \rho \leq 1$. This is because ν^* is translation invariant and invariant with respect to the generator L_0 for AEP. Recall that the generator L_0 acts on a local function f by, see [11],

$$L_0 f(\eta) = \sum_{x,y\in\mathbb{Z}} p(y-x)\eta(x)\left(1-\eta(y)\right)\left(f(\eta^{x,y}) - f(\eta)\right). \tag{5.6}$$

Lemma 5.2. *Let $\bar{\nu}$ be an invariant measure with respect to the generator L. Any weak limit ν^* of the Cesàro means of $\bar{\nu}$ under translation:*

$$\nu^* = \lim_{k\to\infty} \nu^*_{n_k} = \lim_{k\to\infty} \frac{1}{n_k}\sum_{i=1}^{n_k} \tau_i \bar{\nu}, \tag{5.7}$$

is translation invariant and invariant with respect to the generator L_0 for AEP. That is, for any local function f,

$$\langle \nu^*, \tau_x f\rangle = \langle \nu^*, f\rangle, \tag{5.8}$$

$$\langle \nu^*, L_0 f\rangle = 0, \tag{5.9}$$

where L_0 is translation invariant. In particular, there is a probability measure w_ρ on $[0,1]$, such that

$$\nu^* = \int \mu_\rho dw_\rho. \tag{5.10}$$

Proof. By Theorem VIII.3.9 [18], we only need to show translation invariance and invariance ((5.8), (5.9)) to get (5.10). The proofs for both are similar.

For any local function f, which is a bounded function on $\{0,1\}^{\mathbb{Z}}$ depending on finitely many ξ_x,

$$\begin{aligned}
\langle \nu^*_{n_k}, \tau_1 f\rangle =& \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \tau_i\bar{\nu}, \tau_1 f\rangle \\
=& \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \tau_{i+1}\bar{\nu}, f\rangle \\
=& \langle \nu^*_{n_k}, f\rangle + O_f\left(\frac{1}{n_k}\right).
\end{aligned}$$

Also, as $\bar{\nu}$ is invariant with respect to L and $L_0\tau_i = \tau_i L_0$, we can compare (5.1) with (5.6) and get,

$$\begin{aligned}
\langle \nu^*_{n_k}, L_0 f\rangle =& \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \tau_i\bar{\nu}, L_0 f\rangle \\
=& \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \bar{\nu}, L_0(\tau_i f)\rangle \\
=& \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \bar{\nu}, L(\tau_i f)\rangle + \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \bar{\nu}, (L_0 - L)(\tau_i f)\rangle \\
=& O_f\left(\frac{1}{n_k}\right).
\end{aligned}$$

In the last line, since f is local, $(L_0 - L)(\tau_i f)$ is non-zero for finitely many i. Taking limits as $n_k \to \infty$, we get (5.8) and (5.9). $\square$

The third lemma says if an invariant measure $\bar{\nu}$ has a current with a zero mean and some weak limit ν^* of its Cesàro means under translation is a Bernoulli measure μ_0 with density 0, the densities of positive sites are identically 0 for $\bar{\nu}$.

Lemma 5.3. *Let $\bar{\nu}$ be an invariant measure with respect to the generator L, and ν^* be a weak limit of its Cesàro means defined in* (5.7)*. If $\langle \bar{\nu}, C_{-1,1}\rangle = 0$ and $\langle \nu^*, \eta_x\rangle = 0$ for some x (which implies for all x since ν^* is translation invariant), we have $\langle \bar{\nu}, \eta_x\rangle = 0$ for all $x > 0$.*

Proof. We will divide the proof into 3 steps.

S1. Define a quantity G_i:

With identities $p(y,x) = p(x,y) + p(y,x) - p(x,y)$ and $\eta_x(1-\eta_y) - \eta_y(1-\eta_x) = \eta_x - \eta_y$, from (5.2), we get

$$\begin{aligned}\langle \bar{\nu}, C_{i,i+1}\rangle =& \langle \bar{\nu}, \sum_{x\le i, i+1\le y} p(y-x)(\eta_x - \eta_y)\rangle \\ &+ \langle \bar{\nu}, \sum_{x\le i, i+1\le y} (p(y-x) - p(x-y))\eta_y(1-\eta_x)\rangle.\end{aligned}$$

Therefore, by Lemma 5.1, we have, for $i \ge R$, $\langle \bar{\nu}, C_{i,i+1}\rangle = 0$, and

$$\begin{aligned}&\sum_{x\le i, i+1\le y} p(y-x)\langle \bar{\nu}, \eta_y - \eta_x\rangle \\ =& \sum_{x\le i, i+1\le y} (p(y-x) - p(x-y))\langle \bar{\nu}, \eta_y(1-\eta_x)\rangle.\end{aligned} \tag{5.11}$$

The choice for $i \ge R$ is to avoid $x, y = 0$ for any term inside the sum.

Notice that there is some symmetry on the left hand side of (5.11), which allows us to rewrite (5.11) as a backward difference for some sequence $(G_i)_{i\ge R}$

$$\sum_{x\le i, i+1\le y} p(y-x)\langle \bar{\nu}, \eta_y - \eta_x\rangle = G_{i+1} - G_i. \tag{5.12}$$

We will prove (5.12). Indeed, we can expand the left hand side of (5.11), and rearrange terms according to $\langle \bar{\nu}, \eta_{i+j}\rangle$, for $j = -(R-1), -(R-2), \dots, R$. We will get $2R$ terms with coefficients b_j,

$$\sum_{x\le i, i+1\le y} p(y-x)\langle \bar{\nu}, \eta_y - \eta_x\rangle = \sum_{j=-(R-1)}^{R} b_j \langle \bar{\nu}, \eta_{i+j}\rangle,$$

where b_j can be computed explicitly as

$$b_j = \begin{cases} \sum_{k=j}^{R} p(k) & , \text{ for } j \ge 1 \\ -\sum_{k=-R}^{j-1} p(-k) & , \text{ for } j \le 0 \end{cases}. \tag{5.13}$$

The coefficients b_j are "odd" in the sense that

$$b_{-(j-1)} = -b_j, \text{ for } j = 1, \dots, R. \tag{5.14}$$

From (5.14), we can find $2R+1$ "even" numbers with boundary conditions $a_R = a_{-R} = 0$,

$$a_{-j} = a_j, \text{ for } j = 0, 1, \dots, R, \tag{5.15}$$

and rewrite b_j as a (negative) forward difference

$$b_j = a_{j-1} - a_j, \text{ for } j = -(R-1), \dots, R. \tag{5.16}$$

We can also express a_j in terms of $p(.)$ explicitly as

$$a_j = \sum_{k=|j|+1}^{R} (k - |j|)\, p(k)., \text{ for } |j| = 0, 1, \dots, R. \tag{5.17}$$

One can see (5.16) by working on an example. For example, when $R = 2$, we have 4 "odd" terms,

$$-b_2, -b_1, b_1, b_2,$$

and we can find 5 "even terms" $0, b_2, b_2 + b_1, b_2, 0$ and write the 4 odd terms as

$$0 - b_2, b_2 - (b_2 + b_1), (b_2 + b_1) - b_2, b_2 - 0.$$

In fact, (5.16) is a direct consequence of the symmetry (5.14), and it does not rely on the explicit expressions (5.13), (5.17). From (5.16), we can apply the summation by parts formula to the left hand side of (5.11) and get (5.12),

$$\begin{aligned}\sum_{j=-(R-1)}^{R} b_j \langle \bar{\nu}, \eta_{i+j} \rangle &= \sum_{j=-(R-1)}^{R} (a_{j-1} - a_j) \langle \bar{\nu}, \eta_{i+j} \rangle \\ &= \sum_{j=-(R-1)}^{R-1} a_j \langle \bar{\nu}, \eta_{i+1+j} \rangle - \sum_{j=-(R-1)}^{R-1} a_j \langle \bar{\nu}, \eta_{i+j} \rangle \\ &= G_{i+1} - G_i, \end{aligned} \tag{5.18}$$

for all $i \geq R$, which is the forward difference of a sequence (G_i). The sequence (G_i) is unique up to a constant, and we can use the last equality of (5.18) and express G_i in the matrix form,

$$G_i = \sum_{j:|j| \leq R-1} a_j \langle \bar{\nu}, \eta_{i+j} \rangle = A v_i, \tag{5.19}$$

where A is a row vector with $2R - 1$ positive entries $a_j = \sum_{k=|j|+1}^{R} (k - |j|)\, p(k)$, for $|j| \leq R - 1$, and v_i is a column vector with $2R - 1$ nonnegative entries $\langle \bar{\nu}, \eta_{i+j} \rangle$, for $|j| \leq R - 1$.

S2. Convergence of $(G_i)_{i \geq R}$:

By the assumption $p(k) \geq p(-k)$ for $k > 0$, we have the right hand side of (5.11) is positive. Also, (5.19) implies that G_i is bounded uniformly for $i \geq R$. Therefore, we get the monotone convergence of $(G_i)_{i \geq R}$:

$$G_i \uparrow c, \text{ as } i \uparrow \infty. \tag{5.20}$$

S3. From $\langle \nu^*, \eta_x \rangle = 0$ to $\langle \bar{\nu}, \eta_x \rangle = 0$:

As the Cesáro limit of a sequence is the same as its limit when both limits exist, by the definition (5.7) of ν^*, (5.19), and (5.20), we get $c = 0$ from linearity. With strictly positive entries in A, we get, for $i \geq R$,

$$G_i = A v_i = 0,$$

and all entries in v_i are 0. In particular, $\langle \bar{\nu}, \eta_{i+j} \rangle = 0$, for all indices $i + j$ with $i + j \geq R - (R - 1) = 1$. □

We should notice that to write G_i in forms of (5.19), we need $i \geq R$. It is because we don't want terms involving $p(0, x)$ or $p(x, 0)$. This condition holds for sites sufficiently right to the origin. We will see similar conditions in Theorem 5.4 and Lemma 6.1 involved.

5.2 Proof of positive currents in AEP with a blockage

The theorem below will be proved in section 6.3. It says, if the initial configuration has no particles after some point $x > 0$, ν^* is dominated by $\mu_{\frac{1}{2}}$, in the sense of (5.21). Let's recall from section 3 that the mean of empirical measures ν_t is defined by its action on local functions $\langle \nu_t, f\rangle = \frac{1}{t}\mathbb{E}^{\nu_0,0}\left[\int_0^t f(\eta_s)\,ds\right]$ for some initial measure ν_0.

Theorem 5.4. *Consider the AEP on lattice $\mathbb{Z}$ with a blockage at the origin and $p(.)$ has a positive mean $\sum z \cdot p(z) > 0$. Let $\bar{\nu}$ be a weak limit of the mean of empirical measures $\bar{\nu}_{T_n}$, and ν^* be defined via a subsequence mentioned in* (5.7)*. If there is an $x > R$ such that $\langle \nu_0, \eta_y\rangle = 0$ for all $y \geq x$, we will have, for any finite set $A \subset \mathbb{Z}$,*

$$\langle \nu^*, \prod_{x\in A} \eta_x\rangle \leq \langle \mu_{\frac{1}{2}}, \prod_{x\in A} \eta_x\rangle = 2^{-|A|}. \tag{5.21}$$

Proof. See Corollary 6.4. □

Theorem 5.5 is the main result of sections 5, 6. It says the current through bond $(-1, 1)$ is strictly positive for the AEP on $\mathbb{Z}$ when the initial measure is the step measure $\mu_{1,0}$. We will prove it by contradiction.

Theorem 5.5. *Suppose $p(.,.)$ satisfy assumptions at the beginning of subsection 5.1. For the AEP on lattice $\mathbb{Z}$ with a blockage at the origin, there is a lower bound $C_1 > 0$ for the current through bond $(-1, 1)$,*

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}^{\mu_{1,0},0}[N_t] = \liminf_{t\to\infty} \langle \nu_t, C_{-1,1}\rangle = C_1 > 0. \tag{5.22}$$

Proof. Let N_t be the (net) number of particles jumping through bond $(-1, 1)$ by time t, which is the same as (3.3) when the tagged particle is not moving. Under the initial measure $\mu_{1,0}$, there are no particles on the positive axis, we can see that N_t is the same as the number of particles on the positive axis at time t, and therefore $N_t \geq 0$. Together with the fact that $N_t - \int_0^t C_{-1,1}(\eta_s)\,ds$ is a $\mathbb{P}^{\mu_{1,0},0}$- martingale (see Chapter 6.2 [13]), we get that

$$C_1 = \liminf_{t\to\infty}\langle \nu_t, C_{-1,1}\rangle = \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}^{\mu_{1,0},0}[N_t] \geq 0.$$

Suppose $C_1 = 0$. By tightness, there is an invariant measure $\bar{\nu}$ with a zero current $\langle \bar{\nu}, C_{-1,1}\rangle = 0$. By Lemma 5.1, $\langle \bar{\nu}, C_{x,x+1}\rangle = 0$, for $x \geq R$. We have

$$\begin{aligned}\langle \nu^*_{n_k}, C_{R,R+1}\rangle &= \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \tau_i\bar{\nu}, C_{R,R+1}\rangle \\ &= \frac{1}{n_k}\sum_{i=1}^{n_k}\langle \bar{\nu}, C_{R+i,R+i+1}\rangle = 0.\end{aligned}$$

Then, for any weak limit $\nu^* = \lim_{k\to\infty} \nu^*_{n_k} = \lim_{k\to\infty} \frac{1}{n_k}\sum_{i=1}^{n_k} \tau_i\bar{\nu}$,

$$\langle \nu^*, C_{R,R+1}\rangle = 0. \tag{5.23}$$

On the other hand, by Lemma 5.2, ν^* is a mixture of Bernoulli measures (on $\{0,1\}^{\mathbb{Z}}$), that is, $\nu^* = \int \mu_\rho dw_\rho$ for some probability measure w_ρ. A computation shows

$$\langle \mu_\rho, C_{R,R+1}\rangle = \rho(1-\rho)\sum_{i\leq R, j\geq R+1}(p(i,j) - p(j,i)) = \rho(1-\rho)w, \tag{5.24}$$

where w is the mean drift for $p(.,.)$

$$w = \sum_{i \le R, j \ge R+1} (p(i,j) - p(j,i)) = \sum_k k \cdot p(k). \tag{5.25}$$

By assumption 2 for $p(.,.)$, this sum (5.24) is strictly positive unless $\rho = 0$ or 1. As a consequence, ν^* is a convex combination of μ_0 and μ_1

$$\nu^* = c_0\mu_0 + c_1\mu_1, \tag{5.26}$$

with $c_0 + c_1 = 1$.

By Theorem 5.4, we have $c_1 \le 2^{-|A|}$, for any finite set $A \subset \mathbb{Z}$. This implies $c_1 = 0$ and $c_0 = 1$. Then, by Lemma 5.3, we have, for $x > 0$,

$$\langle \nu^*, \eta_x \rangle = 0, \text{ and } \langle \bar{\nu}, \eta_x \rangle = 0. \tag{5.27}$$

By the particle-hole duality, (i.e. viewing holes as particles, viewing particles as holes, and reversing $\mathbb{Z} \setminus \{0\}$, we can get the dynamics of holes the same as the dynamic of particles in the AEP with a blockage at site 0), we get a result like (5.27): for $x < 0$,

$$\langle \bar{\nu}, \eta_x \rangle = 1. \tag{5.28}$$

(5.27) and (5.28) imply the current $\langle \bar{\nu}, C_{-1,1} \rangle$ is strictly positive, which is a contradiction. □

6 AEP on half-line with creation and annihilation

To show Theorem 5.4, we will consider an auxiliary process: the AEP on the half-line with creation and annihilation. This model has a long history and was studied by Liggett in [16] and [17]. We will use some results from [16] and [17] to show the estimate (5.21) in Theorem 5.4.

6.1 Comparison between AEP on half-line with creation and AEP with a blockage

We first describe the AEP on the half-line with only creation formally as follows. Particles move according to asymmetric exclusion process on half-line $[1, \infty)$ with jump rates $p(x,y) = p(y-x)$. If a positive site $y > 0$ is vacant, a particle is created at y with a rate $\sum_{x \le 0} p(y-x)$. Also, no particles are allowed to jump out of the positive half-line. Alternatively, if we consider the AEP on $\mathbb{Z}$ with an immediate creation of particles on $(-\infty, 0]$ when sites are vacant, the dynamic restricted to the positive axis is the same as the dynamic of the AEP on the half-line with creation.

The first lemma connects the AEP with a blockage at site 0 with the AEP on the half-line with creation. Denote by η_t the AEP with a blockage at site 0, which has a probability measure P; denote by ζ_t the AEP on the half-line with creation, which has a probability measure Q.

Lemma 6.1. *Suppose AEP with a blockage at site 0 starts from the initial measure $\mu_{1,0}$ and the AEP on the half-line with creation starts from the Bernoulli measure μ_0 on positive axis. Then, for any finite subset $A \subset \mathbb{Z}_+$, and any $t \ge 0$,*

$$P(\eta_t(x+R) = 1, \textit{for all } x \in A) \le Q(\zeta_t(x) = 1, \textit{for all } x \in A), \tag{6.1}$$

where R is the range of jump rates $p(.)$ as defined at the beginning of section 5. We use R to avoid sites too close to the origin.

Proof. In the AEP with a blockage, we use independent exponential clocks with rates $p(x, x+z)$ to indicate times of potential jumps from a site x to a site $x+z$. These clocks also help us to interpret movements of holes. When a potential jump from site x to $x+z$ occurs, a hole at site x can interchange with another hole at site $x+z$ (even though the interchanging doesn't affect the configuration), but its jump to a site $x+z$ occupied by a particle is suppressed. Then, we can obtain an intermediate process ϕ_t by labeling holes and particles in the AEP with a blockage as different classes of "particles" and suppressing certain jumps. In this intermediate process, there are three classes of particles, we label each class by $1, 2$, or 3. Holes and particles in the AEP with a blockage are labeled according to the following rules:

a. a particle in the AEP with a blockage is always a first-class particles and labeled "1" in the intermediate process;

b. a hole in the AEP with a blockage at any time is either a second-class particle or a third-class particle;

c. a hole becomes a second-class particle once it visits or starts from a site on $(-\infty, R]$, and its label becomes "2";

d. a hole is always a third-class particle if it never visits or starts from a site on $(-\infty, R]$, and its label stays "3".

We will also suppress a jump from site x to $x+z$ (in addition to those jumps suppressed due to the target site $x+z$ already occupied by a particle in the P-process)

$$\text{if } x \text{ has a third-class particle and } x+z \text{ has a second-class or third-class particle.} \tag{6.2}$$

(6.2) does not affect the P-process because both the second-class and third-class "particles" are holes, but under (6.2), only jumps from a site with a particle of a larger label to a site with a particle of a smaller label is allowed. We will denote by $\tilde{P}$ the probability measure corresponding to ϕ_t.

See Figure 5 for an example. ϕ_{t_0} is the a configuration at time $t_0 > 0$ with a specific labeling of three classes of particles. In particular, the hole at the site 4 is labeled a second-class particle. ϕ_{t_1} is the configuration after a (first-class) particle jumps from -1 to 1, a (second-class) particle jumps from 2 to 3, and a (second-class) particle jumps from 4 to 6 in ϕ_{t_0}; ϕ_{t_2} is a configuration at a general time t_2.

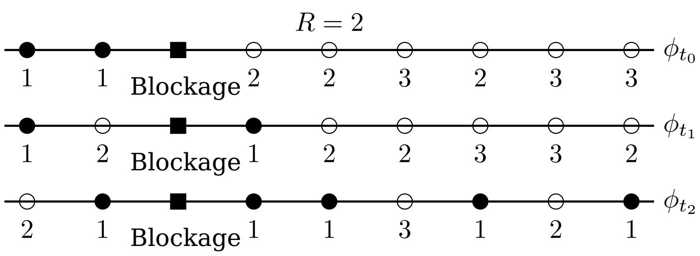

Figure 5: The AEP with a Blockage and 3 Classes of Particles

The intermediate process ϕ_t connects both the P-process and Q-process. On one hand, it follows from the rules that the first-class particles in ϕ_t correspond to particles in the P-process. We have that for any finite subset $A \subset \mathbb{Z}_+$, $t \geq 0$

$$\tilde{P}(\phi_t(x) = 1, \text{for all } x \in A + R) = P(\eta_t(x) = 1, \text{for all } x \in A + R). \tag{6.3}$$

On the other hand, the dynamics of the third-class particles (on (R,∞)) in ϕ_t are identical to the dynamics of holes in the Q-process (on $(0,\infty)$) when the Q-process has an initial measure μ_0. We can see this because a third-class particle is not created; a third-class particle is affected by either being moved from a site $y > R$ to a new site $x > R$, when the site x is occupied previously by a non-third-class particle and a potential jump from x to y occurs, or being removed from the system due to a jump from a site $x \le R$ to y. This is the same as a hole at a site $y - R$ in the Q-process: a hole at site $y - R$ is affected by either being moved to a new site $x - R > 0$, when the site x is occupied previously by a particle, and a jump from site $x - R$ to $y - R$ occurs, or a hole is affected by being removed from the system due to a jump from a site $x - R \le 0$ to y. Therefore, together with the initial measure $\mu_{1,0}$ for the P-process, we can get that for any finite subset $A \subset \mathbb{Z}_+$, $t \ge 0$

$$\tilde{P}(\phi_t(x) \neq 3, \text{for all } x \in A + R) = Q(\zeta_t(x) \neq 0, \text{for all } x \in A). \tag{6.4}$$

As a consequence of (6.3) and (6.4),

$$\begin{aligned} P(\eta_t(x) = 1, \text{for all } x \in A + R) \le & \tilde{P}(\phi_t(x) = 1 \text{ or } 2, \text{for all } x \in A + R) \\ = & Q(\zeta_t(x) = 1, \text{for all } x \in A). \end{aligned}$$

□

6.2 Couplings in the AEP with creation and annihilation

By the above lemma, we can study the asymptotic behavior of the AEP on half-line with only creation. The main theorem of this section is Theorem 6.3. The proof of Theorem 6.3 can be derived from results in [17], with stochastic orderings (couplings). We start with some notion and results from [16] and [17].

Consider a subset $D_{m,n} = \{m, m+1, \ldots, n\} \subset \mathbb{Z}$, for $m \le n \le \infty$, the configuration space on $D_{m,n}$ is $\mathbb{X}_{m,n} = \{0,1\}^{D_{m,n}}$, and a probability measure $v_{m,n}$ on $\mathbb{X}_{m,n}$. We can extend $v_{m,n}$ to a measure on $\mathbb{X}_{-\infty,\infty} = \{0,1\}^{\mathbb{Z}}$ by taking product measure: let $\lambda, \rho \in [0,1]$, we can have

$$v_{m,n;\lambda,\rho} = \mu_\lambda^{-\infty,m-1} \otimes v_{m,n} \otimes \mu_\rho^{n+1,\infty}, \tag{6.5}$$

$$v_{m,\infty;\lambda} = \mu_\lambda^{-\infty,m-1} \otimes v_{m,\infty}, \tag{6.6}$$

where $\mu_\lambda^{-\infty,m-1}$ is a Bernoulli measure with density λ on $\mathbb{X}_{-\infty,m-1} = \{0,1\}^{\{i:i<m\}}$ and $\mu_\rho^{n+1,\infty}$ is a Bernoulli measure with density ρ on $\mathbb{X}_{n+1,\infty} = \{0,1\}^{\{i:i>n\}}$. With this extension, we can compare measures on different $\mathbb{X}_{m,n}$ with partial orders on the space of measures on $\mathbb{X}_{-\infty,\infty}$.

We first define partial orders on the space of configurations $\mathbb{X}_{-\infty,\infty}$

$$\eta \ge \xi \Leftrightarrow \eta(x) \ge \xi(x) \text{ for all } x \in \mathbb{Z}. \tag{6.7}$$

Then we can define partial orders on the space of probability measures via stochastic ordering:

$$\nu \ge \mu \Leftrightarrow \langle \nu, f\rangle \ge \langle \mu, f\rangle \text{for all } f \text{ increasing (with respect to (6.7))}. \tag{6.8}$$

We will consider the AEP with creation and annihilation on both a finite system and an infinite system. The former is a process on $\mathbb{X}_{m,n}$ with a generator $\Omega_{m,n}^{\lambda;\rho}$ and a semigroup

$S^{\lambda,\rho}_{m,n}$. $\Omega^{\lambda,\rho}_{m,n}$ acts on a local function f by

$$\begin{aligned}\Omega^{\lambda,\rho}_{m,n}f(\eta) = &\sum_{x<m,y\in D_{m,n}} \left(p(x,y)\lambda\left(1-\eta(y)\right)+p(y,x)(1-\lambda)\eta(y)\right)\left(f(\eta^y)-f(\eta)\right)\\ &+\sum_{x\in D_{m,n},y>n} \left(p(x,y)\eta(x)(1-\rho)+p(y,x)\rho\left(1-\eta(x)\right)\right)\left(f(\eta^x)-f(\eta)\right)\\ &+\sum_{x,y\in D_{m,n}} p(x,y)\eta(x)\left(1-\eta(y)\right)\left(f(\eta^{x,y})-f(\eta)\right),\end{aligned} \tag{6.9}$$

where

$$\eta^x(z) = \begin{cases} 1-\eta(x) & , \text{ if } z=x \\ \eta(z) & , \text{ otherwise} \end{cases}.$$

And the latter is a process on $\mathbb{X}_{m,\infty}$ with a generator $\Omega^{\lambda}_{m,\infty}$ and a semigroup $S^{\lambda}_{m,\infty}$. $\Omega^{\lambda}_{m,\infty}$ acts on a local function f by

$$\begin{aligned}\Omega^{\lambda}_{m,\infty}f(\eta) = &\sum_{x<m,y\geq m} \left(p(x,y)\lambda\left(1-\eta(y)\right)+p(y,x)(1-\lambda)\eta(y)\right)\left(f(\eta^y)-f(\eta)\right)\\ &+\sum_{x,y\geq m} p(x,y)\eta(x)\left(1-\eta(y)\right)\left(f(\eta^{x,y})-f(\eta)\right).\end{aligned} \tag{6.10}$$

6.3 Liggett's results and their consequences

Below are results from [16] and [17]. Particularly, the monotonicity in the first part of Lemma 6.2 guarantees interchanging of limits. Recall $\mu^{m,n}_\rho$ is a Bernoulli measure on $\mathbb{X}_{m,n}$ with density ρ.

Lemma 6.2. *Assume* $1 \geq \lambda \geq \rho \geq 0$*, and* $m \leq n \leq \infty$*. Let* $\nu^{\lambda,\rho}_{m,n}(t) := \mu^{-\infty,m-1}_{\lambda} \otimes \left(\mu^{m,n}_\rho S^{\lambda,\rho}_{m,n}(t)\right) \otimes \mu^{n+1,\infty}_\rho$*. Then we have,*

1. *In the sense of* (6.8)*, the probability measure* $\nu^{\lambda,\rho}_{m,n}(t)$ *is increasing in parameters* m,n,t,λ *and* ρ*.*
2. *Let* $\bar{\nu}_{m,n;\lambda,\rho} = \lim_{t\uparrow\infty} \nu^{\lambda,\rho}_{m,n}(t)$*.* $\bar{\nu}_{m,n;\lambda,\rho}$ *converges to a unique limit* $\bar{\nu}_{m;\lambda,\rho}$ *as* n *goes to* ∞*. And* $\bar{\nu}_{m;\lambda,\rho} = \lim_{t\uparrow\infty} \mu^{-\infty,m-1}_{\lambda} \otimes \left(\mu^{m,\infty}_\rho S^{\lambda}_{m,\infty}(t)\right)$*.*
3. *For* $n-m>2R$*, the current in* $D_{m,n}$ *has two lower bounds:*
$$\langle \bar{\nu}_{m,n;\lambda,\rho}, C_{x,x+1}\rangle \geq w\cdot\max\{\lambda(1-\lambda),\rho(1-\rho)\} \tag{6.11}$$
where $w=\sum_{|k|\leq R} kp(k)$*,see* (2.5)*.*

Proof. The first part of Lemma 6.2 is proved in Theorems 2.4, 2.13 in [16]. The second part is a consequence of the monotonicity in parameters from the first part and the Trotter Theorem, see Proposition 2.2 in [16]. We only show the last equality:

$$\begin{aligned}\lim_{t\uparrow\infty} \mu^{-\infty,m-1}_{\lambda} \otimes \left(\mu^{m,\infty}_\rho S^{\lambda}_{m,\infty}(t)\right) &= \lim_{t\uparrow\infty} \mu^{-\infty,m-1}_{\lambda} \otimes \left(\lim_{n\uparrow\infty} \left(\mu^{m,n}_\rho S^{\lambda,\rho}_{m,n}(t)\right) \otimes \mu^{n+1,\infty}_\rho\right)\\ &= \lim_{t\uparrow\infty}\lim_{n\uparrow\infty} \mu^{-\infty,m-1}_{\lambda} \otimes \left(\mu^{m,n}_\rho S^{\lambda,\rho}_{m,n}(t)\right) \otimes \mu^{n+1,\infty}_\rho\\ &= \lim_{t\uparrow\infty}\lim_{n\uparrow\infty} \nu^{\lambda,\rho}_{m,n}(t)\\ &= \lim_{n\uparrow\infty}\lim_{t\uparrow\infty} \nu^{\lambda,\rho}_{m,n}(t) = \bar{\nu}_{m;\lambda,\rho}.\end{aligned}$$

In particular, the first line is by Proposition 2.2 [16], and the interchanging of limits in the fourth line is by the monotonicity in parameters n,t from the first part. The third part of Lemma 6.2 is by the proof of Proposition 2.6 in [17]. It is a consequence of the monotonicity of $\bar{\nu}_{m,n;\lambda,\rho}$ in m, n and a direct computation of currents at two boundaries $C_{m-1,m}$ and $C_{n,n+1}$. Indeed, we can compute $\langle \bar{\nu}_{m,n;\lambda,\rho}, C_{m-1,m} \rangle$,

$$\begin{aligned}\langle \bar{\nu}_{m,n;\lambda,\rho}, C_{m-1,m} \rangle &= \sum_{x<m\leq y} (p(x,y)\lambda\langle \bar{\nu}_{m,n;\lambda,\rho}, 1-\eta_y\rangle - p(y,x)(1-\lambda)\langle \bar{\nu}_{m,n;\lambda,\rho}, \eta_y\rangle) \\ &\geq \sum_{x<m\leq y} (p(x,y)\lambda\langle \bar{\nu}_{y+1,n;\lambda,\rho}, 1-\eta_y\rangle - p(y,x)(1-\lambda)\langle \bar{\nu}_{y+1,n;\lambda,\rho}, \eta_y\rangle) \\ &= \lambda(1-\lambda) \sum_{x<m\leq y} (p(x,y) - p(y,x)) = w \cdot \lambda(1-\lambda),\end{aligned}$$

where we use that $\bar{\nu}_{m,n;\lambda,\rho}$ and $\bar{\nu}_{y+1,n;\lambda,\rho}$ are product measures in the first line and the third line, we use that $\bar{\nu}_{m,n;\lambda,\rho}$ is increasing in m, n in the second line, and we use (2.5) to get the last equality. Repeating this for $\langle \bar{\nu}_{m,n;\lambda,\rho}, C_{n,n+1} \rangle$ we get

$$\begin{aligned}\langle \bar{\nu}_{m,n;\lambda,\rho}, C_{n,n+1} \rangle &= \sum_{x<n+1\leq y} (p(x,y)(1-\rho)\langle \bar{\nu}_{m,n;\lambda,\rho}, \eta_x\rangle - p(y,x)\rho\langle \bar{\nu}_{m,n;\lambda,\rho}, 1-\eta_x\rangle) \\ &\geq \sum_{x<n+1\leq y} (p(x,y)(1-\rho)\langle \bar{\nu}_{m,x-1;\lambda,\rho}, \eta_x\rangle - p(y,x)\rho\langle \bar{\nu}_{m,x-1;\lambda,\rho}, 1-\eta_x\rangle) \\ &= w \cdot \rho(1-\rho).\end{aligned}$$

Then, we can use the same argument as Lemma 5.1. From a direct computation, we get that for $x = m, \dots, n$,

$$\Omega^{\lambda,\rho}_{m,n} \eta_x = C_{x-1,x} - C_{x,x+1}.$$

From the first point, $\bar{\nu}_{m,n;\lambda,\rho}$ is the limiting measure $\bar{\nu}_{m,n;\lambda,\rho} = \lim_{t\uparrow\infty} \nu^{\lambda,\rho}_{m,n}(t)$, and therefore it is invariant with respect to $\Omega^{\lambda,\rho}_{m,n}$. Taking expectation, we obtain that the expected values of currents are constant for all $x = m-1, \dots, n$

$$\langle \bar{\nu}_{m,n;\lambda,\rho}, C_{x,x+1} \rangle = \langle \bar{\nu}_{m,n;\lambda,\rho}, C_{m-1,m} \rangle,$$

which implies (6.11) from the lower bounds. □

The main theorem of this section says the AEP on half-line with creation has a limiting measure. When translated along the positive direction, the limiting measure converges to the Bernoulli measure $\mu_{\frac{1}{2}}$ in the Cesàro sense. This corresponds to the limiting measure of usual AEP being the Bernoulli measure $\mu_{\frac{1}{2}}$ when the initial measure is the step measure $\mu_{1,0}$.

Theorem 6.3. *Assume the AEP on half-line with creation has the initial configuration with only holes in positive sites. Let m_t be measures on $\{0,1\}^{\mathbb{Z}_+}$ with $\langle m_t, \prod_{x\in A} \eta_x \rangle = Q(\zeta_t(x) = 1, \text{for all } x \in A)$ for any finite subset $A \subset \mathbb{Z}_+$. Then we have the following,*

$$\lim_{t\to\infty} m_t = \bar{m} \text{ exists} \tag{6.12}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \langle \bar{m}, \prod_{x\in A+i} \eta_x \rangle = 2^{-|A|}. \tag{6.13}$$

Proof. Assume $\lambda \geq \rho$, from Lemma 6.2, $\bar{\nu}_{m;\lambda,\rho}$ is the limiting measure of $\nu^{\lambda,\rho}_{m,n}(t)$ as t, n go to ∞. It is also increasing in m, λ, ρ. Therefore, we can define a unique limiting measure $\mu(\lambda,\rho)$, which is also increasing in λ and ρ,

$$\mu(\lambda,\rho) = \lim_{m\to\infty} \bar{\nu}_{-m;\lambda,\rho}. \tag{6.14}$$

It is also the same as the limit of the Cesàro means of $\bar{\nu}_{m;\lambda,\rho}$ under translation:

$$\tau_i \bar{\nu}_{m;\lambda,\rho} = \lim_{n\to\infty} \tau_i \bar{\nu}_{m,n;\lambda,\rho} = \lim_{n\to\infty} \bar{\nu}_{m-i,n-i;\lambda,\rho} = \bar{\nu}_{m-i;\lambda,\rho},$$
$$\mu(\lambda,\rho) = \lim_{N\to\infty} \tau_N \bar{\nu}_{m;\lambda,\rho} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \tau_i \bar{\nu}_{m;\lambda,\rho}.$$

On the other hand, we see that the limit $\mu(\lambda,\rho)$ is translation invariant (due to Cesàro mean) and invariant with respect to L_0, following similar arguments in Lemma 5.2. Indeed, for any local function f on $\{0,1\}^{\mathbb{Z}}$ $(\Omega^{\lambda}_{m,\infty}\tau_i)f$ is well-defined and

$$(\Omega^{\lambda}_{m,\infty}\tau_i)f = (L_0\tau_i)f$$

for $i \geq C(m,f) > 0$. Then, we get for i large,

$$\begin{aligned}\langle \bar{\nu}_{m-i;\lambda,\rho}, L_0 f\rangle =& \langle \tau_i \bar{\nu}_{m;\lambda,\rho}, L_0 f\rangle = \langle \bar{\nu}_{m;\lambda,\rho}, (L_0\tau_i)f\rangle \\ =& \langle \bar{\nu}_{m;\lambda,\rho}, (\Omega^{\lambda}_{m,\infty}\tau_i)f\rangle = \langle \bar{\nu}_{m;\lambda,\rho}, \Omega^{\lambda}_{m,\infty}(\tau_i f)\rangle = 0.\end{aligned}$$

where the last equality is a consequence of the point 2 in Lemma 6.2: we see that $\bar{\nu}_{m;\lambda,\rho}$ is the limiting measure, and therefore it is invariant with respect to $\Omega^{\lambda}_{m,\infty}$ by Theorem B7, [15]. Taking limit as i goes to ∞, we get $\langle \mu(\lambda,\rho), L_0 f\rangle = 0$. Therefore, $\mu(\lambda,\rho)$ is a mixture of Bernoulli measures. For any Bernoulli measure μ_ρ, by (5.24), $\langle \mu_\rho, C_{R,R+1}\rangle = w\rho(1-\rho)$. As a consequence, we get an upper bound, for any $\lambda \geq \rho$,

$$\langle \mu(\lambda,\rho), C_{R,R+1}\rangle \leq \frac{1}{4}w, \tag{6.15}$$

and equality holds if and only if $\mu(\lambda,\rho) = \mu_{\frac{1}{2}}$.

The lower bound (6.11) in Lemma 6.2 indicates $\langle \mu\left(\frac{1}{2},0\right), C_{R,R+1}\rangle \geq \frac{1}{4}w$, $\langle \mu\left(1,\frac{1}{2}\right), C_{R,R+1}\rangle \geq \frac{1}{4}w$. We see that $\mu\left(\frac{1}{2},0\right)$ and $\mu\left(1,\frac{1}{2}\right)$ are Bernoulli measures with the same density $\frac{1}{2}$,

$$\mu\left(\frac{1}{2},0\right) = \mu\left(1,\frac{1}{2}\right) = \mu_{\frac{1}{2}}.$$

Together with monotonicity in λ,ρ, we get for $\lambda \geq \frac{1}{2} \geq \rho$,

$$\mu_{\frac{1}{2}} = \mu\left(\frac{1}{2},0\right) \leq \mu(\lambda,\rho) \leq \mu\left(1,\frac{1}{2}\right) = \mu_{\frac{1}{2}}. \tag{6.16}$$

We can conclude the proof by letting $\lambda = 1, \rho = 0$, and identifying m_t as the restriction of $\nu^{1,0}_{0,\infty}(t)$ on $\mathbb{X}_{0,\infty}$. Taking weak limits (again by Lemma 6.2), we get (6.13)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \langle \bar{m}, \prod_{x\in A+i} \eta_x\rangle = \langle \mu(1,0), \prod_{x\in A}\eta_x\rangle = 2^{-|A|}. \qquad \square$$

We give the proof of Theorem 5.4 as a corollary of Theorem 6.3.

Corollary 6.4. *(proof of Theorem 5.4) Let $\bar{\nu}$ be a weak limit of the mean of empirical measures $\bar{\nu}_{T_n}$, and ν^* be a weak limit of the Cesàro means of $\bar{\nu}$ under translation* (5.7)*. Then for any finite set $A \subset \mathbb{Z}$,*

$$\langle \nu^*, \prod_{x\in A}\eta_x\rangle \leq 2^{-|A|}. \tag{6.17}$$

Proof. Consider some weak limit $\bar{\nu}$ of the means of the empirical measure for the P-process defined by (3.2) along some sequence (t_n). By (6.1) and (6.12), we have that, for any $A \subset \mathbb{Z}_+$

$$\begin{aligned}\langle \bar{\nu}, \prod_{x\in A} \eta_{x+R} \rangle &= \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} P(\eta_s(x+R) = 1, \text{for all } x \in A)\, ds \\ &\le \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} Q(\eta_s(x) = 1, \text{for all } x \in A)\, ds \\ &= \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \langle m_s, \prod_{x\in A} \eta_x \rangle\, ds = \langle \bar{m}, \prod_{x\in A} \eta_x \rangle.\end{aligned}$$

Then, for $i > 0$,

$$\langle \tau_i \bar{\nu}, \prod_{x\in A} \eta_{x+R} \rangle = \langle \bar{\nu}, \prod_{x\in A} \eta_{x+R+i} \rangle \le \langle \bar{m}, \prod_{x\in A} \eta_{x+i} \rangle.$$

Therefore, by (6.13) and (5.7), the definition of ν^*,

$$\langle \nu^*, \prod_{x\in A} \eta_{x+R} \rangle \le \lim_{N_k\to\infty} \frac{1}{N_k} \sum_{i=1}^{N_k} \langle \bar{m}, \prod_{x\in A} \eta_{x+i} \rangle = 2^{-|A|}.$$

We can extend the inequality to any subset A of $\mathbb{Z}$ since ν^* is translation invariant by Lemma 5.2. □

7 Proofs of Theorem 2.1 and Theorem 2.3

In this section, we prove Theorems 2.1 and 2.3. Let's start with the proof of Theorem 2.3, and we will see that the proof of Theorem 2.1 follows similar arguments.

Proof. (Theorem 2.3) We divide the proof into two steps.

Step1. Existence of $q(.)$ and ergodic measure ν_e for the environment process ξ_t:

By Theorem 5.5, we can define $C_1 := \liminf_{t\to\infty} \frac{1}{t} \mathbb{E}^{\mu_{1,0,0}}[N_t] > 0$. Then by Theorem 4.4, for any nearest-neighbor $q(.)$, we have $C_0 := C_1 - (q(1) + q(-1))$, such that

$$\liminf_{t\to\infty} \frac{1}{t} \mathbb{E}^{\mu_{1,0,q}}[N_t] \ge C_0.$$

As a consequence, by Lemma 3.2, there is an invariant measure $\bar{\nu}$ for the environment process ξ_t, such that

$$\liminf_{t\to\infty} \mathbb{E}^{\mu_{1,0,q}}\left[\frac{D_t}{t}\right] = \langle \bar{\nu}, f \rangle \ge \frac{q(1)}{p(2)} C_0 - (q(-1) - q(1)), \tag{7.1}$$

where

$$f(\xi) = q(1)(1 - \xi_1) - q(-1)(1 - \xi_{-1}).$$

We can choose $q(-1) > q(1)$, to obtain a strict positive lower bound for (7.1).

On the other hand, the collection of invariant measures satisfying (7.1) forms a nonempty closed convex compact set by tightness. Then, there is an extremal point ν_e, which is ergodic for the environment process ξ_t, and ν_e also satisfies (7.1)

$$\langle \nu_e, f \rangle \ge \frac{q(1)}{p(2)} C_0 - (q(-1) - q(1)) > 0. \tag{7.2}$$

Step2. The positive speed of the tagged particle:

We can use $\mathbb{P}^{\nu_e,q}-$ martingales,(see (3.7),(3.8))

$$M_t = D_t - \int_0^t f(\xi_s)\,ds,$$

where M_t is a martingale with quadratic variance of order t. As ν_e is invariant and ergodic for the environment process ξ_t, we apply Ergodic Theorem, and get

$$\lim_{t\to\infty} \frac{D_t}{t} = \langle \nu_e, f\rangle > 0, \quad \mathbb{P}^{\nu_e,q} - a.s. \tag{7.3}$$

□

In the case when the tagged particle only has pure left jumps, following arguments in Step 2 of the above proof, we only need to show that $\liminf_{t\to\infty} \frac{1}{t}\int_0 t f(\xi_s)\,ds \le c$ for some $c < 0$.

Proof. (Theorem 2.1) We first use arguments similar to the proof of Theorem 4.4. Since jump rates $p(.)$ satisfy A1, we can use Theorem 4.2 to get a coupling

$$\vec{X}_t = (\vec{X}_0, \tilde{L}_R, p, q) \succeq (\vec{X}_0, \tilde{L}, p, 0) = \vec{Y}_t.$$

Then under some joint distribution, we have (4.20)

$$F(\vec{X}_0) - F(\vec{X}_t) \ge F(\vec{Y}_0) - F(\vec{Y}_t) \text{ , a.s.}$$

for the decreasing function $F(\vec{X}) = \max\{i : X_i \le -1\}$. When the tagged particle does not jump to the right, each side of (4.20) is identical to the number of red particles through bond $(-1,1)$ by time t (integrated current through bond (-1,1)). The above inequality (4.20) is equivalent to

$$N_{\vec{X}}(t) \ge N_{\vec{Y}}(t) \quad \text{a.s.,} \tag{7.4}$$

which implies that

$$\liminf_{t\to\infty} \frac{1}{t} N_{\vec{X}}(t) \ge \liminf_{t\to\infty} \frac{1}{t} N_{\vec{Y}}(t) \text{ ,a.s.} \tag{7.5}$$

It is worth noting that due to non-nearest-neighbor jumps of the tagged particle, $N_{\vec{X}}(t)$ in (7.4) is different from the $N_{\vec{X}}(t)$ described before (4.22) because a left jump of the tagged particle can increase $N_{\vec{X}}(t)$ when there is a red particle between the tagged particle and its target site (see Figure 3 for instance). For more details on $N_{\vec{X}}(t)$, see point 1 in Remark 4.5.

We can use the the Kingman Subadditive Ergodic Theorem and Theorem 5.5 to get that the right hand side of (7.5) is a positive constant C_1,

$$\liminf_{t\to\infty} \frac{1}{t} N_{\vec{Y}}(t) = \lim_{t\to\infty} \frac{1}{t} N_{\vec{Y}}(t) = \liminf_{t\to\infty} \frac{1}{t} \mathbb{E}^{\mu_{1,0},0}[N_t] =: C_1 > 0. \tag{7.6}$$

Indeed, due to the step initial measure $\mu_{1,0}$, we can label particles initially as $\vec{Y}_0 = (Y_i)_{i\in\mathbb{Z}}$, where

$$Y_i = \begin{cases} i-1 & \text{if } i \le 0 \\ +\infty & \text{if } i > 0 \end{cases}.$$

At any fixed time t, we can get a new (random) configuration $\vec{Y}_t' \ge \vec{Y}_t$ from $\vec{Y}_t$ by "increasing" all the red particles in $\vec{Y}_t$ that are on the positive axis to $+\infty$, and "increasing" all the other red particles to fill the "rightmost" holes on the negative axis. More precisely, in view of $N_{\vec{Y}}(t) = F(\vec{Y}_0) - F(\vec{Y}_t) \geq 0$, the new configuration $\vec{Y}_t' = (Y_i'(t))_{i\in\mathbb{Z}}$ is

$$Y_i'(t) = \begin{cases} i - 1 + N_{\vec{Y}}(t) & \text{if } i \leq -N_{\vec{Y}}(t) \\ +\infty & \text{if } i > -N_{\vec{Y}}(t) \end{cases},$$

which dominates $\vec{Y}_t = (Y_i(t))_{i\in\mathbb{Z}}$ because for each $i \leq -N_{\vec{Y}}(t) = -\max\{i : Y_i(t) \leq -1\}$,

$$\begin{aligned} Y_i(t) &= \Big(Y_i(t) - Y_{N_{\vec{Y}}(t)}(t)\Big) + Y_{N_{\vec{Y}}(t)}(t) \\ &\leq \big(i - N_{\vec{Y}}(t)\big) + (-1) = Y_i'(t), \end{aligned}$$

and for each $i > -N_{\vec{Y}}(t)$,

$$Y_i(t) \leq +\infty = Y_i'(t).$$

It is also immediate that $\vec{Y}_t'$ is identical to the initial configuration $\vec{Y}_0 = (Y_i)_{i\in\mathbb{Z}}$, but with $N_{\vec{Y}}(t)$ (random) shifts of labels. Hence, we have

$$S_{N_{\vec{Y}}(t)}\vec{Y}_0 = \vec{Y}_t' \geq \vec{Y}_t. \tag{7.7}$$

Then by Theorem 4.2, we can couple two auxiliary processes $\vec{Z}_s, \vec{Y}_{t+s}$ with initial configurations $\vec{Y}_t', \vec{Y}_t$,

$$\vec{Z}_s = \Big(S_{N_{\vec{Y}}(t)}\vec{Y}_0, \tilde{L}, p, 0\Big) \succeq \Big(\vec{Y}_t, \tilde{L}, p, 0\Big) = \vec{Y}_{t+s}. \tag{7.8}$$

Applying the argument for (7.4), we can get the subadditivity for $N_{\vec{Y}}$, for any $s > 0$,

$$N_{\vec{Z}}(s) \geq N_{\vec{Y}}(t+s) - N_{\vec{Y}}(t) \text{ a.s.}, \tag{7.9}$$

where $N_{\vec{Z}}(s)$ has the same distribution as $N_{\vec{Y}}(s)$ because $\vec{Z}_0$ and $\vec{Y}_t$ are the same up to $N_{\vec{Y}}(t)$ (random) shifts of labels, and $N_{\vec{Z}}(s)$, $N_{\vec{Y}}(s)$ are differences of labels, see arguments before (7.4). From (7.8) and (7.9), we can apply the Kingman Subadditive Ergodic Theorem to get the convergence in (7.6), and identify the limit by Theorem 5.5. This is also a proof for the second point in Remark 2.4.

On the other hand, for the environment process of AEP with a driven tagged particle, we can compute $L\xi_{-1}$ by (2.2) and the fact that $p(.)$ is supported on $[-2,2]$, and $q(.)$ is supported on the negative axis. We can bound it above by

$$\begin{aligned} L\xi_{-1} =& (1-\xi_{-1}) \sum_{z\neq 0,-1} p(z)\xi_{-1-z} - \xi_{-1} \sum_{z\neq 0,1} p(z)(1-\xi_{-1+z}) \\ &+ \sum_{z<0} q(z)(1-\xi_z)\left(\xi_{-1+z} - \xi_{-1}\right) \\ \leq& (1-\xi_{-1}) \left(\sum_{z\neq 0,-1} p(z) + q(-1) \right) + \sum_{z<-1} q(z)(1-\xi_z) \\ &- \xi_{-1} \sum_{z>1} p(z)(1-\xi_{-1+z}) \end{aligned} \tag{7.10}$$

Also, we can compute $\hat{C}_{-1,1}$ by adding an extra term to (5.2), which corresponds to the jump of the tagged particle,

$$\begin{aligned} \hat{C}_{-1,1} =& p(2)\xi_{-1}(1-\xi_1) - p(-2)\xi_1(1-\xi_{-1}) + \sum_z q(z)(1-\xi_z)\left(\sum_{z<z'<0} \xi_{z'}\right) \\ \leq& p(2)\xi_{-1}(1-\xi_1) + \sum_{z<-1} q(z)(1-\xi_z)(-z-1). \end{aligned} \tag{7.11}$$

Notice that the negative term in the last inequality of (7.10) is the same as the first term $p(2)\xi_{-1}(1-\xi_1)$ on the right hand side of (7.11). Therefore, we can bound the sum $L\xi_{-1} + \hat{C}_{-1,1}$ by summing the other positive terms in (7.10), (7.11), and bound the sum by a multiple of $f = \sum_{z<0} z \cdot q(z)\,(1-\xi_z)$,

$$\begin{aligned} L\xi_{-1} + \hat{C}_{-1,1} \leq & (1-\xi_{-1})\left(\sum_z p(z) + q(-1)\right) + \sum_{z<-1} q(z)(1-\xi_z) \\ & \leq -\frac{\sum_z p(z)}{q(-1)} f - f = -\frac{C_4}{q(-1)} f, \end{aligned} \tag{7.12}$$

where $C_4 = q(-1) + \sum_z p(z)$.

We can use three $\mathbb{P}^{\mu_{1,0,q}}-$ martingales, (see (3.7), (3.8), and Chapter 6.2 [13])

$$N_{\vec{X}}(t) - \int_0^t \hat{C}_{-1,1}(\xi_s)\,ds, \quad \xi_t(-1) - \int_0^t L\xi_{-1}(\xi_s)\,ds, \quad D_t - \int_0^t f(\xi_s)\,ds, \tag{7.13}$$

which all have quadratic variance of order t. Dividing by t and taking limits, we see from (7.5) and $|\xi_t(-1)| \leq 1$ that, $\mathbb{P}^{\mu_{1,0,q}}$-a.s.,

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{t}\int_0^t \hat{C}_{-1,1}(\xi_s)\,ds &= \liminf_{t\to\infty} \frac{1}{t} N_{\vec{X}}(t) \geq C_1, \\ \lim_{t\to\infty} \frac{1}{t}\int_0^t L\xi_{-1}(\xi_s)\,ds &= \lim_{t\to\infty} \frac{1}{t}\left(\xi_t(-1) - \xi_0(-1)\right) = 0. \end{aligned}$$

Together with (7.12), we get $\mathbb{P}^{\mu_{1,0,q}} -$ a.s.

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t f(\xi_s)\,ds \leq -\frac{q(-1)}{C_4} \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left(L\xi_{-1} + \hat{C}_{-1,1}\right)\,ds \leq -\frac{q(-1)C_1}{C_4}, \tag{7.14}$$

Choosing $c := -C_1 \frac{q(-1)}{q(-1)+\sum_z p(z)} < 0$, we obtain

$$\limsup_{t\to\infty} \frac{D_t}{t} = \limsup_{t\to\infty} \frac{1}{t}\int_0^t f(\xi_s)\,ds \leq c < 0, \quad \mathbb{P}^{\mu_{1,0,q}} - a.s. \tag{7.15}$$

□

We can extend Theorem 2.1 and Theorem 2.3 to the case with more general jump rates $p(.)$, $q(.)$.

Remark 7.1. We can have more general $p(.)$ and $q(.)$. We assume that $p(.)$ satisfies assumptions A'1, and A'3, so that couplings in Theorem 4.2 are still possible by Theorem A.4.

1. To generalize Theorem 2.1, $p(.)$ satisfies additional assumption A'2, and $q(.)$ is supported only on the negative axis with $q(z) > 0$ for all $0 < -z < R$, where $[-R, R]$ contains the support of $p(.)$. Then the displacement D_t satisfies (7.15) for some $c < 0$.

   The proof is similar to that of Theorem 2.1. Once we've shown (7.5) (by the same argument), we can use an inequality similar to (7.12), see (7.20) below, to get (7.15). Indeed, we will have three $\mathbb{P}^{\mu_{1,0,q}}-$ martingales similar to (7.13),

$$\begin{aligned} & N_{\vec{X}}(t) - \int_0^t \hat{C}_{-1,1}(\xi_s)\,ds, \quad \sum_{0<z<R} \xi_t(-z) - \int_0^t L\left(\sum_{0<z<R} \xi_{-z}\right)(\xi_s)\,ds, \\ & D_t - \int_0^t f(\xi_s)\,ds, \end{aligned} \tag{7.16}$$

where $\hat{C}_{-1,1}$ is almost the same as $C_{-1,1}$ from (5.2), except for an extra term due to left jumps of the tagged particle,

$$\hat{C}_{-1,1} = C_{-1,1} + \sum_{z<z'\leq -1} \left(q(z)(1-\xi_z)\xi_{z'}\right), \tag{7.17}$$

and $f(\xi) = \sum_z q(z)z\,(1-\xi_z)$. We can write $C_{-1,1}$ as a difference

$$C_{-1,1} = \sum_{x<0<y} p(y-x)\xi_x\,(1-\xi_y) - \sum_{x<0<y} p(x-y)\xi_y\,(1-\xi_x)\,, \tag{7.18}$$

and compute $L\left(\sum_{0<z<R}\xi_{-z}\right)$ by different jumps due to the tagged particle and red particle,

$$\begin{aligned} L\left(\sum_{0<z<R}\xi_{-z}\right) =& \sum_{k<0} q(k)\,(1-\xi_k)\left(\sum_{0<z<R}\xi_{-z+k} - \sum_{0<z<R}\xi_{-z}\right) \\ &+ \sum_{0<z<R}(1-\xi_{-z})\left(\sum_{k\neq -z} p(k)\xi_{-z-k}\right) \\ &- \sum_{0<z<R}\xi_{-z}\left(\sum_{k\neq z} p(k)(1-\xi_{-z+k})\right). \end{aligned} \tag{7.19}$$

By comparing the positive terms of (7.18) and the last negative term in (7.19), we can bound the positive terms of $C_{-1,1}$ by the negative terms of $L\left(\sum_{0<z<R}\xi_{-z}\right)$ in absolute value. Therefore, $\hat{C}_{-1,1} + L\left(\sum_{0<z<R}\xi_{-z}\right)$ is bounded above by the sum of the positive terms in (7.17) and (7.19),

$$\begin{aligned} &\hat{C}_{-1,1} + L\left(\sum_{0<z<R}\xi_{-z}\right) \\ \leq& \sum_{z<z'\leq -1}\left(q(z)(1-\xi_z)\xi_{z'}\right) + \sum_{z<0} q(z)\,(1-\xi_z)\left(\sum_{0<k<R}\xi_{-k+z}\right) \\ &+ \sum_{0<z<R}(1-\xi_{-z})\left(\sum_{k\neq -z} p(k)\xi_{-z-k}\right). \end{aligned}$$

Since $\sum_{z<z'\leq -1}\xi_{z'} \leq R-2$, and $\sum_{0<k<R}\xi_{-k+z} \leq R-1$ for all $-R<z\leq -1$, we can get an upper bound for the above inequality

$$\begin{aligned} \hat{C}_{-1,1} + L\left(\sum_{0<z<R}\xi_{-z}\right) \leq& \sum_{0<z<R}(1-\xi_{-z})\left((2R-3)\cdot q(-z) + \sum_k p(k)\right) \\ \leq& C_5\sum_{0<z<R}(1-\xi_{-z}) \leq -\frac{C_5}{\min_{-R<z<0} q(z)} f, \end{aligned} \tag{7.20}$$

where $C_5 = (2R-3)\cdot \max_z q(z) + \sum_k p(k)$. (7.20) is an analogue of (7.12), and we can use a similar argument as (7.14) to get (7.15) for some $c<0$.

2. To generalize Theorem 2.3, $p(.)$ are under additional assumption that $p(-k) = p(k)$ for $2\leq k\leq R$, and $p(1) > p(-1)$. Then there exists jump rates $q(.)$ with a negative drift $\sum_z q(z) < 0$ and an ergodic measure ν_e, such that the speed of the tagged particle is positive under $\mathbb{P}^{\nu_e,q}$.

The proof is also similar to the proof of Theorem 2.3 and we outline it below in a different order. Due to the assumptions on the jump rates $p(.)$, we can use Theorem 5.5 to get a positive lower bound for

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}^{\mu_{1,0,0}}\left[N_t\right] = C_1 > 0.$$

Then we will choose $R' = R-1$, and construct jump rates $q(.)$ supported on $[-R', R']$. We can observe the following facts:

(a) When $\sum_z q(z)$ is small enough, we can use (4.25) in the first point of Remark 4.5 to get a positive lower bound for

$$\liminf_{t\to\infty} \frac{1}{t}\mathbb{E}^{\mu_{1,0,q}}\left[N_t\right] = C_1 - C_{R'}\sum_z q(z) > 0.$$

By using a $\mathbb{P}^{\mu_{1,0,q}}$-martingale $N_t - \int_0^t \hat{C}_{-1,1}(\xi_s)\,ds$, we can get a lower bound for

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}^{\mu_{1,0,q}}\left[C_{-1,1}(\xi_s)\right]\,ds = C_1 - 2C_{R'}\sum_z q(z) > 0, \tag{7.21}$$

where $C_{-1,1}$ is the current through bond $(-1,1)$ given by formula (5.2), and $C_{-1,1}$ differs from $\hat{C}_{-1,1}$ by a term of size at most $C_{R'}\sum_z q(z)$.

(b) As $p(k) = p(-k)$ for $k \geq 2$, the current $C_{-1,1}$ through bond $(-1,1)$ is a linear combination of $(1-\eta_i)$ with "odd coefficients" $(b_i)_{0<|i|\leq R-1}$,

$$C_{-1,1} = \sum_{i=1}^{R-1} b_i(1-\eta_i) - \sum_{i=1}^{R-1} b_i(1-\eta_{-i}), \tag{7.22}$$

where by "odd" we mean $b_{-i} = -b_i$, which is differnt from (5.14).

(c) When $q(.)$ is the sum of a multiple of $\left(\frac{b_z}{z}\right)_z$ and an error term $(e(z))_z$, for $1 \leq |z| \leq R'$

$$q(z) = c\cdot\frac{b_z}{z} + e(z) \tag{7.23}$$

for some positive $c > 0$, by (7.22), the function $f = \sum_z zq(z)\,(1-\eta_z)$ is $cC_{-1,1}$ up to an error of size at most

$$\left|f - cC_{-1,1}\right| \leq \sum_z |ze(z)|\,, \tag{7.24}$$

and the drift for jump rates $q(.)$ is

$$\sum_z z\cdot q(z) = \sum_z z\cdot e(z) \tag{7.25}$$

Therefore, by (7.21),(7.24),(7.25), we can choose positive c, $(e(z))_z$ with $\sum_z z\cdot e(z) < 0$ so that $q(.)$ of the form (7.23) has a negative drift $w = \sum_z z\cdot q(z) = \sum_z z\cdot e(z)$, and there is an invariant measure $\bar{\nu}$ for the environment process ξ_t, such that

$$\langle\bar{\nu}, f\rangle = \liminf_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\mathbb{E}^{\mu_{1,0,q}}\left[f(\xi_s)\right]\,ds \geq c\left(C_1 - 2C_{R'}\sum_z q(z)\right) - \sum_z |ze(z)| > 0. \tag{7.26}$$

The invariant measure $\bar{\nu}$ can be obtained as the weak limit of the mean ν_{t_n} of the empirical measure,see (3.2), along some sequence (t_n). We can also obtain an ergodic measure ν_e which also satisfies (7.26). Then, by the step 2 of the proof of Theorem 2.3, we get that under $\mathbb{P}^{\nu_e,q}$, the tagged particle has a positive speed.

8 Ballistic behavior of a fast tagged particle in AEP

In this section, we will prove Theorem 2.2. In this case, both the green tagged particle and red particles have non-nearest-neighbor jump rates and the means of jump rates are positive. This is a scenario different from Theorem 2.3. In particular, the jump rate $\beta = \sum_z q(z)$ of the tagged particle can be larger than the jump rate $\lambda = \sum_z p(z)$ of red particles.

We briefly discuss the steps of the proof. We will modify the auxiliary process introduced in section 4. Instead of labeling red particles and considering their positions relative to the tagged particle, we will also label the tagged particle, and keep track of its label (see (8.1)). With this modified auxiliary process, we can couple the ordered particles (including the tagged particle) in the AEP with the ordered particles in the usual AEP. By investigating the change in labels of tagged particles in both processes, we can compare their positions. We obtain a lower bound for the driven tagged particle in AEP by estimates from the usual AEP.

8.1 Assumptions and labels of the tagged particle

Let's recall assumptions A"1,A"2, and A"3 on jump rates $p(.), q(.)$.

A"1 (Supports) $p(.)$ has a support on $-2,-1,1$; $q(.)$ has a support on $-1,1,2$,

A"2 (Radially decreasing) $p(-1) \geq p(-2)$, $q(1) \geq q(2) > 0$,

A"3 (Dominance and Positive) $q(1) \geq p(1), q(-1) \leq p(-1)$, $w = \sum_z z \cdot p(z) > 0$.

These conditions imply that the tagged particle moves "faster" than a red particle, and that red particles starting from the left of the tagged particle always remain to the left of the tagged particle. We will explain their roles in Remark 8.2. Consider an AEP with a driven tagged particle, we label particles in an ascending order and also keep track of the label I_t of tagged particle. We get a modified auxiliary process $(\vec{X}_t, I_t) = (\vec{X}_0, p, q, I_t) = \big((X_i(t))_{i\in\mathbb{Z}}, I_t\big)$. Its generator $\hat{L}_{p,q}$ is given by its action on a local function F,

$$\begin{aligned}\hat{L}_{p,q}F(\vec{X}, I) = &\sum_{i\neq I, z\in\mathbb{Z}} p(z)\mathbb{1}_{A_{i,z}}(\vec{X}) \left[F(T_{i,z}\vec{X}, \hat{I}_{i,z}(\vec{X}, I)) - F(\vec{X}, I)\right] \\ &+ \sum_z q(z)\mathbb{1}_{A_{I,z}}(\vec{X}) \left[F(T_{I,z}\vec{X}, I_{I,z}(\vec{X})) - F(\vec{X}, I)\right] \end{aligned} \tag{8.1}$$

where $T_{i,z}\vec{X}$ is defined by (4.4),(4.6), and $\hat{I}_{i,z}(\vec{X}, I)$ is defined as

$$\hat{I}_{i,z}(\vec{X}, I) = \begin{cases} I-1, & \text{if } X_i < X_I < X_i + z \\ I+1, & \text{if } X_i + z < X_I < X_i \\ I_{i,z}(\vec{X}), & \text{if } i = I \\ I, & \text{else.} \end{cases} \tag{8.2}$$

For an AEP with a usual tagged particle,i.e., $p(.) = q(.)$, we get a second auxiliary process $(\vec{Y}_t, i_t) = (\vec{Y}_0, p, p, i_t)$ with a generator $\hat{L}_{p,p}$. For convenience, we let initial configuration be the same for both processes, and label the tagged particles with 0, ie.

$$I_0 = i_0 = 0 \tag{8.3}$$

The first lemma says that we can couple two modified auxiliary processes $(\vec{X}_t, I_t)$ and $(\vec{Y}_t, i_t)$ in the sense similar to Definition 4.1., for all $t \geq 0$,

$$X_i(t) \geq Y_i(t), \text{ for all } i \text{ in } \mathbb{Z}. \tag{8.4}$$

Lemma 8.1. *Suppose $p(.), q(.)$ satisfy A"1, A"2 and A"3. For two modified auxiliary processes $(\vec{X}_t, I_t)$ and $(\vec{Y}_t, i_t)$ with generators $\hat{L}_{p,q}$ and $\hat{L}_{p,p}$, there is a joint Ω, such that if* (8.4) *holds for $t = 0$, we have* (8.4) *holds for all $t > 0$, and the marginal condition holds*

$$\Omega F_1\left(\vec{X}, I, \vec{Y}, i\right) = \hat{L}_{p,q} H_1\left(\vec{X}, I\right),$$
$$\Omega F_2\left(\vec{X}, I, \vec{Y}, i\right) = \hat{L}_{p,p} H_2\left(\vec{Y}, i\right),$$

for any local functions $F_1\left(\vec{X}, I, \vec{Y}, i\right) = H_1\left(\vec{X}, I\right)$ and $F_2\left(\vec{X}, I, \vec{Y}, i\right) = H_2\left(\vec{Y}, i\right)$.

Proof. This is proved in Corollary A.5. In this case, we have $R = 2$. □

Remark 8.2. A special case is when both processes have exactly one tagged particle and no red particles. This is a degenerate case because the tagged particles follow continuous time random walks with jump rates $p(.), q(.)$, and $I_t = i_t = 0$ for all $t \geq 0$. Assumptions A"2, and A"3 guarantee that we can couple these two random walks with $X_0(t) \geq Y_0(t)$, for any $t \geq 0$ (without Lemma 8.1). These two assumptions also allow us to generalize the coupling of random walks to other cases described by Lemma 8.1, so that (8.4) holds for all $t \geq 0$. However, (8.4) is only useful if we know the labels I_t, i_t of the tagged particles or their differences $I_t - i_t$. The assumption A"1 does not affect the couplings of two modified auxiliary processes; instead, this assumption implies that I_t is increasing in time t. Together with a law of large number for i_t, we can get the Lemma 8.3 below which implies the signs of the $I_t - i_t$ asymptotically.

The second lemma gives estimates of I_t and i_t with respect to the Bernoulli initial measure μ_ρ.

Lemma 8.3. *Suppose $p(.), q(.)$ satisfy A"1, A"2 and A"3. Let $I_0 = i_0 = 0$, and $\vec{X}_0$ correspond to the initial Bernoulli product measure μ_ρ. The labels I_t, i_t of the tagged particles in the modified processes $(\vec{X}_t, I_t) = (\vec{X}_0, p, q, I_t)$ and $(\vec{Y}_t, i_t) = (\vec{X}_0, p, p, i_t)$ satisfy,*

$$\liminf_{t\to\infty} \frac{I_t}{t} \geq 0, \ \mathbb{P}^{\mu_\rho, q} - a.s.$$

and

$$\lim_{t\to\infty} \frac{i_t}{t} = 0, \ \mathbb{P}^{\mu_\rho, p} - a.s.$$

Proof. Notice that i_t is identical to the integrated current $-N_t$ through bond $(-1, 1)$ in the environment process ξ_t. For a general jump rate $\hat{q}(.)$ supported on $[-2, 2]$, we can obtain the current $\hat{C}_{-1,1}$ by considering the jumps of the red and the tagged particles, and modifying (5.2). Notice that a jump of the tagged particle to the site -2 (relative to the tagged particle) increases the integrated currents N_t by one if there is a particle at the site -1 (relative to the tagged particle), and that a jump to the site 2 decreases N_t by one if there is a particle at the site 1. Therefore, $\hat{C}_{-1,1}$ is

$$\hat{C}_{-1,1} = -p(-2)\xi_1(1-\xi_{-1}) + p(2)\xi_{-1}(1-\xi_1) + \hat{q}(-2)\xi_{-1}(1-\xi_{-2}) + -\hat{q}(2)\xi_1(1-\xi_2), \quad (8.5)$$

which is the compensator of the integrated current N_t. Similar to (3.7) and (3.8), $N_t - \int_0^t \hat{C}_{-1,1}(\xi_s)\, ds$ is a $\mathbb{P}^{\mu_\rho, \hat{q}}$- martingale, and we can obtain

$$\mathbb{E}^{\mu_\rho, \hat{q}}\left[\frac{i_t}{t}\right] = \mathbb{E}^{\mu_\rho, \hat{q}}\left[\frac{-N_t}{t}\right] = \frac{1}{t}\int_0^t \mathbb{E}^{\mu_\rho, \hat{q}}\left[-\hat{C}_{-1,1}(\xi_s)\right] ds.$$

When we take $\hat{q}(.) = p(.)$, the Bernoulli measure μ_ρ is ergodic for ξ_t, and by (8.5), the expectation in the last integral is 0. Therefore, we have that

$$\lim_{t\to\infty} \frac{i_t}{t} = \mathbb{E}^{\mu_\rho, p}\left[\frac{i_t}{t}\right] = 0, \ \mathbb{P}^{\mu_\rho, p} - a.s.$$

For I_t, since $q(-k) = 0$ for all $k \geq 2$, the tagged particle cannot jump from the right side of a red particle to its left side, and I_t does not decrease due to jumps of the tagged particle. Also, that $p(k) = 0$ for all $k \geq 2$ implies that no red particle can jump from the left side of the tagged particle to its right side, which also means, I_t does not decrease due to jumps of the red particles. Therefore, we have that I_t is increasing in time t,

$$I_t = -N_t \geq 0. \tag{8.6}$$

□

8.2 Proof of Theorem 2.2

Now we can prove Theorem 2.2.

Proof. (Theorem 2.2) By Lemma 8.1, there is a joint distribution $\mathbb{P}$, and we have $\vec{X}_t \geq \vec{Y}_t,\ \mathbb{P} - a.s.$ In particular, $X_{I_t} \geq Y_{I_t},\ \mathbb{P} - a.s.$

On the other hand, by Lemma 8.3, under the joint distribution $\mathbb{P}$, which has marginal distributions $\mathbb{P}^{\mu_\rho, q}$ and $\mathbb{P}^{\mu_\rho, p}$,

$$\liminf_{t\to\infty} \frac{I_t - i_t}{t} \geq 0,\ \mathbb{P} - a.s. \tag{8.7}$$

Therefore, for any fixed $\delta > 0$, $I_t \geq \lfloor i_t - \delta \cdot t \rfloor$ for large t, so $Y_{I_t} \geq Y_{\lfloor i_t - \delta \cdot t \rfloor}$. Consider $Y_{i_t} - Y_{\lfloor i_t - \delta \cdot t \rfloor}$. Since the Bernoulli product measure μ_ρ is an ergodic measure for the environment process, $Y_{i_t} - Y_{\lfloor i_t - \delta \cdot t \rfloor}$ is dominated by the sum of $\lceil \delta \cdot t \rceil$ independent geometric random variables with parameter ρ. For each fixed $k > 0$, we can get a sequence $(t_{n,k})_n = (\frac{n}{2^k})_n$ with

$$\limsup_{n\to\infty} \frac{Y_{i_{t_{n,k}}} - Y_{\lfloor i_{t_{n,k}} - \delta \cdot t_n \rfloor}}{t_{n,k}} \leq \frac{\delta}{\rho},\ \mathbb{P} - a.s.$$

Then, we can use a standard interpolation argument to replace "$t_{n,k} \uparrow \infty$" by "$t \uparrow \infty$".

With the law of large numbers for the displacement of a tagged particle in the usual AEP , i.e., when $q(.) = p(.)$, $\lim_{t\to\infty} \frac{Y_{i_t}}{t} = w \cdot (1-\rho)$. We also have

$$\liminf_{t\to\infty} \frac{Y_{I_t}}{t} \geq \liminf_{t\to\infty} \frac{Y_{\lfloor i_t - \delta \cdot t \rfloor}}{t} \geq w \cdot (1-\rho) - \frac{\delta}{\rho},\ \mathbb{P} - a.s.$$

where $w = \sum_z z \cdot p(z) > 0$. This is sufficient to get Theorem 2.2 since $X_{I_t} \geq Y_{I_t}$. □

We can also extend Theorem 2.2 to the case with more general jump rates $p(.)$, $q(.)$.

Remark 8.4. To generalize Theorem 2.2, $p(.)$ satisfies assumptions A'1, A'3 and an additional assumption that for all $k \geq 2$,

$$p(k) = 0, q(-k) = 0. \tag{8.8}$$

It is immediate that under (8.8), red particles starting from the left of the tagged particle always remain to the left of the tagged particle. The proof will be almost the same: we can replace Lemma 8.1 by Corollary A.5 to obtain a coupling because $p(.)$ satisfies assumptions A'1, A'3, and under the assumption (8.8), (A.23) is immediate. Therefore, (8.7) also holds. Because the Bernoulli measure μ_ρ is ergodic for the environment process, under which the term $\hat{C}_{-1,1}$ has a zero expectation, we get that

$$\lim_{t\to\infty} \frac{i_t}{t} = 0.$$

On the other hand, (8.8) on jump rates $p(.), q(.)$ ensures that (8.6) holds, so we get

$$\liminf_{t\to\infty} \frac{I_t - i_t}{t} \geq 0.$$

The rest of the proof follows the same arguments after (8.7).

A Appendix

The generator for the coupled process in Theorem 4.2 is long and consists of several parts. The first lemma allows us to consider different parts separately, and then combine them to get the joint generator. The second lemma provides us some convenient inequalities. The last lemma, Lemma A.3, provides us most parts of the generator, and it is the building block for the construction of the coupling.

Firstly, we observe that these are jump processes. Because the generators are sums of terms corresponding to different jumps for the same type of $G = \tilde{L}, \tilde{L}_L$, or $\tilde{L}_R$, we can combine two pairs of coupled processes, in the sense of adding their generators, to obtain a new pair of coupled processes. The main requirement is that couplings exist for any ordered deterministic initial configurations.

Lemma A.1. *Let Ω_1, Ω_2 be two joint generators for two pairs of auxiliary processes. Suppose that these two pairs of auxiliary processes are coupled via Ω_1, Ω_2 (see Definition 4.1) for any (deterministic) $\vec{W}_0 \geq \vec{X}_0$. That is,*

$$\vec{W}_t \succeq \vec{X}_t, \vec{Y}_t \succeq \vec{Z}_t,$$

where

$$\vec{W}_t = (\vec{W}_0, G, p_1, q_1), \vec{X}_t = (\vec{X}_0, G', p_2, q_2)$$

and

$$\vec{Y}_t = (\vec{W}_0, G, p_1', q_1'), \vec{Z}_t = (\vec{X}_0, G', p_2', q_2').$$

Then, the combined auxiliary processes $\vec{U}_t$ and $\vec{V}_t$, starting from $\vec{W}_0 \geq \vec{X}_0$,

$$\vec{U}_t = (\vec{W}_0, G, p_1 + p_1', q_1 + q_1'), \vec{V}_t = (\vec{X}_0, G', p_2 + p_2', q_2 + q_2'),$$

are also coupled via the joint generator $\Omega = \Omega_1 + \Omega_2$. That is,

$$\vec{U}_t \succeq \vec{V}_t.$$

We can use either $p(.)$ or $p(.,.)$ in this context, and generators G, G' can be the same.

Proof. By assumption, the condition for the marginals is immediate from the forms of the generators (4.8) (4.11) and (4.10). We need to check the first condition.

By arguments in the proof of Theorem 2.5.2 [11], to show $\vec{U}_t \geq \vec{V}_t$, we need to show the closed set $F_0 = \{(\vec{U}, \vec{V}) : \vec{U} \geq \vec{V}\}$ is an absorbing set, which can be checked via showing:

$$\Omega \mathbb{1}_{F_0} \geq 0. \tag{A.1}$$

Indeed, by martingale $\mathbb{1}_{F_0}(\vec{U}_t, \vec{V}_t) - \int_0^t \Omega \mathbb{1}_{F_0}(\vec{U}_s, \vec{V}_s)\, ds$, we get from (A.1), for any $t \geq 0$

$$P(\vec{U}_t \geq \vec{V}_t) = \mathbb{E}\left[\mathbb{1}_{F_0}(\vec{U}_t, \vec{V}_t)\right] \geq \mathbb{E}\left[\mathbb{1}_{F_0}(\vec{U}_0, \vec{V}_0)\right] = 1.$$

Usual interpolation arguments allow us to get $P(Figure3\vec{U}_t \geq \vec{V}_t, \text{ for all } t) = 1$.

Lastly, by the assumption that two pairs of auxiliary processes are coupled via Ω_1, Ω_2 for any $\vec{W}_0 \geq \vec{X}_0$, we get that (without any computation)

$$\Omega_1 \mathbb{1}_{F_0}(\vec{W}_0, \vec{X}_0) \geq 0, \text{ and } \Omega_2 \mathbb{1}_{F_0}(\vec{W}_0, \vec{X}_0) \geq 0,$$

which is sufficient for (A.1). Indeed, if $\vec{W}_0 \not\geq \vec{X}_0$,

$$\mathbb{1}_{F_0}(\vec{W}_0, \vec{X}_0) = 0,$$

and $\Omega \mathbb{1}_{F_0}(\vec{W}_0, \vec{X}_0)$ is a sum of differences, which have the same sign as

$$\mathbb{1}_{F_0}(\vec{W}', \vec{X}') - \mathbb{1}_{F_0}(\vec{W}_0, \vec{X}_0) \geq 0. \qquad \square$$

Secondly, we observe four monotone functions on the configuration space by comparing the configurations before and after the tagged particle jump with shifts of labels or not. See Fig. 2,3 for examples.

Lemma A.2. *Let z>0. If a jump of the tagged particles by z or $-z$ is possible, we have*

$$\Theta_{-z}\vec{X} \geq \vec{X}, \quad S_z \circ \Theta_z \vec{X} \geq \vec{X} \tag{A.2}$$

$$\Theta_z \vec{X} \leq \vec{X}, \quad S_{-z} \circ \Theta_{-z} \vec{X} \leq \vec{X} \tag{A.3}$$

As a consequence, there are two generators $\Omega_{0,R}$ and $\Omega_{0,L}$, such that for any $\vec{X}_0 \geq \vec{Y}_0$, we can couple $\vec{X}_t = (\vec{X}_0, \tilde{L}_R, 0, q) \succeq \vec{Y}_t = (\vec{Y}_0, \tilde{L}_R, 0, 0)$ via $\Omega_{0,R}$, and couple $\vec{W}_t = (\vec{X}_0, \tilde{L}, 0, 0) \succeq \vec{Z}_t = (\vec{Y}_0, \tilde{L}_R, 0, q)$ via $\Omega_{0,L}$.

Proof. We will prove equations (A.2) and define $\Omega_{0,R}$, via which we can couple two auxiliary processes $\vec{X}_t = (\vec{X}_0, \tilde{L}_R, 0, q) \succeq \vec{Y}_t = (\vec{Y}_0, \tilde{L}_R, 0, 0)$ for any initial $\vec{X}_0 \geq \vec{Y}_0$. The other case is similar.

By (4.2) and (4.9), we check coordinates,

$$\begin{aligned} (\Theta_{-z}\vec{X})_i &= X_i + z \geq X_i \\ (S_z \circ \Theta_z \vec{X})_i &= X_{i+z} - z \geq X_i \end{aligned} \tag{A.4}$$

Then it is immediate to see that the generator $\Omega_{0,R}$ defined below works, since under this generator, $\vec{X}_t$ is increasing in t while $\vec{Y}_t$ is constant in t,

$$\begin{aligned} \Omega_{0,R} F(\vec{X}, \vec{Y}) =& \tilde{L}_R F(., \vec{Y}) \left[\vec{X}\right] \\ =& \sum_{y<0} q(y) \mathbb{1}_{B_y}(\vec{X}) \left[F(\Theta_y \vec{X}, \vec{Y}) - F(\vec{X}, \vec{Y})\right] \\ &+ \sum_{y>0} q(y) \mathbb{1}_{B_y}(\vec{X}) \left[F(S_y \circ \Theta_y \vec{X}, \vec{Y}) - F(\vec{X}, \vec{Y})\right]. \end{aligned} \tag{A.5}$$

□

Thirdly, we see that given $\vec{X} \geq \vec{Y}$, whenever the i-th particle in $\vec{Y}$ jumps by $z > 0$, we can move the i-th particle in $\vec{X}$ by $z' \geq 0$, such that ordering is preserved after relabeling, $T_{i,z'}\vec{X} \geq T_{i,z}\vec{Y}$. This is the primary step for constructing couplings in Theorem A.4,and we will prove this in the next lemma. Once we can couple positive jumps of the i-th particle in the slower process by positive jumps of its corresponding particle in the faster process, we only need to assign jump rates according to different pairs z, z'. The assignment is possible by Assumptions A'1,A'3.. See (A.17),(A.18) in Theorem A.4 for assignment in detail.

Lemma A.3. *Assume $\vec{X} \geq \vec{Y}$, and i is fixed. For every z in $(0, R]$, if $\vec{Y} \in A_{i,z}$, then there is a $z' \geq 0$ depending on $\vec{X}, \vec{Y}$, i,and z, such that $\max\{Y_i + z, X_i\} \geq X_i + z' \geq \min\{Y_i + z, X_i\}$ and*

$$\vec{X}' = T_{i,z'}\vec{X} \geq T_{i,z}\vec{Y} = \vec{Y}'. \tag{A.6}$$

The choice of z' can be made so that every nonzero z' corresponds to a unique z in $(0, R]$ satisfying $\vec{Y} \in A_{i,z}$.

Proof. We first describe how to find z', and then we show (A.6) by considering a simple case and the general case. Without losing generality, we assume that $i = 0$ in figures below. Suppose there are exactly k holes in $\vec{Y}$ between Y_i and $Y_i + R$: $H_1, \ldots, H_k$. We label them in a descending order:

$$Y_i < H_k = Y_i + z_k < H_{k-1} = Y_i + z_{k-1} < \cdots < H_1 = Y_i + z_1 \leq Y_i + R \tag{A.7}$$

Step1. Define $z'_l, l = 1, 2, \dots, k$ inductively by,

$$z'_1 = \begin{cases} \max\{z' > 0 : X_i + z' \le Y_i + z_1, \vec{X} \in A_{i,z'}\} & \text{, if exists} \\ 0 & \text{, otherwise} \end{cases} \tag{A.8}$$

$$z'_{l+1} = \begin{cases} \max\{z'_l > z' > 0 : X_i + z' \le Y_i + z_{l+1}, \vec{X} \in A_{i,z'}\} & \text{, if exists} \\ 0 & \text{, otherwise.} \end{cases} \tag{A.9}$$

That is, for $l > 1$, if $z'_l > 0$, $H'_l = X_i + z'_l$ is the right-most hole in $\vec{X}$ which is to the left of both H'_{l-1} in $\vec{X}$ and H_l in $\vec{Y}$. (H'_l might equal H_l, but $H'_l < H'_{l-1}$.) See Figure 6 for an example. In this example, $i = 0$, $R = 8$, $z'_1 = 7$, $z'_2 = 3$, $z'_3 = 1$, $z'_4 = 0$.

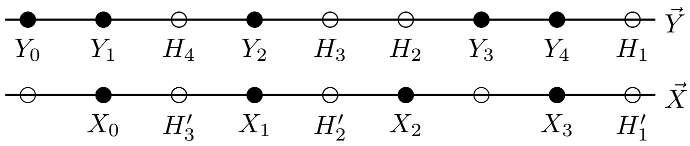

Figure 6: Target Sites z' for X_0

Step2. We consider a simple case first. We assume that $X_i = Y_i$ and the numbers of particles on the fixed interval $[X_i, X_i + R]$ in both $\vec{X}$, $\vec{Y}$ are identical. In view of (4.3), and let

$$I_1 := I_{i,R}\left(\vec{Y}\right) = \max\{s : Y_s \le Y_i + R\} \quad \text{and} \quad I_2 := I_{i,R}\left(\vec{X}\right) = \max\{s : X_s \le X_i + R\},$$

we have

$$I_1 - i + 1 = I_2 - i + 1. \tag{A.10}$$

Then, it is immediate to see that the numbers of holes on $[X_i, X_i + R]$ in both $\vec{X}$ and $\vec{Y}$ are the same as $k = R - I_1 + i$. Following (A.7), (A.8), (A.9), we actually label all the holes in $\vec{X}$ in the descending order,

$$H'_k = X_i + z'_k \le H'_{k-1} = X_i + z'_{k-1} \le \cdots \le H'_1 = X_i + z'_1$$

and pair holes in $\vec{X}$, $\vec{Y}$ with

$$H'_l \le H_l, \text{ for } l = 1, \dots, k. \tag{A.11}$$

We emphasize that, when the numbers of particles on $[X_i, X_i + R]$ in $\vec{X}$, $\vec{Y}$ are the same, because $\vec{X} \ge \vec{Y}$, (A.11) is equivalent to "holes are paired via a vertical line or a southwest line", and (A.11) is also equivalent to $X_j \le Y_j$ for all X_j, Y_j on $[X_i, X_i + R]$. See Figure 7. In this example, $R = 8, i = 0, I_1 = I_2 = 5$.

After a jump, there is a relabeling of holes according to the previous rule. Hence (A.11) is preserved. Indeed, after the jumps to $Y_i + z$ and $X_i + z'$, we delete a line connecting $Y_i + z$ and $X_i + z'$, and add a vertical line connecting the initial positions of X_i and Y_i, see Figure 7. Since only particles on $[X_i, X_i + R]$ are affected by the jumps, and the number of particles on $[X_i, X_i + R]$ are the same for $T_{i,z}\vec{Y}$, $T_{i,z'}\vec{X}$, we can conclude that $T_{i,z}\vec{Y} \le T_{i,z'}\vec{X}$ from the "new" (A.11).

Step3. For the general case, we can assume that $X_i \le Y_i + R$. Otherwise, if $Y_i + R < X_i$, we can easily find that $z' = 0$ and $T_{i,z}\vec{Y} \le \vec{X} = T_{i,0}\vec{X}$.

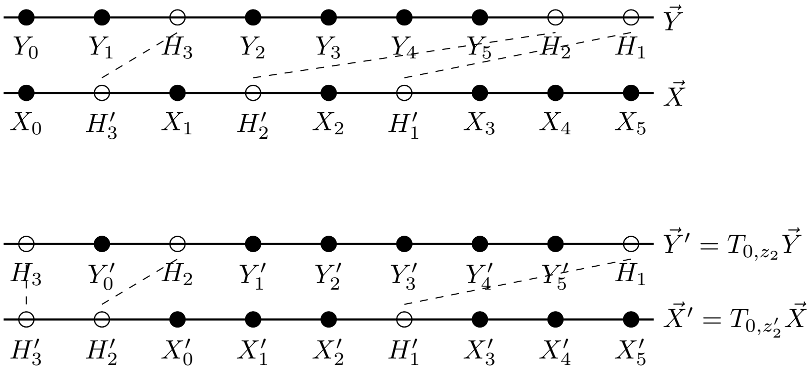

Figure 7: Configurations before and after Jumps z_2, z_2'

This situation is similar to the simple case. We first compare the number of particles in $\vec{Y}$ on the interval $[Y_i, Y_i + R]$ with the number of particles $\vec{X}$ on the interval $[X_i, Y_i + R]$. The right end point is always $Y_i + R$. Let

$$I_1 := I_{i,R}(\vec{Y}), \quad \text{and} \quad I_2 := \max\{s : X_s \leq Y_i + R\}.$$

We get from $\vec{Y} \leq \vec{X}$ that

$$I_2 = \max\{s : X_s \leq Y_i + R\} \leq \max\{s : Y_s \leq Y_i + R\} = I_1. \tag{A.12}$$

Since $X_i + z' \leq Y_i + z \leq Y_i + R$, we see that only particles in $\vec{Y}$ on $[Y_i, Y_i + R]$ and particles in $\vec{X}$ on $[X_i, Y_i + R]$ are affected by the jumps z and z'. Therefore, we only need to show that for every pair z, z',

$$X_j' = (T_{i,z'}\vec{X})_j \geq Y_j' = (T_{i,z}\vec{Y})_j, \text{for all } i \leq j \leq I_1. \tag{A.13}$$

For all $I_2 < j \leq I_1$, (A.13) is immediate since

$$X_j' = X_j > Y_i + R \geq Y_j'.$$

To get (A.13) for all $i \leq j \leq I_2$, we can add artificial particles and holes on $[Y_i, Y_i + R + I_1 - I_2]$ for $\vec{X}$ and $\vec{Y}$ as follows to get two new configurations $\vec{X}''$ and $\vec{Y}''$ (restricted to this interval) with the same number of particles.

(a) Replace all particles on $[Y_i, X_i)$ in $\vec{X}$ with holes. Move the i-th particle in $\vec{X}$ from X_i to Y_i.

(b) Replace all holes on $(Y_i + R, Y_i + R + I_1 - I_2]$ with particles for $\vec{X}$.

(c) Replace all particles on $(Y_i + R, Y_i + R + I_1 - I_2]$ with holes for $\vec{Y}$.

See Figure 8 for an example. In this example, $R = 8$, $I_1 = 6, I_2 = 4$.

The new configurations on $[Y_i, Y_i + R + I_1 - I_2]$ have the same numbers of holes, too. We can pair holes in $\vec{X}''$ and $\vec{Y}''$ in the descending order (uniquely). Holes on $(Y_i + R, X_i + R]$ in $\vec{Y}''$ are the only additional holes added to $\vec{Y}$, and they are added to match the number of holes in $\vec{X}''$ and $\vec{Y}''$. We don't need to consider these additional holes and their corresponding holes in $\vec{X}''$, and therefore, we can keep the labels of the original holes in $\vec{Y}$ and label the additional holes as j-th holes, with non-positive indices $0 \geq j \geq I_2 - I_1 + 1$. We denote by $H_j'' = Y_i + z_j''$ the j-th corresponding hole in $\vec{X}''$, including the ones with

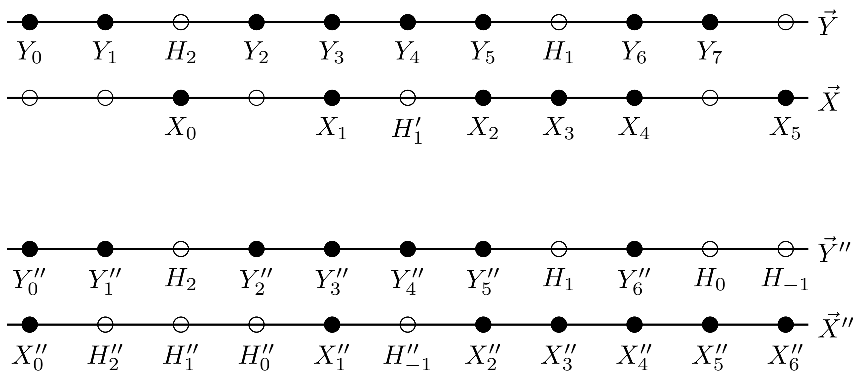

Figure 8: Add artificial particles and holes

non-positive indices. On one hand, we can use the same argument for the simple case to get that, after jumps z and z'', (restricted on $[Y_i, Y_i + R + I_1 - I_2]$)

$$T_{i,z''}\vec{X}'' \geq T_{i,z}\vec{Y}''. \tag{A.14}$$

On the other hand, if $z'_j > 0$, H'_j is the right-most hole to the left of H_j and H'_{j-1}. We can see that the target sites for the particles at X_i and X''_i satisfy

$$X_i + z'_j = H'_j \geq H''_j = Y_i + z''_j \tag{A.15}$$

from induction, (A.8) and (A.9). And if $z'_j = 0$, the target sites for the particles at X_i and X''_i also satisfy (A.15),

$$X_i + z'_j = H'_j = X_i \geq H''_j = Y_i + z''_j.$$

On $[Y_i, Y_i + R + I_1 - I_2]$, we can get $T_{i,z'}\vec{X}$ by moving the i-th particle in $\vec{X}''$ to the site $H' = X_i + z'$ and relabeling. We can also get $T_{i,z''}\vec{X}''$ by moving the i-th particle in $\vec{X}''$ to the site $H'' = Y_i + z''$ and relabeling. Therefore, from (A.15), we get that for $i \leq j \leq I_2$, $H' - Y_i \geq z''$ and

$$(T_{i,z'}\vec{X})_j = (T_{i,H'-Y_i}\vec{X}'')_j \geq (T_{i,z''}\vec{X}'')_j, \tag{A.16}$$

where the last inequality is due to monotonicity in z for $T_{i,z}\vec{X}''$ when the jump z is possible. (Indeed, if jumps $z' \geq z''$ are possible, with relabeling, we can get $T_{i,z'}\vec{X}''$ by first move the i-th particle to $X''_i + z''$, and then move the particle at $X''_i + z''$ to $X_i + z'$. With these two operation, we can derive that $T_{i,z'}\vec{X} \geq T_{i,z''}\vec{X}''$.)

By comparing particles with indices from i to I_2, and using (A.14) and (A.16), we get

$$X'_j = (T_{i,z'}\vec{X})_j \geq (T_{i,z''}\vec{X}'')_j \geq (T_{i,z}\vec{Y}'')_j = Y'_j$$

for all $i \leq j \leq I_2$. □

One can see from the proof of Lemma A.3 that we have more than one way to assign z' to ensure (A.6). We take a convenient one, which helps us to obtain the coupling for Theorem 4.2.

Let $\mathcal{C}_+$ be the class of jump rates $p(.,.)$ with the following properties:

A*1 (Positive) $p(x,y) \geq 0$, if $y > x$; otherwise, $p(x,y) = 0$, A*2 (Finite-range) there is an $R > 0$, such that $p(x, y) = 0$, for all $y - x > R$,

A*3 (Radially Decreasing) for $x, y \neq 0$ and $x < y$, $p(x, y)$ is increasing in x, and decreasing in y,

A*4 (A Blockage at 0) $p(x, y) = 0$ if $x = 0$ or $y = 0$.

Notice that the value of $p(0, y)$ can be any non-negative number as long as $p(x, 0) = 0$ for all x and no particle is at the site 0 initially, since no particles can jump to the site 0. Jump rates $p(.,.)$ from the class $\mathcal{C}_+$ correspond to jumps along the positive direction. To get jumps towards both directions, we combine two jump rates to get $p_c(x, y) = p_+(x, y) + p_-(y, x)$ where both p_+, p_- are from the class $\mathcal{C}_+$. We shall denote the collection of p_c as $\mathcal{C}$.

The main result in the following theorem is the first part, which says we can couple two AEPs with a blockage $\vec{X}_t = (\vec{X}_0, \tilde{L}, p, 0) \succeq \vec{Y}_t = (\vec{Y}_0, \tilde{L}, p_+, 0)$ when they have the same jump rates p_+ from the class $\mathcal{C}_+$. With (4.5), (4.6), (4.7), and Lemma A.1, we can replace p_+ from $\mathcal{C}_+$ by p_c from $\mathcal{C}$. Lastly, we can use Lemmas A.1, A.2 to replace zero jump rates $q(.)$ in either $\vec{X}_t$ or $\vec{Y}_t$ by a nonzero $q(.)$.

Theorem A.4. *Suppose jump rates $p_+(.,.)$, $p_-(.,.)$ are from the class $\mathcal{C}_+$.*

1. *There is a joint generator Ω_+, such that for any $\vec{X}_0 \geq \vec{Y}_0$, we can couple the pair of auxiliary processes $\vec{X}_t = (\vec{X}_0, \tilde{L}, p_+, 0) \succeq \vec{Y}_t = (\vec{Y}_0, \tilde{L}, p_+, 0)$ via Ω_+.*

2. *For combined jump rates $p_c(x, y) = p_+(x, y) + p_-(y, x)$, there is a joint generator Ω, such that for any $\vec{X}_0 \geq \vec{Y}_0$, we can couple the pair of auxiliary processes $\vec{X}_t = (\vec{X}_0, \tilde{L}, p_c, 0) \succeq \vec{Y}_t = (\vec{Y}_0, \tilde{L}, p_c, 0)$ via Ω.*

3. *(Theorem 4.2) Let $q(.) : \mathbb{Z} \setminus \{0\} \to \mathbb{R}_{\geq 0}$, and $p_c(x, y) = p_+(x, y) + p_-(y, x)$. There are generators Ω_R, and Ω_L, such that for any $\vec{X}_0 \geq \vec{Y}_0$, we can couple $\vec{X}_t = (\vec{X}_0, \tilde{L}_R, p_c, q) \succeq \vec{Y}_t = (\vec{Y}_0, \tilde{L}, p_c, 0)$ via Ω_R, and $\vec{W}_t = (\vec{X}_0, \tilde{L}, p_c, 0) \succeq \vec{Z}_t = (\vec{Y}_0, \tilde{L}_L, p_c, q)$ via Ω_L.*

Proof. 1. By Lemma A.3, for any $\vec{X} \geq \vec{Y}$ and $0 < z \leq R$, we can find a $z' = C(\vec{X}, \vec{Y}, i, R, z) \geq 0$, such that

$$T_{i,z'}\vec{X} \geq T_{i,z}\vec{Y}.$$

(One choice for $C(\vec{X}, \vec{Y}, i, R, z)$ is the function constructed inductively in the proof of Lemma A.3.) Therefore, we can assign the jump rates for the i-th particles by following functions:

$$p_{i,s,z}(\vec{X}, \vec{Y}) := \begin{cases} \mathbb{1}_{A_{i,z}}(\vec{Y}) \cdot p(Y_i, Y_i + z) & \text{, if } s = C(\vec{X}, \vec{Y}, i, R, z), \text{ and } \vec{X} \geq \vec{Y}, \\ 0 & \text{, else ,} \end{cases} \tag{A.17}$$

$$p_{i,s,0}(\vec{X}, \vec{Y}) := \begin{cases} \mathbb{1}_{A_{i,s}}(\vec{X}) \left(p(X_i, X_i + s) - \sum_{0<z\leq R} p_{i,s,z}(\vec{X}, \vec{Y}) \right) & \text{, if } s > 0 \\ 0 & \text{, if } s = 0. \end{cases} \tag{A.18}$$

In particular, by (A.17), at most one term in the sum of (A.18) is positive with value $p(Y_i, Y_i + z)$ for some $z \leq R$. By Lemma A.3, we get $X_i \geq Y_i, X_i + s \leq Y_i + z$, which implies $p_{i,s,0} \geq 0$ by Assumption A*3.

Then we define the generator Ω_+ by its action on a local function F by: if $\vec{X} \geq \vec{Y}$,

$$\Omega_+ F(\vec{X},\vec{Y}) = \sum_{i,0<z\leq R,0\leq s\leq R} p_{i,s,z}(\vec{X},\vec{Y}) \left[F(T_{i,s}\vec{X},T_{i,z}\vec{Y}) - F(\vec{X},\vec{Y})\right] \tag{A.19}$$

$$+ \sum_{i,0<s\leq R} p_{i,s,0}(\vec{X},\vec{Y}) \left[F(T_{i,s}\vec{X},\vec{Y}) - F(\vec{X},\vec{Y})\right], \tag{A.20}$$

if $\vec{X} \ngeq \vec{Y}$,

$$\Omega_+ F(\vec{X},\vec{Y}) = \sum_{i,0<z\leq R} \mathbb{1}_{A_{i,z}}(\vec{Y}) p(Y_i, Y_i+z) \left[F(\vec{X},T_{i,z}\vec{Y}) - F(\vec{X},\vec{Y})\right] \tag{A.21}$$

$$+ \sum_{i,0<s\leq R} \mathbb{1}_{A_{i,s}}(\vec{X}) p(X_i, X_i+s) \left[F(T_{i,s}\vec{X},\vec{Y}) - F(\vec{X},\vec{Y})\right] \tag{A.22}$$

(A.19) corresponds to the case in Lemma A.3 when both of the i-th particles in $\vec{X}$ and $\vec{Y}$ jump, while (A.20) corresponds to the case where only the i-th particle in $\vec{X}$ jumps; (A.21),(A.22) correspond to the case where particles in $\vec{X}, \vec{Y}$ jump independently. The rest is to check Ω_+ satisfies Definition 4.1. This is standard:

The initial configuration can always be chosen with $\vec{W} \geq \vec{V}$ almost surely and $\vec{W} \stackrel{d}{=} \vec{X}_0$, $\vec{V} \stackrel{d}{=} \vec{Y}_0$. (See Theorem B9[15])

To show $\vec{W}_t \geq \vec{V}_t$ almost surely, use the same arguments in the proof of Lemma A.1. We want to show the closed set $F_0 = \{(\vec{X},\vec{Y}) : \vec{X} \geq \vec{Y}\}$ is an absorbing set by checking $\Omega_+ \mathbb{1}_{F_0}(\vec{X},\vec{Y}) \geq 0$:

(a) for $\vec{X} \geq \vec{Y}$, by Lemma A.3 and $p_{i,s,z}(\vec{X},\vec{Y}) \geq 0$

$$\Omega_+ \mathbb{1}_{F_0}(\vec{X},\vec{Y}) = \sum_{\substack{i\in\mathbb{Z},0<z\leq R,\\ 0\leq s\leq R}} p_{i,s,z}(\vec{X},\vec{Y}) \left[\mathbb{1}_{F_0}(T_{i,s}\vec{X},T_{i,z}\vec{Y}) - \mathbb{1}_{F_0}(\vec{X},\vec{Y})\right]$$
$$+ \sum_{i\in\mathbb{Z},0<s\leq R} p_{i,s,0}(\vec{X},\vec{Y}) \left[\mathbb{1}_{F_0}(T_{i,s}\vec{X},\vec{Y}) - \mathbb{1}_{F_0}(\vec{X},\vec{Y})\right] = 0.$$

(b) for $\vec{X} \ngeq \vec{Y}$, it's obvious that $\Omega_+ \mathbb{1}_{F_0}(\vec{X},\vec{Y}) \geq 0$ since each term is nonnegative. We only need to show the sum is finite. Notice that only finitely many terms in (A.22) are positive. Since $T_{i,s}$ changes finitely many X_i, if one term $T_{i,s}\vec{X} \geq \vec{Y}$ holds while $\vec{X} \ngeq \vec{Y}$, $T_{i',s'}\vec{X} \geq \vec{Y}$ holds for finitely many pairs i', s'. Similarly, only finitely many terms in (A.21) are positive. Therefore, $\Omega_+ \mathbb{1}_{F_0}(\vec{X},\vec{Y}) \geq 0$.

To show the marginal conditions, we will check for $F_2(\vec{X},\vec{Y}) = H_2(\vec{Y})$, and the other follows directly from $T_{i,0}\vec{X} = \vec{X}$, (A.17) and (A.18). On $F_0^c = \{(\vec{X},\vec{Y}) : \vec{X} \ngeq \vec{Y})\}$, clearly $\Omega_+ H_2(\vec{X},\vec{Y}) = \tilde{L}H_2(\vec{Y})$. We only need for every $\vec{X} \geq \vec{Y}$,

$$\begin{aligned}
\Omega_+ F_2(\vec{X},\vec{Y}) = & \sum_{i\in\mathbb{Z},0<z\leq R,0\leq s\leq R} p_{i,s,z}(\vec{X},\vec{Y}) \left[H_2(T_{i,z}\vec{Y}) - H_2(\vec{Y})\right] \\
& + \sum_{i\in\mathbb{Z},0<s\leq R} p_{i,s,0}(\vec{X},\vec{Y}) \left[H_2(\vec{Y}) - H_2(\vec{Y})\right] \\
= & \sum_{i\in\mathbb{Z},0<z\leq R,0\leq s\leq R} p_{i,s,z}(\vec{X},\vec{Y}) \left[H_2(T_{i,z}\vec{Y}) - H_2(\vec{Y})\right] \\
= & \sum_{i\in\mathbb{Z},0<z\leq R} \left(\sum_{0\leq s\leq R} \mathbb{1}_{\{s=C(\vec{X},\vec{Y},i,R,z)\}}\right) \\
& \cdot \mathbb{1}_{A_{i,z}}(\vec{Y}) p(Y_i, Y_i+z) \left[H_2(T_{i,z}\vec{Y}) - H_2(\vec{Y})\right] \\
= & \sum_{i\in\mathbb{Z},0<z\leq R} \mathbb{1}_{A_{i,z}}(\vec{Y}) p(Y_i, Y_i+z) \left[H_2(T_{i,z}\vec{Y}) - H_2(\vec{Y})\right] = \tilde{L}H_2(\vec{Y}).
\end{aligned}$$

The fourth equality is due to Lemma A.3, which implies that there is exactly one s in $[0,R]$ such that $s = C(\vec{X},\vec{Y},i,R,z)$.

2. The second part is an application of Lemma A.1, the change of variable argument in (4.5), (4.6), (4.7),and the first part.

   Let $\vec{X}_{-,t} = (R(\vec{X}_0), \tilde{L}, \tilde{p}_-, 0)$, where $\tilde{p}_-(x,y) = p_-(y,x)$. Then, $R(\vec{X}_{-,t}) = (\vec{X}_0, \tilde{L}, p_-, 0)$. As $R(.)$ is a map reversing ordering,

   $$\vec{X} \geq \vec{Y} \Leftrightarrow R(\vec{X}) \leq R(\vec{Y}).$$

   By the first part of Theorem A.4, we can couple $\vec{X}_{-,t} = (R(\vec{X}_0), \tilde{L}, \tilde{p}_-, 0) \preceq (R(\vec{Y}_0), \tilde{L}, \tilde{p}_-, 0) = \vec{Y}_{-,t}$ for any $\vec{X}_0 \geq \vec{Y}_0$ via a generator. Therefore, there is a generator Ω_-, via which we can couple $R(\vec{X}_{-,t}) = (\vec{X}_0, \tilde{L}, p_-, 0) \succeq R(\vec{Y}_{-,t}) = (\vec{Y}_0, \tilde{L}, p_-, 0)$ for any $\vec{X}_0 \geq \vec{Y}_0$. Then by Lemma A.1, we get the joint generator $\Omega = \Omega_+ + \Omega_-$.

3. This is a consequence of the second part, Lemma A.1 and Lemma A.2. Take $\Omega_R = \Omega_{0,R} + \Omega$, and $\Omega_L = \Omega_{0,L} + \Omega$. We will show the first case, and the other is similar:

   By the second part of Theorem A.4, we have a generator Ω, via which we can couple auxiliary processes

   $$\vec{X}_t = (\vec{X}_0, \tilde{L}, p_c, 0) \succeq (\vec{Y}_0, \tilde{L}, p_c, 0) = \vec{Y}_t,$$

   for any $\vec{X}_0 \geq \vec{Y}_0$. By Lemma A.2, we can also find a generator $\Omega_{0,R}$ to couple auxiliary processes

   $$\vec{W}_t = (\vec{X}_0, \tilde{L}_R, 0, q) \succeq (\vec{Y}_0, \tilde{L}, 0, 0) = \vec{Z}_t,$$

   for any $\vec{X}_0 \geq \vec{Y}_0$. Notice that $\vec{X}_t$ is also $(\vec{X}_0, \tilde{L}_R, p_c, 0)$. By Lemma A.1, we can use generator $\Omega_R = \Omega_{0,R} + \Omega$ to couple

   $$\vec{U}_t = (\vec{X}_0, \tilde{L}_R, p_c, q) \succeq (\vec{Y}_0, \tilde{L}_R, p_c, 0) = \vec{V}_t$$

   for any $\vec{X}_0 \geq \vec{Y}_0$. □

In the proof of the first part of Theorem A.4, we see $p_{i,s,z}$ and $p_{i,s,0}$ defined by (A.17) and (A.18) are important in constructing the joint generator Ω_+ defined by (A.20)-(A.22).

They require Lemma A.2 and assumption A*3. The Lemma A.2 depends on the finite range R and $\vec{X} \geq \vec{Y}$, while the latter is an assumption on the jump rates. We can easily modify $p_{i,s,z}$, $p_{i,s,0}$ and Ω_+ to couple two modified auxiliary processes defined in section 8.

Corollary A.5. *Let $p(.)$ satisfy assumption A'1, A'3, and $q(.)$ be of range R with an extra condition*

$$\begin{cases} q(k) \geq p(k), & \text{if } k > 0, \\ q(k) \leq p(k), & \text{if } k < 0, \end{cases} \tag{A.23}$$

Then we can find a joint generator $\tilde{\Omega}$ to couple modified auxiliary processes $(\vec{X}_t, I_t) = (\vec{X}_0, p, q, I_t)$ and $(\vec{Y}_t, i_t) = (\vec{Y}_0, p, p, i_t)$ for any initial condition $\vec{X}_0 \geq \vec{Y}_0$, in the sense

$$\begin{aligned} \vec{X}_t \geq \vec{Y}_t&, \text{ for all } t \geq 0, \\ \tilde{\Omega} F_1\left(\vec{X}, I, \vec{Y}, i\right) =& \hat{L}_{p,q} H_1\left(\vec{X}, I\right), \\ \tilde{\Omega} F_2\left(\vec{X}, I, \vec{Y}, i\right) =& \hat{L}_{p,p} H_2\left(\vec{Y}, i\right), \end{aligned}$$

for any local functions $F_1\left(\vec{X}, I, \vec{Y}, i\right) = H_1\left(\vec{X}, I\right)$ and $F_2\left(\vec{X}, I, \vec{Y}, i\right) = H_2\left(\vec{Y}, i\right)$.

Proof. We will give the joint generator $\tilde{\Omega} = \tilde{\Omega}_+ + \tilde{\Omega}_-$ by writing out $\tilde{\Omega}_+$ and $\tilde{\Omega}_-$, which will have the same form in terms of $p_{j,s,z}$. The rest is to check conditions, which follows almost the same arguments as those in the first part of Theorem A.4, and we will omit it.

We first define the modified $\tilde{\Omega}_+$ by modifying $p_{j,s,z}$, $p_{j,s,0}$ from (A.17) and (A.18):

For $R \geq z > 0$, $R \geq s \geq 0$,

$$\tilde{p}_{j,s,z}\left(\vec{X}, I, \vec{Y}, i\right) := \begin{cases} \mathbb{1}_{A_{j,z}}(\vec{Y}) \cdot p(z) & , \text{ if } s = C(\vec{X}, \vec{Y}, j, R, z), \text{ and } \vec{X} \geq \vec{Y} \\ 0 & , \text{ else} \end{cases} \tag{A.24}$$

$$\tilde{p}_{j,s,0}\left(\vec{X}, I, \vec{Y}, i\right) := \begin{cases} \mathbb{1}_{A_{j,s}}(\vec{X}) \left(p(s) - \sum_{0<z\leq R} \tilde{p}_{j,s,z}\left(\vec{X}, I, \vec{Y}, i\right)\right) & , \text{ if } j \neq I \\ \mathbb{1}_{A_{j,s}}(\vec{X}) \left(q(s) - \sum_{0<z\leq R} \tilde{p}_{I,s,z}\left(\vec{X}, I, \vec{Y}, i\right)\right) & , \text{ if } j = I \end{cases} \tag{A.25}$$

where $C(\vec{X}, \vec{Y}, j, R, z)$ is the function constructed in Lemma A.3. If we replace q by p, (A.25) is the same as (A.18). Therefore, it is nonnegative by condition (A.23). Then the generator $\tilde{\Omega}_+$ acts on F is given by: if $\vec{X} \geq \vec{Y}$,

$$\begin{aligned} \tilde{\Omega}_+ F\left(\vec{X}, I, \vec{Y}, i\right) = \sum_{\substack{j\in\mathbb{Z}, 0\leq z\leq R, \\ 0\leq s\leq R}} \tilde{p}_{j,s,z}\left(\vec{X}, I, \vec{Y}, i\right) \Big[& F\left(T_{j,s}\vec{X}, \hat{I}_{j,s}\left(\vec{X}, I\right), T_{j,z}\vec{Y}, \hat{I}_{j,z}\left(\vec{Y}, i\right)\right) \\ & - F\left(\vec{X}, I, \vec{Y}, i\right) \Big], \end{aligned}$$

and if $\vec{X} \ngeq \vec{Y}$,

$$\begin{aligned} \tilde{\Omega}_+ F\left(\vec{X}, I, \vec{Y}, i\right) = & \sum_{j\in\mathbb{Z}, 0<z\leq R} \mathbb{1}_{A_{j,z}}(\vec{Y}) \cdot p(z) \left[F\left(\vec{X}, I, T_{i,z}\vec{Y}, \hat{I}_{j,z}\left(\vec{Y}, i\right)\right) - F\left(\vec{X}, I, \vec{Y}, i\right)\right] \\ & + \sum_{j\neq I, 0<s\leq R} \mathbb{1}_{A_{j,s}}(\vec{X}) \cdot p(s) \left[F\left(T_{i,s}\vec{X}, \hat{I}_{j,s}(\vec{X}, I), \vec{Y}, i\right) - F\left(\vec{X}, I, \vec{Y}, i\right)\right] \\ & + \sum_{0<s\leq R} \mathbb{1}_{A_{I,s}}(\vec{X}) \cdot q(s) \left[F\left(T_{i,s}\vec{X}, I_{I,s}(\vec{X}), \vec{Y}, i\right) - F\left(\vec{X}, I, \vec{Y}, i\right)\right], \end{aligned}$$

where $\hat{I}_{j,z}\left(\vec{X}, I\right)$ is defined by (8.2), which is the same as $I_{I,z}(\vec{X})$ when $j = I$. On the other hand, we can also define $\tilde{\Omega}_-$ in a similar way:

For $R \geq -s > 0$, $R \geq -z \geq 0$,

$$\tilde{p}_{j,s,z}\left(\vec{X}, I, \vec{Y}, i\right) := \begin{cases} \mathbb{1}_{A_{i,z}}(\vec{Y}) \cdot q(s) & \text{, if} j = i, z = -C(R(\vec{Y}), R(\vec{X}), -i, R, -s), \text{and} \vec{X} \geq \vec{Y} \\ \mathbb{1}_{A_{j,z}}(\vec{Y}) \cdot p(s) & \text{, if} j \neq i, z = -C(R(\vec{Y}), R(\vec{X}), -j, R, -s), \text{and} \vec{X} \geq \vec{Y} \\ 0 & \text{, else} \end{cases} \tag{A.26}$$

$$\tilde{p}_{j,0,z}\left(\vec{X}, I, \vec{Y}, i\right) := \mathbb{1}_{A_{j,z}}(\vec{X}) \left(p(z) - \sum_{0<-s\leq R} \tilde{p}_{j,s,z}\left(\vec{X}, I, \vec{Y}, i\right) \right), \tag{A.27}$$

where $R(\vec{X})$ is defined via (4.5). Also, by replacing q by p in (A.26) and using condition (A.23), we see both (A.26) and (A.27) are nonnegative. The generator $\tilde{\Omega}_-$ acts on F is given by: if $\vec{X} \geq \vec{Y}$,

$$\begin{aligned} \tilde{\Omega}_- F\left(\vec{X}, I, \vec{Y}, i\right) = \sum_{\substack{j, 0\leq -s\leq R, \\ 0\leq -z\leq R}} \tilde{p}_{j,s,z}\left(\vec{X}, I, \vec{Y}, i\right) \Big[& F\left(T_{j,s}\vec{X}, \hat{I}_{j,s}\left(\vec{X}, I\right), T_{j,z}\vec{Y}, \hat{I}_{j,z}\left(\vec{Y}, i\right)\right) \\ & - F\left(\vec{X}, I, \vec{Y}, i\right) \Big], \end{aligned}$$

and if $\vec{X} \not\geq \vec{Y}$,

$$\begin{aligned} \tilde{\Omega}_- F\left(\vec{X}, I, \vec{Y}, i\right) = & \sum_{j\in\mathbb{Z}, 0<-z\leq R} \mathbb{1}_{A_{j,z}}(\vec{Y}) \cdot p(z) \left[F\left(\vec{X}, I, T_{i,z}\vec{Y}, \hat{I}_{j,z}\left(\vec{Y}, i\right)\right) - F\left(\vec{X}, I, \vec{Y}, i\right) \right] \\ & + \sum_{j\neq I, 0<-s\leq R} \mathbb{1}_{A_{j,s}}(\vec{X}) \cdot p(s) \left[F\left(T_{i,s}\vec{X}, \hat{I}_{j,s}(\vec{X}, I), \vec{Y}, i\right) - F\left(\vec{X}, I, \vec{Y}, i\right) \right] \\ & + \sum_{0<-s\leq R} \mathbb{1}_{A_{I,s}}(\vec{X}) \cdot q(s) \left[F\left(T_{I,s}\vec{X}, I_{I,s}(\vec{X}), \vec{Y}, i\right) - F\left(\vec{X}, I, \vec{Y}, i\right) \right]. \end{aligned}$$

We can see both $\tilde{\Omega}_+$ and $\tilde{\Omega}_-$ have the same form in terms of $\tilde{p}_{j,s,z}$. Then, we obtain $\tilde{\Omega}$ by $\tilde{\Omega} = \tilde{\Omega}_+ + \tilde{\Omega}_-$. □

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