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On a class of recursive estimators for spatially dependent observations

Mohamed El Machkouri and Lucas Reding

Laboratoire de Mathématiques Raphaël Salem (UMR CNRS 6085), Université de Rouen Normandie, France
e-mail: mohamed.elmachkouri@univ-rouen.fr; lucas.reding@univ-rouen.fr

Abstract: We investigate the mean squared error and the asymptotic normality for a class of recursive kernel estimators based on a sample of spatially dependent observations. Our main result provides sufficient conditions for a spatial version of a recursive estimator introduced by Hall and Patil (1994) to satisfy a central limit theorem. The results are stated for strongly mixing random fields in the sense of Rosenblatt (1956) and for weakly dependent random fields in the sense of Wu (2005).

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Contents

1. Introduction

In statistics, the nonparametric estimation of probability density functions of continuous random variables is a basic and central problem. From a given sample of observations, the main goal for a practitionner is to understand the mecanism from which the observations have been generated. In the last several decades, this question has attracted much attention among statisticians since it is of considerable interest in many applied fields such as forecasting, computer vision and machine learning. Among the plethora of nonparametric density estimators is the kernel density estimator introduced by Parzen [31] and Rosenblatt [34] which received considerable attention in nonparametric estimation for time series. More precisely, if $(X_1, ..., X_n)$ is a sample (observations) drawn from some univariate distribution with an unknown probability density f with respect to the Lebesgue measure on $\mathbb{R}$ then the Parzen-Rosenblatt density estimator of f is defined for any positive integer n and any x in $\mathbb{R}$ by

$$f_n^{PR}(x) = \frac{1}{nh_n} \sum_{i=1}^{n} \mathrm{K}\left(\frac{x - X_i}{h_n}\right) \tag{1.1}$$

where K is a density function and the bandwidth h_n is a positive parameter which converges to zero such that nh_n goes to infinity. The bandwidth h_n is the most dominant parameter in the kernel density estimator since it controls its amount of smoothness. In fact, if h_n is small then the variance of the estimator is large while the bias is small. This leads to a nonsmooth estimated density. On the other, if h_n is large then the estimated density will be much smoother (small variance) but with a large bias leading to an unsatisfactory estimation. So, in practice, a trade-of between the variance and the bias must be found and the number of publications which are devoted to this crucial question in the literature is very extensive and is still a subject of many works in the statistic community (see for example [8], [17], [22], [38]). From a theoretical but also practical point of view, it is important to investigate asymptotic properties of density estimators when the number n of observations goes to infinity. For example, the consistency and the asymptotic normality of the estimator are very important in order to get pointwise estimation and confidence intervals for the target density f. In his seminal paper, Parzen [31] proved that when the observations $(X_1, ..., X_n)$ are i.i.d. and the bandwidth h_n goes to zero such that nh_n goes to infinity then $(nh_n)^{1/2}(f_n^{PR}(x_0) - \mathbb{E}[f_n^{PR}(x_0)])$ converges in distribution to the normal law with zero mean and variance $f(x_0)\int_{\mathbb{R}} \mathrm{K}^2(t)dt$ as n goes to infinity and this result was extended by Wu and Mielniczuk [44] for causal linear processes with i.i.d. innovations and by Dedecker and Merlevède [11] for strongly mixing sequences. Previously, Bosq, Merlevède and Peligrad [6] established a central limit theorem for the kernel density estimator f_n when the sequence $(X_i)_{i\in\mathbb{Z}}$ is assumed to be strongly mixing but the bandwidth parameter h_n is assumed to satisfy $h_n \geq Cn^{-1/3}\log n$ (for some positive constant C) which is stronger than the bandwidth parameter assumption in [11], [31] and [44].

In many situations, practicians are also interested by the relationship between some predictors and a response. This is a natural question and a very important task in statistics. The objective is to find a relation between a pair of random variables X (predictor) and Y (response) using a given sample $(X_i, Y_i)_{1\leqslant i\leqslant n}$ drawn from the unknown law of (X, Y). A very popular tool to handle this problem is the kernel regression estimator introduced by Nadaraya [30] and Watson [41]. More formally, let N be a positive integer and assume that $(X_i, Y_i)_{1\leqslant i\leqslant n}$ are identically distributed $\mathbb{R}^N \times \mathbb{R}$-valued sequence of random variables such that $Y_i = R(X_i, \eta_i)$ where R is an unknown functional and $(\eta_i)_{i\in\mathbb{Z}}$ are i.i.d. $\mathbb{R}^N$-valued random variables with zero mean and finite variance and independent of $(X_i)_{i\in\mathbb{Z}}$. Let f be the marginal density function of X_0. If r is the (unknown) regression function defined for any x in $\mathbb{R}^N$ by $r(x) = \mathbb{E}\left[R(x, \eta_0)\right]$ if $f(x) \neq 0$ and $r(x) = \mathbb{E}[Y_0]$ if $f(x) = 0$ then the Nadaraya-Watson regression estimator r_n^{NW} of r is defined for any x in $\mathbb{R}^N$ by

$$r_n^{NW}(x) = \begin{cases} \dfrac{\sum_{i=1}^n Y_i \mathrm{K}((x - X_i)/h_n)}{\sum_{i=1}^n \mathrm{K}((x - X_i)/h_n)} & \text{if } \sum_{i=1}^n \mathrm{K}((x - X_i)/h_n) \neq 0 \\ n^{-1} \displaystyle\sum_{i=1}^n Y_i & \text{else.} \end{cases} \tag{1.2}$$

The literature on the asymptotic properties of r_n^{NW} for time series is very expansive. One can refer to Lu and Cheng [24], Masry and Fan [26], Robinson [33], Roussas [36] and many references therein. Kernel nonparametric methods are still very popular and fairly well established in the statistical community but despite their power, the data streams problem, which refers to data sets that continuously and rapidly grow over time, present new challenges. In order to handle such data sets, several recursive versions of the Parzen-Rosenblatt estimator (1.1) have been introduced (see for example [3], [12], [20], [43], [47]). For example, if $(w_k)_{k \geqslant 1}$ is a nonincreasing sequence of positive real numbers satisfying $\sum_{k \geqslant 1} w_k = \infty$ and $(h_k)_{k \geqslant 1}$ is a sequence of positive real numbers going to 0 as n goes to infinity (bandwidth parameters) then the resursive kernel density estimator f_n^{HP} of Hall and Patil [20] is defined by

$$f_n^{HP}(x) = \frac{1}{\sum_{k=1}^n w_k} \sum_{i=1}^n \frac{w_i}{h_i^d} K\left(\frac{x - X_i}{h_i}\right). \tag{1.3}$$

This estimator is recursive in the sense that it satisfies

$$f_{n+1}^{HP}(x) = (1 - \gamma_{n+1}) f_n^{HP}(x) + \gamma_{n+1} \tilde{f}_{n+1}(x) \tag{1.4}$$

where $\gamma_n := \frac{w_n}{\sum_{i=1}^n w_i}$ and $\tilde{f}_{n+1}(x) := \frac{1}{h_{n+1}^d} K\left(\frac{x - X_{n+1}}{h_{n+1}}\right)$.

Such a property endows recursive estimators with a decisive computational advantage because they can be easily updated as new data items arrive over time. More precisely, in order to obtain the estimation $f_{n+1}^{HP}(x)$ at time $n+1$, using the recursive equation (1.4), it is sufficient to combine the estimation $f_n^{HP}(x)$ at time n (which is known at time $n+1$) with the estimation $\tilde{f}_{n+1}(x)$ at time $n+1$ based on the single observation X_{n+1}. In fact, a non-recursive estimator must be fully recomputed whenever a new observation is collected. This clearly represents a drawback in a data stream context compared to the recursive approach. The class (1.3) contains the recursive estimators introduced by Wolverton and Wagner [43] and Deheuvels [12] but also a renormalized version of the one introduced by Wegman and Davies [42] and another class of estimators introduced by Amiri [3]. It contains also the (non-recursive) Parzen-Rosenblatt estimator (1.1) when $h_i = h_n$ and $w_i = 1$ for any $1 \leqslant i \leqslant n$. In this work, our aim is to investigate asymptotic properties for a spatial version of the Hall and Patil estimator (1.3) in terms of mean squared error and asymptotic normality under weak and strong dependence conditions. The first studies that focused on a recursive version of the Parzen-Rosenblatt estimator were presented by [12], [43] and [47] and later by [20], [25], [29] and many others. Actually, many papers in the literature are devoted to the asymptotic properties of recursive kernel density and regression estimators for i.i.d. observations. There are also some published papers on the asymptotic properties of recursive kernel density and regression estimators for dependent (weakly dependent and strongly mixing) data. One can refer for example to [4], [7], [18], [19], [25], [36], [39], [40] and others.

In our context, we deal with spatial data which is modelized using finite realizations of dependent random fields indexed by $\mathbb{Z}^d$ where d is a positive integer. More precisely, let N be a positive integer and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a stationary $\mathbb{R}^N$-valued random field $(X_k)_{k\in\mathbb{Z}^d}$ such that the law μ_0 of X_0 is absolutely continuous with respect to the Lebesgue measure λ_N on $\mathbb{R}^N$ and we denote by f the probability density function of μ_0 with respect to λ_N. Given two sub-σ-algebras $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{F}$, recall that the α-mixing coefficient introduced by Rosenblatt [35] is defined by

$$\alpha(\mathcal{U}, \mathcal{V}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|\,,\ A \in \mathcal{U},\ B \in \mathcal{V}\}.$$

Let p be fixed in $[1, +\infty]$. The strong mixing coefficients $(\alpha_{1,p}(n))_{n\geqslant 0}$ associated to $(X_k)_{k\in\mathbb{Z}^d}$ are defined by

$$\alpha_{1,p}(n) = \sup\left\{\alpha(\sigma(X_k), \mathcal{F}_\Gamma),\ k \in \mathbb{Z}^d,\ \Gamma \subset \mathbb{Z}^d,\ |\Gamma| \leqslant p,\ \rho(\Gamma, \{k\}) \geqslant n\right\}$$

where $|\Gamma|$ is the number of elements in Γ, the collection $\mathcal{F}_\Gamma$ is the σ-algebra $\sigma(X_k\,;\,k \in \Gamma)$ and the distance ρ is defined for any subsets Γ_1 and Γ_2 of $\mathbb{Z}^d$ by $\rho(\Gamma_1, \Gamma_2) = \min\{|u - v|,\ u \in \Gamma_1,\ v \in \Gamma_2\}$ with $|u - v| = \max_{1\leqslant \ell\leqslant d} |u_\ell - v_\ell|$ for any $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ in $\mathbb{Z}^d$. We say that the random field $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing if $\lim_{n\to\infty} \alpha_{1,p}(n) = 0$. Moreover, we are going to consider also Bernoulli fields defined for any $k \in \mathbb{Z}^d$ by

$$X_k = G\left(\varepsilon_{k-u}\,;\,u \in \mathbb{Z}^d\right) \tag{1.5}$$

where $G : (\mathbb{R}^m)^{\mathbb{Z}^d} \to \mathbb{R}^N$ is a measurable function, $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ are i.i.d. $\mathbb{R}^m$-valued random variables and m is a positive integer. The class of random fields that (1.5) represents is huge and it includes many commonly used linear and nonlinear processes (see Wu [46] for a review). Let $(\varepsilon^{'}_k)_{k\in\mathbb{Z}^d}$ be an i.i.d. copy of $(\varepsilon_k)_{k\in\mathbb{Z}^d}$ and let X^*_k be the coupled version of X_k defined by $X^*_k = G\left(\varepsilon^*_{k-u}\,;\,u \in \mathbb{Z}^d\right)$ where $\varepsilon^*_k = \varepsilon_k$ if $k \neq 0$ and $\varepsilon^*_0 = \varepsilon^{'}_0$. Note that X^*_k is obtained from X_k by replacing ε_0 by its copy ε'_0. For any positive integer ℓ and any $\mathbb{R}^\ell$-valued random variable $Z \in \mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ with $p > 0$, we denote $\|Z\|_p := \mathbb{E}\left[\|Z\|^p\right]^{1/p}$ where $\|\,.\,\|$ is the Euclidian norm on $\mathbb{R}^\ell$. Following Wu [45] and El Machkouri et al. [16], we define the physical dependence measure coefficient $\delta_{k,p} := \|X_k - X^*_k\|_p$ as soon as X_k is p-integrable for $p \geqslant 2$. Physical dependence measure should be seen as a measure of the dependence of the function G (defined in (1.5)) in the coordinate zero. In some sense, it quantifies the degree of dependence of outputs on inputs in physical systems and provide a natural framework for a limit theory for stationary random fields (see [16]). In particular, it gives mild and easily verifiable conditions (see condition (H2)(ii) below) because it is directly related to the data-generating mechanism.

2. Main results

Let $\Lambda_0=\emptyset$, $s_0=0\in\mathbb{Z}^d$ and $\Lambda_n=\{s_1,...,s_n\}\subset\mathbb{Z}^d$ for $n\geqslant 1$. Let $(w_{s_n})_{n\geqslant 1}$ and $(h_{s_n})_{n\geqslant 1}$ be two nonincreasing sequences of positive real numbers such that $(w_{s_n}h_{s_n}^{-N})_{n\geqslant 1}$ is nondecreasing, h_{s_n} goes to 0 as n goes to infinity and $\sum_{n\geqslant 1}w_{s_n}=\infty$. Let also $K:\mathbb{R}^N\to\mathbb{R}_+$ be a function (called a kernel) such that $\int_{\mathbb{R}^N}K(t)dt=1$ and $\sup_{x\in\mathbb{R}^N}K(x)<\infty$. Assume that K is Lipschitz and satisfies $\lim_{\|x\|\to\infty}\|x\|K(x)=0$, $\int_{\mathbb{R}^N}\|u\|^2K(u)du<\infty$ where $\|\,.\,\|$ is the usual norm on $\mathbb{R}^N$ and $\int_{\mathbb{R}^N}u_iK(u)du=0$ for any $1\leqslant i\leqslant N$. Let $\Phi:\mathbb{R}\to\mathbb{R}$ be a measurable function such that $\mathbb{E}\left[|\Phi(Y_0)|^{2+\theta}\right]<\infty$ and $\mathbb{E}\left[|\Phi(Y_0)|^{2+\theta}K_{s_n}(x,X_0)\right]\leqslant Ch_{s_n}^N$ for some $\theta>0$ and $C>0$ and assume that $u\mapsto\mathbb{E}[|\Phi(Y_0)|^2|X_0=u]$ is continuous. One can notice that $\mathbb{E}[|\Phi(Y_0)|^{2+\theta}K_{s_n}(x,X_0)]\leqslant Ch_{s_n}^N$ is satisfied when $u\mapsto\mathbb{E}[|\Phi(Y_0)|^{2+\theta}|X_0=u]$ is continuous (see Lemma 2 below). Let $(\eta_k)_{k\in\mathbb{Z}^d}$ be i.i.d. $\mathbb{R}^N$-valued random variables with zero mean and finite variance and independent of $(X_k)_{k\in\mathbb{Z}^d}$ and consider the regression model $Y_{s_i}=R(X_{s_i},\eta_{s_i})$ for any $1\leqslant i\leqslant n$ where R is an unknown functional. For any $x\in\mathbb{R}^N$, we denote $f_\Phi(x)=r_\Phi(x)f(x)$ where $r_\Phi(x)=\mathbb{E}[\Phi(Y_0)|X_0=x]=\mathbb{E}[\Phi(R(x,\eta_0))]$ if $f(x)\neq 0$ and $r_\Phi(x)=\mathbb{E}[\Phi(Y_0)]$ if $f(x)=0$ and we consider the estimator $f_{n,\Phi}$ of f_Φ defined by

$$f_{n,\Phi}(x)=(\sum_{i=1}^n w_{s_i})^{-1}\sum_{j=1}^n w_{s_j}h_{s_j}^{-N}\Phi(Y_{s_j})K_{s_j}(x,X_{s_j}) \tag{2.1}$$

where $K_{s_j}(x,v)=K((x-v)/h_{s_j})$ for any $v\in\mathbb{R}^N$ and any $1\leqslant j\leqslant n$. One can notice that if $\Phi(u)=1$ for any $u\in\mathbb{R}$ then $f_{n,\Phi}$ reduces to the spatial version $f_{n,\mathbf{1}}$ of the recursive kernel density estimator of f introduced by Hall and Patil [20] and defined for any $x\in\mathbb{R}^N$ by

$$f_{n,\mathbf{1}}(x)=(\sum_{i=1}^n w_{s_i})^{-1}\sum_{j=1}^n w_{s_j}h_{s_j}^{-N}K_{s_j}(x,X_{s_j}). \tag{2.2}$$

Moreover, for particular choices of the weights $(w_{s_n})_{n\geqslant 1}$, the estimator (2.2) reduces to the recursive estimators introduced by [3], [4], [12] or [43]. In particular, one can check that $f_{n,\Phi}$ satisfies the following recursive equation

$$f_{n,\Phi}(x)=(1-\rho_n)f_{n-1,\Phi}(x)+\rho_n h_{s_n}^{-N}\Phi(Y_{s_n})K_{s_n}(x,X_{s_n}) \tag{2.3}$$

where $\rho_n=\frac{w_{s_n}}{\sum_{i=1}^n w_{s_i}}$. Equation (2.3) is the spatial version of the recursive equation (1.4). It lays emphasis on that the update of $f_{n,\Phi}$ at time n can be done from $f_{n-1,\Phi}$ at time $n-1$ and the new single observation X_{s_n}. This is a definitive advantage over the spatial version of the non-recursive Parzen-Rosenblatt estimator f_n^{PR} defined by (1.1) since it is necessary to consider the whole sample $(X_{s_1}, ..., X_{s_n})$ in order to compute f_n^{PR} at any time n. In this work, we consider also the following class of spatial semi-recursive kernel regression estimator $r_{n,\Phi}$ of r_Φ defined for any x in $\mathbb{R}^N$ by

$$r_{n,\Phi}(x) = \begin{cases} \dfrac{f_{n,\Phi}(x)}{f_{n,\mathbf{1}}(x)} & \text{if } \sum_{j=1}^n w_{s_j} h_{s_j}^{-N} K_{s_j}(x, X_{s_j}) \neq 0 \\ n^{-1} \displaystyle\sum_{i=1}^n Y_{s_i} & \text{else} \end{cases} \tag{2.4}$$

which contains the first two semi-recursive kernel regression estimators introduced by Ahmad and Lin [2] and Devroye and Wagner [13] for time series (i.e. for $d = 1$) but also the class of semi-recursive kernel regression estimators considered by Amiri [4]. Since $r_{n,\Phi}$ is defined from $f_{n,\Phi}$ and $f_{n,\mathbf{1}}$, it inherits the good properties in term of computation time of the recursive estimators $f_{n,\Phi}$ and $f_{n,\mathbf{1}}$ and consequently, in a data stream setting, it has a decisive advantage over the spatial version of the non-recursive Nadaraya-Watson estimator defined by (1.2).

Now, we are going to present our main contributions. For $j \in \{2, 4\}$, we adopt the notation

$$\nu_j(\theta) = \mathbb{1}_{\|\Phi\|_\infty < \infty} + \frac{\theta}{j + \theta} \mathbb{1}_{\|\Phi\|_\infty = \infty} \tag{2.5}$$

and for any sequences $(a_n)_{n \geqslant 1}$ and $(b_n)_{n \geqslant 1}$ of real positive numbers, we denote $a_n \trianglelefteq b_n$ if and only if there exists $\kappa > 0$ (not depending on n) such that $a_n \leqslant \kappa b_n$. Recall that $(w_{s_n})_{n \geqslant 1}$ and $(h_{s_n})_{n \geqslant 1}$ are two nonincreasing sequences of positive real numbers such that $(w_{s_n} h_{s_n}^{-N})_{n \geqslant 1}$ is nondecreasing, h_{s_n} goes to 0 as n goes to infinity and $\sum_{n \geqslant 1} w_{s_n} = \infty$ and keep in mind that $K : \mathbb{R}^N \to \mathbb{R}_+$ is a function (kernel) such that $\int_{\mathbb{R}^N} K(t) dt = 1$ and $|K|_\infty := \sup_{t \in \mathbb{R}^N} K(t) < \infty$.

For any integer $n \geqslant 1$ and any $(p, q) \in \mathbb{Z}^2$, we denote also

$$A_{n,p,q} = (n h_{s_n}^p w_{s_n}^q)^{-1} \sum_{i=1}^n h_{s_i}^p w_{s_i}^q$$

and we consider the following assumptions:

(H1) There exists $(\beta_{0,1}, \beta_{-N,2}) \in (\mathbb{R}_+^*)^2$ such that $\lim_{n \to \infty} A_{n,0,1} = \beta_{0,1}$ and $\lim_{n \to \infty} A_{n,-N,2} = \beta_{-N,2}$.

(H2) There exist $\theta > 0$ such that $\mathbb{E}[|\Phi(Y_0)|^{2+\theta}] < \infty$ and $\mathbb{E}[|\Phi(Y_0)|^{2+\theta} K_{s_n}(x, X_0)] \trianglelefteq h_{s_n}^N$, and $\tau \in]1 - \nu_4(\theta), 1]$ such that $\lim_{n \to \infty} n h_{s_n}^{N\left(1 + \frac{d\nu_2(\theta)(\nu_4(\theta) + \tau - 1)}{d\nu_2(\theta) + (d-1)(\nu_4(\theta) + \tau - 1)}\right)} = \infty$ and $h_{s_n}^{N(1-\tau)} \sum_{i=1}^n w_{s_i}^2 \trianglelefteq n w_{s_n}^2$. Moreover, one of the following condition holds:

- (i) $(X_k)_{k \in \mathbb{Z}^d}$ is strongly mixing and $\sum_{k \in \mathbb{Z}^d} |k|^{\frac{d\nu_2(\theta)}{\nu_4(\theta) + \tau - 1}} \alpha_{1,\infty}^{\nu_2(\theta)}(|k|) < \infty$,
- (ii) $(X_k)_{k \in \mathbb{Z}^d}$ is of the form (1.5) and $\sum_{k \in \mathbb{Z}^d} |k|^{\frac{d(N + 2\nu_2(\theta) + 2N(\nu_4(\theta) + \tau - 1))}{2N(\nu_4(\theta) + \tau - 1)}} \delta_{k,2}^{\nu_2(\theta)} < \infty$.

(H3) (i) The function $u \mapsto \mathbb{E}[|\Phi(Y_0)|^2|X_0 = u]$ is continuous.

(ii) The function f_Φ is twice differentiable with bounded second derivatives.

(iii) For any $k \in \mathbb{Z}^d\backslash\{0\}$, the law of (X_0, X_k) is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N \times \mathbb{R}^N$ and there exists $c > 0$ such that $\sup_{k\in\mathbb{Z}^d\backslash\{0\}} |f_{0,k}(x, y) - f(x)f(y)| \leqslant c$ for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^N$ where $f_{0,k}$ is the joint density function of (X_0, X_k).

Assumptions (H1) and (H3) are classical in the context of recursive kernel estimators (see [4], [25], [28], [42] and many others). In (H2), we assume that the bandwidth parameter h_{s_n} satisfies a condition sligthly stronger than the usual minimal condition assumed in the non recursive i.i.d. setting (i.e. $nh_{s_n}^N \to \infty$). However, this fact seems to be inherent to the case of recursive estimators since a condition like $nh_{s_n}^{N(1+\varepsilon)} \to \infty$ for some $\varepsilon > 0$ is assumed in many contributions for dependent data (see for example [4], [1], [25], [28] or [42]).

For any x in $\mathbb{R}^N$, we denote

$$\sigma_\Phi^2(x) := \beta_{0,1}^{-2}\beta_{-N,2}\mathbb{E}[|\Phi(Y_0)|^2|X_0 = x]f(x)\int_{\mathbb{R}^N} K^2(t)dt. \tag{2.6}$$

Our first result gives the asymptotic variance of the estimator $f_{n,\Phi}$ defined by (2.1).

Proposition 1. *Assume that (H1) and (H3) hold and there exists $\theta > 0$ such that $\mathbb{E}[|\Phi(Y_0)|^{2+\theta}] < \infty$ and $\mathbb{E}[|\Phi(Y_0)|^{2+\theta}K_{s_n}(x, X_0)] \trianglelefteq h_{s_n}^N$. If there exists $\tau \in]1 - \nu_4(\theta), 1]$ such that $h_{s_n}^{N(1-\tau)}\sum_{i=1}^n w_{s_i}^2 \trianglelefteq nw_{s_n}^2$ and one of the following conditions is satisfied:*

(i) $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing and $\sum_{k\in\mathbb{Z}^d} |k|^{\frac{d\nu_2(\theta)}{\nu_4(\theta)+\tau-1}}\alpha_{1,1}^{\nu_2(\theta)}(|k|) < \infty$

(ii) $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5) and $\sum_{k\in\mathbb{Z}^d} |k|^{\frac{d(N+2\nu_2(\theta)+2N(\nu_4(\theta)+\tau-1))}{2N(\nu_4(\theta)+\tau-1)}}\delta_{k,2}^{\nu_2(\theta)} < \infty$

where $\nu_2(\theta)$ and $\nu_4(\theta)$ are defined by (2.5) then for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} |nh_{s_n}^N\mathbb{V}[f_{n,\Phi}(x)] - \sigma_\Phi^2(x)| = 0 \tag{2.7}$$

where $\sigma_\Phi^2(x)$ is defined by (2.6).

We obtain also the convergence to zero of the mean square error of $f_{n,\Phi}$.

Proposition 2. *Assume that f_Φ is twice differentiable with bounded second derivatives.*

Then, for any $x \in \mathbb{R}^N$, $|\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x)| \trianglelefteq (\sum_{i=1}^n w_{s_i})^{-1}\sum_{i=1}^n w_{s_i}h_{s_i}^2 = o(1)$. So, if $\max\{A_{n,0,1}^{-1}, A_{n,2,1}\} \trianglelefteq 1$ then $|\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x)| \trianglelefteq h_{s_n}^2$ and, under assumptions of Proposition 1, we get $\mathbb{E}[(f_{n,\Phi}(x) - f_\Phi(x))^2] \trianglelefteq n^{-\frac{4}{4+N}}$ for $h_{s_n} = n^{-\frac{1}{4+N}}$.

The main contribution of this paper is the following central limit theorem.

Theorem 1. *If (H1), (H2) and (H3) hold then for any $x \in \mathbb{R}^N$,*

$$\sqrt{nh_{s_n}^N}\left(f_{n,\Phi}(x) - \mathbb{E}[f_{n,\Phi}(x)]\right) \xrightarrow[n\to\infty]{Law} \mathcal{N}(0, \sigma_\Phi^2(x))$$

where $\sigma_\Phi^2(x)$ is defined by (2.6).

One can notice that Theorem 1 is an extension of Theorem 1 in [1] where the case of strongly mixing time series is considered. In fact, with our notations, if $d = 1$ and $\Phi(u) = 1$ for any $u \in \mathbb{R}$ then $f_{n,\Phi}$ reduces to the recursive kernel density estimator $f_{n,\mathbf{1}}$ introduced by Hall and Patil [20]. In this case, we have $\nu_2(\theta) = \nu_4(\theta) = 1$ and (H2)(ii) holds as soon as $\sum_{k>0} k^{1/\tau}\alpha_{1,\infty}(k) < \infty$ and $\lim_{n\to\infty} nh_{s_n}^{N\left(1+\frac{d\nu_2(\theta)(\nu_4(\theta)+\tau-1)}{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}\right)} = \lim_{n\to\infty} nh_{s_n}^{N(1+\tau)} = \infty$ which are exactly the conditions imposed in Theorem 1 in [1]. Using Theorem 1, we derive the asymptotic normality for the recursive estimator $r_{n,\Phi}$ defined by (2.4).

Theorem 2. *Assume that (H1), (H2) and (H3) hold. If f is Lipschitz and twice differentiable with bounded second derivatives then for any $x \in \mathbb{R}^N$ such that $f(x) > 0$,*

$$\sqrt{nh_{s_n}^N}\left(r_{n,\Phi}(x) - \frac{\mathbb{E}[f_{n,\Phi}(x)]}{\mathbb{E}[f_{n,\mathbf{1}}(x)]}\right) \xrightarrow[n\to\infty]{Law} \mathcal{N}\left(0, \tilde{\sigma}_\Phi^2(x)\right),$$

with $\tilde{\sigma}_\Phi^2(x) = \frac{V(x)\beta_{-N,2}}{f(x)\beta_{0,1}^2}\int_{\mathbb{R}^N} \mathrm{K}^2(t)dt$ and $V(x) = \mathbb{E}[|\Phi(Y_0)|^2|X_0 = x] - r_\Phi^2(x)$.

Theorem 2 is also an extension of Theorem 2.1 in [36] where the asymptotic normality of the semi-recursive kernel regression estimator for time series (i.e. $d = 1$) introduced by Ahmad and Lin [2] is obtained under more restrictive conditions on the bandwidth parameter and the strong mixing coefficients. Using Theorem 2 and Proposition 2, the condition $nh_{s_n}^{N+4} \to 0$ can be imposed for the control of the bias of the estimator and leads immediately to the following result.

Theorem 3. *Assume that (H1), (H2) and (H3) hold. If f is Lipschitz and twice differentiable with bounded second derivatives, $nh_{s_n}^{N+4} \to 0$ and $A_{n,2,1} \trianglelefteq 1$, then for any $x \in \mathbb{R}^N$ such that $f(x) > 0$,*

$$\sqrt{nh_{s_n}^N}\left(r_{n,\Phi}(x) - r_\Phi(x)\right) \xrightarrow[n\to\infty]{Law} \mathcal{N}\left(0, \tilde{\sigma}_\Phi^2(x)\right),$$

where $\tilde{\sigma}_\Phi^2(x)$ is defined in Theorem 2.

3. Preliminary lemmas

This section is devoted to the presentation of several technical lemmas and propositions which are key tools in the proof of the main contributions in section 4. For any real x, we also define $\lceil x\rceil = \lfloor x\rfloor + 1$, where $\lfloor x\rfloor$ is the largest integer less or equal than x.

Lemma 1. *Let $(a_k)_{k\in\mathbb{Z}^d}$ be a family of real numbers such that a_{s_n} goes to some value $a\in\mathbb{R}$ as n goes to infinity. If $\lim_{n\to\infty} A_{n,-N,2}=\beta_{-N,2}\in\mathbb{R}$ then*

$$\lim_{n\to\infty}\frac{h_{s_n}^N}{nw_{s_n}^2}\sum_{i=1}^{n}\frac{w_{s_i}^2 a_{s_i}}{h_{s_i}^N}=a\beta_{-N,2}.$$

Proof of Lemma 1. For any positive integers i and n, we denote $b_{i,n}=w_{s_i}^2h_{s_n}^N/(nh_{s_i}^N w_{s_n}^2)$ if $i\leqslant n$ and $b_{i,n}=0$ otherwise. Since $(h_{s_n})_{n\geqslant 1}$ is nonincreasing and $(w_{s_n}h_{s_n}^{-N})_{n\geqslant 1}$ is nondecreasing, for $i\leqslant n$, we have

$$b_{i,n}=\frac{h_{s_i}^N}{nh_{s_n}^N}\times\frac{w_{s_i}^2/h_{s_i}^{2N}}{w_{s_n}^2/h_{s_n}^{2N}}\leqslant\frac{h_{s_1}^N}{nh_{s_n}^N}\xrightarrow[n\to+\infty]{}0.$$

Moreover, $\sum_{i=1}^{+\infty}b_{i,n}=A_{n,-N,2}\to\beta_{-N,2}\in\mathbb{R}$ as $n\to+\infty$. So, by Toeplitz's lemma (see Lemma 3 in [25]), we get

$$\lim_{n\to\infty}\sum_{i=1}^{n}b_{i,n}a_{s_i}=a\beta_{-N,2}.$$

The proof Lemma 1 is complete. □

The following lemma will be usefull in order to compute the asymptotic variance of the estimator $f_{n,\Phi}$ (see Propositions 1 and 4).

Lemma 2. *Let $x\in\mathbb{R}^N$ be fixed and let $\Psi_1:\mathbb{R}\to\mathbb{R}$ and $\Psi_2:\mathbb{R}\to\mathbb{R}$ be two functions. If $u\mapsto\mathbb{E}[\Psi_1(Y_0)|X_0=u]$ is continuous and the conditions $\sup_{t\in\mathbb{R}^N}|\Psi_2(K(t))|<\infty$, $\lim_{\|t\|\to\infty}\|t\|\,|\Psi_2(K(t))|=0$ and $\int_{\mathbb{R}^N}|\Psi_2(K(t))|\,dt<\infty$ are satisfied then*

$$\lim_{n\to\infty}h_{s_n}^{-N}\mathbb{E}[\Psi_1(Y_0)\Psi_2(K_{s_n}(x,X_0))]=\mathbb{E}[\Psi_1(Y_0)|X_0=x]f(x)\int_{\mathbb{R}^N}\Psi_2(K(v))\,dv.$$

Proof of Lemma 2. Let $x\in\mathbb{R}^N$ and let n be a positive integer. It is obvious that

$$\mathbb{E}[\Psi_1(Y_0)\Psi_2(K_{s_n}(x,X_0))]=h_{s_n}^N\int_{\mathbb{R}^N}\mathbb{E}[\Psi_1(Y_0)\,|\,X_0=x-vh_{s_n}]\Psi_2(K(v))\,f(x-vh_{s_n})dv.$$

By Theorem 1A in [31], we derive

$$\lim_{n\to\infty}h_{s_n}^{-N}\mathbb{E}[\Psi_1(Y_0)\,\Psi_2(K_{s_n}(x,X_0))]=\mathbb{E}[\Psi_1(Y_0)\,|\,X_0=x]f(x)\int_{\mathbb{R}^N}\Psi_2(K(v))\,dv. \quad (3.1)$$

The proof of Lemma 2 is complete. □

For any $\ell \in \{1,2\}$, any $1 \leqslant i \leqslant n$ and any sequence $(m_n)_{n\geqslant 1}$ of positive integers, we define

$$\Delta_{s_i}^{(\ell)} = \frac{\Phi_\ell(Y_{s_i})K_{s_i}(x,X_{s_i}) - \mathbb{E}[\Phi_\ell(Y_0)K_{s_i}(x,X_0)]}{h_{s_i}^{N/2}} \quad \text{and} \quad \overline{\Delta}_{s_i}^{(\ell)} = \mathbb{E}[\Delta_{s_i}^{(\ell)}|\mathcal{H}_{i,m_n}] \tag{3.2}$$

where $\mathcal{H}_{i,m_n} = \sigma(\eta_{s_i}, \varepsilon_{s_i-k}; |k| \leqslant m_n)$ and $\Phi_\ell : \mathbb{R} \to \mathbb{R}$ is a measurable function.

Lemma 3. *Let $\ell \in \{1,2\}$ and $\theta > 0$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_n}(x,X_0)] \trianglelefteq h_{s_n}^N$ then $\mathbb{E}[|\Phi_\ell(Y_0)|^p K_{s_n}(x,X_0)] \trianglelefteq h_{s_n}^N$ for any $0 < p < 2+\theta$. Moreover, if (H3)(iii) holds then $\sup_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}}(h_{s_i}h_{s_j})^{-N}\mathbb{E}[K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})] \trianglelefteq 1$.*

Proof of Lemma 3. If $0 < p < 2+\theta$ then

$$\mathbb{E}[|\Phi_\ell(Y_0)|^p K_{s_i}(x,X_0)] \leqslant \mathbb{E}[K_{s_i}(x,X_0)] + \mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_i}(x,X_0)].$$

Since $\mathbb{E}[|\Phi_\ell(Y_0)|^r K_{s_i}(x,X_0)] \trianglelefteq h_{s_n}^N$ for $r \in \{0, 2+\theta\}$, we get $\mathbb{E}[|\Phi_\ell(Y_0)|^p K_{s_i}(x,X_0)] \trianglelefteq h_{s_n}^N$. In the other part, using (H3)(iii), for any $1 \leqslant i,j \leqslant n$ such that $i \neq j$ we have

$$\begin{aligned}\mathbb{E}[K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})] \trianglelefteq &\int_{\mathbb{R}^N} K_{s_i}(x,u)du \int_{\mathbb{R}^N} K_{s_j}(x,v)dv \\ &+ \mathbb{E}[K_{s_i}(x,X_0)]\mathbb{E}[K_{s_j}(x,X_0)] \\ \trianglelefteq &(h_{s_i}h_{s_j})^N.\end{aligned}$$

The proof of Lemma 3 is complete. □

Lemma 4. *If $(p,q) \in \{1,2\}^2$ and $\theta > 0$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}] < \infty$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_n}(x,X_0)|] \trianglelefteq h_{s_n}^N$ for any $\ell \in \{p,q\}$ then*

$$\sup_{\substack{1\leqslant i,j\leqslant n\\ i\neq j}} (h_{s_i}h_{s_j})^{\frac{-N\gamma}{2}}\mathbb{E}[|\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}|] \trianglelefteq 1 \tag{3.3}$$

$$\textit{where} \quad \gamma = \begin{cases} 1 & \textit{if} \quad \max(\|\Phi_p\|_\infty, \|\Phi_q\|_\infty) < \infty \\ \frac{\theta}{4+\theta} & \textit{if} \quad \min(\|\Phi_p\|_\infty, \|\Phi_q\|_\infty) = \infty \\ \frac{\theta}{2+\theta} & \textit{else.}\end{cases} \tag{3.4}$$

Proof of Lemma 4. Let i and j be two positive integers such that $i \neq j$ and $(p,q) \in \{1,2\}^2$ and let $\theta > 0$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_n}(x,X_0)] \trianglelefteq h_{s_n}^N$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}] < \infty$ for any $\ell \in \{p,q\}$. Keeping in mind the notations $\Delta_{s_i}^{(p)}$ and $\Delta_{s_j}^{(q)}$ defined by (3.2), we have the following bound

$$\begin{aligned}(h_{s_i}h_{s_j})^{\frac{N}{2}}\mathbb{E}[|\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}|] \leqslant &\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})] \\ &+ 3\mathbb{E}[|\Phi_p(Y_0)|K_{s_i}(x,X_0)]\mathbb{E}[|\Phi_q(Y_0)|K_{s_j}(x,X_0)].\end{aligned} \tag{3.5}$$

Note that the second term of the right hand side of (3.5) can be dealt with using Lemma 3. Therefore, we focus on the first part of the right hand side only. Let $L>1$ be fixed then

$$\begin{aligned}
&\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]\\
&\quad=\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|\,\mathbb{1}_{|\Phi_p(Y_{s_i})|\leqslant L}\,\mathbb{1}_{|\Phi_q(Y_{s_j})|\leqslant L}K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]\\
&\qquad+\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|\,\mathbb{1}_{|\Phi_p(Y_{s_i})|\leqslant L}\,\mathbb{1}_{|\Phi_q(Y_{s_j})|>L}K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]\\
&\qquad+\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|\,\mathbb{1}_{|\Phi_p(Y_{s_i})|>L}\,\mathbb{1}_{|\Phi_q(Y_{s_j})|\leqslant L}K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]\\
&\qquad+\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|\,\mathbb{1}_{|\Phi_p(Y_{s_i})|>L}\,\mathbb{1}_{|\Phi_q(Y_{s_j})|>L}K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})].
\end{aligned}$$

Using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]\\
&\leqslant (L\wedge\|\Phi_p\|_\infty)\,(L\wedge\|\Phi_q\|_\infty)\,\mathbb{E}[K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]\\
&\quad+\sqrt{\mathbb{E}[|\Phi_p(Y_0)|^2K^2_{s_i}(x,X_0)]}\sqrt{\mathbb{E}[|\Phi_q(Y_0)|^2\,\mathbb{1}_{|\Phi_q(Y_0)|>L}K^2_{s_j}(x,X_0)]}\\
&\quad+\sqrt{\mathbb{E}[|\Phi_p(Y_0)|^2\,\mathbb{1}_{|\Phi_p(Y_0)|>L}K^2_{s_i}(x,X_0)]}\sqrt{\mathbb{E}[|\Phi_q(Y_0)|^2K^2_{s_j}(x,X_0)]}\\
&\quad+\sqrt{\mathbb{E}[|\Phi_p(Y_0)|^2\,\mathbb{1}_{|\Phi_p(Y_0)|>L}K^2_{s_i}(x,X_0)]}\sqrt{\mathbb{E}[|\Phi_q(Y_0)|^2\,\mathbb{1}_{|\Phi_q(Y_0)|>L}K^2_{s_j}(x,X_0)]}.
\end{aligned}$$

Since $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}]<\infty$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_n}(x,X_0)]\trianglelefteq h_{s_n}^N$ for any $\ell\in\{p,q\}$, we apply Lemma 3 and we get

$$\begin{aligned}
&\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]\\
&\qquad\trianglelefteq (L\wedge\|\Phi_p\|_\infty)\,(L\wedge\|\Phi_q\|_\infty)(h_{s_i}h_{s_j})^N+L^{-\theta/2}(h_{s_i}h_{s_j})^{N/2}.
\end{aligned}\tag{3.6}$$

Optimizing (3.6) with respect to L, we derive

$$\frac{\mathbb{E}[|\Phi_p(Y_{s_i})\Phi_q(Y_{s_j})|K_{s_i}(x,X_{s_i})K_{s_j}(x,X_{s_j})]}{h_{s_i}^{N/2}h_{s_j}^{N/2}}
\trianglelefteq\begin{cases}
\left(h_{s_i}h_{s_j}\right)^{\frac{N\theta}{2(2+\theta)}} & \text{if}\quad \|\Phi_p\|_\infty<\infty \text{ and } \|\Phi_q\|_\infty=\infty\\
\left(h_{s_i}h_{s_j}\right)^{\frac{N\theta}{2(2+\theta)}} & \text{if}\quad \|\Phi_p\|_\infty=\infty \text{ and } \|\Phi_q\|_\infty<\infty\\
\left(h_{s_i}h_{s_j}\right)^{N/2} & \text{if}\quad \|\Phi_p\|_\infty<\infty \text{ and } \|\Phi_q\|_\infty<\infty\\
\left(h_{s_i}h_{s_j}\right)^{\frac{N\theta}{2(4+\theta)}} & \text{if}\quad \|\Phi_p\|_\infty=\infty \text{ and } \|\Phi_q\|_\infty=\infty.
\end{cases}\tag{3.7}$$

Combining (3.5), (3.7) and Lemma 3, we obtain (3.3). The proof of Lemma 4 is complete. □

Lemma 5. *Let $(\Xi_k)_{k\in\mathbb{Z}^d}$ be a family of non negative real numbers.*

If $\sum_{k\in\mathbb{Z}^d}|k|^{\frac{d\ell_2}{\ell_1}}\Xi_k<\infty$ for some positive constants ℓ_1 and ℓ_2 then there exists a sequence $(m_n)_{n\geqslant1}$ of positive integers satisfying

$$\lim_{n\to\infty}m_n=+\infty,\qquad \lim_{n\to\infty}m_n^dh_{s_n}^{\ell_1}=0\quad \textit{and}\quad \lim_{n\to\infty}h_{s_n}^{-\ell_2}\sum_{|k|>m_n}\Xi_k=0.$$

Notice that if $\ell_1 \leqslant N$ and $nh_{s_n}^N \to \infty$ then $m_n^d = o(n)$.

Proof of Lemma 5. Let ℓ_1, ℓ_2 and r be positive constants such that $r > \ell_2/\ell_1$ and let $(m_n)_{n\geqslant 1}$ be the sequence defined for any integer $n \geqslant 1$ by

$$m_n = \max\left\{v_n, \left\lceil h_{s_n}^{-\ell_1/d}\left(\sum_{|k|>v_n}|k|^{\frac{d\ell_2}{\ell_1}}\Xi_k\right)^{\frac{1}{dr}}\right\rceil\right\} \quad \text{and} \quad v_n = \lfloor h_{s_n}^{-\ell_1/(2d)}\rfloor.$$

Since $v_n \to \infty$, we have $m_n \to \infty$ as n goes to infinity. Moreover,

$$m_n^d h_{s_n}^{\ell_1} \trianglelefteq \max\left\{h_{s_n}^{\ell_1/2}, \left(\sum_{|k|>v_n}|k|^{\frac{d\ell_2}{\ell_1}}\Xi_k\right)^{\frac{1}{r}} + h_{s_n}^{\ell_1}\right\} \xrightarrow[n\to\infty]{} 0.$$

Since $v_n \leqslant m_n$, we have

$$m_n^d h_{s_n}^{\ell_1} \geqslant \left(\sum_{|k|>m_n}|k|^{\frac{d\ell_2}{\ell_1}}\Xi_k\right)^{\frac{1}{r}}.$$

Since $\sum_{k\in\mathbb{Z}^d}|k|^{\frac{d\ell_2}{\ell_1}}\Xi_k < \infty$ and $r > \ell_2/\ell_1$, we get

$$h_{s_n}^{-\ell_2}\sum_{|k|>m_n}\Xi_k \leqslant \left(m_n^d h_{s_n}^{\ell_1}\right)^{-\frac{\ell_2}{\ell_1}}\sum_{|k|>m_n}|k|^{\frac{d\ell_2}{\ell_1}}\Xi_k \leqslant \left(\sum_{|k|>m_n}|k|^{\frac{d\ell_2}{\ell_1}}\Xi_k\right)^{1-\frac{\ell_2}{\ell_1 r}} \xrightarrow[n\to\infty]{} 0.$$

The proof of Lemma 5 is complete. □

Lemma 6. *Let $\ell \in \{1, 2\}$ and $\theta > 0$ be fixed such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}] < \infty$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_n}(x, X_0)] \trianglelefteq h_{s_n}^N$ then $\|\Delta_{s_i}^{(\ell)}\|_{2+\theta}^2 \trianglelefteq h_{s_i}^{\frac{-N\theta}{2+\theta}}$ where $\Delta_{s_i}^{(\ell)}$ is given by* (3.2).

Proof of Lemma 6. Let $\theta > 0$ and $\ell \in \{1, 2\}$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}] < \infty$ and let $1 \leqslant i \leqslant n$, we have

$$\|\Delta_{s_i}^{(\ell)}\|_{2+\theta}^2 \leqslant \frac{2\,\|\Phi_\ell(Y_0)K_{s_i}(x, X_0)\|_{2+\theta}^2}{h_{s_i}^N} + \frac{2\left(\mathbb{E}[|\Phi_\ell(Y_0)|K_{s_i}(x, X_0)]\right)^2}{h_{s_i}^N}.$$

Since $\sup_{t\in\mathbb{R}^N}|K(t)| < \infty$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}K_{s_n}(x, X_0)] \trianglelefteq h_{s_n}^N$, we get $\mathbb{E}[|\Phi_\ell(Y_0)K_{s_i}(x, X_0)|^{2+\theta}] \trianglelefteq h_{s_i}^N$. Moreover, using Lemma 3, we have $\mathbb{E}[|\Phi_\ell(Y_0)|K_{s_i}(x, X_0)] \trianglelefteq h_{s_i}^N$ and we obtain $\|\Delta_{s_i}^{(\ell)}\|_{2+\theta}^2 \trianglelefteq h_{s_i}^{-\frac{\theta N}{2+\theta}}$. The proof of Lemma 6 is complete. □

Proposition 3. *Let M be a positive integer and let $x \in \mathbb{R}^N$. If $(X_k)_{k\in\mathbb{Z}^d}$ is of the form* (1.5) *and $\Phi : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $\|\Phi(Y_0)\|_{2+\theta} < \infty$*

for some $\theta \in]0, +\infty]$ *then for any positive integer* n *and any family* $(c_k)_{k\in\Lambda_n}$ *of real numbers and any* $(p, q) \in [2, +\infty[\times]0, +\infty]$ *such that* $p + q \leqslant 2 + \theta$*, we have*

$$\left\|\sum_{i=1}^{n} c_{s_i} W_i\right\|_p \leq 8pM^d |K|_\infty^{\frac{p}{p+q}} C(p,q) \left(\sum_{i=1}^{n} c_{s_i}^2\right)^{\frac{1}{2}} h_{s_n}^{\frac{-q}{p+q}} \sum_{|j|>M} \delta_{j,p}^{\frac{q}{p+q}},$$

where

$$W_i := \Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i})|\,\mathcal{H}_{i,M}], \tag{3.8}$$

$$\begin{aligned} C(p,q) =& 2^{\frac{2p+q}{p+q}} \left\|\Phi(Y_0)\right\|_{p+q} \|\mathrm{K}\|_{\mathrm{Lip}}^{\frac{q}{p+q}} \\ &+ |K|_\infty^{\frac{q}{p+q}} \left\| \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ x\neq y}} \frac{|\Phi(R(x,\eta_0)) - \Phi(R(y,\eta_0))|}{\|x-y\|} \right\|_p \end{aligned}$$

and

$$\mathcal{H}_{i,M} = \sigma\left(\eta_{s_i}, \varepsilon_{s_i-k}\,; |k| \leqslant M\right) \quad \textit{and} \quad \|K\|_{\mathrm{Lip}} = \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ x\neq y}} \frac{|K(x) - K(y)|}{\|x-y\|}.$$

Proof of Proposition 3. Let M be a positive integer and let x in $\mathbb{R}^N$ and $1 \leqslant i \leqslant n$ be fixed. Recall that $Y_{s_i} = R(X_{s_i}, \eta_{s_i})$ and let $W_i = \Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i})|\,\mathcal{H}_{i,M}]$ where $\mathcal{H}_{i,M} = \sigma\left(\eta_{s_i}, \varepsilon_{s_i-k}\,; |k| \leqslant M\right)$. We follow the same lines as in the proof of Proposition 1 in [16]. Let $2 \leqslant p < 2 + \theta$ and denote by H_i the measurable function such that $W_i = \mathrm{H}_i(\mathcal{H}_{i,\infty})$ with $\mathcal{H}_{i,\infty} = \sigma\left(\eta_{s_i}, \varepsilon_{s_i-k}\,; k \in \mathbb{Z}^d\right)$. Let τ be a bijection from $\mathbb{Z}$ to $\mathbb{Z}^d$ and ℓ in $\mathbb{Z}$ be fixed. We define the projection operator P_ℓ by $P_\ell f = \mathbb{E}[f|\mathcal{F}_\ell] - \mathbb{E}[f|\mathcal{F}_{\ell-1}]$ for any integrable function f, where $\mathcal{F}_\ell = \sigma\left(\varepsilon_{\tau(j)}; j \leq \ell\right)$. One can notice that the operator P_ℓ depends on the bijection τ. The proof of the following technical result is postponned to section 5.

Lemma 7. *Almost surely, it holds that* $\mathbb{E}[W_i|\mathcal{F}_{\ell-1}] = \mathbb{E}[H_i(\mathcal{H}_{i,\infty}^{(\ell)})|\mathcal{F}_\ell]$ *with* $\mathcal{H}_{i,\infty}^{(\ell)} = \sigma\left(\eta_{s_i}, \varepsilon'_{\tau(\ell)}, \varepsilon_{s_i-k}\,; k \in \mathbb{Z}^d\backslash\{s_i - \tau(\ell)\}\right)$.

Using Lemma 7, we obtain

$$\|P_\ell W_i\|_p = \|\mathbb{E}[\mathrm{H}_i(\mathcal{H}_{i,\infty})|\mathcal{F}_\ell] - \mathbb{E}[\mathrm{H}_i(\mathcal{H}_{i,\infty}^{(\ell)})|\mathcal{F}_\ell]\|_p \leqslant \|\mathrm{H}_i(\mathcal{H}_{i,\infty}) - \mathrm{H}_i(\mathcal{H}_{i,\infty}^{(\ell)})\|_p. \tag{3.9}$$

Now, denoting $\mathcal{H}_{i,M}^{(\ell)} = \sigma\left(\eta_{s_i}, \varepsilon'_{\tau(\ell)}, \varepsilon_{s_i-k}\,; k \in \mathbb{Z}^d\backslash\{s_i - \tau(\ell)\} \text{ and } |k| \leqslant M\right)$, we have

$$\begin{aligned} H_i(\mathcal{H}_{i,\infty}) &= \Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi(Y_{s_i})K_{s_i}(x, X_{s_i})|\,\mathcal{H}_{i,M} \vee \mathcal{H}_{i,M}^{(\ell)}], \\ H_i(\mathcal{H}_{i,\infty}^{(\ell)}) &= \Phi(Y'_{i,\tau(\ell)})K_{s_i}(x, X'_{i,\tau(\ell)}) - \mathbb{E}[\Phi(Y'_{i,\tau(\ell)})K_{s_i}(x, X'_{i,\tau(\ell)})|\,\mathcal{H}_{i,M}^{(\ell)} \vee \mathcal{H}_{i,M}] \end{aligned}$$

where $X^{'}_{i,\tau(\ell)} = G(\varepsilon^{'}_{\tau(\ell)}, \varepsilon_{s_i-k}\,; k \in \mathbb{Z}^d\backslash\{s_i - \tau(\ell)\})$ and $Y^{'}_{i,\tau(\ell)} = R(X^{'}_{i,\tau(\ell)}, \eta_{s_i})$. So, we derive

$$\|P_\ell W_i\|_p \leqslant 2\|\Phi(Y_{s_i})K_{s_i}(x, X_{s_i}) - \Phi(Y^{'}_{i,\tau(\ell)})K_{s_i}(x, X^{'}_{i,\tau(\ell)})\|_p.$$

So, for any $L > 0$, we get the bound

$$\begin{aligned}\|P_\ell W_i\|_p \leqslant & \frac{2L\,\|\mathrm{K}\|_{\mathrm{Lip}}}{h_{s_i}}\|X_{s_i} - X^{'}_{i,\tau(\ell)}\|_p + \frac{4|\mathrm{K}|_\infty}{L^{q/p}}\,\|\Phi(Y_0)\|_{p+q}^{\frac{p+q}{p}} \\ & + 2|\mathrm{K}|_\infty \left\| \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ x\neq y}} \frac{|\Phi(R(x,\eta_0)) - \Phi(R(y,\eta_0))|}{\|x-y\|}\right\|_p \|X_{s_i} - X^{'}_{i,\tau(\ell)}\|_p\end{aligned}$$

where

$$\begin{aligned}\|X_{s_i} - & X^{'}_{i,\tau(\ell)}\|_p \\ & = \|G(\varepsilon_{s_i-k}; k \in \mathbb{Z}^d) - G(\varepsilon^{'}_{\tau(\ell)}, \varepsilon_{s_i-k};\, k \in \mathbb{Z}^d\backslash\{s_i - \tau(\ell)\})\|_p \\ & = \|G(\varepsilon_{s_i-\tau(\ell)-k}; k \in \mathbb{Z}^d) - G(\varepsilon^{'}_0, \varepsilon_{s_i-\tau(\ell)-k};\, k \in \mathbb{Z}^d\backslash\{s_i - \tau(\ell)\})\|_p \\ & = \|X_{s_i-\tau(\ell)} - X^{*}_{s_i-\tau(\ell)}\|_p \\ & = \delta_{s_i-\tau(\ell),p}.\end{aligned}$$

Optimizing this last inequality in L, we get the following bound

$$\|P_\ell W_i\|_p \leqslant 2|\mathrm{K}|_\infty^{\frac{p}{p+q}} C(p,q) h_{s_i}^{\frac{-q}{p+q}} \delta_{s_i-\tau(\ell),p}^{\frac{q}{p+q}}, \tag{3.10}$$

where

$$\begin{aligned}C(p,q) = & 2^{\frac{2p+q}{p+q}}\,\|\Phi(Y_0)\|_{p+q}\,\|\mathrm{K}\|_{\mathrm{Lip}}^{\frac{q}{p+q}} \\ & + |\mathrm{K}|_\infty^{\frac{q}{p+q}} \left\| \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ x\neq y}} \frac{|\Phi(R(x,\eta_0)) - \Phi(R(y,\eta_0))|}{\|x-y\|}\right\|_p.\end{aligned}$$

Now, we are going to obtain another bound for $\|P_\ell W_i\|_p$. Let $\ell \geqslant 0$ and $i \geqslant 1$ be two integers. We denote by $\Gamma_{i,\ell}$ the set of all k in $\mathbb{Z}^d$ such that $|s_i - k| = \ell$ and we define

$$a_\ell := \sum_{j=0}^{\ell} |\Gamma_{i,j}| = (2\ell+1)^d.$$

On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $u = (u_1, \dots, u_d)$ and $v = (v_1, \dots, v_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $u <_{\mathrm{lex}} v$ means that either $u_1 < v_1$ or for some k in $\{2, \dots, d\}$, $u_k < v_k$ and $u_\ell = v_\ell$ for $1 \leqslant \ell < k$. We consider the bijection $\tau_i :]0, +\infty[\cap\mathbb{Z} \to \mathbb{Z}^d$ defined by $\tau_i(1) = s_i$, $\tau_i(u) \in \Gamma_{i,\ell}$ if $a_{\ell-1} < u \leqslant a_\ell$ and $\ell > 0$, and $\tau_i(u) <_{\mathrm{lex}} \tau_i(v)$ if

$a_{\ell-1} < u < v \leqslant a_\ell$ and $\ell > 0$. Let $\mathcal{G}_{i,M} = \sigma\left(\eta_{s_i}, \varepsilon_{\tau_i(j)}\,; 1 \leqslant j \leqslant M\right)$ and recall that $\mathcal{H}_{i,M} = \sigma\left(\eta_{s_i}, \varepsilon_{s_i-k}\,; |k| \leqslant M\right)$. Since $1 \leq j \leq a_M$ if and only if $|s_i - \tau_i(j)| \leq M$, we have $\mathcal{G}_{i,a_M} = \mathcal{H}_{i,M}$. Consequently,

$$W_i = \sum_{\ell > a_M} D_{i,\ell}$$

where $D_{i,\ell} = \mathbb{E}[\Phi(Y_{s_i})\mathrm{K}_{s_i}(x, X_{s_i})|\mathcal{G}_{i,\ell}] - \mathbb{E}[\Phi(Y_{s_i})\mathrm{K}_{s_i}(x, X_{s_i})|\mathcal{G}_{i,\ell-1}]$.

Given that $(D_{i,\ell})_{\ell\geqslant 1}$ is a martingale difference sequence with respect to the filtration $(\mathcal{G}_{i,\ell})_{\ell\geqslant 1}$, we apply Burkholder's inequality ([10], remark 6, page 85) and we obtain

$$\|W_i\|_p \leqslant \left(2p \sum_{\ell > a_M} \|D_{i,\ell}\|_p^2\right)^{1/2}. \tag{3.11}$$

Since $X^{'}_{i,\tau_i(\ell)} = G(\varepsilon^{'}_{\tau_i(\ell)}, \varepsilon_{s_i-k}\,; k \in \mathbb{Z}^d\backslash\{s_i - \tau_i(\ell)\})$ and $Y^{'}_{i,\tau_i(\ell)} = R(X^{'}_{i,\tau_i(\ell)}, \eta_{s_i})$, we have $\mathbb{E}[\Phi(Y_{s_i})\mathrm{K}_{s_i}(x, X_{s_i})|\mathcal{G}_{i,\ell-1}] = [\Phi(Y^{'}_{i,\tau_i(\ell)})\mathrm{K}_{s_i}(x, X^{'}_{i,\tau_i(\ell)})|\mathcal{G}_{i,\ell}]$ and

$$\|D_{i,\ell}\|_p \leqslant \|\Phi(Y_{s_i})\mathrm{K}_{s_i}(x, X_{s_i}) - \Phi(Y^{'}_{i,\tau_i(\ell)})\mathrm{K}_{s_i}(x, X^{'}_{i,\tau_i(\ell)})\|_p.$$

Arguing as before, for any $L > 0$, we derive

$$\begin{aligned}\|D_{i,\ell}\|_p \leqslant & \frac{L\,\|\mathrm{K}\|_{\mathrm{Lip}}}{h_{s_i}}\|X_{s_i} - X^{'}_{i,\tau_i(\ell)}\|_p + \frac{2|\mathrm{K}|_\infty}{L^{q/p}}\,\|\Phi(Y_0)\|_{p+q}^{\frac{p+q}{p}} \\ & + |\mathrm{K}|_\infty \left\| \sup_{\substack{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N \\ x\neq y}} \frac{|\Phi(R(x,\eta_0)) - \Phi(R(y,\eta_0))|}{\|x-y\|}\right\|_p \|X_{s_i} - X^{'}_{i,\tau_i(\ell)}\|_p\end{aligned}$$

with $\|X_{s_i} - X^{'}_{i,\tau_i(\ell)}\|_p = \delta_{s_i-\tau_i(\ell),p}$. Optimizing this last inequality on L and noting that $s_i - \tau_i(\ell) = -\tau_0(\ell)$, we obtain

$$\|D_{i,\ell}\|_p \leqslant |K|_\infty^{\frac{p}{p+q}} C(p,q) h_{s_i}^{-\frac{q}{p+q}} \delta_{-\tau_0(\ell),p}^{\frac{q}{p+q}}.$$

Consequently, we get

$$\begin{aligned}\|P_\ell W_i\|_p \leqslant 2\|W_i\|_p &\leqslant 2\sqrt{2p}|K|_\infty^{\frac{p}{p+q}} C(p,q) h_{s_i}^{-\frac{q}{p+q}} \sum_{\ell > a_M} \delta_{-\tau_0(\ell),p}^{\frac{q}{p+q}} \\ &\leqslant 2\sqrt{2p}|K|_\infty^{\frac{p}{p+q}} C(p,q) h_{s_n}^{-\frac{q}{p+q}} \sum_{|k|>M} \delta_{k,p}^{\frac{q}{p+q}}.\end{aligned} \tag{3.12}$$

Since $(\sum_{i=1}^n c_{s_i} P_\ell W_i)_{\ell\in\mathbb{Z}}$ is a martingale difference sequence with respect to the filtration $(\mathcal{F}_\ell)_{\ell\in\mathbb{Z}}$, the Burkholder inequality (see [10], remark 6, page 85) implies

$$\left\|\sum_{i=1}^n c_{s_i} W_i\right\|_p \leq \left(2p \sum_{\ell\in\mathbb{Z}} \left\|\sum_{i=1}^n c_{s_i} P_\ell W_i\right\|_p^2\right)^{\frac{1}{2}}$$

$$\leq \left(2p\sum_{\ell\in\mathbb{Z}}\left(\sum_{i=1}^{n}|c_{s_i}|\,\|P_\ell W_i\|_p\right)^2\right)^{\frac{1}{2}}. \tag{3.13}$$

Moreover, by the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n}|c_{s_i}|\,\|P_\ell W_i\|_p\right)^2 \leq \sum_{i=1}^{n}c_{s_i}^2\,\|P_\ell W_i\|_p \times \sum_{j=1}^{n}\|P_\ell W_j\|_p. \tag{3.14}$$

Now, keeping in mind that P_ℓ is defined from the bijection τ and using (3.10) and (3.12), we have

$$\begin{aligned}
\sup_{\ell\in\mathbb{Z}}\sum_{i=1}^{n}\|P_\ell W_i\|_p &\leqslant \sup_{\ell\in\mathbb{Z}}\sum_{\substack{1\leqslant i\leqslant n\\ |s_i-\tau(\ell)|\leqslant M}}\|P_\ell W_i\|_p + \sup_{\ell\in\mathbb{Z}}\sum_{\substack{1\leqslant i\leqslant n\\ |s_i-\tau(\ell)|>M}}\|P_\ell W_i\|_p\\
&\leqslant 2\sqrt{2p}M^d|K|_\infty^{\frac{p}{p+q}}C(p,q)h_{s_n}^{-\frac{q}{p+q}}\sum_{|k|>M}\delta_{k,p}^{\frac{q}{p+q}}\\
&\quad + 2|K|_\infty^{\frac{p}{p+q}}C(p,q)h_{s_n}^{-\frac{q}{p+q}}\sum_{\substack{1\leqslant i\leqslant n\\ |s_i-\tau(\ell)|>M}}\delta_{s_i-\tau(\ell),p}^{\frac{q}{p+q}}\\
&\leqslant 2\left(M^d\sqrt{2p}+1\right)|K|_\infty^{\frac{p}{p+q}}C(p,q)h_{s_n}^{-\frac{q}{p+q}}\sum_{|k|>M}\delta_{k,p}^{\frac{q}{p+q}}.
\end{aligned}$$

Similarly, we have also

$$\begin{aligned}
\sup_{1\leqslant i\leqslant n}\sum_{\ell\in\mathbb{Z}}\|P_\ell W_i\|_p &\leqslant \sup_{1\leqslant i\leqslant n}\sum_{\substack{\ell\in\mathbb{Z}\\ |s_i-\tau(\ell)|\leqslant M}}\|P_\ell W_i\|_p + \sup_{1\leqslant i\leqslant n}\sum_{\substack{\ell\in\mathbb{Z}\\ |s_i-\tau(\ell)|>M}}\|P_\ell W_i\|_p\\
&\leqslant 2\sqrt{2p}M^d|K|_\infty^{\frac{p}{p+q}}C(p,q)h_{s_n}^{-\frac{q}{p+q}}\sum_{|k|>M}\delta_{k,p}^{\frac{q}{p+q}}\\
&\quad + 2|K|_\infty^{\frac{p}{p+q}}C(p,q)h_{s_i}^{-\frac{q}{p+q}}\sum_{\substack{\ell\in\mathbb{Z}\\ |s_i-\tau(\ell)|>M}}\delta_{s_i-\tau(\ell),p}^{\frac{q}{p+q}}\\
&\leqslant 2\left(M^d\sqrt{2p}+1\right)|K|_\infty^{\frac{p}{p+q}}C(p,q)h_{s_n}^{-\frac{q}{p+q}}\sum_{|k|>M}\delta_{k,p}^{\frac{q}{p+q}}.
\end{aligned}$$

Combining (3.13) and (3.14) with the last two bound above, we get

$$\left\|\sum_{i=1}^{n}c_{s_i}W_i\right\|_p \leq \sqrt{2p(2(M^d\sqrt{2p}+1))^2}|\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)\left(\sum_{i=1}^{n}c_{s_i}^2\right)^{\frac{1}{2}}h_{s_n}^{\frac{-q}{p+q}}\sum_{|k|>M}\delta_{k,p}^{\frac{q}{p+q}}.$$

Noting that $\sqrt{2p}+1\leqslant 2\sqrt{2p}$, we obtain

$$\left\|\sum_{i=1}^{n}c_{s_i}W_i\right\|_p \leq 8pM^d|\mathrm{K}|_\infty^{\frac{p}{p+q}}C(p,q)\left(\sum_{i=1}^{n}c_{s_i}^2\right)^{\frac{1}{2}}h_{s_n}^{\frac{-q}{p+q}}\sum_{|k|>M}\delta_{k,p}^{\frac{q}{p+q}}.$$

The proof of Proposition 3 is complete. □

4. Proofs of the main results

Now, we denote by $\mathbb{V}(Z)$ the variance of any square-integrable $\mathbb{R}$-valued random variable Z and we consider a sequence $(m_n)_{n\geqslant 1}$ of positive integers. For any $x \in \mathbb{R}$ and any $\ell \in \{1,2\}$, denote

$$f_n^{(\ell)}(x) = f_{n,\Phi_\ell}(x) = (\sum_{i=1}^n w_{s_i})^{-1} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N} \Phi_\ell(Y_{s_i}) K_{s_i}(x, X_{s_i})$$
$$\text{and} \quad \overline{f}_n^{(\ell)}(x) = \mathbb{E}[f_n^{(\ell)}(x)|\mathcal{H}_{i,m_n}] \tag{4.1}$$

where $\Phi_\ell : \mathbb{R} \to \mathbb{R}$ is a measurable function and $\mathcal{H}_{i,m_n} = \sigma(\eta_{s_i}, \varepsilon_{s_i-k}; |k| \leqslant m_n)$. First, we note that Proposition 1 is a particular case of the following result.

Proposition 4. *Assume that (H1) and (H3)(iii) hold. Let $(p,q) \in \{1,2\}^2$ and $x \in \mathbb{R}^N$ be fixed. Let $\theta > 0$ such that $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta} K_{s_n}(x, X_0)] \trianglelefteq h_{s_n}^N$ and $\mathbb{E}[|\Phi_\ell(Y_0)|^{2+\theta}] < \infty$ for any $\ell \in \{p,q\}$. Assume also that the function $u \mapsto \mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)|X_0 = u]$ is continuous. If there exists $\tau \in]1-\gamma, 1]$ such that $h_{s_n}^{N(1-\tau)} \sum_{i=1}^n w_{s_i}^2 \trianglelefteq n w_{s_n}^2$ and one of the following conditions is satisfied:*

(i) $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing and $\sum_{k\in\mathbb{Z}^d} |k|^{\frac{d\tilde{\gamma}}{\gamma+\tau-1}} \alpha_{1,1}^{\tilde{\gamma}}(|k|) < \infty$.

(ii) $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5) and $\sum_{k\in\mathbb{Z}^d} |k|^{\frac{d(N+2\tilde{\gamma}+2N(\gamma+\tau-1))}{2N(\gamma+\tau-1)}} \delta_{k,2}^{\tilde{\gamma}} < \infty$.

where $\tilde{\gamma}$ is defined by

$$\tilde{\gamma} = \begin{cases} 1 & \textit{if} \ \ \max\left(\|\Phi_p\|_\infty, \|\Phi_q\|_\infty\right) < +\infty \\ \frac{\theta}{2+\theta} & \textit{else} \end{cases} \tag{4.2}$$

then

$$\lim_{n\to\infty} n h_{s_n}^N \operatorname{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)]$$
$$= \beta_{0,1}^{-2} \beta_{-N,2} \mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)|X_0 = x] f(x) \int_{\mathbb{R}^N} K^2(t) dt.$$

Proof of Proposition 4. Let $x \in \mathbb{R}^N$ and let $(p,q) \in \{1,2\}^2$ be fixed. Using the notations (3.2) and (4.1), for $\ell \in \{p,q\}$, we have

$$f_n^{(\ell)}(x) - \mathbb{E}\left[f_n^{(\ell)}(x)\right] = (\sum_{i=1}^n w_{s_i})^{-1} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N/2} \Delta_{s_i}^{(\ell)}.$$

Keeping in mind that $A_{n,0,1} = (n w_{s_n})^{-1} \sum_{i=1}^n w_{s_i}$, we get

$$n h_{s_n}^N \operatorname{Cov}\left[f_n^{(p)}(x), f_n^{(q)}(x)\right]$$

$$= nh_{s_n}^N(\sum_{i=1}^n w_{s_i})^{-2}\mathbb{E}\left[\left(\sum_{i=1}^n w_{s_i}h_{s_i}^{-N/2}\Delta_{s_i}^{(p)}\right)\left(\sum_{i=1}^n w_{s_i}h_{s_i}^{-N/2}\Delta_{s_i}^{(q)}\right)\right]$$

$$= \frac{A_{n,0,1}^{-2}h_{s_n}^N}{nw_{s_n}^2}\left(\sum_{i=1}^n w_{s_i}^2 h_{s_i}^{-N}\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_i}^{(q)}] + \sum_{\substack{i,j=1\\i\neq j}}^n w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]\right).$$

So, we obtain

$$\left|nh_{s_n}^N \operatorname{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)] - \frac{h_{s_n}^N A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{i=1}^n w_{s_i}^2 h_{s_i}^{-N}\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_i}^{(q)}]\right|$$
$$\leqslant \frac{A_{n,0,1}^{-2}h_{s_n}^N}{nw_{s_n}^2}\sum_{\substack{i,j=1\\i\neq j}}^n w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]|. \tag{4.3}$$

Moreover, for any $1 \leqslant i \leqslant n$,

$$\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_i}^{(q)}] = h_{s_i}^{-N}\left(\mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)K_{s_i}^2(x,X_0)]\right.$$
$$\left.-\mathbb{E}[\Phi_p(Y_0)K_{s_i}(x,X_0)]\mathbb{E}[\Phi_q(Y_0)K_{s_i}(x,X_0)]\right).$$

Using Lemma 2 and Lemma 3, we derive

$$\lim_{i\to\infty}|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_i}^{(q)}] - \mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)|X_0 = x]f(x)\int_{\mathbb{R}^N}K^2(t)dt| = 0.$$

So, using Lemma 1 and (H1), we derive

$$\lim_{n\to\infty}\frac{h_{s_n}^N A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{i=1}^n h_{s_i}^{-N}w_{s_i}^2\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_i}^{(q)}]$$
$$= \beta_{0,1}^{-2}\beta_{-N,2}\mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)|X_0 = x]f(x)\int_{\mathbb{R}^N}K^2(t)dt. \tag{4.4}$$

Now, we are going to prove that

$$\lim_{n\to\infty}\frac{A_{n,0,1}^{-2}h_{s_n}^N}{nw_{s_n}^2}\sum_{\substack{i,j=1\\i\neq j}}^n w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]| = 0. \tag{4.5}$$

Using Lemma 4, we have $|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]| \trianglelefteq (h_{s_i}h_{s_j})^{\frac{N\gamma}{2}}$ for any $i \neq j$ where γ is defined by (3.4). Moreover, since $\sum_{k\in\mathbb{Z}^d}|k|^{\frac{d\tilde{\gamma}}{\gamma+\tau-1}}\alpha_{1,1}^{\tilde{\gamma}}(|k|) < \infty$, using Lemma 5, there exists a sequence $(m_n)_{n\geqslant 1}$ of positive integers such that

$$\lim_{n\to\infty} m_n^d h_{s_n}^{N(\gamma+\tau-1)} = \lim_{n\to\infty} h_{s_n}^{-N\tilde{\gamma}}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}\alpha_{1,1}^{\tilde{\gamma}}(|k|) = 0. \tag{4.6}$$

So, we have

$$\frac{h_{s_n}^N}{nw_{s_n}^2}\sum_{\substack{i,j=1\\ i\neq j}}^n w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]|$$
$$= \frac{2h_{s_n}^N}{nw_{s_n}^2}\sum_{1\leqslant i<j\leqslant n} w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]| \trianglelefteq E_{1,n}+E_{2,n}$$

where

$$E_{1,n} = \frac{h_{s_n}^N}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\ |s_i-s_j|<m_n}} w_{s_i}w_{s_j}\left(h_{s_i}h_{s_j}\right)^{-N(1-\gamma)/2}$$

and

$$E_{2,n} = \frac{h_{s_n}^N}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\ |s_i-s_j|\geqslant m_n}} w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]|.$$

Since $\gamma\leqslant 1$, using the inequality $2ab\leqslant a^2+b^2$, we have

$$E_{1,n} \leqslant \frac{h_{s_n}^{N\gamma}}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\ |s_i-s_j|<m_n}} w_{s_i}w_{s_j} \leqslant \frac{h_{s_n}^{N\gamma}}{2nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\ |s_i-s_j|<m_n}}\left(w_{s_i}^2+w_{s_j}^2\right)$$
$$\leqslant \frac{m_n^d h_{s_n}^{N\gamma}}{nw_{s_n}^2}\sum_{i=1}^n w_{s_i}^2$$
$$= m_n^d h_{s_n}^{N(\gamma+\tau-1)}\times\frac{h_{s_n}^{N(1-\tau)}}{nw_{s_n}^2}\sum_{i=1}^n w_{s_i}^2.$$

Using (4.6) and keeping in mind that $h_{s_n}^{N(1-\tau)}\sum_{i=1}^n w_{s_i}^2 \trianglelefteq nw_{s_n}^2$, we get $\lim_{n\to\infty}E_{1,n}=0$. Now, we are going to control the term $E_{2,n}$ when $(X_k)_{k\in\mathbb{Z}^d}$ is assumed to be strongly mixing. Using Rio's inequality ([32], Theorem 1.1), we have for any $1\leqslant i<j\leqslant n$,

$$|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]| \leqslant 2\int_0^{2\alpha_{1,1}(|s_i-s_j|)} Q_{\Delta_{s_i}^{(p)}}(u)Q_{\Delta_{s_j}^{(q)}}(u)du$$

where $Q_{\Delta_{s_\ell}^{(p)}}(u) = \inf\{\varepsilon>0\,|\,\mathbb{P}(|\Delta_{s_\ell}^{(p)}|>\varepsilon)\leqslant u\}$ for any $u\in[0,1]$ and any $\ell\in\{i,j\}$.

First, we assume that $\|\Phi_p\|_\infty=\infty$ or $\|\Phi_q\|_\infty=\infty$. In this case, $\tilde{\gamma}=\theta/(2+\theta)$. Using Lemma 6, we have $\|\Delta_{s_r}^{(\ell)}\|_{2+\theta}\trianglelefteq h_{s_r}^{\frac{-N\tilde{\gamma}}{2}}$ for any $\ell\in\{p,q\}$ and any $1\leqslant r\leqslant n$. So, we derive $Q_{\Delta_{s_r}^{(\ell)}}(u)\trianglelefteq u^{-\frac{1}{2+\theta}}h_{s_r}^{\frac{-N\tilde{\gamma}}{2}}$ and

$$|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]| \trianglelefteq 2\left(h_{s_i}h_{s_j}\right)^{\frac{-N\tilde{\gamma}}{2}}\int_0^{2\alpha_{1,1}(|s_i-s_j|)} u^{\frac{-2}{2+\theta}}du$$

$$\trianglelefteq \left(h_{s_i}h_{s_j}\right)^{\frac{-N\tilde{\gamma}}{2}} \alpha_{1,1}^{\tilde{\gamma}}(|s_i - s_j|).$$

Consequently, we get

$$E_{2,n} \trianglelefteq \frac{h_{s_n}^N}{nw_{s_n}^2} \sum_{\substack{1\leqslant i<j\leqslant n\\ |s_i-s_j|>m_n}} w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{\frac{-N(\tilde{\gamma}+1)}{2}} \alpha_{1,1}^{\tilde{\gamma}}(|s_i - s_j|).$$

Using again the inequality $2ab \leqslant a^2 + b^2$, we derive

$$\begin{aligned} E_{2,n} &\trianglelefteq \frac{h_{s_n}^N}{nw_{s_n}^2} \sum_{\substack{1\leqslant i<j\leqslant n\\ |s_i-s_j|>m_n}} \left(w_{s_i}^2 h_{s_i}^{-N(\tilde{\gamma}+1)} + w_{s_j}^2 h_{s_j}^{-N(\tilde{\gamma}+1)}\right) \alpha_{1,1}^{\tilde{\gamma}}(|s_i - s_j|) \\ &\trianglelefteq \left(\frac{h_{s_n}^N}{nw_{s_n}^2} \sum_{i=1}^n \frac{w_{s_i}^2}{h_{s_i}^N}\right) h_{s_n}^{-N\tilde{\gamma}} \sum_{\substack{k\in\mathbb{Z}^d\\ |k|>m_n}} \alpha_{1,1}^{\tilde{\gamma}}(|k|). \end{aligned}$$

Using (H1) and (4.6), we get

$$E_{2,n} \trianglelefteq h_{s_n}^{-N\tilde{\gamma}} \sum_{\substack{k\in\mathbb{Z}^d\\ |k|>m_n}} \alpha_{1,1}^{\tilde{\gamma}}(|k|) \xrightarrow[n\to\infty]{} 0 \tag{4.7}$$

and finally, we obtain (4.5). Now, we deal with the case $\|\Phi_p\|_\infty < \infty$ and $\|\Phi_q\|_\infty < \infty$. In this case, we have $\tilde{\gamma} = 1$. So, noting that $|\Delta_{s_r}^{(\ell)}| \trianglelefteq h_{s_r}^{-N\tilde{\gamma}/2}$ for any $\ell \in \{p, q\}$ and any $1 \leqslant r \leqslant n$, we derive $Q_{\Delta_{s_r}^{(\ell)}}(u) \trianglelefteq h_{s_r}^{-N\tilde{\gamma}/2}$ and $|\mathbb{E}[\Delta_{s_i}^{(p)}\Delta_{s_j}^{(q)}]| \trianglelefteq \left(h_{s_i}h_{s_j}\right)^{-N\tilde{\gamma}/2} \alpha_{1,1}^{\tilde{\gamma}}(|s_i - s_j|)$. Arguing as before, we obtain (4.7) and consequently (4.5) holds.

Finally, combining (4.3), (4.4) and (4.5), we obtain

$$\begin{aligned} \lim_{n\to\infty} nh_{s_n}^N \operatorname{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)] \\ &= \frac{\beta_{-N,2}}{\beta_{0,1}^2} \mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)|X_0 = x] f(x) \int_{\mathbb{R}^N} K^2(t)dt. \end{aligned}$$

Now, we assume that $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5). Since $\sum_{k\in\mathbb{Z}^d} |k|^{\frac{d(N+2\tilde{\gamma}+2N(\gamma+\tau-1))}{2N(\gamma+\tau-1)}} \delta_{k,2}^{\tilde{\gamma}} < \infty$ implies $\sum_{k\in\mathbb{Z}^d} |k|^{\frac{d(N+2\tilde{\gamma})}{2N(\gamma+\tau-1)}} |k|^d \delta_{k,2}^{\gamma} < \infty$ and using Lemma 5, there exists a sequence $(m_n)_{n\geqslant 1}$ of positive integers such that

$$\lim_{n\to\infty} m_n^d h_{s_n}^{N(\gamma+\tau-1)} = \lim_{n\to\infty} h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k\in\mathbb{Z}^d\\ |k|>m_n}} |k|^d \delta_{k,2}^{\tilde{\gamma}} = 0.$$

Keeping in mind (3.2) and (4.1), we have $nh_{s_n}^N \operatorname{Cov}[f_n^{(p)}(x), f_n^{(q)}(x)] = C_{1,n} + C_{2,n} + C_{3,n} + C_{4,n}$ where

$$C_{1,n} = nh_{s_n}^N \operatorname{Cov}[f_n^{(p)}(x) - \overline{f}_n^{(p)}(x), f_n^{(q)}(x) - \overline{f}_n^{(q)}(x)]$$

$$C_{2,n} = nh_{s_n}^N \mathrm{Cov}[f_n^{(p)}(x) - \overline{f}_n^{(p)}(x), \overline{f}_n^{(q)}(x)]$$
$$C_{3,n} = nh_{s_n}^N \mathrm{Cov}[\overline{f}_n^{(p)}(x), f_n^{(q)}(x) - \overline{f}_n^{(q)}(x)]$$
$$C_{4,n} = nh_{s_n}^N \mathrm{Cov}[\overline{f}_n^{(p)}(x), \overline{f}_n^{(q)}(x)].$$

First, we assume that $\|\Phi_p\|_\infty = \infty$ or $\|\Phi_q\|_\infty = \infty$. In this case, $\tilde{\gamma} = \theta/(2+\theta)$. Moreover, we have

$$|C_{1,n}| \leqslant \left\| \frac{h_{s_n}^{N/2} A_{n,0,1}^{-1}}{\sqrt{n} w_{s_n}} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N/2} (\Delta_{s_i}^{(p)} - \overline{\Delta}_{s_i}^{(p)}) \right\|_2 \times \left\| \frac{h_{s_n}^{N/2} A_{n,0,1}^{-1}}{\sqrt{n} w_{s_n}} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N/2} (\Delta_{s_i}^{(q)} - \overline{\Delta}_{s_i}^{(q)}) \right\|_2 .$$

Using (H1) and Proposition 3, we obtain for any $\ell \in \{p,q\}$,

$$\left\| \frac{h_{s_n}^{N/2} A_{n,0,1}^{-1}}{\sqrt{n} w_{s_n}} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N/2} (\Delta_{s_i}^{(\ell)} - \overline{\Delta}_{s_i}^{(\ell)}) \right\|_2 \trianglelefteq m_n^d \left(\frac{h_{s_n}^N}{n w_{s_n}^2} \sum_{i=1}^n \frac{w_{s_i}^2}{h_{s_i}^N} \right)^{1/2} h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > m_n}} \delta_{k,2}^{\tilde{\gamma}} \trianglelefteq h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > m_n}} |k|^d \delta_{k,2}^{\tilde{\gamma}}. \tag{4.8}$$

So, we derive

$$|C_{1,n}| \trianglelefteq \left(h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > m_n}} |k|^d \delta_{k,2}^{\tilde{\gamma}} \right)^2 \xrightarrow[n\to\infty]{} 0.$$

Similarly, we have

$$|C_{2,n}| \leqslant \left\| \frac{h_{s_n}^{N/2} A_{n,0,1}^{-1}}{\sqrt{n} w_{s_n}} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N/2} (\Delta_{s_i}^{(p)} - \overline{\Delta}_{s_i}^{(p)}) \right\|_2 \times \left\| \frac{h_{s_n}^{N/2} A_{n,0,1}^{-1}}{\sqrt{n} w_{s_n}} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N/2} \overline{\Delta}_{s_i}^{(q)} \right\|_2 .$$

From (4.8), we know that

$$\left\| \frac{h_{s_n}^{N/2} A_{n,0,1}^{-1}}{\sqrt{n} w_{s_n}} \sum_{i=1}^n w_{s_i} h_{s_i}^{-N/2} (\Delta_{s_i}^{(p)} - \overline{\Delta}_{s_i}^{(p)}) \right\|_2 \trianglelefteq h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > m_n}} |k|^d \delta_{k,2}^{\tilde{\gamma}}.$$

Moreover, since $\overline{\Delta}_{s_i}^{(q)}$ and $\overline{\Delta}_{s_j}^{(q)}$ are independent as soon as $|s_i - s_j| > 2m_n$, we have

$$\begin{aligned}\left\|\frac{h_{s_n}^{N/2}A_{n,0,1}^{-1}}{\sqrt{n}w_{s_n}}\sum_{i=1}^{n} w_{s_i}h_{s_i}^{-N/2}\overline{\Delta}_{s_i}^{(q)}\right\|_2^2 &= \frac{h_{s_n}^{N}A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N}\|\overline{\Delta}_{s_i}^{(q)}\|_2^2 \\ &\quad + \frac{2h_{s_n}^{N}A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\|s_i-s_j|\leqslant 2m_n}} w_{s_i}w_{s_j}h_{s_i}^{-N/2}h_{s_j}^{-N/2}\mathbb{E}[\overline{\Delta}_{s_i}^{(q)}\overline{\Delta}_{s_j}^{(q)}]\end{aligned}$$

and, keeping in mind $\|\Delta_{s_i}^{(q)}\|_2 \trianglelefteq 1$ (see Lemma 3) and (H1), we have also

$$\frac{h_{s_n}^{N}A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N}\|\overline{\Delta}_{s_i}^{(q)}\|_2^2 \leqslant \frac{h_{s_n}^{N}A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N}\|\Delta_{s_i}^{(q)}\|_2^2 \trianglelefteq 1.$$

Denoting $W_i^{(\ell)} = \Phi_\ell(Y_{s_i})K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi_\ell(Y_{s_i})K_{s_i}(x, X_{s_i})|\mathcal{H}_{i,m_n}]$ where $\mathcal{H}_{i,m_n} = \sigma(\eta_{s_i}, \varepsilon_{s_i-k}; |k| \leqslant m_n)$ for any $1 \leqslant i \leqslant n$ and any $\ell \in \{p, q\}$ and noting that $\Delta_{s_i}^{(\ell)} - \overline{\Delta}_{s_i}^{(\ell)} = h_{s_i}^{-N/2}W_i^{(\ell)}$, we have

$$\begin{aligned}|\mathbb{E}[\overline{\Delta}_{s_i}^{(q)}\overline{\Delta}_{s_j}^{(q)}] - \mathbb{E}[\Delta_{s_i}^{(q)}\Delta_{s_j}^{(q)}]| &\leqslant h_{s_j}^{-N/2}\|\Delta_{s_i}^{(q)}\|_2\|W_j^{(q)}\|_2 + h_{s_i}^{-N/2}\|\Delta_{s_j}^{(q)}\|_2\|W_i^{(q)}\|_2 \\ &\trianglelefteq h_{s_n}^{-N/2}\left(\|W_j^{(q)}\|_2 + \|W_i^{(q)}\|_2\right).\end{aligned}$$

Using (3.12), we obtain

$$|\mathbb{E}[\overline{\Delta}_{s_i}^{(q)}\overline{\Delta}_{s_j}^{(q)}] - \mathbb{E}[\Delta_{s_i}^{(q)}\Delta_{s_j}^{(q)}]| \trianglelefteq h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}} \delta_{k,2}^{\tilde{\gamma}}. \tag{4.9}$$

Consequently, using Lemma 4, we obtain for any $i \neq j$,

$$|\mathbb{E}[\overline{\Delta}_{s_i}^{(q)}\overline{\Delta}_{s_j}^{(q)}]| \trianglelefteq \left(h_{s_i}h_{s_j}\right)^{\frac{N\gamma}{2}} + h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}} \delta_{k,2}^{\tilde{\gamma}}$$

where γ is defined by (3.4). Since $\gamma \leqslant 1$ and using the inequality $2ab \leqslant a^2 + b^2$, we derive

$$\begin{aligned}&\frac{h_{s_n}^{N}A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\|s_i-s_j|\leqslant 2m_n}} w_{s_i}w_{s_j}h_{s_i}^{-N/2}h_{s_j}^{-N/2}|\mathbb{E}[\overline{\Delta}_{s_i}^{(q)}\overline{\Delta}_{s_j}^{(q)}]| \\ &\leqslant \frac{h_{s_n}^{N}A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\|s_i-s_j|\leqslant 2m_n}} w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}\end{aligned}$$

$$\times\left(\left(h_{s_i}h_{s_j}\right)^{N\gamma/2}+h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}\delta_{k,2}^{\tilde{\gamma}}\right)$$

$$\leqslant\frac{h_{s_n}^N A_{n,0,1}^{-2}}{2nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\|s_i-s_j|\leqslant 2m_n}}\frac{w_{s_i}^2+w_{s_j}^2}{h_{s_n}^{N(1-\gamma)}}$$

$$+\frac{h_{s_n}^N A_{n,0,1}^{-2}}{2nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\|s_i-s_j|\leqslant 2m_n}}\left(w_{s_i}^2h_{s_i}^{-N}+w_{s_j}^2h_{s_j}^{-N}\right)h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}\delta_{k,2}^{\tilde{\gamma}}.$$

Using (H1) and keeping in mind that $h_{s_n}^{N(1-\tau)}\sum_{i=1}^n w_{s_i}^2\trianglelefteq nw_{s_n}^2$, we get

$$\frac{h_{s_n}^N A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\|s_i-s_j|\leqslant 2m_n}}w_{s_i}w_{s_j}h_{s_i}^{-N/2}h_{s_j}^{-N/2}|\mathbb{E}[\overline{\Delta}_{s_i}^{(q)}\overline{\Delta}_{s_j}^{(q)}]|$$

$$\trianglelefteq m_n^d h_{s_n}^{N(\gamma+\tau-1)}+h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\tilde{\gamma}}\xrightarrow[n\to\infty]{}0. \tag{4.10}$$

Consequently, we derive

$$\left\|\frac{h_{s_n}^{N/2}A_{n,0,1}^{-1}}{\sqrt{n}w_{s_n}}\sum_{i=1}^n w_{s_i}h_{s_i}^{-N/2}\overline{\Delta}_{s_i}^{(q)}\right\|_2\trianglelefteq 1$$

and

$$|C_{2,n}|\trianglelefteq h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\tilde{\gamma}}\xrightarrow[n\to\infty]{}0.$$

Similarly, one can notice that

$$|C_{3,n}|\trianglelefteq h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\tilde{\gamma}}\xrightarrow[n\to\infty]{}0.$$

Now, we have to control the last term

$$C_{4,n}=\frac{A_{n,0,1}^{-2}h_{s_n}^N}{nw_{s_n}^2}\sum_{i=1}^n w_{s_i}^2h_{s_i}^{-N}\mathbb{E}[\overline{\Delta}_{s_i}^{(p)}\overline{\Delta}_{s_i}^{(q)}]$$

$$+\frac{2A_{n,0,1}^{-2}h_{s_n}^N}{nw_{s_n}^2}\sum_{\substack{1\leqslant i<j\leqslant n\\|s_i-s_j|\leqslant 2m_n}}w_{s_i}w_{s_j}(h_{s_i}h_{s_j})^{-N/2}\mathbb{E}[\overline{\Delta}_{s_i}^{(p)}\overline{\Delta}_{s_j}^{(q)}]$$

We are going to prove that

$$\lim_{n\to\infty} \frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N} \mathbb{E}[\overline{\Delta}_{s_i}^{(p)} \overline{\Delta}_{s_i}^{(q)}] = \frac{\beta_{-N,2}}{\beta_{0,1}^2} \mathbb{E}[\Phi_p(Y_0)\Phi_q(Y_0)|X_0 = x] f(x) \int_{\mathbb{R}^N} K^2(t)dt \tag{4.11}$$

and

$$\lim_{n\to\infty} \frac{2A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \sum_{\substack{1\leqslant i<j\leqslant n \\ |s_i - s_j| \leqslant 2m_n}} w_{s_i} w_{s_j} (h_{s_i} h_{s_j})^{-N/2} \mathbb{E}[\overline{\Delta}_{s_i}^{(p)} \overline{\Delta}_{s_j}^{(q)}] = 0. \tag{4.12}$$

Keeping in mind (3.12) and arguing as in (4.9), we have

$$|\mathbb{E}[\overline{\Delta}_{s_i}^{(p)} \overline{\Delta}_{s_i}^{(q)}] - \mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_i}^{(q)}]| \trianglelefteq h_{s_n}^{-N/2} \left(\|W_i^{(p)}\|_2 + \|W_i^{(q)}\|_2 \right) \trianglelefteq h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>m_n}} |k|^d \delta_{k,2}^{\tilde{\gamma}}.$$

Using (H1), we obtain

$$\frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N} |\mathbb{E}[\overline{\Delta}_{s_i}^{(p)} \overline{\Delta}_{s_i}^{(q)}] - \mathbb{E}[\Delta_{s_i}^{(p)} \Delta_{s_i}^{(q)}]| \trianglelefteq h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>m_n}} |k|^d \delta_{k,2}^{\tilde{\gamma}} \xrightarrow[n\to\infty]{} 0.$$

Using (4.4), we obtain (4.11). Now, arguing as in (4.10), we get

$$\frac{h_{s_n}^N A_{n,0,1}^{-2}}{n w_{s_n}^2} \sum_{\substack{1\leqslant i<j\leqslant n \\ |s_i - s_j| \leqslant 2m_n}} w_{s_i} w_{s_j} (h_{s_i} h_{s_j})^{-N/2} |\mathbb{E}[\overline{\Delta}_{s_i}^{(p)} \overline{\Delta}_{s_j}^{(q)}]| \trianglelefteq m_n^d h_{s_n}^{N(\gamma+\tau-1)} + h_{s_n}^{-\left(\frac{N}{2}+\tilde{\gamma}\right)} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>m_n}} |k|^d \delta_{k,2}^{\tilde{\gamma}} \xrightarrow[n\to\infty]{} 0.$$

So, (4.12) holds.

Finally, if $\|\Phi_p\|_\infty < \infty$ and $\|\Phi_q\|_\infty < \infty$ then $\tilde{\gamma} = 1$ and the proof follows exactly the same lines as above. The proof of Proposition 4 is complete. □

Proof of Proposition 2. Let $x \in \mathbb{R}^d$ and let n be a positive integer. We have

$$\mathbb{E}[(f_{n,\Phi}(x) - f_\Phi(x))^2] = \mathbb{V}(f_{n,\Phi}(x)) + (\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x))^2 .$$

Moreover,

$$|\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x)| = |(\sum_{i=1}^{n} w_{s_i})^{-1} \sum_{i=1}^{n} w_{s_i} \int_{\mathbb{R}^N} K(v) \left(f_\Phi(x - v h_{s_i}) - f_\Phi(x)\right) dv|.$$

Using Taylor's formula, we derive

$$|\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x)| \trianglelefteq (\sum_{i=1}^n w_{s_i})^{-1} \sum_{i=1}^n w_{s_i} h_{s_i}^2.$$

Since $\max\{A_{n,0,1}^{-1}, A_{n,2,1}\} \trianglelefteq 1$, we obtain $|\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x)| \trianglelefteq A_{n,0,1}^{-1} A_{n,2,1} h_{s_n}^2 \trianglelefteq h_{s_n}^2$. Finally, using Proposition 1, we have $\mathbb{V}[f_{n,\Phi}(x)] \trianglelefteq \left(nh_{s_n}^N\right)^{-1}$ and for $h_{s_n} = n^{\frac{-1}{4+N}}$, we get $\mathbb{E}[(f_{n,\Phi}(x) - f_\Phi(x))^2] \trianglelefteq n^{\frac{-4}{4+N}}$. The proof of Proposition 2 is complete. □

Proof of Theorem 1. We are going to split the proof in two parts. In the first part, we deal simultaneously with the strong mixing case and the weakly mixing case (see (4.15) below) whereas, in the second part, the two dependence conditions are investigated separately.

First part

Let n be a positive integer and $x \in \mathbb{R}^N$ be fixed. One can notice that

$$\sqrt{nh_{s_n}^N}\,(f_{n,\Phi}(x) - \mathbb{E}[f_{n,\Phi}(x)]) = \frac{1}{\sqrt{n}} \sum_{i=1}^n U_{s_i}$$

where

$$U_{s_i} = \frac{h_{s_n}^{N/2} w_{s_i} \Delta_{s_i}}{h_{s_i}^{N/2} w_{s_n} A_{n,0,1}} \quad \text{and} \quad \Delta_{s_i} = \frac{\Phi(Y_{s_i}) K_{s_i}(x, X_{s_i}) - \mathbb{E}[\Phi(Y_0) K_{s_i}(x, X_0)]}{h_{s_i}^{N/2}}. \tag{4.13}$$

In the sequel, $(m_n)_{n\geqslant 1}$ is the sequence defined by Lemma 5 which satisfies $m_n^d h_{s_n}^{N(\nu_4(\theta)+\tau-1)} \to 0$ and either $h_{s_n}^{-N\nu_2(\theta)} \sum_{|k|>m_n} \alpha_{1,\infty}^{\nu_2(\theta)}(|k|) \to 0$ (if $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing) or $h_{s_n}^{-\left(\nu_2(\theta)+\frac{N}{2}\right)} \sum_{|k|>m_n} |k|^d \delta_{k,2}^{\nu_2(\theta)} \to 0$ (if $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5)) where $\nu_2(\theta) = \mathbb{1}_{\{\|\Phi\|_\infty < \infty\}} + \frac{\theta}{2+\theta} \mathbb{1}_{\{\|\Phi\|_\infty = \infty\}}$. Moreover, if $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5), we consider the notations

$$\overline{\Delta}_{s_i} = \mathbb{E}[\Delta_{s_i} | \mathcal{H}_{i,m_n}] \quad \text{and} \quad \overline{U}_{s_i} = \mathbb{E}[U_{s_i} | \mathcal{H}_{i,m_n}].$$

where $\mathcal{H}_{i,m_n} = \sigma(\eta_{s_i}, \varepsilon_{s_i - k}; |k| \leqslant m_n)$.

Note that $\overline{U}_{s_i}$ and $\overline{U}_{s_j}$ are independent if $|s_i - s_j| > 2m_n$. Using (H1) and Proposition 3, we derive

$$\frac{1}{\sqrt{n}} \left\| \sum_{i=1}^n \left(U_{s_i} - \overline{U}_{s_i}\right) \right\|_2 \trianglelefteq h_{s_n}^{-\left(\nu_2(\theta)+\frac{N}{2}\right)} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>m_n}} |k|^d \delta_{k,2}^{\nu_2(\theta)} \xrightarrow[n\to+\infty]{} 0. \tag{4.14}$$

From now on, we denote

$$(Z_{s_i}, M_n) = \begin{cases} (U_{s_i}, m_n) & \text{if } (X_i)_{i\in\mathbb{Z}^d} \text{ is strongly mixing} \\ (\overline{U}_{s_i}, 2m_n) & \text{if } (X_i)_{i\in\mathbb{Z}^d} \text{ is of the form (1.5)} \end{cases} \tag{4.15}$$

and it suffices to prove the asymptotic normality of the partial sums $n^{-1/2}\sum_{i=1}^{n} Z_{s_i}$ as n goes to infinity. Let $(\xi_k)_{k\in\mathbb{Z}^d}$ be independent normal random variables independent of $(X_k)_{k\in\mathbb{Z}^d}$ and $(\eta_k)_{k\in\mathbb{Z}^d}$ and such that $\mathbb{E}[\xi_k]=0$ and $\mathbb{E}[\xi_k^2]=\mathbb{E}[Z_k^2]$. Let $1\leqslant i\leqslant n$ and define $T_{s_i}=n^{-1/2}Z_{s_i}$ and $\Xi_{s_i}=n^{-1/2}\xi_{s_i}$. One can notice that $\sum_{i=1}^{n}\Xi_{s_i}$ is a gaussian random variable with zero mean. If $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing then $Z_{s_i}=U_{s_i}$ and

$$\begin{aligned}\mathbb{V}\left(\sum_{i=1}^{n}\Xi_{s_i}\right) &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[U_{s_i}^2]\\ &= \frac{h_{s_n}^N A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{i=1}^{n}\frac{w_{s_i}^2}{h_{s_i}^N}\times\frac{\mathbb{E}[\Phi(Y_0)^2K_{s_i}^2(x,X_0)]-\left(\mathbb{E}[\Phi(Y_0)K_{s_i}(x,X_0)]\right)^2}{h_{s_i}^N}.\end{aligned}$$

Keeping in mind (2.6) and (H1) and using Lemma 1, Lemma 2 and Lemma 3, we get

$$\lim_{n\to+\infty}\mathbb{V}\left(\sum_{i=1}^{n}\Xi_{s_i}\right)=\sigma_\Phi^2(x). \tag{4.16}$$

If $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5) then $Z_{s_i}=\overline{U}_{s_i}$ and applying (3.12), we get

$$\begin{aligned}|\mathbb{E}[\overline{U}_{s_i}^2]-\mathbb{E}[U_{s_i}^2]| &\leqslant 2\left\|U_{s_i}\right\|_2\left\|U_{s_i}-\overline{U}_{s_i}\right\|_2\\ &\trianglelefteq \frac{h_{s_n}^N w_{s_i}^2}{h_{s_i}^N w_{s_n}^2 A_{n,0,1}^2}\times h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\ |k|>m_n}}\delta_{k,2}^{\nu_2(\theta)}.\end{aligned}$$

Using (H1), we derive

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}|\mathbb{E}[\overline{U}_{s_i}^2]-\mathbb{E}[U_{s_i}^2]| &\trianglelefteq \frac{h_{s_n}^N\sum_{i=1}^{n}h_{s_i}^{-N}w_{s_i}^2}{nw_{s_n}^2A_{n,0,1}^2}\times h_{s_n}^{-\left(\nu_2(\theta)+\frac{N}{2}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\ |k|>m_n}}\delta_{k,2}^{\nu_2(\theta)}\\ &\trianglelefteq h_{s_n}^{-\left(\nu_2(\theta)+\frac{N}{2}\right)}\sum_{\substack{k\in\mathbb{Z}^d\\ |k|>m_n}}\delta_{k,2}^{\nu_2(\theta)}\xrightarrow[n\to\infty]{}0.\end{aligned}$$

So, we get also (4.16) when $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5). Let ψ be any measurable function from $\mathbb{R}$ to $\mathbb{R}$. For any $1\leqslant i\leqslant j\leqslant n$, we introduce the notation

$$\psi_{i,j}=\psi\left(\sum_{\ell=1}^{i}T_{s_\ell}+\sum_{\ell=j}^{n}\Xi_{s_\ell}\right).$$

Let $h:\mathbb{R}\to\mathbb{R}$ be a three times continuously differentiable function such that $\max_{0\leqslant i\leqslant 3}\|h^{(i)}\|_\infty\leqslant 1$. Keeping in mind (4.16), it suffices to prove $\lim_{n\to\infty}|L_n|=0$, where

$$L_n:=\mathbb{E}\left[h\left(\sum_{i=1}^{n}T_{s_i}\right)\right]-\mathbb{E}\left[h\left(\sum_{i=1}^{n}\Xi_{s_i}\right)\right].$$

Using Lindeberg's idea [23] (see also [9]), we have

$$L_n = \mathbb{E}[h_{n,n+1} - h_{0,1}] = \sum_{i=1}^{n} \mathbb{E}[h_{i,i+1} - h_{i-1,i}]$$
$$= \sum_{i=1}^{n} \Big(\mathbb{E}[h_{i,i+1} - h_{i-1,i+1}] - \mathbb{E}[h_{i-1,i} - h_{i-1,i+1}] \Big).$$

Applying Taylor's formula, we get

$$L_n = \sum_{i=1}^{n} \bigg(\mathbb{E}\big[T_{s_i} h'_{i-1,i+1} + \frac{1}{2} T_{s_i}^2 h''_{i-1,i+1} + \beta_i\big] - \mathbb{E}\big[\Xi_i h'_{i-1,i+1} + \frac{1}{2} \Xi_{s_i}^2 h''_{i-1,i+1} + \rho_i\big] \bigg),$$

where $|\beta_i| \leqslant T_{s_i}^2\,(1 \wedge |T_{s_i}|)$ and $|\rho_i| \leqslant \Xi_{s_i}^2\,(1 \wedge |\Xi_{s_i}|)$. Since Ξ_i^2 and $h''_{i-1,i+1}$ are independent, $\mathbb{E}[\Xi_{s_i} h'_{i-1,i+1}] = 0$ and $\mathbb{E}[\Xi_{s_i}^2] = n^{-1}\mathbb{E}[Z_{s_i}^2]$, we obtain

$$L_n = \sum_{i=1}^{n} \bigg(\mathbb{E}[T_{s_i} h'_{i-1,i+1}] + \frac{1}{2}\mathbb{E}[\big(T_{s_i}^2 - n^{-1}\mathbb{E}[Z_{s_i}^2]\big)\, h''_{i-1,i+1}] + \mathbb{E}[\beta_i - \rho_i] \bigg).$$

Since, for any $1 \leqslant i \leqslant n$, the random variable ξ_{s_i} is gaussian with zero mean and variance $\mathbb{E}[Z_{s_i}^2]$, we have

$$\mathbb{E}[|\xi_{s_i}|^3] = \sqrt{8/\pi}\,\big(\mathbb{E}[Z_{s_i}^2]\big)^{3/2} \leqslant \sqrt{8/\pi}\,\big(\mathbb{E}[U_{s_i}^2]\big)^{3/2}.$$

Moreover, since $(w_{s_n} h_{s_n}^{-N})_{n\geqslant 1}$ is nondeacreasing, we get

$$\big(\mathbb{E}[U_{s_i}^2]\big)^{3/2} = \frac{h_{s_n}^N w_{s_i}^2}{h_{s_i}^N w_{s_n}^2 A_{n,0,1}^2} \left(\frac{h_{s_n}^N w_{s_i}^2}{h_{s_i}^N w_{s_n}^2 A_{n,0,1}^2} \right)^{1/2} \big(\mathbb{E}[\Delta_{s_i}^2]\big)^{3/2}$$
$$\trianglelefteq \frac{h_{s_n}^N w_{s_i}^2}{h_{s_i}^N w_{s_n}^2 A_{n,0,1}^2} \big(h_{s_i}^N h_{s_n}^{-N}\big)^{1/2} \big(\mathbb{E}[\Delta_{s_i}^2]\big)^{3/2}.$$

Using (H1) and $\mathbb{E}[\Delta_{s_i}^2] \trianglelefteq 1$ (see Lemma 3), we get

$$\sum_{i=1}^{n} \mathbb{E}[|\rho_i|] \trianglelefteq \frac{1}{\sqrt{n h_{s_n}^N}} \times \frac{h_{s_n}^N A_{n,0,1}^{-2}}{w_{s_n}^2 n} \sum_{i=1}^{n} w_{s_i}^2 h_{s_i}^{-N} \big(\mathbb{E}[\Delta_{s_i}^2]\big)^{3/2} \trianglelefteq \big(n h_{s_n}^N\big)^{-1/2} \xrightarrow[n\to+\infty]{} 0.$$

In the other part, if $\|\Phi\|_\infty < \infty$ then $|U_{s_i}| \trianglelefteq h_{s_n}^{-N/2}$ (since $(w_{s_n} h_{s_n}^{-N})_{n\geqslant 1}$ is nondeacreasing) and

$$\sum_{i=1}^{n} \mathbb{E}[|\beta_i|] \leqslant \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}[|Z_{s_i}|^3] \leqslant \frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}[|U_{s_i}|^3]$$
$$\leqslant \frac{1}{\sqrt{n h_{s_n}^N}} \times \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[U_{s_i}^2] \trianglelefteq \big(n h_{s_n}^N\big)^{-1/2} \xrightarrow[n\to+\infty]{} 0.$$

Since $(w_{s_n} h_{s_n}^{-N})_{n\geqslant 1}$ is nondeacreasing, if $\|\Phi\|_\infty = \infty$ then

$$\mathbb{E}[|U_{s_i}|^{2+\theta}] = \left(\frac{h_{s_n}^{N/2} w_{s_i}}{h_{s_i}^{N/2} w_{s_n} A_{n,0,1}}\right)^2 \left(\frac{h_{s_n}^{N/2} w_{s_i}}{h_{s_i}^{N/2} w_{s_n} A_{n,0,1}}\right)^\theta \mathbb{E}[|\Delta_{s_i}|^{2+\theta}]$$
$$\trianglelefteq \frac{h_{s_n}^N w_{s_i}^2}{h_{s_i}^N w_{s_n}^2 A_{n,0,1}^2} \left(h_{s_i}^{\frac{N}{2}} h_{s_n}^{-\frac{N}{2}}\right)^\theta h_{s_i}^{-\frac{N\theta}{2}} = h_{s_n}^{-\frac{N\theta}{2}} \frac{h_{s_n}^N w_{s_i}^2}{h_{s_i}^N w_{s_n}^2 A_{n,0,1}^2}. \quad (4.17)$$

Consequently, if $d_n := \left(nh_{s_n}^N\right)^{\frac{-\theta}{2(\theta+1)}}$ then

$$\sum_{i=1}^n \mathbb{E}[|\beta_i|] \leqslant \frac{1}{d_n^\theta n^{\theta/2}} \times \frac{1}{n}\sum_{i=1}^n \mathbb{E}[|U_{s_i}|^{2+\theta}] + \frac{d_n}{n}\sum_{i=1}^n \mathbb{E}[U_{s_i}^2]$$
$$\trianglelefteq d_n^{-\theta}\left(nh_{s_n}^N\right)^{-\theta/2} \times \frac{A_{n,0,1}^{-2} h_{s_n}^N}{w_{s_n}^2 n}\sum_{i=1}^n \frac{w_{s_i}^2}{h_{s_i}^N} + d_n.$$

Using (H1), we get

$$\sum_{i=1}^n \mathbb{E}[|\beta_i|] \trianglelefteq \frac{1}{d_n^\theta \left(nh_{s_n}^N\right)^{\theta/2}} + d_n = 2d_n \xrightarrow[n\to\infty]{} 0.$$

Now, we have to prove that

$$\lim_{n\to\infty}\sum_{i=1}^n \left(\mathbb{E}[T_{s_i} h'_{i-1,i+1}] + (2n)^{-1}\,\mathbb{E}[(Z_{s_i}^2 - \mathbb{E}[Z_{s_i}^2])h''_{i-1,i+1}]\right) = 0. \quad (4.18)$$

For any $1 \leqslant i < j \leqslant n$ and any function ψ from $\mathbb{R}$ to $\mathbb{R}$, we define also

$$\psi_{i-1,j}^{(M_n)} = \psi\left(\sum_{\substack{\ell=1\\ |s_\ell - s_i|>M_n}}^{i-1} T_{s_\ell} + \sum_{\ell=j}^n \Xi_{s_\ell}\right).$$

Using Taylor's formula, we have

$$T_{s_i} h'_{i-1,i+1} = T_{s_i} h_{i-1,i+1}^{'(M_n)} + T_{s_i}\left(\sum_{\substack{\ell=1\\ |s_\ell - s_i|\leqslant M_n}}^{i-1} T_{s_\ell}\right) h_{i-1,i+1}^{''(M_n)} + \beta'_i$$

with

$$|\beta'_i| \leqslant 2\left|T_{s_i}\left(\sum_{\substack{\ell=1\\ |s_\ell - s_i|\leqslant M_n}}^{i-1} T_{s_\ell}\right)\left(1 \wedge \left|\sum_{\substack{\ell=1\\ |s_\ell - s_i|\leqslant M_n}}^{i-1} T_{s_\ell}\right|\right)\right|. \quad (4.19)$$

In order to obtain (4.18), we have to prove

$$\lim_{n\to\infty}\sum_{i=1}^n \mathbb{E}[T_{s_i} h_{i-1,i+1}^{'(M_n)}] = 0, \quad (4.20)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}\left[T_{s_i}\left(\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1}T_{s_\ell}\right)h_{i-1,i+1}^{''(M_n)}\right]=0 \tag{4.21}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}[|\beta_i'|]=0, \tag{4.22}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(Z_{s_i}^2-\mathbb{E}[Z_{s_i}^2]\right)h_{i-1,i+1}''\right]=0. \tag{4.23}$$

Second part

First, we assume that $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing. We are going to prove (4.20). Since Ξ is independent of T, then $\mathbb{E}\left[T_{s_i}h'\left(\sum_{r=i+1}^{n}\Xi_{s_r}\right)\right]=0$. So, if we define $\mathrm{E}_i^{(n)}=\{1\leqslant j<i\mid |s_j-s_i|>M_n\}$ and π is a one to one map from $[1,|\mathrm{E}_i^{(n)}|]\cap\mathbb{Z}$ to $\mathrm{E}_i^{(n)}$ such that $|s_{\pi(\ell)}-s_i|\leq|s_{\pi(\ell-1)}-s_i|$ then

$$\begin{aligned}\mathbb{E}[T_{s_i}h_{i-1,i+1}^{'(M_n)}]&=\mathbb{E}\left[T_{s_i}\left(h_{i-1,i+1}^{'(M_n)}-h'\left(\sum_{r=i+1}^{n}\Xi_{s_r}\right)\right)\right]\\&=\sum_{\ell=1}^{|\mathrm{E}_i^{(n)}|}\mathrm{Cov}\left[T_{s_i},t_\ell-t_{\ell-1}\right],\end{aligned}$$

where $t_\ell=h'\left(\sum_{r=1}^{\ell}T_{s_{\pi(r)}}+\sum_{r=i+1}^{n}\Xi_{s_r}\right)$ and $\sum_{r=1}^{0}T_{s_{\pi(r)}}=0$. Since $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing, using Rio's inequality ([32], Theorem 1.1) and keeping in mind that $|s_{\pi(\ell)}-s_i|\leq|s_{\pi(\ell-1)}-s_i|$, we get

$$|\mathbb{E}[T_{s_i}h_{i-1,i+1}^{'(M_n)}]|\leqslant 2\sum_{\ell=1}^{|\mathrm{E}_i^{(n)}|}\int_0^{2\alpha_{1,\infty}\left(|s_{\pi(\ell)}-s_i|\right)}Q_{T_{s_i}}(u)Q_{t_\ell-t_{\ell-1}}(u)du.$$

Assume that $\|\Phi\|_\infty<\infty$ and let $u\in]0,1[$ be fixed. Since h' is Lipschitz and $|U_{s_i}|\trianglelefteq h_{s_n}^{-N/2}$ (because $(w_{s_n}h_{s_n}^{-N})_{n\geqslant 1}$ is nondeacreasing), we have $\max\left\{Q_{T_{s_i}}(u),Q_{t_\ell-t_{\ell-1}}(u)\right\}\trianglelefteq(nh_{s_n}^N)^{-1/2}$ and we derive

$$\sum_{i=1}^{n}|\mathbb{E}[T_{s_i}h_{i-1,i+1}^{'(M_n)}]|\trianglelefteq\frac{1}{nh_{s_n}^N}\sum_{i=1}^{n}\sum_{\ell=1}^{|\mathrm{E}_i^{(n)}|}\alpha_{1,\infty}\left(|s_{\pi(\ell)}-s_i|\right)\leqslant h_{s_n}^{-N}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>M_n}}\alpha_{1,\infty}\left(|k|\right).$$

Using Lemma 6, if $\|\Phi\|_\infty=\infty$ then

$$Q_{T_{s_i}}(u)\leqslant\frac{u^{-\frac{1}{2+\theta}}\|U_{s_i}\|_{2+\theta}}{\sqrt{n}}\trianglelefteq\frac{u^{-\frac{1}{2+\theta}}}{\sqrt{n}h_{s_n}^{\frac{N\theta}{2(2+\theta)}}}\times\frac{h_{s_n}^{N/2}w_{s_i}}{h_{s_i}^{N/2}w_{s_n}A_{n,0,1}}$$

and

$$Q_{t_\ell - t_{\ell-1}}(u) \leqslant \frac{u^{-\frac{1}{2+\theta}} \|U_{s_{\pi(\ell)}}\|_{2+\theta}}{\sqrt{n}} \trianglelefteq \frac{u^{-\frac{1}{2+\theta}}}{\sqrt{n} h_{s_n}^{\frac{N\theta}{2(2+\theta)}}} \times \frac{h_{s_n}^{N/2} w_{s_{\pi(\ell)}}}{h_{s_{\pi(\ell)}}^{N/2} w_{s_n} A_{n,0,1}}.$$

Consequently, using again the inequality $2ab \leqslant a^2 + b^2$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} |\mathbb{E}[T_{s_i} h_{i-1,i+1}^{'(M_n)}]| &\trianglelefteq \frac{h_{s_n}^{-\frac{N\theta}{2+\theta}}}{n} \sum_{i=1}^{n} \sum_{\ell=1}^{|E_i^{(n)}|} \frac{h_{s_n}^{N} w_{s_i} w_{s_{\pi(\ell)}}}{h_{s_i}^{N/2} h_{s_{\pi(\ell)}}^{N/2} w_{s_n}^2 A_{n,0,1}^2} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}} \left(|s_{\pi(\ell)} - s_i|\right) \\
&\trianglelefteq h_{s_n}^{-\frac{N\theta}{2+\theta}} \times \frac{A_{n,0,1}^{-2} h_{s_n}^{N}}{n w_{s_n}^2} \sum_{i=1}^{n} \sum_{\ell=1}^{|E_i^{(n)}|} \left(\frac{w_{s_i}^2}{h_{s_i}^{N}} + \frac{w_{s_{\pi(\ell)}}^2}{h_{s_{\pi(\ell)}}^{N}} \right) \alpha_{1,\infty}^{\frac{\theta}{2+\theta}} \left(|s_{\pi(\ell)} - s_i|\right) \\
&= h_{s_n}^{-\frac{N\theta}{2+\theta}} \times \frac{A_{n,0,1}^{-2} h_{s_n}^{N}}{n w_{s_n}^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ |s_j - s_i| > M_n}}^{i-1} \left(\frac{w_{s_i}^2}{h_{s_i}^{N}} + \frac{w_{s_j}^2}{h_{s_j}^{N}} \right) \alpha_{1,\infty}^{\frac{\theta}{2+\theta}} \left(|s_j - s_i|\right) \\
&\leqslant h_{s_n}^{-\frac{N\theta}{2+\theta}} \times \frac{2 A_{n,0,1}^{-2} h_{s_n}^{N}}{n w_{s_n}^2} \sum_{i=1}^{n} \frac{w_{s_i}^2}{h_{s_i}^{N}} \times \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > M_n}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}} \left(|k|\right).
\end{aligned}$$

Using (H1), we get

$$\sum_{i=1}^{n} |\mathbb{E}[T_{s_i} h_{i-1,i+1}^{'(M_n)}]| \trianglelefteq h_{s_n}^{-\frac{N\theta}{2+\theta}} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > M_n}} \alpha_{1,\infty}^{\frac{\theta}{2+\theta}} \left(|k|\right).$$

Finally, we proved that

$$\sum_{i=1}^{n} |\mathbb{E}[T_{s_i} h_{i-1,i+1}^{'(M_n)}]| \trianglelefteq h_{s_n}^{-N\nu_2(\theta)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > M_n}} \alpha_{1,\infty}^{\nu_2(\theta)}(|k|) \xrightarrow[n \to +\infty]{} 0$$

where $\nu_2(\theta) = \mathbb{1}_{\{\|\Phi\|_\infty < \infty\}} + \frac{\theta}{2+\theta} \mathbb{1}_{\{\|\Phi\|_\infty = \infty\}}$. So, (4.20) holds.

The proof of the following lemma is postponed to section 5.

Lemma 8. *It holds that* $\sup_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} \mathbb{E}[|U_{s_i} U_{s_j}|] \trianglelefteq w_{s_i} w_{s_j} w_{s_n}^{-2} h_{s_n}^{N\nu_4(\theta)}$ *where* $\nu_4(\theta) = \mathbb{1}_{\{\|\Phi\|_\infty < \infty\}} + \frac{\theta}{4+\theta} \mathbb{1}_{\{\|\Phi\|_\infty = \infty\}}$.

Since $(X_k)_{k \in \mathbb{Z}^d}$ is strongly mixing, we have $Z_{s_i} = U_{s_i}$ for any $1 \leqslant i \leqslant n$. Moreover, using (4.19), we have

$$\sum_{i=1}^{n} \mathbb{E}\left[|\beta_i'|\right] \leqslant 2 \sum_{i=1}^{n} \mathbb{E}\left[\frac{|Z_{s_i}|}{\sqrt{n}} \left(\sum_{\substack{j=1 \\ |s_j - s_i| \leqslant M_n}}^{i-1} \frac{|Z_{s_j}|}{\sqrt{n}} \right) \right] = \frac{2}{n} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ |s_j - s_i| \leqslant M_n}}^{i-1} \mathbb{E}[|U_{s_i} U_{s_j}|] \tag{4.24}$$

and

$$\sum_{i=1}^{n}\left|\mathbb{E}\left[T_{s_i}\left(\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1}T_{s_\ell}\right)h_{i-1,i+1}^{''(M_n)}\right]\right|\leqslant\sum_{i=1}^{n}\mathbb{E}\left[\frac{|Z_{s_i}|}{\sqrt{n}}\left(\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}\frac{|Z_{s_j}|}{\sqrt{n}}\right)\right]$$

$$=\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}\mathbb{E}[|U_{s_i}U_{s_j}|]. \quad (4.25)$$

Using Lemma 8 and the inequality $2ab\leqslant a^2+b^2$, we get

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}\mathbb{E}[|U_{s_i}U_{s_j}|]\trianglelefteq\frac{M_n^d h_{s_n}^{N\nu_4(\theta)}}{nw_{s_n}^2}\sum_{i=1}^{n}w_{s_i}^2+\frac{h_{s_n}^{N\nu_4(\theta)}}{nw_{s_n}^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}w_{s_j}^2.$$

Moreover,

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}w_{s_j}^2=\sum_{j=1}^{n-1}w_{s_j}^2\sum_{\substack{i=j+1\\|s_i-s_j|\leqslant M_n}}^{n}1\leqslant M_n^d\sum_{i=1}^{n}w_{s_i}^2.$$

Consequently, using (H1), we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}\mathbb{E}\left[|U_{s_i}U_{s_j}|\right]\trianglelefteq\frac{M_n^d h_{s_n}^{N\nu_4(\theta)}}{nw_{s_n}^2}\sum_{i=1}^{n}w_{s_i}^2$$

$$=M_n^d h_{s_n}^{N(\nu_4(\theta)+\tau-1)}\times\frac{h_{s_n}^{N(1-\tau)}}{nw_{s_n}^2}\sum_{i=1}^{n}w_{s_i}^2$$

$$\trianglelefteq m_n^d h_{s_n}^{N(\nu_4(\theta)+\tau-1)}\xrightarrow[n\to+\infty]{}0. \quad (4.26)$$

Combining (4.24), (4.25) and (4.26), we obtain (4.21) and (4.22).

Now, it suffices to prove (4.23). Let $\beta\geqslant 1$ be a positive integer. For any $1\leqslant j\leqslant n$, the notation $\mathbb{E}_\beta[Z_{s_j}]$ will stand for the conditional expectation of Z_{s_j} with respect to the σ-algebra $\sigma\left(Z_{s_\ell}\,;\,\ell<j \text{ and } |s_\ell-s_j|\geqslant\beta\right)$. Then,

$$\frac{1}{n}\sum_{i=1}^{n}|\mathbb{E}[(Z_{s_i}^2-\mathbb{E}[Z_{s_i}^2])h''_{i-1,i+1}]|\leqslant I_1+I_2,$$

where

$$I_1=\frac{1}{n}\sum_{i=1}^{n}|\mathbb{E}[\left(Z_{s_i}^2-\mathbb{E}_\beta[Z_{s_i}^2]\right)h''_{i-1,i+1}]|$$

and

$$I_2=\frac{1}{n}\sum_{i=1}^{n}|\mathbb{E}[(\mathbb{E}_\beta[Z_{s_i}^2]-\mathbb{E}[Z_{s_i}^2])h''_{i-1,i+1}]|.$$

The next result can be found in [27].

Lemma 9. *Let $\mathcal{U}$ and $\mathcal{V}$ be two σ-algebras and let X be a random variable which is measurable with respect to $\mathcal{U}$.*

If $1 \leqslant p \leqslant r \leqslant \infty$, then $\|\mathbb{E}[X|\mathcal{V}] - \mathbb{E}[X]\|_p \leqslant 2(2^{1/p}+1)\,(\alpha(\mathcal{U},\mathcal{V}))^{\frac{1}{p}-\frac{1}{r}}\,\|X\|_r$.

Since $(X_k)_{k\in\mathbb{Z}^d}$ is strongly mixing, we have $Z_{s_i} = U_{s_i}$ for any $1 \leqslant i \leqslant n$. If $\|\Phi\|_\infty = \infty$ then using Lemma 9 with $p = 1$ and $r = (2+\theta)/2$, we derive

$$I_2 \leqslant \frac{1}{n}\sum_{i=1}^{n} \|\mathbb{E}_\beta[U_{s_i}^2] - \mathbb{E}[U_{s_i}^2]\|_1 \leqslant \frac{6\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta)}{n}\sum_{i=1}^{n}\|U_{s_i}\|_{2+\theta}^2.$$

From Lemma 6, we have

$$\|U_{s_i}\|_{2+\theta}^2 \trianglelefteq \frac{h_{s_n}^{-\frac{N\theta}{2+\theta}} h_{s_n}^N w_{s_i}^2}{h_{s_i}^N w_{s_n}^2 A_{n,0,1}^2}.$$

Consequently, using (H1), we get

$$I_2 \trianglelefteq h_{s_n}^{-\frac{N\theta}{2+\theta}}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta) \times \frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2}\sum_{i=1}^{n}\frac{w_{s_i}^2}{h_{s_i}^N} \trianglelefteq h_{s_n}^{-\frac{N\theta}{2+\theta}}\alpha_{1,\infty}^{\frac{\theta}{2+\theta}}(\beta).$$

Similarly, if $\|\Phi\|_\infty < \infty$ then $|\Delta_{s_i}| \trianglelefteq h_{s_i}^{-N/2}$ and since $(w_{s_n} h_{s_n}^{-N})_{n\geqslant 1}$ is nondecreasing, we have $|U_{s_i}| \trianglelefteq h_{s_n}^{-N/2}$ and

$$I_2 \leqslant 6\alpha_{1,\infty}(\beta) \times \frac{1}{n}\sum_{i=1}^{n}\|U_{s_i}\|_\infty^2 \trianglelefteq h_{s_n}^{-N}\alpha_{1,\infty}(\beta).$$

Finally, keeping in mind that $\nu_2(\theta) = \mathbb{1}_{\{\|\Phi\|_\infty<\infty\}} + \frac{\theta}{2+\theta}\mathbb{1}_{\{\|\Phi\|_\infty=\infty\}}$, it means that

$$I_2 \leqslant \frac{1}{n}\sum_{i=1}^{n}\|\mathbb{E}_\beta[U_{s_i}^2] - \mathbb{E}[U_{s_i}^2]\|_1 \trianglelefteq h_{s_n}^{-N\nu_2(\theta)}\alpha_{1,\infty}^{\nu_2(\theta)}(\beta). \tag{4.27}$$

Now, we make the choice

$$\beta = \left\lceil h_{s_n}^{\frac{-N\nu_2(\theta)(\nu_4(\theta)+\tau-1)}{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}}\right\rceil. \tag{4.28}$$

Consequently, since $\sum_{k\in\mathbb{Z}^d}|k|^{\frac{d\nu_2(\theta)}{\nu_4(\theta)+\tau-1}}\alpha_{1,\infty}^{\nu_2(\theta)}(|k|) < \infty$ means $\sum_{\ell=1}^{\infty}\ell^{\frac{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}{\nu_4(\theta)+\tau-1}}\alpha_{1,\infty}^{\nu_2(\theta)}(\ell) < \infty$, we derive

$$I_2 \trianglelefteq \beta^{\frac{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}{\nu_4(\theta)+\tau-1}}\alpha_{1,\infty}^{\nu_2(\theta)}(\beta) \xrightarrow[n\to\infty]{} 0.$$

Since $h_{i-1,i+1}^{''(\beta)}$ is $\sigma(\Xi_{s_\ell}, Z_{s_k}; 1 \leqslant k < i, |s_k - s_i| \geqslant \beta, \ell \in \mathbb{N}^*)$-measurable, we find that $\mathbb{E}[(Z_{s_i}^2 - \mathbb{E}_\beta[Z_{s_i}^2])h_{i-1,i+1}^{''(\beta)}] = 0$ and consequently

$$\mathbb{E}\left[\left(Z_{s_i}^2 - \mathbb{E}_\beta[Z_{s_i}^2]\right)h_{i-1,i+1}''\right] = \mathbb{E}\left[\left(Z_{s_i}^2 - \mathbb{E}_\beta[Z_{s_i}^2]\right)\left(h_{i-1,i+1}'' - h_{i-1,i+1}^{''(\beta)}\right)\right].$$

Keeping in mind that

$$I_1 = \frac{1}{n}\sum_{i=1}^{n} |\mathbb{E}[\big(Z_{s_i}^2 - \mathbb{E}_\beta[Z_{s_i}^2]\big)\, h''_{i-1,i+1}]|,$$

we obtain

$$I_1 \leqslant \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(2\wedge\left|\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\frac{Z_{s_j}}{\sqrt{n}}\right|\right)\big(Z_{s_i}^2+\mathbb{E}_\beta[Z_{s_i}^2]\big)\right].$$

Let $L>0$, then

$$\begin{aligned}
I_1 \leqslant{}& \frac{L}{n^{3/2}}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\mathbb{E}[|Z_{s_j}||Z_{s_i}|\,\mathbb{1}_{|Z_{s_i}|\leqslant L}] + \frac{2}{n}\sum_{i=1}^{n}\mathbb{E}[Z_{s_i}^2\,\mathbb{1}_{|Z_{s_i}|>L}]\\
&+\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(2\wedge\left|\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\frac{Z_{s_j}}{\sqrt{n}}\right|\right)\big(\mathbb{E}_\beta[Z_{s_i}^2]-\mathbb{E}\left[Z_{s_i}^2\right]\big)\right]\\
&+\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(2\wedge\left|\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\frac{Z_{s_j}}{\sqrt{n}}\right|\right)\right]\mathbb{E}[Z_{s_i}^2].
\end{aligned}$$

Since $Z_{s_i}=U_{s_i}$ for any $1\leqslant i\leqslant n$, we derive

$$\begin{aligned}
I_1 \leqslant{}& \frac{L}{n^{3/2}}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\mathbb{E}[|U_{s_i}U_{s_j}|] + \frac{2}{n}\sum_{i=1}^{n}\mathbb{E}[U_{s_i}^2\,\mathbb{1}_{|U_{s_i}|>L}]\\
&+\frac{2}{n}\sum_{i=1}^{n}\|\mathbb{E}_\beta[U_{s_i}^2]-\mathbb{E}[U_{s_i}^2]\|_1+\frac{1}{n}\sum_{i=1}^{n}\left\|\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\frac{U_{s_j}}{\sqrt{n}}\right\|_2\mathbb{E}[U_{s_i}^2].
\end{aligned}$$

Arguing as in (4.26) and Lemma 9, we derive

$$\begin{aligned}
I_1 \trianglelefteq{}& \frac{\beta^d L h_{s_n}^{N(\nu_4(\theta)+\tau-1)}}{\sqrt{n}} + \frac{L^{-\theta}}{n}\sum_{i=1}^{n}\mathbb{E}[|U_{s_i}|^{2+\theta}] + h_{s_n}^{-N\nu_2(\theta)}\alpha_{1,\infty}^{\nu_2(\theta)}(\beta)\\
&+\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[U_{s_i}^2]\left\|\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\frac{U_{s_j}}{\sqrt{n}}\right\|_2.
\end{aligned}$$

Using (4.17), (4.28) and (H1), we get

$$I_1 \trianglelefteq \frac{\beta^d L h_{s_n}^{N(\nu_4(\theta)+\tau-1)}}{\sqrt{n}} + L^{-\theta} h_{s_n}^{\frac{-N\theta}{2}} + \beta^{\frac{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}{\nu_4(\theta)+\tau-1}} \alpha_{1,\infty}^{\nu_2(\theta)}(\beta)$$
$$+ \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[U_{s_i}^2] \left\| \sum_{\substack{j=1 \\ |s_j-s_i|\leqslant\beta}}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \right\|_2 .$$

Moreover,

$$\left\| \sum_{\substack{j=1 \\ |s_j-s_i|\leqslant\beta}}^{i-1} \frac{U_{s_j}}{\sqrt{n}} \right\|_2^2 = \frac{1}{n}\left(\sum_{\substack{j=1 \\ |s_j-s_i|\leqslant\beta}}^{i-1} \mathbb{E}[U_{s_j}^2] + \sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell}}^{i-1} \mathbb{E}[U_{s_j}U_{s_\ell}] \right).$$

If $\|\Phi\|_\infty = \infty$ then using (H1) and (4.17), for $L' = \left(\beta^{-d} n h_{s_n}^{-\frac{N\theta}{2}}\right)^{\frac{1}{2+\theta}}$, we have

$$\frac{1}{n}\sum_{\substack{j=1 \\ |s_j-s_i|\leqslant\beta}}^{i-1} \mathbb{E}[U_{s_j}^2] \leqslant \frac{\beta^d L'^2}{n} + \frac{L'^{-\theta}}{n}\sum_{j=1}^{n}\mathbb{E}[|U_{s_j}|^{2+\theta}] \trianglelefteq \frac{\beta^d L'^2}{n} + L'^{-\theta} h_{s_n}^{-\frac{N\theta}{2}}$$
$$= 2\left(\frac{\beta^d}{n h_{s_n}^N}\right)^{\frac{\theta}{2+\theta}}.$$

If $\|\Phi\|_\infty < \infty$ then $|U_{s_i}| \trianglelefteq h_{s_n}^{-N/2}$ (since $|\Delta_{s_i}| \trianglelefteq h_{s_i}^{-N/2}$ and $(w_{s_n} h_{s_n}^{-N})_{n\geqslant 1}$ is nondecreasing) and consequently

$$\frac{1}{n}\sum_{\substack{j=1 \\ |s_j-s_i|\leqslant\beta}}^{i-1} \mathbb{E}[U_{s_j}^2] \trianglelefteq \frac{\beta^d}{n h_{s_n}^N}.$$

Finally, it means that

$$\frac{1}{n}\sum_{\substack{j=1 \\ |s_j-s_i|\leqslant\beta}}^{i-1} \mathbb{E}[U_{s_j}^2] \trianglelefteq \left(\frac{\beta^d}{n h_{s_n}^N}\right)^{\nu_2(\theta)}$$
$$\trianglelefteq \left(n h_{s_n}^{N\left(1+\frac{d\nu_2(\theta)(\nu_4(\theta)+\tau-1)}{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}\right)} \right)^{-\nu_2(\theta)} \xrightarrow[n\to\infty]{} 0. \quad (4.29)$$

Moreover, we have

$$\frac{1}{n}\sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell}}^{i-1} |\mathbb{E}[U_{s_j}U_{s_\ell}]| \trianglelefteq \frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell}}^{i-1} \frac{w_{s_j} w_{s_\ell} |\mathbb{E}[\Delta_{s_j}\Delta_{s_\ell}]|}{(h_{s_j} h_{s_\ell})^{N/2}}$$

$$\trianglelefteq \frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \left(\sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell,\,|s_\ell-s_j|\leqslant M_n}}^{i-1} \frac{w_{s_j} w_{s_\ell}}{\left(h_{s_j} h_{s_\ell}\right)^{\frac{N(1-\nu_4(\theta))}{2}}} + \sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell,\,|s_\ell-s_j|> M_n}}^{i-1} \frac{w_{s_j} w_{s_\ell} \alpha_{1,\infty}^{\nu_2(\theta)}\left(|s_j-s_\ell|\right)}{\left(h_{s_j} h_{s_\ell}\right)^{\frac{N(1+\nu_2(\theta))}{2}}} \right)$$

$$\trianglelefteq \frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell,\,|s_\ell-s_j|\leqslant M_n}}^{i-1} \left(\frac{w_{s_j}^2}{h_{s_j}^{N(1-\nu_4(\theta))}} + \frac{w_{s_\ell}^2}{h_{s_\ell}^{N(1-\nu_4(\theta))}} \right) + \frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell,\,|s_\ell-s_j|> M_n}}^{i-1} \left(\frac{w_{s_j}^2}{h_{s_j}^{N(1+\nu_2(\theta))}} + \frac{w_{s_\ell}^2}{h_{s_\ell}^{N(1+\nu_2(\theta))}} \right) \alpha_{1,\infty}^{\nu_2(\theta)}\left(|s_j-s_\ell|\right).$$

Since $\nu_4(\theta) \leqslant 1$, we have

$$\frac{1}{n} \sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell}}^{i-1} \left|\mathbb{E}[U_{s_j} U_{s_\ell}]\right| \trianglelefteq \frac{M_n^d h_{s_n}^{N\nu_4(\theta)}}{n w_{s_n}^2 A_{n,0,1}^2} \sum_{i=1}^n w_{s_i}^2 + \left(\frac{A_{n,0,1}^{-2} h_{s_n}^N}{n w_{s_n}^2} \sum_{j=1}^n \frac{w_{s_j}^2}{h_{s_j}^N} \right) \times h_{s_n}^{-N\nu_2(\theta)} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>M_n}} \alpha_{1,\infty}^{\nu_2(\theta)}\left(|k|\right).$$

Using (H1) and (H2), we derive

$$\frac{1}{n} \sum_{\substack{j,\ell=1 \\ \max\{|s_j-s_i|,|s_\ell-s_i|\}\leqslant\beta \\ j\neq\ell}}^{i-1} \left|\mathbb{E}[U_{s_j} U_{s_\ell}]\right| \trianglelefteq m_n^d h_{s_n}^{N(\nu_4(\theta)+\tau-1)} + h_{s_n}^{-N\nu_2(\theta)} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>M_n}} \alpha_{1,\infty}^{\nu_2(\theta)}\left(|k|\right).$$

Consequently, denoting

$$\varepsilon_n^2 := \left(n h_{s_n}^{N\left(1+\frac{d\nu_2(\theta)(\nu_4(\theta)+\tau-1)}{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}\right)} \right)^{-\nu_2(\theta)} + m_n^d h_{s_n}^{N(\nu_4(\theta)+\tau-1)} + h_{s_n}^{-N\nu_2(\theta)} \sum_{\substack{k\in\mathbb{Z}^d \\ |k|>m_n}} \alpha_{1,\infty}^{\nu_2(\theta)}\left(|k|\right) \xrightarrow[n\to\infty]{} 0,$$

we get

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[U_{s_i}^2]\left\|\sum_{\substack{j=1\\|s_j-s_i|\leqslant\beta}}^{i-1}\frac{U_{s_j}}{\sqrt{n}}\right\|_2 \trianglelefteq \underbrace{\frac{A_{n,0,1}^{-2}h_{s_n}^N}{nw_{s_n}^2}\sum_{i=1}^{n}\frac{w_{s_i}^2\,\mathbb{E}[\Delta_{s_i}^2]}{h_{s_i}^N}}_{\trianglelefteq 1}\times\varepsilon_n \trianglelefteq \varepsilon_n$$

and finally,

$$I_1 \trianglelefteq \frac{\beta^d L h_{s_n}^{N(\nu_4(\theta)+\tau-1)}}{\sqrt{n}} + L^{-\theta}h_{s_n}^{\frac{-N\theta}{2}} + \beta^{\frac{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}{\nu_4(\theta)+\tau-1}}\alpha_{1,\infty}^{\nu_2(\theta)}(\beta)+\varepsilon_n.$$

Optimizing in L, we get

$$I_1 \trianglelefteq \frac{\beta^{\frac{d\theta}{1+\theta}}h_{s_n}^{\frac{N\theta(\nu_4(\theta)+\tau-1)}{1+\theta}}}{\left(nh_{s_n}^N\right)^{\frac{\theta}{2(1+\theta)}}} + \beta^{\frac{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}{\nu_4(\theta)+\tau-1}}\alpha_{1,\infty}^{\nu_2(\theta)}(\beta)+\varepsilon_n.$$

Since, $\beta^d \trianglelefteq h_{s_n}^{\frac{-dN\nu_2(\theta)(\nu_4(\theta)+\tau-1)}{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}}$, we derive

$$I_1 \trianglelefteq \left(nh_{s_n}^N\right)^{\frac{-\theta}{2(1+\theta)}} h_{s_n}^{\frac{N\theta(d-1)(\nu_4(\theta)+\tau-1)^2}{(1+\theta)(d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1))}} + \beta^{\frac{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}{\nu_4(\theta)+\tau-1}}\alpha_{1,\infty}^{\nu_2(\theta)}(\beta) + \varepsilon_n \xrightarrow[n\to+\infty]{} 0.$$

So, we obtain (4.23).

Now, we assume that $(X_k)_{k\in\mathbb{Z}^d}$ is of the form (1.5). As before, we have to prove (4.20), (4.21), (4.22) and (4.23). Now, we have $Z_{s_i}=\overline{U}_{s_i}$ for any $1\leqslant i\leqslant n$. Moreover, $\overline{U}_{s_i}$ ans $\overline{U}_{s_j}$ are independent as soon as $|s_i-s_j|>M_n$ where $M_n=2m_n$. So, (4.20) holds since

$$\mathbb{E}[T_{s_i}h_{i-1,i+1}^{'(M_n)}] = n^{-1/2}\mathbb{E}\left[\overline{U}_{s_i}h'\left(\sum_{\substack{\ell=1\\|s_\ell-s_i|>M_n}}^{i-1}\frac{\overline{U}_{s_\ell}}{\sqrt{n}}+\sum_{\ell=i+1}^{n}\frac{\xi_{s_\ell}}{\sqrt{n}}\right)\right]=0.$$

Arguing as in (4.24) and (4.25), we have

$$\sum_{i=1}^{n}\mathbb{E}[|\beta_i'|]\leqslant\frac{2}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}\mathbb{E}[|\overline{U}_{s_i}\overline{U}_{s_j}|]$$

and

$$\sum_{i=1}^{n}\left|\mathbb{E}\left[T_{s_i}\left(\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1}T_{s_\ell}\right)h_{i-1,i+1}^{''(M_n)}\right]\right|\leqslant\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n}}^{i-1}\mathbb{E}[|\overline{U}_{s_i}\overline{U}_{s_j}|].$$

The proof of the following lemma is postponed to section 5.

Lemma 10. *For any positive integer n,*

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n\\i\neq j}}^{n}\mathbb{E}[|\overline{U}_{s_i}\overline{U}_{s_j}|] \trianglelefteq m_n^d h_{s_n}^{N(\nu_4(\theta)+\tau-1)} + h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\nu_2(\theta)} \xrightarrow[n\to\infty]{} 0.$$

Consequently, we obtain (4.21) and (4.22). In order to finish the proof, it suffices to prove (4.23). Let $L>0$ be fixed. Keeping in mind that $\overline{U}_{s_i}$ and $\overline{U}_{s_j}$ are independent if $|s_i-s_j|>M_n$, we have

$$\mathbb{E}\left[\left(Z_{s_i}^2-\mathbb{E}\left[Z_{s_i}^2\right]\right)h_{i-1,i+1}^{''(M_n)}\right] = \mathbb{E}\left[\left(\overline{U}_{s_i}^2-\mathbb{E}\left[\overline{U}_{s_i}^2\right]\right)h''\left(\sum_{\substack{\ell=1\\|s_\ell-s_i|>M_n}}^{i-1}\frac{\overline{U}_{s_\ell}}{\sqrt{n}}+\sum_{\ell=i+1}^{n}\frac{\xi_{s_\ell}}{\sqrt{n}}\right)\right]=0.$$

So, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left|\mathbb{E}\left[\left(Z_{s_i}^2-\mathbb{E}\left[Z_{s_i}^2\right]\right)h''_{i-1,i+1}\right]\right| \\
&=\frac{1}{n}\sum_{i=1}^{n}\left|\mathbb{E}\left[\left(\overline{U}_{s_i}^2-\mathbb{E}[\overline{U}_{s_i}^2]\right)\left(h''_{i-1,i+1}-h_{i-1,i+1}^{''(M_n)}\right)\right]\right| \\
&\leqslant \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(2\wedge\left|\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1}\frac{\overline{U}_{s_\ell}}{\sqrt{n}}\right|\right)\left(\overline{U}_{s_i}^2+\mathbb{E}[\overline{U}_{s_i}^2]\right)\right] \\
&\leqslant \frac{L}{n^{3/2}}\sum_{i=1}^{n}\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1}\mathbb{E}[|\overline{U}_{s_\ell}\overline{U}_{s_i}|]+\frac{2}{n}\sum_{i=1}^{n}\mathbb{E}[\overline{U}_{s_i}^2\,\mathbb{1}_{|\overline{U}_{s_i}|>L}] \\
&\qquad+\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[2\wedge\left|\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1}\frac{\overline{U}_{s_\ell}}{\sqrt{n}}\right|\right]\mathbb{E}[\overline{U}_{s_i}^2] \\
&\leqslant \frac{L}{\sqrt{n}}\times\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{n}\mathbb{E}[|\overline{U}_{s_\ell}\overline{U}_{s_i}|]+\frac{2L^{-\theta}}{n}\sum_{i=1}^{n}\mathbb{E}[|U_{s_i}|^{2+\theta}] \\
&\qquad+\frac{1}{n}\sum_{i=1}^{n}\left\|\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1}\frac{\overline{U}_{s_\ell}}{\sqrt{n}}\right\|_2\mathbb{E}[U_{s_i}^2].
\end{aligned}$$

Moreover,

$$\left\| \sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1} \frac{\overline{U}_{s_\ell}}{\sqrt{n}} \right\|_2^2 = \frac{1}{n}\left(\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1} \mathbb{E}[\overline{U}_{s_\ell}^2] + \sum_{\substack{\ell,j=1\\ \max(|s_\ell-s_i|,|s_j-s_i|)\leqslant M_n\\ \ell\neq j}}^{i-1} \mathbb{E}[\overline{U}_{s_\ell}\overline{U}_{s_j}] \right).$$

Noting that $\mathbb{E}[\overline{U}_{s_\ell}^2] \leqslant \mathbb{E}[U_{s_\ell}^2]$ and using (4.29), we get

$$\frac{1}{n}\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1} \mathbb{E}[\overline{U}_{s_\ell}^2] \trianglelefteq \left(nh_{s_n}^{N\left(1+\frac{d\nu_2(\theta)(\nu_4(\theta)+\tau-1)}{d\nu_2(\theta)+(d-1)(\nu_4(\theta)+\tau-1)}\right)}\right)^{-\nu_2(\theta)} \xrightarrow[n\to\infty]{} 0.$$

Using Lemma 10 and keeping in mind that $\overline{U}_{s_\ell}$ and $\overline{U}_{s_j}$ are independent if $|s_\ell - s_j| > M_n$, we have

$$\frac{1}{n}\sum_{\substack{\ell,j=1\\ \max(|s_\ell-s_i|,|s_j-s_i|)\leqslant M_n\\ \ell\neq j}}^{i-1} |\mathbb{E}[\overline{U}_{s_\ell}\overline{U}_{s_j}]| \leqslant \frac{1}{n}\sum_{\ell=1}^{n}\sum_{\substack{j=1\\|s_\ell-s_j|\leqslant M_n\\ \ell\neq j}}^{n} |\mathbb{E}[\overline{U}_{s_\ell}\overline{U}_{s_j}]| \xrightarrow[n\to\infty]{} 0.$$

Consequently,

$$\lim_{n\to+\infty}\sup_{1\leqslant i\leqslant n}\left\| \sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1} \frac{\overline{U}_{s_\ell}}{\sqrt{n}} \right\|_2 = 0$$

and

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}\left\| \sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1} \frac{\overline{U}_{s_\ell}}{\sqrt{n}} \right\|_2 \mathbb{E}[U_{s_i}^2]$$

$$\leqslant \lim_{n\to+\infty}\underbrace{\frac{h_{s_n}^N A_{n,0,1}^{-2}}{nw_{s_n}^2}\sum_{i=1}^{n}\frac{w_{s_i}^2\,\mathbb{E}[\Delta_{s_i}^2]}{h_{s_i}^N}}_{\trianglelefteq 1} \times \sup_{1\leqslant i\leqslant n}\left\| \sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{i-1} \frac{\overline{U}_{s_\ell}}{\sqrt{n}} \right\|_2 = 0.$$

Using (4.17) and (H1), we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[(Z_{s_i}^2-\mathbb{E}[Z_{s_i}^2])h''_{i-1,i+1}] \trianglelefteq \frac{L}{\sqrt{n}}\times\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{n}\mathbb{E}[|\overline{U}_{s_\ell}\overline{U}_{s_i}|]+L^{-\theta}h_{s_n}^{\frac{-N\theta}{2}}+o(1).$$

Optimizing in L, we get

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[(Z_{s_i}^2-\mathbb{E}[Z_{s_i}^2])h''_{i-1,i+1}] \trianglelefteq (nh_{s_n}^N)^{\frac{-\theta}{2(1+\theta)}}\left(\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{\ell=1\\|s_\ell-s_i|\leqslant M_n}}^{n}\mathbb{E}[|\overline{U}_{s_\ell}\overline{U}_{s_i}|]\right)^{\frac{\theta}{1+\theta}}+o(1).$$

Using Lemma 10, we obtain (4.23). The proof of Theorem 1 is complete. □

In the proof of Theorem 1, the asymptotic normality of the estimator $f_{n,\Phi}$ is obtained using the Lindeberg's method based on the stability of the standard normal law. This approach seems to be superior to the so-called Bernstein's method (see for example [5] and [21]) since it allows us to obtain mild conditions on the weak and strong dependent coefficients of the considered random field. This fact is of theoretical importance and has already been observed in [1], [14] and [15].

Proof of Theorem 2. Let n be a positive integer and $x \in \mathbb{R}^N$ such that $f(x) > 0$. Then,

$$r_{n,\Phi}(x) - \frac{\mathbb{E}[f_{n,\Phi}(x)]}{\mathbb{E}[f_{n,\mathbf{1}}(x)]} = \frac{(f_{n,\Phi}(x) - \mathbb{E}[f_{n,\Phi}(x)])\mathbb{E}[f_{n,\mathbf{1}}(x)] - (f_{n,\mathbf{1}}(x) - \mathbb{E}[f_{n,\mathbf{1}}(x)])\mathbb{E}[f_{n,\Phi}(x)]}{f_{n,\mathbf{1}}(x)\mathbb{E}[f_{n,\mathbf{1}}(x)]}.$$

Using Proposition 1 and Proposition 2, we obtain that $f_{n,\mathbf{1}}(x)$ converges in probability to $f(x)$ and $\frac{\mathbb{E}[f_{n,\Phi}(x)]}{\mathbb{E}[f_{n,\mathbf{1}}(x)]}$ converges to $r_\Phi(x)$ as $n \to \infty$. So, using Slutsky's lemma, it is sufficient to prove

$$\lambda_1\sqrt{nh_{s_n}^N}(f_{n,\Phi}(x) - \mathbb{E}[f_{n,\Phi}(x)]) + \lambda_2\sqrt{nh_{s_n}^N}(f_{n,\mathbf{1}}(x) - \mathbb{E}[f_{n,\mathbf{1}}(x)]) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, \rho^2_{\lambda_1,\lambda_2}(x))$$

where

$$\rho^2_{\lambda_1,\lambda_2}(x) = (\lambda_1^2\mathbb{E}[|\Phi(Y_0)|^2|X_0 = x] + 2\lambda_1\lambda_2 r_\Phi(x) + \lambda_2^2)\beta_{0,1}^{-2}\beta_{-N,2}f(x)\int_{\mathbb{R}^N} K^2(t)dt$$

for any $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Let $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ be fixed. Then,

$$\lambda_1\sqrt{nh_{s_n}^N}\,(f_{n,\Phi}(x) - \mathbb{E}[f_{n,\Phi}(x)]) + \lambda_2\sqrt{nh_{s_n}^N}\,(f_{n,\mathbf{1}}(x) - \mathbb{E}[f_{n,\mathbf{1}}(x)]) = \sqrt{nh_{s_n}^N}\left(f_{n,\overline{\Phi}}(x) - \mathbb{E}[f_{n,\overline{\Phi}}(x)]\right)$$

where $\overline{\Phi}(x) = \lambda_1\Phi(x) + \lambda_2$ for any x in $\mathbb{R}$. Since $u \mapsto \mathbb{E}[|\Phi(Y_0)|^2|X_0 = u]$ is continuous, one can notice that $u \mapsto \mathbb{E}[|\overline{\Phi}(Y_0)|^2|X_0 = u]$ is continuous. Moreover, since $\mathbb{E}[|\Phi(Y_0)|^{2+\theta}] < \infty$ and $\mathbb{E}[|\Phi(Y_0)|^{2+\theta}K_{s_n}(x, X_0)] \trianglelefteq h_{s_n}^N$, we have $\mathbb{E}[|\overline{\Phi}(Y_0)|^{2+\theta}] < \infty$ and $\mathbb{E}[|\overline{\Phi}(Y_0)|^{2+\theta}K_{s_n}(x, X_0)] \trianglelefteq h_{s_n}^N$. Consequently, using Theorem 1, we get the result. The proof of Theorem 2 is complete. □

One can notice that the asymptotic normality of the regression estimator $r_{n,\Phi}$ obtained in the proof of Theorem 2 is a direct consequence of Theorem 1.

In some sense, it means that Theorem 1 is quite a deep result since it contains both the asymptotic normality of the kernel density estimator $f_{n,\mathbf{1}}$ and that of the regression estimator $r_{n,\mathbf{1}}$.

Proof of Theorem 3. Let $x \in \mathbb{R}^N$ such that $f(x) > 0$. Then, according to Theorem 2, we have

$$\sqrt{nh_{s_n}^N}\left(r_{n,\Phi}(x) - \frac{\mathbb{E}[f_{n,\Phi}(x)]}{\mathbb{E}[f_{n,\mathbf{1}}(x)]}\right) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0, \tilde{\sigma}_\Phi^2(x)\right),$$

where $\tilde{\sigma}_\Phi^2(x) = \frac{V(x)\beta_{-N,2}}{f(x)\beta_{0,1}^2}\int_{\mathbb{R}^N} K^2(t)dt$ and $V(x) = \mathbb{E}[|\Phi(Y_0)|^2|X_0 = x] - r_\Phi^2(x)$. Applying Proposition 2, we have

$$|\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x)| \trianglelefteq h_{s_n}^2 \quad \text{and} \quad |\mathbb{E}[f_{n,\mathbf{1}}(x)] - f(x)| \trianglelefteq h_{s_n}^2.$$

Recall that $r_\Phi(x) = \frac{f_\Phi(x)}{f(x)}$. Then, for n sufficiently large, we have

$$\begin{aligned}\left|\frac{\mathbb{E}[f_{n,\Phi}(x)]}{\mathbb{E}[f_{n,\mathbf{1}}(x)]} - r_\Phi(x)\right| &= \frac{|(\mathbb{E}[f_{n,\Phi}(x)] - f_\Phi(x))f(x) - (\mathbb{E}\left[f_{n,\mathbf{1}}(x)\right] - f(x))f_\Phi(x)|}{f(x)\mathbb{E}\left[f_{n,\mathbf{1}}(x)\right]} \\ &\trianglelefteq h_{s_n}^2.\end{aligned}$$

Finally, using Slutsky's lemma, we obtain

$$\sqrt{nh_{s_n}^N}\left(r_{n,\Phi}(x) - r_\Phi(x)\right) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}\left(0, \tilde{\sigma}_\Phi^2(x)\right).$$

The proof of Theorem 3 is complete. □

5. Appendix

Proof of Lemma 7. First, we note that $\mathbb{E}\left[W_i|\mathcal{F}_{\ell-1}\right]$ is $\mathcal{F}_\ell$-measurable. It suffices to show that for every $A \in \mathcal{F}_\ell$, we have $\mathbb{E}\left[\mathbb{E}\left[W_i|\mathcal{F}_{\ell-1}\right]\mathbf{1}_A\right] = \mathbb{E}\left[H_i(\mathcal{H}_{i,\infty}^{(\ell)})\mathbf{1}_A\right]$. One can notice that the collections $\mathcal{P} = \left\{A\cap B \,|\, A \in \mathcal{F}_{\ell-1}, B \in \sigma(\epsilon_{\tau(\ell)})\right\}$ and $\Lambda = \left\{A \in \mathcal{F}_\ell \,|\, \mathbb{E}\left[\mathbb{E}\left[W_i|\mathcal{F}_{\ell-1}\right]\mathbf{1}_A\right] = \mathbb{E}\left[H_i(\mathcal{H}_{i,\infty}^{(\ell)})\mathbf{1}_A\right]\right\}$ are respectively a π-system and a λ-system which satisfy $\sigma(\mathcal{P}) = \mathcal{F}_\ell$. Since $(\varepsilon_{\tau(j)})_{j\in\mathbb{Z}}$ and $(\varepsilon_{\tau(j)}^{(\ell)})_{j\in\mathbb{Z}}$ are identically distributed where $\varepsilon_{\tau(j)}^{(\ell)} = \varepsilon_{\tau(j)}$ if $j \neq \ell$ and $\varepsilon_{\tau(\ell)}^{(\ell)} = \varepsilon'_{\tau(\ell)}$, it holds for every $C \in \mathcal{F}_{\ell-1}$ that $\mathbb{E}\left[\mathbf{1}_C W_i\right] = \mathbb{E}\left[\mathbf{1}_C H_i(\mathcal{H}_{i,\infty}^{(\ell)})\right]$. Then, if $A = A_1 \cap A_2 \in \mathcal{P}$ with $A_1 \in \mathcal{F}_{\ell-1}$ and $A_2 \in \sigma(\epsilon_{\tau(\ell)})$, we obtain

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left[W_i|\mathcal{F}_{\ell-1}\right]\mathbf{1}_A\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{A_1}W_i|\mathcal{F}_{\ell-1}\right]\mathbf{1}_{A_2}\right] &= \mathbb{E}\left[\mathbf{1}_{A_1}H_{i,\infty}(\mathcal{H}_{i,\infty}^{(\ell)})\right]\mathbb{E}\left[\mathbf{1}_{A_2}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{A_1}H_{i,\infty}(\mathcal{H}_{i,\infty}^{(\ell)})\mathbf{1}_{A_2}\right] \\ &= \mathbb{E}\left[H_{i,\infty}(\mathcal{H}_{i,\infty}^{(\ell)})\mathbf{1}_A\right].\end{aligned}$$

So, we obtain $A \in \Lambda$ and finally $\mathcal{P} \subset \Lambda$. Applying Dynkin's lemma, we get the desired result. The proof of Lemma 7 is complete. □

Proof of Lemma 8. Let $1 \leqslant i, j \leqslant n$ such that $i \neq j$. From Lemma 4, we have $\mathbb{E}[|\Delta_{s_i}\Delta_{s_j}|] \trianglelefteq \left(h_{s_i}h_{s_j}\right)^{\frac{N\nu_4(\theta)}{2}}$. Keeping in mind (4.13), we have

$$\mathbb{E}[|U_{s_i}U_{s_j}|] = \frac{h_{s_n}^N w_{s_i} w_{s_j} \mathbb{E}[|\Delta_{s_i}\Delta_{s_j}|]}{(h_{s_i}h_{s_j})^{\frac{N}{2}} w_{s_n}^2 A_{n,0,1}^2} \trianglelefteq \frac{A_{n,0,1}^{-2} h_{s_n}^N w_{s_i} w_{s_j}}{w_{s_n}^2 (h_{s_i}h_{s_j})^{N(1-\nu_4(\theta))/2}}.$$

Since $\nu_4(\theta) \leqslant 1$ and using (H1), we derive $\mathbb{E}[|U_{s_i}U_{s_j}|] \trianglelefteq w_{s_i} w_{s_j} w_{s_n}^{-2} h_{s_n}^{N\nu_4(\theta)}$.

The proof of Lemma 8 is complete. □

Proof of Lemma 10. For any $1 \leqslant i, j \leqslant n$ such that $i \neq j$, we have

$$|\mathbb{E}[|\overline{U}_{s_i}\overline{U}_{s_j}|] - \mathbb{E}[|U_{s_i}U_{s_j}|]| \leqslant \|U_{s_j}\|_2 \|U_{s_i} - \overline{U}_{s_i}\|_2 + \|U_{s_i}\|_2 \|U_{s_j} - \overline{U}_{s_j}\|_2$$

and for any $1 \leqslant \ell \leqslant n$, using (3.12), we get

$$\|U_{s_\ell}\|_2 = \frac{h_{s_n}^{N/2} w_{s_\ell} \|\Delta_{s_\ell}\|_2}{h_{s_\ell}^{N/2} w_{s_n} A_{n,0,1}} \trianglelefteq \frac{h_{s_n}^{N/2} w_{s_\ell}}{h_{s_\ell}^{N/2} w_{s_n} A_{n,0,1}}$$

and

$$\|U_{s_\ell} - \overline{U}_{s_\ell}\|_2 = \frac{h_{s_n}^{N/2} w_{s_\ell} \|\Delta_{s_\ell} - \overline{\Delta}_{s_\ell}\|_2}{h_{s_\ell}^{N/2} w_{s_n} A_{n,0,1}} \trianglelefteq \frac{h_{s_n}^{N/2} w_{s_\ell} h_{s_n}^{-(\frac{N}{2}+\nu_2(\theta))}}{h_{s_\ell}^{N/2} w_{s_n} A_{n,0,1}} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > M_n}} \delta_{k,2}^{\nu_2(\theta)}.$$

Consequently, we get

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} \sum_{\substack{j=1 \\ |s_j - s_i| \leqslant M_n \\ i \neq j}}^{n} |\mathbb{E}[|\overline{U}_{s_i}\overline{U}_{s_j}|] - \mathbb{E}[|U_{s_i}U_{s_j}|]| \\
&\quad \trianglelefteq \frac{h_{s_n}^N A_{n,0,1}^{-2}}{n w_{s_n}^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ |s_j - s_i| \leqslant M_n \\ i \neq j}}^{n} \frac{w_{s_i} w_{s_j}}{h_{s_i}^{N/2} h_{s_j}^{N/2}} \times h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > M_n}} \delta_{k,2}^{\nu_2(\theta)} \\
&\quad \trianglelefteq \frac{h_{s_n}^N A_{n,0,1}^{-2}}{n w_{s_n}^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ |s_j - s_i| \leqslant M_n}}^{n} \left(\frac{w_{s_i}^2}{h_{s_i}^N} + \frac{w_{s_j}^2}{h_{s_j}^N}\right) \times h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > M_n}} \delta_{k,2}^{\nu_2(\theta)} \\
&\quad \leqslant 2M_n^d \left(\frac{h_{s_n}^N A_{n,0,1}^{-2}}{n w_{s_n}^2} \sum_{i=1}^{n} \frac{w_{s_i}^2}{h_{s_i}^N}\right) h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > M_n}} \delta_{k,2}^{\nu_2(\theta)}.
\end{aligned}$$

Using (H1), we obtain

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_j-s_i|\leqslant M_n\\i\neq j}}^{i-1}|\mathbb{E}[|\overline{U}_{s_i}\overline{U}_{s_j}|]-\mathbb{E}[|U_{s_i}U_{s_j}|]|\trianglelefteq m_n^d h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}\delta_{k,2}^{\nu_2(\theta)}$$

$$\leqslant h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\nu_2(\theta)}.$$

Using Lemma 8, we obtain

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_i-s_j|\leqslant M_n\\i\neq j}}^{n}\mathbb{E}\left[|\overline{U}_{s_i}\overline{U}_{s_j}|\right]\\
&\quad\trianglelefteq\frac{h_{s_n}^{N\nu_4(\theta)}}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_i-s_j|\leqslant M_n\\i\neq j}}^{n}\frac{w_{s_i}w_{s_j}}{w_{s_n}^2}+h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\nu_2(\theta)}\\
&\quad\trianglelefteq\frac{h_{s_n}^{N\nu_4(\theta)}}{nw_{s_n}^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_i-s_j|\leqslant M_n}}^{n}(w_{s_i}^2+w_{s_j}^2)+h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\nu_2(\theta)}\\
&\quad\trianglelefteq\frac{m_n^d h_{s_n}^{N\nu_4(\theta)}}{nw_{s_n}^2}\sum_{i=1}^{n}w_{s_i}^2+h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\nu_2(\theta)}.
\end{aligned}$$

Using (H2), we derive

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\|s_i-s_j|\leqslant M_n\\i\neq j}}^{n}\mathbb{E}\left[|\overline{U}_{s_i}\overline{U}_{s_j}|\right]\\
&\qquad\trianglelefteq m_n^d h_{s_n}^{N(\nu_4(\theta)+\tau-1)}+h_{s_n}^{-\left(\frac{N}{2}+\nu_2(\theta)\right)}\sum_{\substack{k\in\mathbb{Z}^d\\|k|>m_n}}|k|^d\delta_{k,2}^{\nu_2(\theta)}\xrightarrow[n\to\infty]{}0.
\end{aligned}$$

The proof of Lemma 10 is complete. □

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