Electron. Commun. Probab. **27** (2022), article no. 25, 1–15.
https://doi.org/10.1214/22-ECP465
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The compact interface property for the stochastic heat equation with seed bank*

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Abstract

We investigate the compact interface property in a recently introduced variant of the stochastic heat equation that incorporates dormancy, or equivalently seed banks. There individuals can enter a dormant state during which they are no longer subject to spatial dispersal and genetic drift. This models a state of low metabolic activity as found in microbial species. Mathematically, one obtains a memory effect since mass accumulated by the active population will be retained for all times in the seed bank. This raises the question whether the introduction of a seed bank into the system leads to a qualitatively different behaviour of a possible interface. Here, we aim to show that nevertheless in the stochastic heat equation with seed bank compact interfaces are retained through all times in both the active and dormant population. We use duality and a comparison argument with partial functional differential equations to tackle technical difficulties that emerge due to the lack of the martingale property of our solutions which was crucial in the classical non seed bank case.

Keywords: Compact interface; stochastic heat equation; duality; dormancy; seed bank.
MSC2020 subject classifications: 60H15; 35R10.
Submitted to ECP on August 9, 2021, final version accepted on April 16, 2022.

1 Introduction and main result

One of the simplest spatial models for the evolution of the frequency of a bi-allelic population under the influence of random genetic drift is given by

$$\partial_t u(t,x) = \frac{\Delta}{2} u(t,x) + \sqrt{u(t,x)(1-u(t,x))}\dot{W}(t,x), \tag{1.1}$$

where $W = (W(t,x))_{t\geq 0, x\in\mathbb{R}}$ is a Gaussian white noise process. This equation is called the stochastic heat equation with Wright-Fisher noise introduced by Shiga in [16]. Here, $u(t,x)$ models the frequency of one of the two types at space time point $(t,x) \in [0,\infty[\times\mathbb{R}$. Heuristically, one can interpret the model as individuals migrating among a continuum of colonies in a diffusive way. Moreover, reproduction is subject to random genetic drift with variance $u(t,x)(1-u(t,x))$.

This model has been studied extensively in the past (see e.g. [12] and [19]). Remarkably, it turns out that the so called *compact interface property* distinguishes the

*The author gratefully acknowledges financial support from the Berlin Mathematical School (BMS) and the IRTG 2544 “Stochastic Analysis in Interaction”.

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stochastic model from the deterministic heat equation. More precisely, define for a function $f\colon \mathbb{R} \to \mathbb{R}$ with $0 \le f(x) \le 1$ for all $x \in \mathbb{R}$, $f(x) \to 1$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$ the left and right edge as follows:

$$\begin{aligned} L(f) &= \inf\{x \in \mathbb{R} |\, f(x) < 1\}, \\ R(f) &= \sup\{x \in \mathbb{R} |\, f(x) > 0\}. \end{aligned}$$

We then say that Equation (1.1) exhibits the compact interface property if for some initial condition $0 \le u(0,\cdot) \le 1$ with $L(u(0,\cdot)) > -\infty$ and $R(u(0,\cdot)) < \infty$ it follows that

$$\begin{aligned} L(u(t,\cdot)) &> -\infty, \\ R(u(t,\cdot)) &< \infty \end{aligned}$$

for all $t > 0$ almost surely. For Equation (1.1) this was shown in [19]. However, the property has also been studied in the context of super Brownian motion (cf. [10]) and more general equations such as the symbiotic branching model (see [17]). In contrast, for a solution $\tilde{u}$ of the deterministic heat equation with the same initial condition $u(0,\cdot)$ we have

$$L(\tilde{u}(t,\cdot)) = -\infty,\ R(\tilde{u}(t,\cdot)) = \infty.$$

Biologically, this can be interpreted as a finite zone to which the entire interaction between the two types is confined to.

Recently, in the context of microbial species, an additional evolutionary mechanism in the form of *dormancy*, or equivalently *seed banks*, has raised considerable attention in population genetics (see e.g. [11], [18]). Mathematically, this mechanism has been incorporated into the classical (non-spatial) Wright Fisher model and investigated in [2] and [3]. There, dormancy and resuscitation are modeled in the form of classical migration between an active and an inactive state. Corresponding discrete-space population genetic models have also very recently been introduced in [9].

For the case of a continuous spatial structure, the following system of SPDEs was established in [5] to allow individuals to retreat into a seed bank, where spatial dispersal and random genetic drift are absent:

$$\begin{aligned} \partial_t u(t,x) &= \frac{\Delta}{2} u(t,x) + c(v(t,x) - u(t,x)) + \sqrt{u(t,x)(1-u(t,x))}\dot{W}(t,x), \\ \partial_t v(t,x) &= c'(u(t,x) - v(t,x)). \end{aligned} \tag{1.2}$$

Here, $c, c' > 0$ are the seed bank migration rates. This equation admits unique in law weak solutions when started from Heaviside initial conditions and satisfies a moment duality to a system of "on/off" coalescing Brownian motions. This object is a coalescing Brownian motion where spatial movement and coalescence can be switched on and off at rates c' and c, respectively.

Moreover, note that the following reformulation of Equation (1.2) as a stochastic partial delay differential equation is crucial in both proofs and heuristic considerations:

$$\begin{aligned} \partial_t u(t,x) &= \frac{\Delta}{2} u(t,x) + c\left(e^{-c't} v(0,x) + c' \int_0^t e^{-c'(t-s)} u(s,x)\,\mathrm{d}s - u(t,x)\right) \\ &\qquad + \sqrt{u(t,x)(1-u(t,x))}\dot{W}(t,x), \\ v(t,x) &= e^{-c't} v(0,x) + c' \int_0^t e^{-c'(t-s)} u(s,x)\,\mathrm{d}s. \end{aligned} \tag{1.3}$$

We now investigate whether in this seed bank model the compact interface property still holds. Note that from the Delay Equation (1.3) it immediately follows that the interface of the dormant component v is increasing in time. One may think of this as a memory effect introduced by the seed bank since mass the active population u accumulated is retained through all times. This is in stark contrast to the classical non seed bank case where the interface can shrink and move freely in space and time. Similarly, this memory effect leads to an upwards drift for the active component u, albeit the situation is less clear compared to the dormant population due to the presence of the noise. Intuitively, this would then suggest that the interface becomes larger after the introduction of a seed bank – raising the question whether it becomes too large to retain its compactness.

In this paper, we show that this is indeed *not* the case and the compact interface property holds at all times, almost surely. As a byproduct we will also provide on/off versions of well-known statements such as the Feynman-Kac formula.

For the proof of the main result we use a comparison argument with deterministic differential equations originating from the theory of super Brownian motion as in [19], [7] and [8]. Note however that their arguments rely heavily on the fact that the corresponding SPDE solutions are martingales. This is not true in our case due to the presence of the seed bank drift term. For the stochastic F-KPP Equation this was tackled in [13] by using the Girsanov theorem for SPDEs. Since the seed bank drift term does not satisfy the prerequisites of said theorem, we resort to duality and comparison with a partial *functional* differential equation instead of a classical PDE to overcome these difficulties.

The following theorem is the main result of this paper:

Theorem 1.1. *Let* $u_0 = v_0 = \mathbb{1}_{]-\infty,0]}$ *and* (u, v) *be the solution of Equation* (1.2) *with* $c, c' > 0$ *corresponding to these initial conditions. Then, almost surely, we have*

$$\begin{aligned} L(u(t,\cdot)) &> -\infty, & L(v(t,\cdot)) &> -\infty, \\ R(u(t,\cdot)) &< \infty, & R(v(t,\cdot)) &< \infty, \end{aligned}$$

for all $t \geq 0$.

The result of this paper seems to open up some interesting and challenging lines of further research.

For example, in [4] and [19] it was shown that the interface of the classical stochastic heat equation and the symbiotic branching model have non-trivial scaling limits. This raises the question whether this remains true for the stochastic heat equation with seed bank and how the limit compares to the previous ones. However, we would like to point out that showing tightness for Equation (1.2) – even in a weaker topology like the Meyer Zheng topology – seems to be more challenging than in the previous cases due to the lack of the martingale property for the solutions.

Moreover, it seems to be natural to investigate the extension of the main result to the stochastic F-KPP Equation with seed bank. This would enable more in-depth study of the"right marker speed" $\lim_{t\to\infty} R(u(t,\cdot))/t$ which was shown to exist and be strictly positive for the classical stochastic F-KPP Equation in [6].

2 The stochastic heat equation with seed bank

We recall some basic results regarding Equation (1.2) from [5]. The proofs and further additional motivation may be found there as well.

Theorem 2.1. *Let* $u_0 = v_0 = \mathbb{1}_{]-\infty,0]}$. *Then, there exists a weak solution* (u, v) *of Equation* (1.2) *with* $u(t,\cdot) \in C(\mathbb{R}, [0,1])$ *and* $v(t,\cdot) \in B(\mathbb{R}, [0,1])$ *for all* $t > 0$*, almost surely. This solution is unique in law and has the following integral representation:*

$$u(t,x) = G_t u_0(x) + c \int_0^t \int_{\mathbb{R}} G(t-s,x,y)(v(s,y) - u(s,y))\, dx\, ds$$

$$+\int_0^t \int_{\mathbb{R}} G(t-s,x,y)\sqrt{u(s,y)(1-u(s,y))}\,W(\mathbf{d}x,\mathbf{d}s), \tag{2.1}$$

$$v(t,x) = v_0(x) + c'\int_0^t u(s,x) - v(s,x)\,\mathbf{d}x. \tag{2.2}$$

Here, $G(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right)$ *is the heat kernel for* $t \geq 0$ *and* $x,y \in \mathbb{R}$ *and* $(G_t)_{t\geq 0}$ *denotes the heat semigroup given by*

$$G_t f(x) = \int_{\mathbb{R}} G(t,x,y)f(y)\,\mathbf{d}x$$

for $f \in B(\mathbb{R})$ *and* $x \in \mathbb{R}$*.*

As mentioned earlier, this equation is dual to a process which is defined as follows:

Definition 2.2. *We denote by* $M = (M_t)_{t\geq 0}$ *an on/off coalescing Brownian motion taking values in* $\bigcup_{k\in\mathbb{N}_0}\left(\mathbb{R}\times\{\boldsymbol{a},\boldsymbol{d}\}\right)^k$ *and starting at* $M_0 = ((x_1,\sigma_1),\cdots,(x_n,\sigma_n)) \in \left(\mathbb{R}\times\{\boldsymbol{a},\boldsymbol{d}\}\right)^n$ *for some* $n \in \mathbb{N}$*. Here, the marker* $\boldsymbol{a}$ *(resp.* $\boldsymbol{d}$*) means that the corresponding particle is active (resp. dormant). The process evolves according to the following rules:*

- *Active particles, i.e. particles with the marker* $\boldsymbol{a}$*, move in* $\mathbb{R}$ *according to independent Brownian motions.*
- *Pairs of active particles coalesce according to the following mechanism:*
  - *We define for each pair of particles labelled* (α,β) *their intersection local time* $L^{\alpha,\beta} = (L_t^{\alpha,\beta})_{t\geq 0}$ *as the local time of* $M^\alpha - M^\beta$ *at* 0 *which we assume to only increase whenever both particles carry the marker* $\boldsymbol{a}$*.*
  - *Whenever the intersection local time exceeds the value of an independent exponential clock with rate* $1/2$*, the two involved particles coalesce into a single particle.*
- *Independently, each active particle switches to a dormant state at rate* c *by switching its marker from* $\boldsymbol{a}$ *to* $\boldsymbol{d}$*.*
- *Dormant particles do not move or coalesce.*
- *Independently, each dormant particle switches to an active state at rate* c' *by switching its marker from* $\boldsymbol{d}$ *to* $\boldsymbol{a}$*.*

Moreover, denote by $I = (I_t)_{t\geq 0}$ *and* $J = (J_t)_{t\geq 0}$ *the (time dependent) index set of active and dormant particles of* M*, respectively, and let* N_t *be the random number of particles at time* $t \geq 0$ *so that* $M_t = (M_t^1,\cdots,M_t^{N_t})$*.*

Next, we recall a moment duality between the solution to Equation (1.2) and the previously defined on/off coalescing Brownian motion $M = (M_t)_{t\geq 0}$.

Theorem 2.3. *Let* (u,v) *be a solution to the system* (1.2) *with initial conditions* $u_0, v_0 \in B(\mathbb{R})$*. Then, we have for any initial state* $M_0 = ((x_1,\sigma_1),\cdots,(x_n,\sigma_n)) \in \left(\mathbb{R}\times\{\boldsymbol{a},\boldsymbol{d}\}\right)^n$*,* $n \in \mathbb{N}$ *and* $t \geq 0$

$$\mathbb{E}\left[\prod_{\beta\in I_0} u(t,M_0^\beta)\prod_{\gamma\in J_0} v(t,M_0^\gamma)\right] = \mathbb{E}\left[\prod_{\beta\in I_t} u_0(M_t^\beta)\prod_{\gamma\in J_t} v_0(M_t^\gamma)\right].$$

Finally, we provide a delay representation of the v component in terms of the u component, which will become useful later on.

Theorem 2.4. *Let* (u,v) *be a solution to the system* (1.2) *with initial conditions* $u_0, v_0 \in B(\mathbb{R})$*. Then, we have*

$$v(t,x) = e^{-c't}v_0(x) + c'e^{-c't}\int_0^t e^{c's}u(s,x)\,\mathbf{d}s.$$

3 Proof of Theorem 1.1

Proposition 3.1. *Let* (u, v) *be a solution of* (1.2) *with initial conditions* $u_0 = v_0 = \mathbb{1}_{]-\infty,0]}$. *Then, for all* $t > 1$, *there exists a map* $\eta(t, b)$ *integrable in* b *on* $[0, \infty)$ *such that*

$$\mathbb{P}\left(\sup_{0\le s\le t}\sup_{x\in[b,\infty[} u(s, x) > 0\right) \le \eta(t, b)$$

for all $b \ge 0$.

Remark 3.2. We note that the preceding result and Theorem 1.1 may also be shown for more general initial conditions u_0, v_0 satisfying $R(u_0), R(v_0) \le 0$ with little additional effort. For simplicity and brevity we focus only on the Heaviside case in what follows.

Proof. The proof follows the general structure of [19, Proposition 3.2] and [13, Lemma 2.6]. Let $b > 0$ be arbitrary but fixed. We begin by taking some bounded $\psi \in L^2(\mathbb{R})\cap C^1(\mathbb{R})$ such that $0 \le \psi \le 1$ and $\{x\colon \psi(x) > 0\} = (0, \infty)$. Moreover, define $\psi_b(x) := \psi(x - b)$ and the stopping times

$$\begin{aligned}\tau_b &:= \inf\{t \ge 0\colon \exists x \ge b/2 \text{ s.t. } u(t, x) \ge 1/2\},\\ \sigma_b &:= \inf\{t \ge 0\colon \langle u(t,\cdot), \psi_b\rangle > 0\}.\end{aligned}$$

The main idea is to show that, for each $t > 0$, there exists some map $\eta(t, b)$ with the aforementioned properties such that

$$\mathbb{P}(\sigma_b \le t) \le \eta(t, b).$$

Then, the statement of the proposition follows immediately.

Next, fix $t > 1$, $\lambda > 0$ and apply Ito's formula to see that for $0 \le s \le t$

$$\begin{aligned}
&\exp\left(-\langle u(s,\cdot), h^\lambda(s,\cdot)\rangle - \langle v(s,\cdot), k^\lambda(s,\cdot)\rangle - \lambda\int_0^s \langle u(r,\cdot), \psi_b\rangle\,\mathrm{d}r\right)\\
&\quad= \exp\Big(-\langle u_0, h^\lambda(0,\cdot)\rangle - \langle v_0, k^\lambda(0,\cdot)\rangle\Big)\\
&\qquad+\int_0^s \exp\left(-\langle u(s',\cdot), h^\lambda(s',\cdot)\rangle - \langle v(s',\cdot), k^\lambda(s',\cdot)\rangle - \lambda\int_0^{s'} \langle u(r,\cdot), \psi_b\rangle\,\mathrm{d}r\right)\\
&\qquad\quad\times\Big(\langle u(s',\cdot), -\tfrac{\Delta}{2}h^\lambda(s',\cdot) - \partial_{s'}h^\lambda(s',\cdot) - \lambda\psi_b\rangle - c\langle v(s',\cdot) - u(s',\cdot), h^\lambda(s',\cdot)\rangle\\
&\qquad\quad\quad-\langle v(s',\cdot), \partial_{s'}k^\lambda(s',\cdot)\rangle - c'\langle u(s',\cdot) - v(s',\cdot), k^\lambda(s',\cdot)\rangle\\
&\qquad\quad\quad+\frac{1}{2}\langle u(s',\cdot)(1-u(s',\cdot)), (h^\lambda(s',\cdot))^2\rangle\Big)\,\mathrm{d}s' + H_s\\
&\quad= \exp\Big(-\langle u_0, h^\lambda(0,\cdot)\rangle - \langle v_0, k^\lambda(0,\cdot)\rangle\Big)\\
&\qquad+\int_0^s \exp\left(-\langle u(s',\cdot), h^\lambda(s',\cdot)\rangle - \langle v(s',\cdot), k^\lambda(s',\cdot)\rangle - \lambda\int_0^{s'} \langle u(r,\cdot), \psi_b\rangle\,\mathrm{d}r\right)\\
&\qquad\quad\times\left\langle -\frac{1}{4}u(s',\cdot) + \frac{1}{2}u(s',\cdot)(1-u(s',\cdot)), (h^\lambda(s',\cdot))^2\right\rangle\mathrm{d}s' + H_s.
\end{aligned}\tag{3.1}$$

Here, H is a continuous local martingale and we choose $(h^\lambda, k^\lambda) = (h^\lambda(s,\cdot), k^\lambda(s,\cdot))_{0\le s\le t}$ as the time reversed versions[1] of the solution $(\phi^\lambda, \varphi^\lambda) = (\phi^\lambda(s,\cdot), \varphi^\lambda(s,\cdot))_{0\le s\le t}$ to the system of PDEs given by

$$\partial_s\phi^\lambda(s, x) = \frac{\Delta}{2}\phi^\lambda(s, x) - \frac{1}{4}(\phi^\lambda(s, x))^2 + c'\varphi^\lambda(s, x) - c\phi^\lambda(s, x) + \lambda\psi_b(x),$$

[1] This means that we set $h^\lambda(s, x) = \phi^\lambda(t - s, x)$ and $k^\lambda(s, x) = \varphi^\lambda(t - s, x)$ for $(s, x) \in [0, t] \times \mathbb{R}$.

$$\partial_s \varphi^\lambda(s,x) = c\phi^\lambda(s,x) - c'\varphi^\lambda(s,x) \tag{3.2}$$

on $[0,t]\times\mathbb{R}$ with initial condition $\phi^\lambda(0,\cdot) = \varphi^\lambda(0,\cdot) \equiv 0$. By integration by parts (see Theorem 2.4), we see that

$$\varphi^\lambda(s,x) = ce^{-c's}\int_0^s e^{c's'}\phi^\lambda(s',x)\,\mathrm{d}s' \tag{3.3}$$

for any $0 \le s \le t$ and $x \in \mathbb{R}$. Hence, substituting this into the original equation, we may interpret Equation (3.2) as a partial functional differential equation with the same initial condition via

$$\begin{aligned}\partial_s \phi^\lambda(s,x) = \frac{\Delta}{2}\phi^\lambda(s,x) - \frac{1}{4}(\phi^\lambda(s,x))^2 \\ + c\left(c'\int_0^s e^{-c'(s-s')}\phi^\lambda(s',x)\,\mathrm{d}s' - \phi^\lambda(s,x)\right) + \lambda\psi_b(x).\end{aligned} \tag{3.4}$$

By Lemma 4.6, this equation has a unique positive $C_b^{1,\infty}([0,T]\times\mathbb{R}) \cap B([0,T],L^2(\mathbb{R}))$-valued solution. Thus, using the square integrability of h^λ, the process $H = (H_s)_{0\le s\le t}$ given by

$$\begin{aligned}H_s = \int_0^s\int_{\mathbb{R}} \exp\left(-\langle u(s',\cdot),h^\lambda(s',\cdot)\rangle - \langle v(s',\cdot),k^\lambda(s',\cdot)\rangle - \lambda\int_0^{s'}\langle u(r,\cdot),\psi_b\rangle\,\mathrm{d}r\right) \\ \sqrt{u(s',y)(1-u(s',y))}h^\lambda(s',y)\,W(\mathrm{d}y,\mathrm{d}s')\end{aligned}$$

for $0 \le s \le t$ is actually a true martingale.

Next, consider the map ξ given by

$$\xi(s,x) = \frac{\alpha}{(x-b)^2}\mathbb{1}_{(-\infty,b)}(x)$$

for $0 \le s \le t$ and $\alpha, x \in \mathbb{R}$. Note that the right hand side actually does not depend on s. Then, ξ satisfies the partial functional differential inequality

$$\partial_s\xi(s,x) \ge \frac{\Delta}{2}\xi(s,x) - \frac{1}{4}(\xi(s,x))^2 + c\left(c'e^{-c's}\int_0^s e^{c's'}\xi(s',x)\,\mathrm{d}s' - \xi(s,x)\right) + \lambda\psi_b(x) \tag{3.5}$$

outside the support of ψ_b on $(-\infty,b)$. Indeed, we have for $x < b$

$$\begin{aligned}&\partial_s\xi(s,x) - \frac{\Delta}{2}\xi(s,x) + \frac{1}{4}\xi^2(s,x) - c\left(c'e^{-c's}\int_0^s e^{c's'}\xi(s',x)\,\mathrm{d}s' - \xi(s,x)\right) - \lambda\psi_b(x) \\ &= \frac{\alpha(\alpha-12)}{4(x-b)^4} - c(1-e^{-c's})\frac{\alpha}{(x-b)^2} + c\frac{\alpha}{(x-b)^2} \\ &\ge 0\end{aligned}$$

if α is large enough. By a comparison theorem (e.g. a slight modification of [1, Theorem 4.II][2]) and Equation (3.3), this implies that

$$h^\lambda(s,x) \le \frac{\alpha}{(x-b)^2}, \tag{3.6}$$

[2]Note that for each $\lambda > 0$ the map h^λ is uniformly bounded. Hence, for $\epsilon > 0$ small enough we will have $\xi(s,x) = \frac{\alpha}{(x-b)^2} \ge h^\lambda(s,x)$ on $[0,t]\times[b-\epsilon,b[$. Moreover, since ξ is bounded on $[0,t]\times]-\infty,b-\epsilon]$, we only require the Lipschitz condition on a compact interval. We can thus apply the comparison theorem on $[0,t]\times]-\infty,b-\epsilon]$ to get (3.6).

$$
\begin{aligned}
k^\lambda(s,x) &\le c e^{-c's} \int_0^s e^{c's'} \frac{\alpha}{(x-b)^2}\, \mathrm{d}s' \\
&\le \frac{c\alpha}{c'(x-b)^2}
\end{aligned} \tag{3.7}
$$

for all $s \le t, x < b$.

Then, on the set $\{\sigma_b < t \wedge \tau_b\}$ we have

$$
\langle u(t\wedge\tau_b,\cdot), h^\lambda(t\wedge\tau_b,\cdot)\rangle + \langle v(t\wedge\tau_b,\cdot), k^\lambda(t\wedge\tau_b,\cdot)\rangle + \lambda \int_0^{t\wedge\tau_b} \langle u(r,\cdot),\psi_b\rangle\, \mathrm{d}r \to \infty
$$

as $\lambda \to \infty$. This implies, since $0 \le u \le \frac{1}{2}$ holds on $[0,\tau_b \wedge t] \times [b/2,\infty[$, that

$$
\begin{aligned}
&\mathbb{P}(\sigma_b \le t\wedge\tau_b) \\
&\le \lim_{\lambda\to\infty} \mathbb{E}\Bigg[1 - \exp\Bigg(- \langle u(t\wedge\tau_b,\cdot), h^\lambda(t\wedge\tau_b,\cdot)\rangle - \langle v(t\wedge\tau_b,\cdot), k^\lambda(t\wedge\tau_b,\cdot)\rangle \\
&\qquad -\lambda \int_0^{t\wedge\tau_b} \langle u(s,\cdot),\psi_b\rangle\, \mathrm{d}s \Bigg)\Bigg] \\
&\le 1 - \mathbb{E}\left[\exp\left(-\langle u_0, h^\infty(0,\cdot)\rangle - \langle v_0, k^\infty(0,\cdot)\rangle\right)\right] \\
&\qquad + \mathbb{E}\left[\int_0^{t\wedge\tau_b} \left\langle \frac{1}{4} u(s,\cdot)\mathbb{1}_{]-\infty,b/2[}, (h^\infty(s,\cdot))^2 \right\rangle \mathrm{d}s\right] \\
&\le \langle u_0, h^\infty(0,\cdot)\rangle + \langle v_0, k^\infty(0,\cdot)\rangle + \mathbb{E}\left[\int_0^{t\wedge\tau_b} \left\langle \frac{1}{4} u(s,\cdot)\mathbb{1}_{]-\infty,b/2[}, (h^\infty(s,\cdot))^2 \right\rangle \mathrm{d}s\right].
\end{aligned}
$$

In the penultimate inequality we used Equation (3.1) and that

$$
\frac{1}{4}u(r,x) - \frac{1}{2}u(r,x)(1-u(r,x)) \le 0
$$

on $[0,\tau_b \wedge t] \times [b/2,\infty[$. Moreover, we have set $h^\infty := \lim_{\lambda\to\infty} h^\lambda$ and $k^\infty := \lim_{\lambda\to\infty} k^\lambda$. To see that these limits exist on $]-\infty, b/2]$ note that h^λ is increasing in λ by another application of a comparison theorem and that h^λ is bounded on $]-\infty,b/2]$ by Equation (3.6). To get an analogous result for k^∞ we may use Equation (3.3) and the dominated convergence theorem. Thus, using (3.6), Lemma 4.5 and (3.3), we see for any $b \ge 4\sqrt{t}$ and some constant $C_1(t) > 0$ that

$$
\begin{aligned}
&\mathbb{P}(\sigma_b < t\wedge\tau_b) \\
&\le C_1(t)\left(\int_{-\infty}^0 \exp\left(-\frac{(x-b)^2}{20t}\right) \mathrm{d}x + \int_{-\infty}^{b/2} \frac{\alpha^2}{(x-b)^4}\, \mathrm{d}x\right) \\
&\le C_1(t)\left(\exp(-b^2/20t) + \frac{8\alpha^2}{3b^3}\right).
\end{aligned}
$$

For the last inequality we also used the standard Gaussian tail bound

$$
\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, \mathrm{d}y \le \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \tag{3.8}
$$

for $x \ge 0$. Hence, if we now show that for some $C(t) > 0$

$$
\mathbb{P}(\tau_b \le t) \le C(t) \exp\left(-\frac{b^2}{8t}\right),
$$

our claim is proven.

For this purpose, using Equation (2.1), we see that

$$\begin{aligned}\mathbb{P}(\tau_b \le t) = \mathbb{P}\Bigg(\exists x \in [b/2,\infty[, s \le t\colon \int_0^t \int_{\mathbb{R}} G(t-s,x,y)c(v(s,y)-u(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ + G_s u(0,\cdot)(x) + \int_0^t \int_{\mathbb{R}} G(t-s,x,y)\sqrt{(1-u(s,y))u(s,y)}\,W(\mathrm{d}s,\mathrm{d}y) \ge \frac{1}{2}\Bigg) \\ \le \mathbb{P}\left(\exists x \in [b/2,\infty[, s \le t\colon G_s u(0,\cdot)(x) + N_s(x) \ge \frac{1}{2}\right).\end{aligned}$$

Here, we set

$$\begin{aligned}N_t(x) := \int_0^t \int_{\mathbb{R}} G(t-s,x,y)c(v(s,y)+u(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}} G(t-s,x,y)\sqrt{(1-u(s,y))u(s,y)}\,W(\mathrm{d}s,\mathrm{d}y) \qquad (3.9)\end{aligned}$$

for $x \in \mathbb{R}, t \ge 0$. Since $G_s u(0,\cdot)(x) = G_s \mathbb{1}_{]-\infty,0]}(x) \to 0$ as $x \to \infty$, there exists some $b_0 \ge 4\sqrt{t}$ such that we have

$$\frac{1}{2} - G_s u(0,\cdot)(x) \ge \delta$$

for any $\frac{1}{2} > \delta > 0$, $b \ge b_0$ and $x \ge b/2$. Fix $\delta > 0$. Then, by Lemma 4.3 and Gaussian tail bounds, we have that

$$\begin{aligned}\mathbb{P}(\tau_b \le t) &\le \mathbb{P}(\exists x \in [b/2,\infty[, s \le t\colon N_s(x) \ge \delta) \\ &\le C(t,\delta)\left(\int_{]-\infty,0]} G(t,b/2,z)\,\mathrm{d}z + \int_{[b/2,\infty[}\int_{]-\infty,0]} G(t,x,z)\,\mathrm{d}z\,\mathrm{d}x\right) \\ &\le C(t,\delta)\exp\left(-\frac{b^2}{8t}\right)\end{aligned}$$

for any b large enough and some $C(t,\delta) > 0$ as desired. □

Corollary 3.3. *In the setting of Proposition 3.1 we have*

$$\mathbb{E}\left[\sup_{0\le s\le t}\big|R(v(s,\cdot))\big|\right], \mathbb{E}\left[\sup_{0\le s\le t}\big|R(u(s,\cdot))\big|\right] < \infty$$

and

$$\mathbb{E}\left[\sup_{0\le s\le t}\big|L(v(s,\cdot))\big|\right], \mathbb{E}\left[\sup_{0\le s\le t}\big|L(u(s,\cdot))\big|\right] < \infty$$

for every $t > 1$. *In particular, the statement of Theorem 1.1 holds true.*

Proof. For the right edge and b large enough we have by Proposition 3.1 that

$$\mathbb{P}\left(\sup_{0\le s\le t} R(u(s,\cdot)) > b\right) \le \eta(t,b).$$

Now, note that $(1-u(t,-x), 1-v(t,-x))$ also solves Equation (1.2) with initial condition (u_0, v_0). Hence, we obtain

$$\mathbb{P}\left(\sup_{0\le s\le t} R(1-u(s,-\cdot)) > b\right) = \mathbb{P}\left(-\inf_{0\le s\le t} L(u(s,\cdot)) > b\right) \le \eta(t,b).$$

Since $R(u(s,\cdot)) \ge L(u(s,\cdot))$ holds, we get

$$\mathbb{P}\left(\sup_{0\le s\le t} |R(u(s,\cdot))| > b\right) \le \eta(t,b)$$

which implies

$$\mathbb{E}\left[\sup_{0\le s\le t} |R(u(s,\cdot))|\right] < \infty.$$

By symmetry, we get an analogous result for the left edge.

For the v component we have by the delay representation (see Theorem 2.4) that

$$v(t,x) = e^{-c't}v_0(x) + \int_0^t c' e^{-c'(t-s)} u(s,x)\,\mathrm{d}s.$$

Hence, we have that $v(s,x) = 0$ for every $0 \le s \le t$ and $x > \sup_{0\le s\le t} |R(u(s,\cdot))|$. This implies

$$\sup_{0\le s\le t} R(v(s,\cdot)) \le \sup_{0\le s\le t} |R(u(s,\cdot))|.$$

Similarly, we have that

$$\inf_{0\le s\le t} L(v(s,\cdot)) \ge -\sup_{0\le s\le t} |L(u(s,\cdot))|.$$

Combining the preceding two equations, we obtain the desired result for v. □

4 Auxiliary results

Here, we provide all the calculations required for the preceding section.

Define for $t \ge 0$ and $x \in \mathbb{R}$ the quantities

$$\begin{aligned}
D_t(x) &:= c\int_0^t \int_{\mathbb{R}} G(t-s,x,y)u(s,y)\,\mathrm{d}y\,\mathrm{d}s,\\
E_t(x) &:= c\int_0^t \int_{\mathbb{R}} G(t-s,x,y)v(s,y)\,\mathrm{d}y\,\mathrm{d}s,\\
M_t(x) &:= \int_0^t \int_{\mathbb{R}} G(t-s,x,y)\sqrt{(1-u(s,y))u(s,y)}\,W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

Note that this implies the relation

$$N_t(x) = D_t(x) + E_t(x) + M_t(x)$$

for $N_t(x)$ as defined in Equation (3.9), $t \ge 0$ and $x \in \mathbb{R}$.

Lemma 4.1. *In the setting of Proposition 3.1 we have the existence of a constant $C(p) > 0$ such that*

$$\mathbb{E}\left[\left|N_t(x) - N_t(y)\right|^{2p}\right] \le C(p)(t^{1/2}(|x-y| \wedge t^{1/2})^{p-1} + t(t^{1/2}|x-y| \wedge t)^{2p-1})$$

$$\times \int_{]-\infty,0]} G(t,x,z) + G(t,y,z)\, \mathrm{d}z,$$

$$\mathbb{E}\left[\left|N_t(x) - N_s(x)\right|^{2p}\right] \leq C(p)(t^{1/2}|t-s|^{(p-1)/2} + t(t|t-s|)^{(2p-1)/2} + t|t-s|^{2p-1})$$
$$\times \int_{]-\infty,0]} G(t,x,z) + G(s,x,z)\, \mathrm{d}z$$

for all $p \geq 1$*,* $b > 1$*,* $0 \leq s \leq t$ *and* $x, y \in [b/2, \infty[$*.*

Remark 4.2. A similar proposition can be found in [19] and [13]. However, the introduction of the seed bank drift term poses technical difficulties which we tackle by using a duality technique.

Proof. We only verify the first inequality, the second one can be completed in a similar manner. Note that the following bound on the heat kernel is well-known for all $t \geq 0$ and $x, y \in \mathbb{R}$:

$$\int_0^t \int_{\mathbb{R}} (G(t-s,x,z) - G(t-s,y,z))^2\, \mathrm{d}z\, \mathrm{d}s \leq C(|x-y| \wedge t^{1/2}). \tag{4.1}$$

By the Burkholder-Davis-Gundy and Hölder inequality, the fact that $0 \leq u \leq 1$ and Equation (4.1), we get

$$\begin{aligned}
&\mathbb{E}\left[\left|M_t(x) - M_t(y)\right|^{2p}\right] \\
&\leq C(p)\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} (G(t-s,x,z) - G(t-s,y,z))^2 u(s,z)(1-u(s,z))\, \mathrm{d}z\, \mathrm{d}s\right)^p\right] \\
&\leq C(p)(|x-y| \wedge t^{1/2})^{p-1}\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} (G(t-s,x,z) - G(t-s,y,z))^2 u(s,z)\, \mathrm{d}z\, \mathrm{d}s\right] \\
&\leq C(p)(|x-y| \wedge t^{1/2})^{p-1} \int_0^t (t-s)^{-1/2} \int_{\mathbb{R}} (G(t-s,x,z) + G(t-s,y,z))\mathbb{E}[u(s,z)]\, \mathrm{d}z\, \mathrm{d}s.
\end{aligned} \tag{4.2}$$

Now, recall that $u_0 = v_0 = \mathbb{1}_{]-\infty,0]}$ and denote by $(B_t)_{t\geq 0}$ an on/off and by $(\tilde{B}_t)_{t\geq 0}$ a *standard* Brownian motion. Then, we have by Theorem 2.3 that

$$\begin{aligned}
\mathbb{E}[u(s,z)] &= \mathbb{P}_{(0,\boldsymbol{a})}(B_s \geq z) \\
&= \int_0^s \mathbb{P}(\tilde{B}_{s-r} \geq z)\, \mathrm{d}\mathbb{P}_J(r) \\
&= \int_0^s \int_{]-\infty,0]} G(s-r,z,w)\, \mathrm{d}w\, \mathrm{d}\mathbb{P}_J(r)
\end{aligned}$$

for $s \geq 0, z \in \mathbb{R}$. Here, J denotes the random time during which the on/off Brownian motion is switched off on $[0,s]$. Note that this quantity is independent of the movement of the Brownian motion $\tilde{B}$. Thus, by the semigroup property of the heat kernel and Equation (4.2), we have

$$\begin{aligned}
&\mathbb{E}\left[\left|M_t(x) - M_t(y)\right|^{2p}\right] \\
&\leq C(p)(|x-y| \wedge t^{1/2})^{p-1} \\
&\qquad \int_0^t (t-s)^{-1/2} \int_{]-\infty,0]} \int_0^s (G(t-r,x,w) + G(t-r,y,w))\, \mathrm{d}\mathbb{P}_J(r)\, \mathrm{d}w\, \mathrm{d}s
\end{aligned}$$

$$\le C(p)(|x-y|\wedge t^{1/2})^{p-1}\int_0^t (t-s)^{-1/2}\int_{]-\infty,0]}\int_0^s (G(t,x,w)+G(t,y,w))\,\mathrm{d}\mathbb{P}_J(r)\,\mathrm{d}w\,\mathrm{d}s$$
$$\le C(p)(|x-y|\wedge t^{1/2})^{p-1}t^{1/2}\int_{]-\infty,0]} G(t,x,w)+G(t,y,w)\,\mathrm{d}w,$$

where we also used that $x,y\ge 0$ in the penultimate inequality.

Similarly, using Hölder's inequality, Equation (4.1) and [14, Lemma 5.2] (using $\beta=1$ and $\lambda'=0$ there), we obtain for some $C(p)>0$

$$\begin{aligned}
&\mathbb{E}\left[\left|D_t(x)-D_t(y)\right|^{2p}\right]\\
&\quad\le c^{2p}\left(\int_0^t\int_{\mathbb{R}} |G(t-s,x,z)-G(t-s,y,z)|\,\mathrm{d}z\,\mathrm{d}s\right)^{2p-1}\\
&\qquad\mathbb{E}\left[\int_0^t\int_{\mathbb{R}} |G(t-s,x,z)-G(t-s,y,z)|u(s,z)^{2p}\,\mathrm{d}z\,\mathrm{d}s\right]\\
&\quad\le C(p)(t^{1/2}|x-y|\wedge t)^{2p-1}\mathbb{E}\left[\int_0^t\int_{\mathbb{R}} G(t-s,x,z)+G(t-s,y,z)u(s,z)\,\mathrm{d}z\,\mathrm{d}s\right]\\
&\quad\le C(p)(t^{1/2}|x-y|\wedge t)^{2p-1}t\int_{]-\infty,0]} G(t,x,z)+G(t,y,z)\,\mathrm{d}z.
\end{aligned}$$

The exact same calculation works for E_t since the only difference in the duality relation is that we start the on/off Brownian motion in the dormant state. ☐

This enables us to obtain a bound on the size of N.

Lemma 4.3. *In the setting of Proposition 3.1 we have the existence of some constant $C>0$ such that*

$$\begin{aligned}
&\mathbb{P}\left(\left|N_s(x)\right|\ge\varepsilon \text{ for some } x\in]b/2,\infty[, s\in[0,t]\right)\\
&\qquad\le C\varepsilon^{-18}t^{29}\left(\int_{]-\infty,0]} G(t,b/2,z)\,dz+\int_{[b/2,\infty[}\int_{]-\infty,0]} G(t,x,z)\,dz\,dx\right)
\end{aligned}$$

for all $t>1, b>2$ and $1\ge\varepsilon>0$.

Proof. The proof is the exact same as [19, Lemma 3.1], i.e. one replaces the interval $]A,\infty[$ by $]b/2,\infty[$ and the bounds from the first half of his lemma by our bounds from Lemma 4.1. ☐

Lemma 4.4 (On/off Feynman-Kac)**.** *Let $(\phi^\lambda,\varphi^\lambda)$ be the solution to the PDE (3.2) and $t\ge 0$. Denote by $B=(B_r)_{r\ge 0}$ an on/off Brownian motion starting in an active state and $I\subseteq[0,t]$ the union of random time intervals in which the Brownian path is active. Then, we have the stochastic representation*

$$\phi^\lambda(s,x)=\mathbb{E}_{(x,\boldsymbol{a})}\left[\int_{I\cap[0,s]}\lambda\psi_b(B_r)e^{-\int_{I\cap[0,r]}\phi^\lambda(s-u,B_u)\,du}\,dr\right]$$

for all $0\le s\le t$ and $x\in\mathbb{R}$.

Proof. Set $J=[0,t]\setminus I$ and consider for $s\in[0,t]$ the quantity

$$\hat{M}_s=E_s\phi^\lambda(t-s,B_s)\mathbb{1}_I(s)+E_s\varphi^\lambda(t-s,B_s)\mathbb{1}_J(s),$$

where

$$E_s := \exp\left(-\frac{1}{4}\int_{[0,s]\cap I} \phi^\lambda(t-r,B_r)\,\mathrm{d}r\right).$$

Then, by applying the Ito formula on the random time intervals between jumps and adding and subtracting the compensator of the jumps, we see

$$\begin{aligned}
\hat{M}_s = \hat{M}_0 &+ \int_{[0,s]\cap I}\left(\frac{\Delta}{2}\phi^\lambda(t-r,B_r) + \dot{\phi}^\lambda(t-r,B_r) - \frac{1}{4}\phi^\lambda(t-r,B_r)^2\right)E_r\,\mathrm{d}r \\
&+ c\int_{I\cap[0,s]}(\varphi^\lambda(t-r,B_r) - \phi^\lambda(t-r,B_r))E_r\,\mathrm{d}r \\
&+ c'\int_{J\cap[0,s]}(\phi^\lambda(t-r,B_r) - \varphi^\lambda(t-r,B_r))E_r\,\mathrm{d}r \\
&+ \int_{J\cap[0,s]}\dot{\varphi}^\lambda(t-r,B_r)E_r\,\mathrm{d}r + \tilde{M}_s
\end{aligned}$$

for $s \geq 0$ and some local martingale $\tilde{M} = (\tilde{M}_s)_{0\leq s\leq t}$. Since ϕ^λ and φ^λ are bounded and solve the system (3.2), we see that

$$\mathbb{E}_{(x,\boldsymbol{a})}[\hat{M}_s] = \mathbb{E}_{(x,\boldsymbol{a})}[\hat{M}_0] - \mathbb{E}_{(x,\boldsymbol{a})}\left[\int_{[0,s]\cap I}\lambda\psi_b(B_r)E_r\,\mathrm{d}r\right].$$

In particular, for $s = t$ we see that

$$0 = \mathbb{E}_{(x,\boldsymbol{a})}[\hat{M}_t] = \phi^\lambda(t,x) - \mathbb{E}_{(x,\boldsymbol{a})}\left[\int_{[0,t]\cap I}\lambda\psi_b(B_r)E_r\,\mathrm{d}r\right]$$

since we start in an active state. This gives the desired result. □

Lemma 4.5. *Let* $(\phi^\lambda, \varphi^\lambda)$ *be the solution to the PDE* (3.2) *and* $t \geq 0$. *Then, we have the existence of a constant* $K > 0$ *such that*

$$\phi^\lambda(s,x) \leq \frac{K}{t}\exp\left(-\frac{(b-x)^2}{20t}\right)$$

for all $s \leq t$, $b \geq 4\sqrt{t}$, $x < b - 2\sqrt{t}$ *and* $\lambda > 0$.

Proof. This lemma is the on/off version of [7, Lemma 3.5]. Set for $x < b$ and $r \in (x,b)$

$$\tau := \inf\{t \geq 0 | B_t \geq r\},$$

where $B = (B_t)_{t\geq 0}$ denotes an on/off Brownian motion started in x. Using the strong Markov property, that the support of ψ_b is $]b,\infty[$ and Lemma 4.4, we obtain

$$\begin{aligned}
&\phi^\lambda(s,x) \\
&\quad= \mathbb{E}_{(x,\boldsymbol{a})}\left[\int_{[0,s]\cap I}\lambda\psi_b(B_r)E_r\,\mathrm{d}r\right] \\
&\quad= \mathbb{E}_{(x,\boldsymbol{a})}\left[\int_{[0,s]\cap I}\lambda\psi_b(B_r)E_r\,\mathrm{d}r\mathbb{1}_{\{\tau\geq s\}}\right] + \mathbb{E}_{(x,\boldsymbol{a})}\left[\int_{[0,s]\cap I}\lambda\psi_b(B_r)E_r\,\mathrm{d}r\mathbb{1}_{\{\tau\leq s\}}\right] \\
&\quad= \mathbb{E}_{(x,\boldsymbol{a})}\left[\int_{[\tau,s]\cap I}\lambda\psi_b(B_r)E_r\,\mathrm{d}r\mathbb{1}_{\{\tau\leq s\}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{(x,\boldsymbol{a})}\Bigg[\mathbb{1}_{\{\tau\le s\}}\exp\left(-\int_{[0,\tau]\cap I}\phi^\lambda(s-u,B_u)\,\mathrm{d}u\right)\\
&\qquad \times \mathbb{E}\left[\int_{[\tau,s]\cap I}\lambda\psi_b(B_r)\exp\left(-\int_{[\tau,r]\cap I}\phi^\lambda(s-u,B_u)\,\mathrm{d}u\right)\,\mathrm{d}r\,\middle|\,\mathcal{F}_\tau\right]\Bigg]\\
&= \mathbb{E}_{(x,\boldsymbol{a})}\Bigg[\mathbb{1}_{\{\tau\le s\}}\exp\left(-\int_{[0,\tau]\cap I}\phi^\lambda(s-u,B_u)\,\mathrm{d}u\right)\\
&\qquad \times \mathbb{E}\left[\int_{[0,s-\tau]\cap I}\lambda\psi_b(B_{r+\tau})\exp\left(-\int_{[\tau,r+\tau]\cap I}\phi^\lambda(s-u,B_u)\,\mathrm{d}u\right)\,\mathrm{d}r\,\middle|\,\mathcal{F}_\tau\right]\Bigg]\\
&= \mathbb{E}_{(x,\boldsymbol{a})}\Bigg[\mathbb{1}_{\{\tau\le s\}}\exp\left(-\int_{[0,\tau]\cap I}\phi^\lambda(s-u,B_u)\,\mathrm{d}u\right)\\
&\qquad \times \mathbb{E}\left[\int_{[0,s-\tau]\cap I}\lambda\psi_b(B_{r+\tau})\exp\left(-\int_{[0,r]\cap I}\phi^\lambda(s-\tau-u,B_{u+\tau})\,\mathrm{d}u\right)\,\mathrm{d}r\,\middle|\,\mathcal{F}_\tau\right]\Bigg]\\
&\le \mathbb{E}_{(x,\boldsymbol{a})}\left[\mathbb{1}_{\{\tau\le s\}}\mathbb{E}_{(B_\tau,\boldsymbol{a})}\left[\int_{[0,s-\tau]\cap I}\lambda\psi_b(B_r)\exp\left(-\int_{[0,r]\cap I}\phi^\lambda(s-\tau-u,B_u)\,\mathrm{d}u\right)\mathrm{d}r\right]\right]\\
&= \mathbb{E}_{(x,\boldsymbol{a})}[\mathbb{1}_{\{\tau\le s\}}\phi^\lambda(s-\tau,B_\tau)]
\end{aligned}
$$

for any $s\le t$ and $\lambda>0$. Now, by the bound (3.6) and the observation that $B_\tau=r$, we have

$$\mathbb{E}_{(x,\boldsymbol{a})}[\mathbb{1}_{\{\tau\le s\}}\phi^\lambda(s-\tau,B_\tau)]\le\frac{\alpha}{(r-b)^2}\mathbb{P}_{(x,\boldsymbol{a})}(\tau\le s)$$

for some constant $\alpha>0$. Then, set

$$\tilde\tau:=\inf\{t\ge0|\tilde B_t\ge r\}$$

for a standard Brownian motion $\tilde B=(\tilde B_t)_{t\ge0}$ started in x and note that

$$\mathbb{P}_{(x,\boldsymbol{a})}(\tau\le s)\le\mathbb{P}_x(\tilde\tau\le s).$$

Thus, using the reflection principle as in [19, Proposition 3.2], we see that

$$\mathbb{P}_{(x,\boldsymbol{a})}(\tau\le s)\le\mathbb{P}_x(\tilde\tau\le s)=2\mathbb{P}_0(\tilde B_s\ge r-x).$$

Then, assuming in addition that $b\ge4\sqrt{t}$, $x\le b-2\sqrt{t}$ and setting $r=b-\sqrt{t}$, we finally obtain

$$\phi^\lambda(s,x)\le\frac{K}{t}\mathbb{P}\left(\tilde B_s\ge\frac{b-x}{2}\right)\le\frac{K}{t}\exp\left(-\frac{(b-x)^2}{20t}\right)$$

for some constant $K>0$. □

Lemma 4.6. *Let $\lambda>0$, $b>0$, $T>0$ and ψ_b be as in the proof of Proposition 3.1. Then, the partial functional differential equation given by*

$$\partial_s\phi^\lambda(s,x)=\frac{\Delta}{2}\phi^\lambda(s,x)-\frac{1}{4}(\phi^\lambda(s,x))^2$$

$$+ c\left(c' e^{-c's}\int_0^s e^{c's'}\phi^\lambda(s',x)\,ds' - \phi^\lambda(s,x)\right) + \lambda\psi_b(x) \qquad (4.3)$$

with initial condition $\phi^\lambda(0,\cdot)\equiv 0$ *has a unique* $C_b^{1,2}([0,T]\times\mathbb{R})\cap B([0,T],L^2(\mathbb{R}))$*-valued positive solution.*

Proof. We begin by considering the *linear* partial functional differential equation given by

$$\partial_s\phi_1^\lambda(s,x) - \frac{\Delta}{2}\phi_1^\lambda(s,x) = c\left(c' e^{-c's}\int_0^s e^{c's'}\phi_1^\lambda(s',x)\,\mathrm{d}s' - \phi_1^\lambda(s,x)\right) + \lambda\psi_b(x). \qquad (4.4)$$

Note that the right hand side satisfies a linear growth and Lipschitz bound. Thus, by classical theory for Delay PDEs (see e.g. [20] or simply by Picard iteration), we obtain global existence of a solution ϕ_1^λ taking values in $C_b^{1,2}([0,T]\times\mathbb{R})\cap B([0,T],L^2(\mathbb{R}))$. Now, choose for each $\lambda>0$ the map ϕ_1^λ as an upper and the constant zero map (considered as a solution of the homogeneous version of Equation (4.3)) as a lower solution. Then, [15, Theorem 2.1][3] yields existence and uniqueness of a solution ϕ^λ to Equation (4.3) with

$$0\le\phi^\lambda(s,x)\le\phi_1^\lambda(s,x)$$

for all $0\le s\le t$ and $x\in\mathbb{R}$. In particular, we also obtain that ϕ^λ takes values in the space $C_b^{1,2}([0,T]\times\mathbb{R})\cap B([0,T],L^2(\mathbb{R}))$. □

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Acknowledgments. The author wishes to thank Andras Tobias and Jochen Blath for useful discussion and comments.

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