Electron. Commun. Probab. **27** (2022), article no. 35, 1–13.
https://doi.org/10.1214/22-ECP475
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Stochastic approximation of the paths of killed Markov processes conditioned on survival*

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Abstract

Reinforced processes are known to provide a stochastic representation for the quasi-stationary distribution of a given killed Markov process – describing the killed Markov process at fixed time instants. In this paper we shall adapt the construction to provide a pathwise description. We also obtain a stochastic approximation for the quasi-limiting distributions of reducible killed Markov processes as a corollary.

Keywords: killed Markov processes; reinforced processes; urn processes.
MSC2020 subject classifications: 60J10; 60J85.
Submitted to ECP on February 9, 2022, final version accepted on June 27, 2022.

1 Introduction

We consider in discrete time a killed Markov process $(X_t)_{0\leq t<\tau_\partial}$ on the state space $\chi \cup \partial$, evolving in χ until the killing time $\tau_\partial = \inf\{t > 0 : X_t \in \partial\}$, after which time it remains in the cemetery state (we assume without loss of generality that ∂ is a one-point set). Note that whilst we specify that time is discrete, the results of this paper may be applied in continuous time by discretising time.

In general, one is interested in the law of the killed Markov process (with initial condition $X_0 \sim \mu \in \mathcal{P}(\chi)$) conditioned on survival,

$$\mathcal{L}_\mu(X_t|\tau_\partial > t),$$

and the long-time limits of this law,

$$\mathcal{L}_\mu(X_t|\tau_\partial > t)(\cdot) \to \pi(\cdot) \quad \text{as} \quad t \to \infty.$$

A general criterion for the existence and uniqueness of these limits is given by [6, Assumption (A)]. In general, these limits correspond to quasi-stationary distributions (QSDs) π, which satisfy

$$\mathcal{L}_\pi(X_t|\tau_\partial > t)(\cdot) = \pi(\cdot) \quad \text{for all} \quad 0 \leq t < \infty.$$

Aldous, Flannery and Palacios [1] introduced a method for simulating QSDs based on reinforced processes. The reinforced process, $(Y_t)_{0\leq t<\infty}$, is obtained by running a copy of X_t until it is killed,

$$(Y_t)_{0\leq t<\tau_\partial} = (X_t)_{0\leq t<\tau_\partial}. \tag{1.1}$$

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At this killing time, Y_t jumps to a point sampled independently from the empirical measure of the history,

$$Y_{\tau_\partial} \sim \frac{1}{\tau_\partial} \sum_{s=0}^{\tau_\partial - 1} \delta_{Y_s}(\cdot). \tag{1.2}$$

This is then repeated inductively, so that at the n^{th} killing time τ_∂^n, Y_t jumps to a point sampled independently from the empirical measure of the history of Y_t up to that killing time,

$$Y_{\tau_\partial^n} \sim \frac{1}{\tau_\partial^n} \sum_{s=0}^{\tau_\partial^n - 1} \delta_{Y_s}(\cdot) ds, \tag{1.3}$$

and continues evolving like a copy of X_t as before. In [1] they established that these provide an approximation method for the QSDs of irreducible killed Markov chains when the state space is finite and time is discrete, proving that

$$\frac{1}{t} \sum_{s=0}^{t-1} \delta_{Y_s}(\cdot) ds \overset{a.s.}{\to} \pi(\cdot) \quad \text{as} \quad t \to \infty. \tag{1.4}$$

This was made quantitative for a more general version of this algorithm by Benaim and Cloez [3], and has since been extended to a quite general setting in [7] and [2]. These results do not, however, apply to reducible killed Markov processes. We shall obtain in Corollary 2.4 a stochastic approximation for the quasi-limiting distributions of reducible killed Markov processes for given initial condition as a corollary of our main result, Theorem 2.3.

Note, however, that QSDs only describe killed Markov processes at fixed time instants. One may also be interested in obtaining pathwise information, so we may seek to approximate

$$\mathcal{L}_\mu((X_s)_{0 \le s \le t} | \tau_\partial > t)(\cdot)$$

for finite t. In this note, we shall demonstrate how the construction of reinforced processes may be adapted to provide such a pathwise approximation. Since our result (Theorem 2.3) is restricted to discrete time, for continuous time processes we obtain (for any $m < \infty$) an approximation for

$$\mathcal{L}_\mu((X_0, X_{\frac{t}{m}}, X_{\frac{2t}{m}}, \ldots, X_t) | \tau_\partial > t).$$

One may also consider the Q-process, which provides a pathwise description of $(X_t)_{0 \le t < \tau_\partial}$ conditioned never to be killed, a definition of which is given in [6, Theorem 3.1].

A second method for approximating QSDs is given by the Fleming-Viot process, a particle system introduced by Burdzy, Holyst and March in [5]. They considered the case whereby the killed Markov process is Brownian motion in an open, bounded domain, killed instantaneously upon contact with the boundary. They established in [5] that this particle system provides an approximation method for both the distribution of this killed Brownian motion conditioned on survival at fixed instants of time, and the corresponding QSD. This was later extended to a general setting by Villemonais [9]. In [4], Bieniek and Burdzy established that the Fleming-Viot process also provides for the distribution of the path of a killed Markov process conditioned to survive over a fixed time interval. Since then, the present author established in [8, Corollary 5.1] that the Fleming-Viot process also provides a representation for the Q-process when the killed Markov process is a normally reflected diffusion in a compact domain, killed at position-dependent Poisson rate (this result may be extended to a more general setting, subject to overcoming an additional difficulty if the state space is non-compact).

Thus, whilst both reinforced processes and the Fleming-Viot process are known to provide an approximation method for the QSDs of killed Markov processes – thereby describing killed Markov processes at fixed time instants – prior to the present paper only the Fleming-Viot process was known to provide a pathwise description. In the present paper, we shall show how the construction of reinforced processes may be adapted to provide such a pathwise description.

Whereas reinforced processes are constructed by sampling a spatial location from the history, we adapt this construction by sampling both the spatial and temporal location (or, with some probability, sampling a killed Markov process started from some fixed initial distribution at time 0 instead). We refer to the resultant constructions as reinforced path processes. A more precise definition is given by the following.

Throughout this paper we abuse notation by writing $[a, b]$ for $[a, b] \cap \mathbb{N}$, for all $a, b \in \mathbb{N}$. For paths f and g on the time intervals I and J respectively such that $f = g$ on $I \cap J$, we define

$$f \oplus g : I \cup J \ni t \mapsto \begin{cases} f(t), & t \in I \\ g(t), & t \in J \end{cases} . \tag{1.5}$$

Definition 1.1 (Reinforced Path Process)**.** *We fix a time horizon $0 < T \leq \infty$, a renewal probability $0 < p < 1$ and initial condition $\mu \in \mathcal{P}(\chi)$. The reinforced path process $(u_n)_{n=1}^{\infty}$ is defined by inductively sampling triples $u_n = (t_b^n, f^n, t_d^n)$, whereby t_b^n is the n^{th} birth time, t_d^n is the n^{th} killing time, and f^n is the n^{th} path from time 0 to time t_d^n. Whilst u_n is considered to be "alive" only between times t_b^n and t_d^n, its path f^n is defined prior to time t_b^n – it includes an "ancestral path". The first triple $u_1 = (t_b^1, f^1, t_d^1)$ is defined by taking a copy $(X_t)_{0 \leq t < \tau_\partial}$ of the killed Markov process with initial condition $X_0 \sim \mu$, and defining $u_1 = (t_b^1, f^1, t_d^1)$ to be $(0, (X_{t \wedge (\tau_\partial - 1)})_{0 \leq t \leq T}, \tau_\partial \wedge (T + 1))$.*

Given $u_1, \dots, u_n$ we inductively define $u_{n+1} = (t_b^{n+1}, f^{n+1}, t_d^{n+1})$ as follows. With probability p, we take another independent copy $(X_t)_{0 \leq t < \tau_\partial}$ of the killed Markov process with initial condition $X_0 \sim \mu$, and define $u_{n+1} = (t_b^{n+1}, f^{n+1}, t_d^{n+1})$ to be $(0, (X_{t \wedge (\tau_\partial - 1)})_{0 \leq t \leq T}, \tau_\partial \wedge (T + 1))$, in which case we say that we "renew". Otherwise, with probability $1 - p$, we choose $m \in [1, n]$ independently with probability

$$\frac{t_d^m - t_b^m}{\sum_{1 \leq \ell \leq n} (t_d^\ell - t_b^\ell)}.$$

Given our choice of m, we then choose $t' \in [t_b^m, t_d^m - 1]$ independently at random. Given the choice of m and t', we independently take $(X_t)_{0 \leq t < \tau_\partial}$ to be a copy of the killed Markov process with initial condition $X_0 = f^m(t')$, and set

$$u_{n+1} = (t_b^{n+1}, f^{n+1}, t_d^{n+1}) := (t', f^m_{|_{[0,t']}} \oplus (X_{(t - t') \wedge (\tau_\partial - 1)})_{t' \leq t \leq T}, (t' + \tau_\partial) \wedge (T + 1)).$$

Clearly, one may formulate the same construction in continuous time, but in this paper we shall only consider discrete time. We depict the reinforced path process corresponding to Brownian motion killed at the boundary of a bounded interval in Figure 1.

Structure of the paper

We will provide a statement of Theorem 2.3 in Section 2, which establishes that reinforced path processes provide an approximation for $\mathcal{L}_\mu((X_s)_{0 \leq s \leq t} | \tau_\partial > t)(\cdot)$, thus giving a pathwise description of killed Markov processes conditioned on survival. We will then use this in Corollary 2.4 to provide an approximation for the quasi-limiting distributions of reducible killed Markov processes.

We provide a proof of Theorem 2.3 in Section 3.

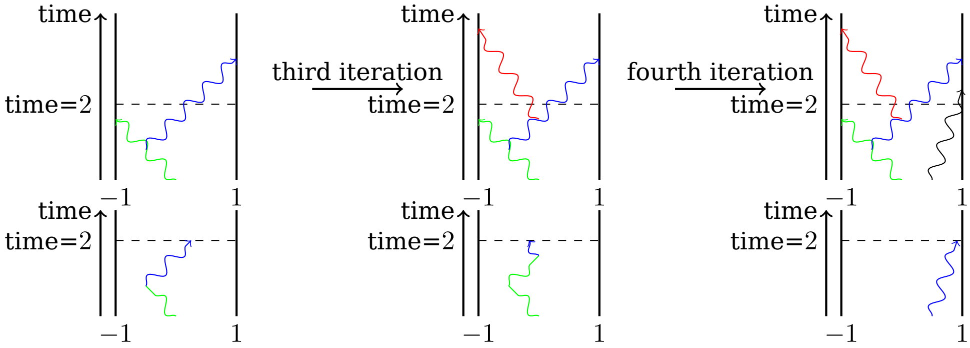

Figure 1: The killed Markov process here is Brownian motion in $(-1,1)$, killed instantaneously at $\{-1,1\}$. The third and fourth iteration are shown above, with the first, second, third and fourth paths in green, blue, red and black respectively. We may see that in the second and third iteration we sample from the history, whilst with the fourth we begin a new particle from time 0 - we renew. While each path is considered to be alive only between the time it is born (the sampled time) and the time it is killed at the boundary, it also carries with it information of its ancestral path from time 0 to the time it is born. Thus the three paths corresponding to time 2 after 4 iterations are shown below (the first path is not alive at time 2), with the path while they are alive in blue and the ancestral path before they are born in green.

2 Statement of results

For any topological space $\mathcal{T}$, we write $\mathcal{M}(\mathcal{T})$ and $\mathcal{P}(\mathcal{T})$ for the set of Borel measures (respectively Borel probability measures) on $\mathcal{T}$, equipped with the topology of weak convergence of measures.

We assume that (χ,d) is a metric space. We define $K:\chi\rightarrow\mathcal{M}(\chi)$ to be a submarkovian transition kernel, which defines the discrete-time killed Markov process $(X_t)_{0\leq t<\tau_\partial}$. We fix an initial condition $\mu\in\mathcal{P}(\chi)$, (possibly infinite) time horizon $T\in\mathbb{N}\cup\{\infty\}$ and renewal probability $0<p\leq 1$. We take a reinforced path process $(u_n)_{n=1}^{\infty}=((t_b^n,f^n,t_d^n))_{n=1}^{\infty}$ – that is a solution to Definition 1.1 – corresponding to these choices.

We write $\bar{\chi}$ for the completion of χ. We define $C_b(\chi)$, $C_0(\bar{\chi})$ and $C_0(\chi)$ to be the space of bounded, continuous functions on χ (respectively continuous functions on $\bar{\chi}$ which vanish on $\partial\chi:=\bar{\chi}\setminus\chi$, and the restriction to χ of elements of $C_0(\bar{\chi})$), all equipped with the uniform norm.

We impose the following assumption.

Assumption 2.1. *The completion $\bar{\chi}$ is compact. Moreover we assume that $(\chi\ni x\mapsto Kf(x):=K(x,f))\in C_0(\chi)$ for all $f\in C_b(\chi)$. Furthermore if $T<\infty$ we assume that $\mathbb{P}_\mu(\tau_\partial>T)>0$, whilst if $T=\infty$ we assume that $\mathbb{P}_\mu(\tau_\partial>t)>0$ for all $t<\infty$.*

Thus K defines a contraction operator

$$K:\chi\ni x\mapsto Kf(x):=K(x,f)\in C_0(\chi)\subseteq C_b(\chi)$$

with spectral radius $r(K)$. If $T=\infty$ then we impose the following additional assumption.

Assumption 2.2. *The operator $K:C_b(\chi)\rightarrow C_b(\chi)$ is compact, and $\mathbb{P}_x(\tau_\partial=\infty)=0$ for all $x\in\chi$.*

Note that if $\bar{\chi}$ weren't compact, we would need to impose a Lyapunov condition (see Page 9), which in this context would be complicated and likely hard to verify. Moreover it wouldn't be typical for K to be compact, so Assumption 2.2 would have to be replaced.

Theorem 2.3. *There exists a unique solution, Z, to*

$$z = p\sum_{s=0}^{T}\mathbb{P}_{\mu}(\tau_{\partial} > s)\Big[1-\frac{1-p}{z}\Big]^{-(s+1)}, \quad z\in[1,\infty). \tag{2.1}$$

We define the coefficients $(\gamma_t)_{0\leq t\leq T}$ (or $(\gamma_t)_{0\leq t<\infty}$ if $T=\infty$) to be

$$\gamma_t := p\mathbb{P}_{\mu}(\tau_{\partial} > t)\Big[1-\frac{1-p}{Z}\Big]^{-(t+1)}, \tag{2.2}$$

whereby Z is the unique solution to (2.1). Then for any $t\in[0,T]$ (or $t\in\mathbb{N}$ if $T=\infty$) we have

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}(t_b^n\leq t<t_d^n)\delta_{f^n_{[0,t]}}(\cdot)\overset{\mathcal{M}(F([0,t];\chi))}{\to}\gamma_t\mathcal{L}_{\mu}((X_s)_{0\leq s\leq t}|\tau_{\partial}>t)(\cdot) \quad \textit{a.s.} \quad \textit{as} \quad N\to\infty. \tag{2.3}$$

Note that if we have a continuous-time killed Markov process $(X_t)_{0\leq t<\tau_{\partial}}$, Theorem 2.3 can be applied to obtain, for any $m\in\mathbb{N}_{>0}$ and $t\in\mathbb{R}_{>0}$, a stochastic approximation for

$$\mathcal{L}_{\mu}((X_0, X_{\frac{t}{m}}, X_{\frac{2t}{m}},\dots,X_t)|\tau_{\partial}>t).$$

Choosing p

We ask how p should be chosen, firstly considering the case whereby $T<\infty$. We assume that the killed Markov process satisfies the consequence of [10, Theorem 2.1]. Thus we assume that there exists a right eigenfunction ϕ, positive and bounded on χ, such that

$$r(K)^{-t}\mathbb{P}_{\mu}(\tau_{\partial}>t)\to\mu(\phi) \quad \text{exponentially quickly, uniformly over all} \quad \mu\in\mathcal{P}(\chi). \tag{2.4}$$

Note that if γ_T is small, then by considering the mass on both sides of (2.3) we see that for large n a small proportion of the $u_n s$ will be alive at time T, leading to a large variance. If, on the other hand, γ_t is small for some $t<T$, then all $f^n s$ at time T will share a small number of ancestral paths at time t, leading to a large variance. Thus it is reasonable that we should seek to maximise $\min_{0\leq t\leq T}\gamma_t$. Therefore by (2.4) it is reasonable to choose p such that

$$1-\frac{1-p}{Z}=r(K). \tag{2.5}$$

Straightforward algebra shows that this is achieved by

$$p=[1+(1-r(K))\sum_{s=0}^{T}\mathbb{P}_{\mu}(\tau_{\partial}>s)r(K)^{-(s+1)}]^{-1}=\frac{r(K)}{(1-r(K))\mu(\phi)T}+\mathcal{O}_{T\to\infty}\Big(\frac{1}{T^2}\Big). \tag{2.6}$$

We now turn to the $T=\infty$ case. We seek to maximise $\min_{t\leq H}\gamma_t$, for some large time horizon $H<\infty$. We have that $\mathbb{E}_x[\tau_{\partial}]$ is uniformly bounded by (2.4), giving a uniform bound on Z by considering the mass on both sides of (2.3). Thus $\min_{t\leq H}\gamma_t=\mathcal{O}_{H\to\infty}(\frac{1}{H})$. We try

$$1-\frac{1-p}{Z}=r(K)e^{\frac{A}{H}}, \quad \text{for some} \quad A>0, \tag{2.7}$$

since (2.5) would give $Z=\infty$. We solve for $p(H)$, obtaining $\min_{t\leq H}\gamma_t\geq\frac{cAe^{-A}}{\mu(\phi)H(1-r(K))}$ for some constant $c>0$, the optimal scaling in H. Thus we choose $A=1$, corresponding to

$$p(H)=\frac{r(K)}{(1-r(K))\mu(\phi)H}+\mathcal{O}_{H\to\infty}\Big(\frac{1}{H^2}\Big). \tag{2.8}$$

Approximation of the QLDs of reducible killed Markov processes

If a killed Markov process is reducible, even if the state space is finite, it is an open problem to determine which QSDs (if any at all) will be obtained from the reinforced processes $(Y_t)_{0\leq t<\infty}$ described in the introduction, by taking the limit (1.4). We suppose that for some $\mu, \pi \in \mathcal{P}(\chi)$ we have

$$\mathcal{L}_{\mu}(X_t|\tau_{\partial} > t) \to \pi(\cdot) \quad \text{as} \quad t \to \infty.$$

We also impose Assumptions 2.1 and 2.2. The following corollary, which came from a discussion of the author with Michel Benaïm, allows us to obtain the quasi-limiting distribution $\pi(\cdot)$. We construct for $0 < p \leq 1$ a reinforced process with renewal $(Y^p_t)_{0\leq t<\infty}$ as follows. We firstly take a copy of the killed Markov process $(X_t)_{0\leq t<\tau_{\partial}}$ with initial condition $X_0 \sim \mu$, and define $(Y^p_t)_{0\leq t<\tau_{\partial}}$ as in (1.1). At this killing time, with probability p we "renew", sampling

$$Y^p_{\tau_{\partial}} \sim \mu(\cdot). \tag{2.9}$$

Otherwise, with probability $1-p$, we sample from the empirical measure of the history,

$$Y^p_{\tau_{\partial}} \sim \frac{1}{\tau_{\partial}} \sum_{t=0}^{\tau_{\partial}-1} \delta_{Y^p_t}(\cdot), \tag{2.10}$$

as in (1.2). The process Y^p_t then continues evolving like a copy of $(X_t)_{0\leq t<\tau_{\partial}}$ up to its next killing time. This is then repeated inductively, so that at each killing time we sample from μ with probability p, otherwise sampling from the empirical measure of the history as in (1.3). We are then able to obtain the quasi-limiting distribution $\pi(\cdot)$ from $(Y^p_t)_{0\leq t<\infty}$, for small $0 < p \ll 1$.

Corollary 2.4. *For given $0 < p \leq 1$ we let Z^p be the unique solution to (2.1) for $T = \infty$ and $(\gamma^p_t)_{0\leq t<\infty}$ be the coefficients thereby defined in (2.2). Then we have*

$$\frac{1}{t}\sum_{s=0}^{t-1} \delta_{Y^p_s}(\cdot) \overset{a.s.}{\to} \pi^p(\cdot) \quad \textit{as} \quad t \to \infty, \tag{2.11}$$

whereby

$$\pi^p(\cdot) := \sum_{t=0}^{\infty} \frac{\gamma^p_t}{Z^p} \mathcal{L}_{\mu}(X_t|\tau_{\partial} > t)(\cdot) \to \pi(\cdot) \quad \textit{as} \quad p \to 0. \tag{2.12}$$

3 Proof of Theorem 2.3

We recall that $\bar{\chi}$ is the completion of χ, which by assumption is compact. We extend K to $\bar{\chi}$ by setting $K(x, \cdot) = 0$ for $x \in \partial\chi = \bar{\chi} \setminus \chi$, labelling this extended submarkovian kernel as K by abuse of notation. For (discrete and finite) time intervals $[t_1, t_2] \subseteq \mathbb{N}$, we write $F([t_1, t_2]; \bar{\chi})$ for the set of functions $[t_1, t_2] \to \bar{\chi}$, which we equip with the uniform metric

$$d_{F;[t_1,t_2]}(f, g) := \sup_{t\in[t_1,t_2]} d(f(t), g(t)).$$

We defer for later the proof of the following proposition.

Proposition 3.1. *There exists a unique solution, Z, to (2.1).*

We will establish convergence by formulating the reinforced path process as an urn process, and applying [7, Theorem 1]. We use the terminology given in [7, Section 1.1] throughout.

To identify the limit as being of the desired form, the following observation shall be crucial. We fix $0 \leq t_1 \leq t_2$. Take $(Y_u)_{0\leq u\leq t_1}$ and $(Z_u)_{t_1\leq u\leq t_2}$ such that $(Y_u)_{0\leq u\leq t_1} \sim$

$\mathcal{L}_\mu((X_u)_{0\leq u\leq t_1}|\tau_\partial > t_1)$ and, conditionally on $(Y_u)_{0\leq u\leq t_1}$, $(Z_u)_{t_1\leq u\leq t_2} \sim \mathcal{L}_{Y_{t_1}}((X_{u-t_1})_{t_1\leq u\leq t_2})$. Then we have

$$Y \oplus Z \sim \mathcal{L}_\mu((X_u)_{0\leq u\leq t_2}|\tau_\partial > t_1). \tag{3.1}$$

The urn process formulation

We must distinguish between the $T < \infty$ and $T = \infty$ cases. If $T = \infty$ we fix arbitrary $\bar{T} \in \mathbb{N}$ and define $*$ to be a point distinguished from $\mathbb{N}$. We define

$$E_t := \{t\} \times F([0, t]; \bar{\chi}) \quad \text{for all} \quad t \in \mathbb{N}, \quad E_* := \{*\} \times \bar{\chi}.$$

We then define

$$E := \cup_{t\in[0,T]} E_t \quad \text{if} \quad T < \infty, \quad E := \cup_{t\in[0,T]} E_t \cup E_* \quad \text{if} \quad T = \infty,$$

which we equip with the metric d_E defined by

$$d_E((t, f), (s, g)) := \begin{cases} 1, & t \neq s \\ 1 \wedge d_{F;[0,t]}(f, g), & t = s \leq T \\ 1 \wedge d(f, g), & t = s = * \end{cases},$$

under which E is a compact metric space. We recall that $(u_n)_{n=1}^\infty = ((t_b^n, f^n, t_d^n))_{n=1}^\infty$ is our reinforced path process. We define

$$m_N(\cdot) := \sum_{n=1}^N v_n(\cdot) \in \mathcal{M}(E) \quad \text{whereby}$$
$$v_n(\cdot) := \begin{cases} \sum_{t=t_b^n}^{t_d^n-1} \mathbb{1}(t \leq T)\delta_{(t, f^n_{|_{[0,t]}})}(\cdot), & T < \infty \\ \sum_{t=t_b^n}^{t_d^n-1} \left(\mathbb{1}(t \leq \bar{T})\delta_{(t, f^n_{|_{[0,t]}})}(\cdot) + \mathbb{1}(t > \bar{T})\delta_{(*, f^n(t))}(\cdot)\right), & T = \infty \end{cases}.$$

If $T < \infty$, we define $\Gamma = T + 1$. If $T = \infty$, on the other hand, then $(\{x \in \chi : K^t 1(x) = 1\})_{t=1}^\infty$ is a descending sequence of compact sets, whose intersection must be empty by Assumption 2.2. Therefore $||K^t||_{\text{op}} = \sup_{x\in\chi} K^t 1(x) < 1$ for some $t < \infty$ large enough, so that the spectral radius of K, $r(K)$, is less than 1. Thus

$$\limsup_{t\to\infty} (\sup_{x\in\chi} \mathbb{P}_x(\tau_\partial > t))^{\frac{1}{t}} = \limsup_{t\to\infty} (||K^t||_{\text{op}})^{\frac{1}{t}} = r(K) < 1, \tag{3.2}$$

so that $\sup_{x\in\chi} \mathbb{E}_x[\tau_\partial] < \infty$. Therefore we may define $\Gamma := 1 + \sup_{x\in\chi} \mathbb{E}_x[\tau_\partial]$ in the case that $T = \infty$.

We observe that $(m_N)_{N\geq 1}$ is a measure-valued Polya process with:

- Initial composition

$$m_1 := \begin{cases} \sum_{t=0}^{\tau_\partial - 1} \delta_{(t, (X_s^\mu)_{0\leq s\leq t})} \\ \sum_{t=0}^{\tau_\partial - 1} \left(\delta_{(t, (X_s^\mu)_{0\leq s\leq t})} \mathbb{1}(t \leq \bar{T}) + \delta_{(*, X_t^\mu)} \mathbb{1}(t > \bar{T})\right) \end{cases},$$

  whereby $(X_t^\mu)_{0\leq t<\tau_\partial}$ is an independent copy of the killed Markov process with submarkovian transition kernel K and initial condition $X_0^\mu \sim \mu$.
- Independent and identically distributed random replacement kernels

$$E \ni (t, f) \mapsto R^{(n)}((t, f); .) \in \mathcal{M}(E), \quad 1 \leq n < \infty,$$

  defined as follows. We take for each n, independently of each other and everything else, a Bernoulli random variable $B \sim \text{Ber}(p)$, a copy $(X_t^\mu)_{0\leq t<\tau_\partial}$ of the killed

Markov process with submarkovian transition kernel K and initial condition $X_0 \sim \mu$, and a family of copies $\{(X_t^x)_{0\le t<\tau_\partial} : x \in \chi\}$ of the same killed Markov process with initial conditions $X_0^x = x$. Note that this last definition makes sense since there exists a probability space $(\Omega, \mathbb{P})$ and a measurable function $F : (\bar{\chi} \sqcup \partial) \times \Omega \to \bar{\chi} \sqcup \partial$ such that for all $x \in \bar{\chi} \sqcup \partial$,

$$X^x : \Omega \ni \omega \mapsto F(x, \omega)$$

is a random variable with distribution $X^x \sim K(x, \cdot)$.

When $T < \infty$ we define the random kernel

$$R^{(n)}((t,f);\cdot) := \begin{cases} \sum_{s=0}^{\tau_\partial - 1} \mathbb{1}(s \le T)\delta_{(s,(X_u^\mu)_{0\le u\le s})}(\cdot), & B = 1 \\ \sum_{s=0}^{\tau_\partial - 1} \mathbb{1}(t+s \le T)\delta_{(t+s, f\oplus (X_{u-t}^{f(t)})_{t\le u\le t+s})}(\cdot), & B = 0 \end{cases}.$$

We adopt the convention that $* + s := * > \bar{T}$ for $s \in \mathbb{N}$ and $f(*) := f$ for $(*, f) \in E_*$. When $T = \infty$ we define the random kernel $R^{(n)}$ as

$$R^{(n)}((t,f);\cdot) := \begin{cases} \sum_{s=0}^{\tau_\partial - 1} \left[\mathbb{1}(s \le \bar{T})\delta_{(s,(X_u^\mu)_{0\le u\le s})}(\cdot) + \mathbb{1}(s > \bar{T})\delta_{(*,X_s)}\right], & B = 1 \\ \sum_{s=0}^{\tau_\partial - 1} \left[\mathbb{1}(t+s \le \bar{T})\delta_{(t+s, f\oplus (X_{u-t}^{f(t)})_{t\le u\le t+s})}(\cdot) \right. \\ \left. +\mathbb{1}(t+s > \bar{T})\delta_{(*,X_s^{f(t)})}\right], & B = 0 \end{cases}.$$

These random kernels $R^{(n)}$ have common expectation given by the (deterministic) kernel $R : E \to \mathcal{M}(E)$, which in the $T < \infty$ case is given by

$$\begin{aligned} R((t,f);\cdot) := \mathbb{E}[R^{(n)}((t,f);\cdot)] &= p\sum_{s=0}^{T} \mathcal{L}_\mu((s,(X_u)_{0\le u\le s})|\tau_\partial > s)(\cdot)\mathbb{P}_\mu(\tau_\partial > s) \\ &\quad + (1-p)\sum_{s=0}^{T-t} \mathcal{L}_{f(t)}((t+s, f\oplus (X_{u-t})_{t\le u\le t+s})|\tau_\partial > s)(\cdot)\mathbb{P}_{f(t)}(\tau_\partial > s). \end{aligned}$$

In the $T = \infty$ case the kernel $R : E \to \mathcal{M}(E)$ is given by

$$\begin{aligned} R((t,f);\cdot) := \mathbb{E}[R^{(n)}((t,f);\cdot)] &= p\sum_{s=0}^{\infty} \Big[\mathbb{1}(s > \bar{T})\mathcal{L}_\mu((*,X_s)|\tau_\partial > s)(\cdot)\mathbb{P}_\mu(\tau_\partial > s) \\ &\quad + \mathbb{1}(s \le \bar{T})\mathcal{L}_\mu((s,(X_u)_{0\le u\le s})|\tau_\partial > s)(\cdot)\mathbb{P}_\mu(\tau_\partial > s)\Big] \\ &\quad + (1-p)\sum_{s=0}^{\infty} \Big[\mathbb{1}(t+s > \bar{T})\mathcal{L}_{f(t)}((*,X_s)|\tau_\partial > s)(\cdot)\mathbb{P}_{f(t)}(\tau_\partial > s) \\ &\quad + \mathbb{1}(t+s \le \bar{T})\mathcal{L}_{f(t)}((t+s, f\oplus (X_{u-t})_{t\le u\le t+s})|\tau_\partial > s)(\cdot)\mathbb{P}_{f(t)}(\tau_\partial > s)\Big]. \end{aligned}$$

- Non-negative weight kernel

$$P : E \ni (t,f) \mapsto \frac{1}{\Gamma}\delta_{(t,f)}(\cdot) \in \mathcal{M}(E). \tag{3.3}$$

We therefore define the (random) kernels $Q^{(n)}$ and (deterministic) kernel Q as

$$Q^{(n)}((t,f),\cdot) := \int_E P(z;\cdot)R^{(n)}((t,f);dz) = \frac{1}{\Gamma}R^{(n)}((t,f);\cdot), \quad Q((t,f),\cdot) := \frac{1}{\Gamma}R((t,f);\cdot). \tag{3.4}$$

We claim that m_N satisfies Assumptions [7, $T_{>0}$, (A1)-(A4)]. It is immediate from the definition of $R^{(n)}((t,f),\cdot)$ that Assumption [7, $T_{>0}$] is satisfied. We observe, by considering only the component of the kernel R corresponding to "renewals", that

$$0 < c_1 := \frac{p\mathbb{E}_\mu[\tau_\partial \wedge T]}{\Gamma} \le Q((t,f),E) \le 1 \quad \text{for all} \quad (t,f) \in E.$$

Assumption [7, (A1)] is therefore satisfied with this c_1.

Since $\bar{\chi}$ is compact, E is compact, so that Assumptions [7, (A2), (i) and (ii)] are satisfied with $V \equiv 1$, $\theta = \frac{c_1}{2}$ and $K = 1$. Note that if $\bar{\chi}$ were not compact, E would not be compact, so that rather than being trivial, the existence of the Lyapunov function V would need to be an additional theorem assumption. For Assumption [7, (A2), (iii)], we take $r = \tilde{p}$ sufficiently large ($\tilde{p}$ being the p of [7, (A2), (iii)]). By considering the mass of $R^{(1)}((t,f),\cdot) = \Gamma Q^{(1)}((t,f),\cdot)$, we observe that it suffices to prove that $\mathcal{L}_x(\tau_\partial \wedge (T+1))$ has uniformly (over all $x \in \chi$) bounded r^{th} moment. If $T < \infty$ this is trivial, whereas if $T = \infty$ this is implied by (3.2).

Assumption 2.1 implies that $\bar{\chi} \ni x \mapsto K(x,\cdot) \in \mathcal{M}(\bar{\chi})$ is continuous, so we see that

$$F([0,t];\bar{\chi}) \ni f \mapsto \mathcal{L}_{f(t)}(f \oplus (X_{u-t})_{t \le u \le s} | \tau_\partial > s)\mathbb{P}_{f(t)}(\tau_\partial > s) \in \mathcal{M}(F([0,s];\bar{\chi}))$$

is continuous for all $t \le s$. This then implies Assumption [7, (A4)], by summing over $s \ge t$.

We must therefore verify [7, Assumption (A3)], that is we must establish the following proposition.

Proposition 3.2. *The E-valued continuous-time killed Markov process $(Y^c_t)_{t<\tau_\partial^{Y^c}}$ with submarkovian infinitesimal generator $Q - Id$ converges uniformly to quasi-equilibrium.*

We defer for later the proof of Proposition 3.2.

Identifying the limit

Thus $(Y^c_t)_{t<\tau_\partial^{Y^c}}$ admits a unique QSD, η. Since we have verified Assumption [7, (A3)], we may invoke [7, Theorem 1], giving that

$$\frac{m_N}{N}(\cdot) \overset{\mathcal{M}(E)}{\to} \eta R(\cdot) \quad \text{almost surely.} \tag{3.5}$$

Since QSDs of $(Y_t)_{t<\tau_\partial^Y}$ correspond to solutions of

$$\alpha(\cdot) = \frac{\alpha Q(\cdot)}{\alpha Q(E)} = \frac{\alpha R(\cdot)}{\alpha R(E)}, \quad \alpha \in \mathcal{P}(E), \tag{3.6}$$

η is the unique solution to (3.6).

We assume for the time being that $T < \infty$ and write

$$\tilde{\eta}(\cdot) := \sum_{t=0}^{T} \frac{\gamma_t}{Z} \mathcal{L}_\mu((t,(X_s)_{0\le s\le t}) | \tau_\partial > t) \in \mathcal{P}(E). \tag{3.7}$$

We use (3.1) and straightforward algebra to calculate

$$\begin{aligned}
\tilde{\eta}R(\cdot) &= p\sum_{s=0}^{T} \mathbb{P}_\mu(\tau_\partial > s)\mathcal{L}_\mu((s,(X_u)_{0\le u\le s})|\tau_\partial > s) + (1-p)\sum_{s=0}^{T}\sum_{t=0}^{T} \mathbb{1}(s \ge t)\frac{\gamma_t}{Z} \\
&\int_{F([0,t])} \mathcal{L}_{f(t)}((s, f \oplus (X_{u-t})_{t\le u\le s})|\tau_\partial > s)(\cdot)\mathbb{P}_{f(t)}(\tau_\partial > s) d\mathcal{L}_\mu((X_u)_{0\le u\le t}|\tau_\partial > t)(df) \\
&\overset{(3.1)}{=} p\sum_{s=0}^{T} \mathbb{P}_\mu(\tau_\partial > s)\mathcal{L}_\mu((s,(X_u)_{0\le u\le s})|\tau_\partial > s) \\
&+ (1-p)\sum_{s=0}^{T}\sum_{t=0}^{T} \mathbb{1}(s\ge t)\frac{\gamma_t}{Z}\mathbb{P}_\mu(\tau_\partial > s|\tau_\partial > t)\mathcal{L}_\mu((s,(X_u)_{0\le u\le s})|\tau_\partial > s)(\cdot) \\
&= \sum_{s=0}^{T} p\mathbb{P}_\mu(\tau_\partial > s)\Big[1 + \frac{1-p}{Z}\sum_{t=0}^{s}\Big(1 - \frac{1-p}{Z}\Big)^{-(t+1)}\Big]\mathcal{L}_\mu((s,(X_u)_{0\le u\le s})|\tau_\partial > s)(\cdot) \\
&= \sum_{s=0}^{T} p\mathbb{P}_\mu(\tau_\partial > s)\Big[1 - \frac{1-p}{Z}\Big]^{-(s+1)}\mathcal{L}_\mu((s,(X_u)_{0\le u\le s})|\tau_\partial > s)(\cdot) = Z\tilde{\eta}(\cdot)
\end{aligned}$$

Therefore $\tilde{\eta}(\cdot)$ is the solution to (3.6), hence $\eta = \tilde{\eta}$. Therefore

$$\eta R(\cdot) = Z\tilde{\eta}(\cdot) = \sum_{t=0}^{T} \gamma_t \mathcal{L}_\mu((t,(X_u)_{0\leq u\leq t})|\tau_\partial > t).$$

Combining this with (3.5) we have

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}(t_b^n \leq t < t_d^n)\delta_{f^n_{[0,t]}}(\cdot) \overset{\mathcal{M}(F([0,t];\bar{\chi}))}{\to} \gamma_t \mathcal{L}_\mu((X_s)_{0\leq s\leq t}|\tau_\partial > t)(\cdot) \quad \text{a.s.} \quad \text{as} \quad N\to\infty. \tag{3.8}$$

Note that convergence in $\mathcal{M}(F([0,t];\bar{\chi}))$ of measures supported on $F([0,t];\chi)$ to a measure supported on $F([0,t];\chi)$ implies convergence in $\mathcal{M}(F([0,t];\chi))$ (this is easy to prove using the Portmanteau theorem). Since both sides of (3.8) are supported on $F([0,t];\chi)$, we have (2.3) in the $T<\infty$ case. In the $T=\infty$ case we consider

$$\tilde{\eta}(\cdot) := \sum_{t=0}^{\bar{T}} \frac{\gamma_t}{Z}\mathcal{L}_\mu((t,(X_u)_{0\leq u\leq t}|\tau_\partial > t)(\cdot) + \sum_{t=\bar{T}+1}^{\infty}\frac{\gamma_t}{Z}\mathcal{L}_\mu(X_t|\tau_\partial > t)(\cdot),$$

and repeat the above calculation to obtain (2.3) for all $t\leq \bar{T}$. Since $\bar{T}\in\mathbb{N}$ was arbitrary, we have (2.3) for all $t\in\mathbb{N}$.

We have left only to prove Propositions 3.1 and 3.2. We begin with Proposition 3.2.

Proof of Proposition 3.2

We seek to check that $(Y_t^c)_{t<\tau_\partial^{Y^c}}$ satisfies [6, (A1) and (A2)], which implies Proposition 3.2 by [6, Theorem 2.1]. The former is immediate. We have left to check [6, (A2)].

We claim that it is sufficient to show that the E-valued discrete-time killed Markov process $(Y_n)_{n<\tau_\partial^Y}$ with submarkovian kernel Q satisfies [6, (A1) and (A2)]. To see this, note that Q would then have a bounded non-negative right-eigenfunction as given by [6, Proposition 2.3], which must then be a bounded non-negative right-eigenfunction for $Q-\mathrm{Id}$. Furthermore, since $(Y_t^c)_{t<\tau_\partial^{Y^c}}$ satisfies [6, (A1)], this may then be combined with the existence of the bounded, non-negative right eigenfunction to see that $(Y_t^c)_{t<\tau_\partial^{Y^c}}$ satisfies [6, (A2)].

Thus it is sufficient to check that $(Y_n)_{n<\tau_\partial^Y}$ satisfies [6, (A1) and (A2)]. It is trivial that it satisfies [6, (A1)] with

$$\nu(\cdot) := \frac{1}{\mathbb{E}_\mu[\tau_\partial\wedge(T+1)]}\sum_{s=0}^{T}\mathcal{L}_\mu((s,(X_u)_{0\leq u\leq s})|\tau_\partial > s)(\cdot)\mathbb{P}_\mu(\tau_\partial > s),$$

in the language of [6, (A1)]. It is left to check [6, (A2)], for this same $\nu(\cdot)$. We separate the $T<\infty$ and $T=\infty$ cases.

The $T<\infty$ case

We note that we can write $Q = Q_0 + Q_1 + Q_2$ whereby

$$\delta_{(t,f)}Q_0(\cdot) := c\nu(\cdot),\quad c := \frac{p\mathbb{E}_\mu[\tau_\partial\wedge(T+1)]}{\Gamma},$$
$$Q_1 := \frac{1-p}{\Gamma}\mathrm{Id},\quad \delta_{(t,f)}Q_2(\cdot)\quad \text{is supported on}\quad \{(t',f')\in E : t'\geq t+1\}.$$

We write

$$\delta_{(t,f)}Q^n 1 = \delta_{(t,f)}[(Q_1+Q_2)^n + \sum_{m=0}^{n-1}(Q_1+Q_2)^m Q_0 Q^{n-m-1}]1$$
$$= \delta_{(t,f)}(Q_1+Q_2)^n 1 + c\sum_{m=0}^{n-1}[\delta_{(t,f)}(Q_1+Q_2)^m 1][\nu Q^{n-m-1}1].$$

We observe that Q_1 and Q_2 commute, and that $Q_2^{T+1}=0$. Thus

$$(Q_1+Q_2)^m 1 = \sum_{k=0}^{T+1}\binom{m}{k}\left(\frac{1-p}{\Gamma}\right)^{m-k}Q_2^k 1 \leq C(m^{T+1}+1)\Big(\frac{1-p}{\Gamma}\Big)^m$$

for some $C<\infty$. We now observe for $0\leq m\leq n-1$ that

$$\nu Q^n 1 \geq \nu(Q_0+Q_1)^{m+1}Q^{n-m-1}1 \geq \Big(c+\frac{1-p}{\Gamma}\Big)^{m+1}\nu Q^{n-m-1}1.$$

Therefore combining the above we have

$$\delta_{(t,f)}Q^n 1 \leq C(n^{T+1}+1)\Big(\frac{1-p}{\Gamma}\Big)^n$$
$$+\Big[c\sum_{m=0}^{n-1}C(m^{T+1}+1)\Big(\frac{1-p}{\Gamma}\Big)^m\Big(c+\frac{1-p}{\Gamma}\Big)^{-(m+1)}\Big]\nu Q^n 1$$
$$\leq C'\sum_{m=0}^{n}\Big[(m^{T+1}+1)\Big(\frac{1-p}{\Gamma}\Big)^m\Big(c+\frac{1-p}{\Gamma}\Big)^{-(m+1)}\Big]\nu Q^n 1,$$

for some $C'<\infty$. Therefore there exists a constant M, independent of (t,f) and n, such that $\delta_{(t,f)}Q^n 1\leq M\nu Q^n 1$ for all $(t,f)\in E$ and $n\geq 1$, which implies that $(Y_t)_{t<\tau_\partial^Y}$ satisfies [6, (A2)].

The $T=\infty$ case

Since the spectral radius of K is less than 1, (3.2), the following operator is well-defined,

$$G := \sum_{n=0}^{\infty}K^n : C_b(\chi)\to C_b(\chi).$$

We observe that if $(Y_n)_{n<\tau_\partial^Y} = ((t^n,f^n))_{n<\tau_\partial^Y}$, then $(Z_n)_{n<\tau_\partial^Z} := (f^n(t^n))_{n<\tau_\partial^Y}$ is a killed Markov chain on $\bar{\chi}$ with submarkovian kernel

$$S(x,\cdot) := \frac{p}{\Gamma}G(\mu,\cdot)+\frac{1-p}{\Gamma}G(x,\cdot).$$

Since K is compact, $S(x,\cdot)-\frac{1-p}{\Gamma}\delta_x(\cdot)$ defines a positive compact operator on $C_b(\chi)$ with positive spectral radius. Thus by the Krein-Rutman theorem, there exists a non-negative right eigenfunction $\phi\in C_b(\chi)$, which must then be a right eigenfunction for S. Since $(Z_n)_{n<\tau_\partial^Y}$ also satisfies [6, (A1)], being minorised by $\frac{p}{\Gamma}G(\mu,\cdot)$, it must then satisfy [6, (A2)]: there exists $C<\infty$ such that

$$\mathbb{P}_x(\tau_\partial^Z>n)\leq C\mathbb{P}_{\frac{G(\mu,\cdot)}{G(\mu,1)}}(\tau_\partial^Z>n) \tag{3.9}$$

for all $n\geq 0$ and $x\in\chi$. Since $(\tilde{t},\tilde{f})\sim\nu$ implies that $\tilde{f}(\tilde{t})\sim\frac{G(\mu,\cdot)}{G(\mu,1)}$, (3.9) then implies that

$$\mathbb{P}_{(t,f)}(\tau_\partial^Y>n)\leq C\mathbb{P}_\nu(\tau_\partial^Y>n).\quad\square$$

Proof of Proposition 3.1

We consider for $T<\infty$ the function

$$f_T:[1,\infty)\ni z\mapsto z-\sum_{s=0}^{T}\mathbb{P}_{\mu}(\tau_{\partial}>s)p\Big[1-\frac{1-p}{z}\Big]^{-(s+1)}\in\mathbb{R}. \tag{3.10}$$

Solutions to $f_T(z)=0$, $z\in[1,\infty)$ correspond to solutions to (2.1). We calculate for $T<\infty$,

$$f_T(1)\leq 1-\mathbb{P}_{\mu}(\tau_{\partial}>0)=0,\quad f_T'(z)\geq 1\quad\text{for all}\quad z\in[1,\infty)\quad\text{and}\quad\lim_{z\to\infty}f_T(z)=\infty,$$

so that there exists a unique solution to (2.1).

We now consider the $T=\infty$ case. We let $r:=r(K)$ be the spectral radius of K, defining $\frac{1}{r}:=+\infty$ in the case that $r(K)=0$. We claim that

$$\liminf_{t\to\infty}k^t\mathbb{P}_{\mu}(\tau_{\partial}>t)>0\quad\text{for}\quad\frac{1}{r}\leq k<\infty,\quad\lim_{t\to\infty}k^t\mathbb{P}_{\mu}(\tau_{\partial}>t)=0\quad\text{for}\quad k<\frac{1}{r}. \tag{3.11}$$

We have this for $k\neq\frac{1}{r}$ by (3.2). If $r=0$ then the $k=\frac{1}{r}$ case is vacuous. Otherwise, the Krein-Rutman theorem implies the existence of a positive right eigenfunction $\phi\in C_0(\chi)$, with eigenvalue $r:=r(K)$. This implies that

$$r^{-t}\mathbb{P}_{\mu}(\tau_{\partial}>t)=r^{-t}\mu K^t 1\geq\frac{r^{-t}\mu K^t\phi}{||\phi||_{\infty}}=\frac{\mu(\phi)}{||\phi||_{\infty}}>0\quad\text{for all}\quad t\geq 0,$$

implying the $k=\frac{1}{r}$ case. Thus $\sum_{s=0}^{\infty}\mathbb{P}_{\mu}(\tau_{\partial}>s)p[1-\frac{1-p}{z}]^{-(s+1)}$ is finite for

$$z\in(z_0,\infty)\cap[1,\infty)\quad\text{whereby}\quad z_0:=\frac{1-p}{1-r}<\infty,$$

so that we may define f_{∞} on $(z_0,\infty)\cap[1,\infty)$ similarly to (3.10). Moreover (3.11) gives that f_{∞} is continuous and strictly increasing. It also gives that $f_{\infty}(z)\to-\infty$ as $z\downarrow z_0$ if $z_0\geq 1$. If $z_0<1$ (which is the case if $r=0$) we observe that $f_{\infty}(1)\leq 0$ as with $T<\infty$. Therefore we have the existence of a unique solution to (2.1) as with $T<\infty$. □

This concludes the proof of Theorem 2.3. □

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Acknowledgments. The author would like to thank Michel Benaïm for useful discussions on the convergence (or lack thereof) of reinforced processes, leading in particular to Corollary 2.4. The author also thanks the anonymous referee, whose feedback has improved this paper.

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