

Gaussian fluctuations and a law of the iterated logarithm for Nerman’s martingale in the supercritical general branching process*

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Abstract

In his, by now, classical work from 1981, Nerman made extensive use of a crucial martingale $(W_t)_{t \geq 0}$ to prove convergence in probability, in mean and almost surely, of supercritical general branching processes (also known as Crump-Mode-Jagers branching processes) counted with a general characteristic. The martingale terminal value W figures in the limits of his results.

We investigate the rate at which the martingale, now called *Nerman’s martingale*, converges to its limit W . More precisely, assuming the existence of a Malthusian parameter $\alpha > 0$ and $W_0 \in L^2$, we prove a functional central limit theorem for $(W - W_{t+s})_{s \in \mathbb{R}}$, properly normalized, as $t \rightarrow \infty$. The weak limit is a randomly scaled time-changed Brownian motion. Under an additional technical assumption, we prove a law of the iterated logarithm for $W - W_t$.

Keywords: asymptotic fluctuations; functional central limit theorem; law of the iterated logarithm; Nerman’s martingale; supercritical general branching process.

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1 Introduction

The general (Crump-Mode-Jagers) branching process is a classical model for an evolving population. The process starts with one initial ancestor at time 0 which produces

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offspring at the times of a point process ξ on $(0, \infty)$. Every other individual in the process reproduces according to an independent copy of ξ shifted by the individual's time of birth. The model contains a variety of other models such as Galton-Watson processes, age-dependent branching processes, Bellman-Harris processes and Sevast'yanov processes. Coupled with a random characteristic the general branching process offers a lot of flexibility in modelling and allows considerations of, for instance, the number of individuals in the population in some random phase of life or having some random age-dependent property. We refer to [17] for a textbook introduction.

The weak and strong laws of large numbers for supercritical general branching processes counted with a random characteristic were given by Nerman [22]. Assuming the existence of some $\alpha > 0$ such that $m(\alpha) = 1$ where m is the Laplace transform of the intensity measure of the reproduction point process ξ , these laws exhibit exponential growth of the order $e^{\alpha t}$ of the process, i.e., α is a Malthusian exponent. Key to the proof of these results is a crucial martingale $(W_t)_{t \geq 0}$, nowadays called Nerman's martingale. The martingale limit W appears in the aforementioned weak and strong laws. Further, the rate of convergence of the martingale to its limit is relevant for the rate of convergence in the weak law of large numbers for the general branching process [16].

Recently, Janson [19] studied the fluctuations of supercritical general branching processes in the case where ξ is concentrated on a lattice. A natural first step towards extending Janson's results to the non-lattice case is to investigate the asymptotic fluctuations of Nerman's martingale around its limit. In the paper at hand, we address this problem by proving a functional central limit theorem with a deterministic scaling for $W - W_t$. This functional limit theorem is complemented by a law of the iterated logarithm.

The results of the present paper are analogous to those for Biggins' martingale in the branching random walk [13]. Fluctuations of the latter martingale, also at complex parameters, have received a lot of attention lately [14, 15, 24]. The endmost paper [24] is in the more general context of weighted branching processes. Predecessors of these results are central limit theorems for the classical martingale in the Galton-Watson process [10, 11, 12]. There are further rate-of-convergence results for multitype Galton-Watson processes, we refrain from providing references here and refer to the discussion in [21] instead.

Closely related to Biggins' martingale in the branching random walk is the derivative martingale, the fluctuations of which have been addressed in [6]. The counterpart for the derivative martingale in branching Brownian motion is contained in [21]. Rate-of-convergence results for more complicated branching processes, including branching diffusions and superprocesses, can be found in [23] and the references therein.

2 Model and assumptions

We begin by introducing the standard Ulam-Harris notation in the context of the general branching process. We mainly follow [18]. Let $\mathcal{I} := \bigcup_{n \in \mathbb{N}_0} \mathbb{N}^n$ be the infinite Ulam-Harris tree where $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{N}^0 = \{\emptyset\}$ contains only the empty tuple, which we denote by \emptyset . We identify individuals in a population with their descent, which is encoded by elements of \mathcal{I} . For instance, \emptyset is the label of the ancestor, and if $u = (u_1, \dots, u_m) \in \mathcal{I}$, then u is the u_m th child of the u_{m-1} th child of \dots of the u_1 th child of the ancestor \emptyset . We abbreviate $u = (u_1, \dots, u_m)$ by $u_1 \dots u_m$ and set $|u|$ for the generation of u . Here, $|u| = m$. Similarly, if $v = (v_1, \dots, v_n)$, we write uv for $(u_1, \dots, u_m, v_1, \dots, v_n)$. Further, if $k \leq m$, we write $u|_k$ for $u_1 \dots u_k$, the ancestor of u in the k th generation. If $u = v|_m$ for some $0 \leq m \leq |v|$, i.e., when u is an ancestor of v , then we write $u \preceq v$, and say that v stems from u . For a subset F of \mathcal{I} , we say that v stems

from F , and in this case write $F \preceq v$, if $u \preceq v$ for some $u \in F$. For $F, L \subseteq \mathcal{I}$, we write $F \preceq L$ if every $v \in L$ stems from some $u \in F$.

2.1 The model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a family $(\xi_u)_{u \in \mathcal{I}}$ of independent, identically distributed (i.i.d.) point processes on $(0, \infty)$ is defined. Formally, each ξ_u is an integer-valued mapping $\xi_u : \Omega \times \mathcal{B}((0, \infty)) \rightarrow [0, \infty]$ such that

- for fixed $\omega \in \Omega$, $\xi_u(\omega, \cdot) : \mathcal{B}((0, \infty)) \rightarrow [0, \infty]$ is a measure,
- whereas, for each Borel set $B \in \mathcal{B}((0, \infty))$, the map $\xi_u(\cdot, B) : \Omega \rightarrow [0, \infty]$ is a random variable.

Here, $\mathcal{B}((0, \infty))$ is the Borel σ -algebra of $(0, \infty)$. We write $\xi_u = \sum_{k=1}^{N(u)} \delta_{X_k(u)}$ for $u \in \mathcal{I}$ where $N(u) := \xi_u((0, \infty))$ is a random variable taking values in $\mathbb{N}_0 \cup \{\infty\}$. For convenience, we abbreviate ξ_\emptyset to ξ , $N(\emptyset)$ to N , $X_k(\emptyset)$ to X_k etc. Further, we define $S(u)$, the time of birth of individual u , recursively via

$$S(\emptyset) := 0 \quad \text{and} \quad S(uk) := S(u) + X_k(u) \quad \text{for } u \in \mathcal{I} \text{ and } k \in \mathbb{N}$$

with the convention that $X_k(u) := \infty$ if $k > N(u)$. If $u \in \mathcal{I}$ with $S(u) = \infty$, then individual u is considered never born. We define $\mathcal{G}_n := \{u \in \mathbb{N}^n : S(u) < \infty\}$ to be the n th generation individuals, $n \in \mathbb{N}_0$. Further, $\mathcal{G} := \bigcup_{n \in \mathbb{N}_0} \mathcal{G}_n$ denotes the set of all individuals that are ever born. We set

$$\mathcal{I}_t := \{u \in \mathcal{I} : S(u|_{|u|-1}) \leq t < S(u) < \infty\}$$

to be the coming generation at time t , that is, \mathcal{I}_t is the set of particles born after t whose parents were born at or before time t .

Later on, different filtrations will be important. We call a subset $L \subseteq \mathcal{I}$ a *line* if $u \not\preceq v$ for all $u, v \in L$ with $u \neq v$. For any line L , we define $\mathcal{F}_L := \sigma(\xi_u : L \not\preceq u)$. The σ -algebra \mathcal{F}_L contains all information about the individuals up to and including the line L in the genealogical tree, but is independent of all information that comes after that, i.e., after crossing the line L , for instance, the relative birth times of individuals descended from an element of L . Of particular importance are the σ -algebras with $L = \mathbb{N}^n$, the n th generation,

$$\mathcal{F}_n := \mathcal{F}_{\mathbb{N}^n} = \sigma(\xi_u : |u| < n), \quad n \in \mathbb{N}_0. \tag{2.1}$$

The family $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ forms a filtration of (Ω, \mathcal{F}) . We set $\mathcal{F}_\infty := \sigma(\mathcal{F}_n : n \in \mathbb{N})$. The second filtration is the counterpart of the first when the n th generation is replaced by the coming generation at time t . To formally introduce it, we first recall the notion of an *optional line*. An optional line $\mathcal{J} \subseteq \mathcal{I}$ is a random line with the property that, for every deterministic line $L \subseteq \mathcal{I}$, it holds that $\{\mathcal{J} \preceq L\} \in \mathcal{F}_L$. For instance, for every $t \geq 0$, the coming generation at time t , \mathcal{I}_t , is an optional line. Indeed, for every line $L \subseteq \mathcal{I}$, we have

$$\{\mathcal{I}_t \preceq L\} = \bigcap_{v \in L} \{\mathcal{I}_t \preceq v\} = \bigcap_{v \in L} \{S(v) > t\} \in \mathcal{F}_L.$$

We follow Jagers on p. 190 of [18] and define, for an optional line \mathcal{J} ,

$$\mathcal{F}_{\mathcal{J}} := \{A \in \mathcal{F}_\infty : A \cap \{\mathcal{J} \preceq L\} \in \mathcal{F}_L \text{ for all lines } L \subseteq \mathcal{I}\}.$$

A key result for us is the strong Markov branching property at optional lines, Theorem 4.14 in [18]. Finally, for $t \geq 0$, we set $\mathcal{H}_t := \mathcal{F}_{\mathcal{I}_t}$.

2.2 Basic assumptions

Let $\mu(\cdot) := \mathbb{E}[\xi(\cdot)]$ be the intensity measure of the reproduction point processes. It is a measure on $\mathcal{B}((0, \infty))$. Throughout the paper, we assume that μ is not concentrated on any lattice $h\mathbb{N}_0$, $h > 0$. This assumption is for convenience only, all results have lattice counterparts. We define

$$m(\theta) := \int_{(0, \infty)} e^{-\theta t} \mu(dt), \quad \theta \geq 0. \tag{2.2}$$

The function m is the Laplace transform of the intensity measure μ .

Consider the following assumptions:

(A1) The process is *supercritical*, i.e., $\mathbb{E}[N] = \mu((0, \infty)) > 1$.

(A2) There exists a *Malthusian parameter* $\alpha > 0$, i.e., an $\alpha > 0$ satisfying

$$m(\alpha) = \int_{(0, \infty)} e^{-\alpha t} \mu(dt) = 1. \tag{2.3}$$

(A3) The (right) derivative at $\theta = \alpha$ of the Laplace transform m is finite, i.e.,

$$m'(\alpha) := - \int_{(0, \infty)} t e^{-\alpha t} \mu(dt) \in (-\infty, 0). \tag{2.4}$$

(A4) The (random) Laplace transform of ξ at $\theta = \alpha$ has positive and finite variance, i.e.,

$$0 < \sigma^2 := \mathbb{E} \left[\left(\sum_{k=1}^N e^{-\alpha X_k} - 1 \right)^2 \right] = \mathbb{E} \left[\left(\int_{(0, \infty)} e^{-\alpha t} \xi(dt) - 1 \right)^2 \right] < \infty. \tag{2.5}$$

(A5) There exists a nonincreasing Lebesgue integrable function $g : [0, \infty) \rightarrow (0, \infty)$ such that

$$\mathbb{E} \left[\sup_{t \geq 0} \frac{\sum_{k=1}^N e^{-\alpha X_k} \mathbb{1}_{[0, X_k)}(t)}{g(t)} \right] < \infty. \tag{2.6}$$

Throughout the paper, we shall assume that (A1) through (A3) hold. Assumption (A1) guarantees that the survival set \mathcal{S} defined by

$$\mathcal{S} = \{\mathcal{G}_n \neq \emptyset \text{ for all } n \in \mathbb{N}_0\} \tag{2.7}$$

satisfies $\mathbb{P}(\mathcal{S}) > 0$. While assumption (A4) is required for the central limit theorem for Nerman's martingale, (A5) is additionally used in the proof of the law of the iterated logarithm.

2.3 Nerman's martingale

Recall that \mathcal{I}_t is the coming generation at time t , i.e., the collection of labels of individuals born after time t whose parents were born up to (and including) time t . Put

$$W_t := \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)}, \quad t \geq 0. \tag{2.8}$$

The family $(W_t, \mathcal{H}_t)_{t \geq 0}$ is a nonnegative martingale (Proposition 2.4 in [22]), called Nerman's martingale. It converges almost surely (a. s.) as $t \rightarrow \infty$ to a finite limit $W \geq 0$ (Corollary 2.5 in [22]). For later use, we stipulate $W_t := 1$ for $t < 0$. The martingale

is a pure jump process almost surely taking values in the Skorokhod space $D(\mathbb{R})$ of right-continuous real-valued functions with left limits at every point (càdlàg functions). With probability one, there are only finitely many jumps on every given compact set. Indeed, the martingale jumps at $t \geq 0$ only if, for some $n \in \mathbb{N}$, there are $u_1, \dots, u_n \in \mathcal{G}$ and $S(u_j) = t$ for $j = 1, \dots, n$. In this case

$$\Delta W_t := W_t - W_{t-} = \sum_{j=1}^n e^{-\alpha S(u_j)} \left(\sum_{k \geq 1} e^{-\alpha X_k(u_j)} - 1 \right).$$

Almost surely, there are only finitely many u with $S(u) \leq t$ for any $t \geq 0$. Indeed, by the many-to-one lemma [25, Section 1.3], there is a zero-delayed random walk $(S_n)_{n \in \mathbb{N}_0}$ on \mathbb{R} with increment law $\mathbb{P}(S_n - S_{n-1} \in dx) = e^{-\alpha x} \mu(dx)$, $\mathbb{E}[S_1] = -m'(\alpha) > 0$ and

$$\begin{aligned} \mathbb{E}[N_t] &= \sum_{n \geq 0} \mathbb{E} \left[\sum_{|u|=n} \mathbb{1}_{\{S(u) \leq t\}} \right] \leq e^{\alpha t} \sum_{n \geq 0} \mathbb{E} \left[\sum_{|u|=n} e^{-\alpha S(u)} \mathbb{1}_{\{S(u) \leq t\}} \right] \\ &= e^{\alpha t} \sum_{n \geq 0} \mathbb{P}(S_n \in (0, t]) < \infty. \end{aligned}$$

The process $(N_t)_{t \geq 0}$ is also a natural example of a process the limit of which is given by a constant multiple of Nerman's martingale. Indeed, if (A1) through (A5) hold ((A4) is not needed), Theorem 5.4 of [22] implies that $e^{-\alpha t} N_t \rightarrow \frac{1}{-\alpha m'(\alpha)} W$ a. s. as $t \rightarrow \infty$.

2.4 The connection with Biggins' martingale

Nerman's martingale is related to the corresponding Biggins martingale $(Z_n, \mathcal{F}_n)_{n \in \mathbb{N}_0}$, where

$$Z_n = \sum_{|u|=n} e^{-\alpha S(u)}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

Since the Biggins martingale is nonnegative, it converges a. s. to a finite limit $Z \geq 0$ with $\mathbb{E}[Z] \leq 1$. Further, it holds that $W = Z$ a. s. by Theorem 3.3 in [8].

What is more, if (A1) through (A4) hold, then

$$\frac{m(2\alpha)}{m(\alpha)^2} = m(2\alpha) < 1.$$

and in view of Theorem 1 in [3] or Theorem 2.1 in [20] it holds $Z_n \rightarrow Z$ in L^2 . In particular, $\mathbb{E}[W] = \mathbb{E}[Z] = 1$ and $\text{Var}[W] = \text{Var}[Z] = \mathbb{E}[(Z - 1)^2] < \infty$. Using the fact that martingale increments are uncorrelated, we calculate $\text{Var}[W]$ as follows:

$$\begin{aligned} \sigma_W^2 := \text{Var}[W] &= \mathbb{E}[(Z - 1)^2] = \sum_{n \geq 0} \mathbb{E}[(Z_{n+1} - Z_n)^2] \\ &= \sum_{n \geq 0} \mathbb{E} \left[\sum_{|u|=n} e^{-2\alpha S(u)} \right] \mathbb{E}[(Z_1 - 1)^2] = \frac{\sigma^2}{1 - m(2\alpha)}, \end{aligned} \tag{2.10}$$

where $\sigma^2 = \mathbb{E}[(Z_1 - 1)^2]$.

2.5 Fluctuations of Nerman's martingale

Recall the notation $\mathcal{F}_\infty := \sigma(\mathcal{F}_n : n \in \mathbb{N}_0)$ and denote by $\mathcal{L}(X)$ the law (distribution) of a random variable X . If $X, X_t, t \geq 0$ are real-valued random variables, we write

$$\mathcal{L}(X_t | \mathcal{H}_t) \xrightarrow{w} \mathcal{L}(X | \mathcal{F}_\infty) \quad \text{in } \mathbb{P}\text{-probability as } t \rightarrow \infty \tag{2.11}$$

(in words, 'the distribution of X_t given \mathcal{H}_t converges weakly to the distribution of X given \mathcal{F}_∞ in \mathbb{P} -probability') if $\mathbb{E}[f(X_t) | \mathcal{H}_t] \xrightarrow{p} \mathbb{E}[f(X) | \mathcal{F}_\infty]$ as $t \rightarrow \infty$ for every bounded

continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$. Notice that (2.11) implies $X_t \xrightarrow{d} X$ as $t \rightarrow \infty$, where \xrightarrow{d} denotes convergence in distribution in \mathbb{R} .

Theorem 2.1 gives the asymptotic fluctuations of $(W_t)_{t \geq 0}$.

Theorem 2.1. *Suppose that (A1) through (A4) hold. Then*

$$\mathcal{L}(e^{\alpha t/2}(W - W_t) \mid \mathcal{H}_t) \xrightarrow{d} \mathcal{L}\left(\left(\frac{\sigma^2}{-\alpha m'(\alpha)} W\right)^{1/2} \cdot X \mid \mathcal{F}_\infty\right) \tag{2.12}$$

in probability as $t \rightarrow \infty$ where X is standard normal and independent of \mathcal{F}_∞ .

Convergence in distribution of $e^{\alpha t/2}(W - W_t)$ as $t \rightarrow \infty$ can be strengthened to convergence in distribution of the stochastic process $(e^{\alpha t/2}(W - W_{t+s}))_{s \in \mathbb{R}}$ as $t \rightarrow \infty$ in the Skorokhod space $D(\mathbb{R})$, equipped with the J_1 -topology (Chapter 12 in [5]). We write ' \Rightarrow ' to denote convergence in distribution of random elements in this space.

Theorem 2.2. *Suppose that (A1) through (A4) hold. Then*

$$(e^{\alpha t/2}(W - W_{t+s}))_{s \in \mathbb{R}} \Rightarrow \left(\left(\frac{\sigma^2}{-\alpha m'(\alpha)} W\right)^{1/2} B_{e^{-\alpha s}}\right)_{s \in \mathbb{R}} \text{ as } t \rightarrow \infty \tag{2.13}$$

where $(B_s)_{s \geq 0}$ is a standard Brownian motion independent of W .

Finally, we deal with the almost sure fluctuations of Nerman's martingale, namely, we formulate a law of the iterated logarithm.

Theorem 2.3. *If (A1) through (A5) hold, then, a. s. on the survival set \mathcal{S} ,*

$$\limsup_{t \rightarrow \infty} \frac{e^{\alpha t/2}}{\sqrt{\log t}}(W - W_t) = \left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W\right)^{1/2}, \tag{2.14}$$

$$\liminf_{t \rightarrow \infty} \frac{e^{\alpha t/2}}{\sqrt{\log t}}(W - W_t) = -\left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W\right)^{1/2}. \tag{2.15}$$

Remark 2.4. It can be checked that $\mathbb{E}[(W - W_t)^2] = \text{Var}[W]\mathbb{E}[\sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)}]$. Hence, by Lemma 3.1 below, $e^{\alpha t}\mathbb{E}[(W - W_t)^2]$ converges as $t \rightarrow \infty$ to a positive constant. Thus, $\log t$ in (2.14) and (2.15) can be replaced by the asymptotically equivalent function $\log(|\log \mathbb{E}[(W - W_t)^2]|)$. This demonstrates that Theorem 2.3 is indeed a law of the iterated logarithm.

3 Proofs of the main results

We start this section with some basic notation and discussions.

3.1 Preliminaries

The shift operators

Suppose that ψ is a function of $(\xi_v)_{v \in \mathcal{I}}$, all offspring point processes. For a given $u \in \mathcal{I}$, we write $[\psi]_u$ for the very same function but applied to $((\xi_{uv})_{v \in \mathcal{I}})$. In other words, $[\cdot]_u$ is a shift operator that shifts the ancestor to u . For instance, we have $[Z_1]_u = \sum_{|v|=1} e^{-\alpha X_v(u)}$. Further, $[Z]_u = \lim_{n \rightarrow \infty} [Z_n]_u$ a. s. and $[W]_u = \lim_{t \rightarrow \infty} [W_t]_u$ a. s. are the limits of the shifted martingales. Here,

$$[W_t]_u = \sum_{v \in [\mathcal{I}]_u} e^{-\alpha(S(uv) - S(u))}, \quad t \geq 0.$$

If ψ is a function of a real variable t and $(\xi_v)_{v \in \mathcal{I}}$, i.e., $\psi_t = f(t, (\xi_v)_{v \in \mathcal{I}})$ for some function f , then we write $[\psi]_u \circ t$ for $f(t, (\xi_{uv})_{v \in \mathcal{I}})$. This is particularly useful when t is replaced by a function of $S(u)$, for instance, in this notation, we have $[\psi]_u \circ (t - S(u)) = f(t - S(u), (\xi_{uv})_{v \in \mathcal{I}})$.

Crump-Mode-Jagers processes

For a function ψ as above we may now define Z_t^ψ the general branching process counted with a characteristic ψ (or the Crump-Mode-Jagers processes) by setting

$$Z_t^\psi := \sum_{u \in \mathcal{I}} [\psi \cdot]_u \circ (t - S(u)). \tag{3.1}$$

The classical Nerman's result states that, under suitable assumptions,

$$Z_t^\psi \rightarrow \frac{W}{-m'(\alpha)} \times \int \mathbb{E}[\psi_t] e^{-\alpha t} dt \quad \text{a.s.}$$

Recursive decomposition

With this notation, one deduces the following decomposition for W_{t+r} , valid for $t, r \geq 0$,

$$W_{t+r} = \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} [W_{r+t-\cdot}]_u \circ S(u). \tag{3.2}$$

Passing to the limit as $r \rightarrow \infty$, we infer (after a careful inspection, see Section 14 in [4] or Lemma 4.2 in [1])

$$W = \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} [W]_u \quad \text{a.s.} \tag{3.3}$$

if (A1) through (A3) hold and $\mathbb{E}[Z_1 \log^+ Z_1] < \infty$.

Nerman's martingale as an L^2 -martingale

If (A1) through (A4) hold, then, according to the discussion preceding (2.10), $W = Z \in L^2$. Further, $W_t = \mathbb{E}[W | \mathcal{H}_t]$ a.s. for all $t \geq 0$, i.e., $(W_t)_{t \geq 0}$ is an L^2 -bounded martingale and hence convergent in L^2 (with limit W , of course). We write $v_t := \text{Var}[W_t] = \mathbb{E}[(W_t - 1)^2]$ for the variance of W_t , $t \in \mathbb{R}$. Since $((W_t - 1)^2)_{t \geq 0}$ is a right-continuous submartingale, the function $t \mapsto v_t$ is nondecreasing and right-continuous. We now identify v_t for some relevant values of t . Trivially, $v_t = 0$ for $t < 0$. Further, since $W_0 = Z_1$, we have $v_0 = \text{Var}[Z_1] = \sigma^2$. From $W_t \rightarrow W$ in L^2 and (2.10) we finally deduce

$$v_\infty := \lim_{t \rightarrow \infty} \mathbb{E}[(W_t - 1)^2] = \mathbb{E}[(W - 1)^2] = \sigma_W^2 = \frac{\sigma^2}{1 - m(2\alpha)}. \tag{3.4}$$

Moreover, Doob's maximal L^2 -inequality (with $p = 2$) gives

$$\mathbb{E} \left[\sup_{s \geq 0} (W_s - 1)^2 \right] \leq 4 \mathbb{E}[(W - 1)^2], \tag{3.5}$$

i.e., $M_t := \sup_{0 \leq s \leq t} |W_s - 1| \in L^2$.

3.2 Fluctuations of Nerman's martingale: proofs

We start with an auxiliary result derived from Nerman's law of large numbers for the general branching process.

Lemma 3.1. *Suppose that (A1) through (A3) hold. Then*

$$e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \xrightarrow{\mathbb{P}} \frac{1 - m(2\alpha)}{-\alpha m'(\alpha)} W \quad \text{as } t \rightarrow \infty. \tag{3.6}$$

More generally, if $f \in D(\mathbb{R})$ is a nonnegative bounded function, then, as $t \rightarrow \infty$,

$$e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} f(t-S(u)) \xrightarrow{\mathbb{P}} \frac{W}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} f(x-X_k) dx \right]. \quad (3.7)$$

The convergence in (3.7) (and thus also in (3.6)) holds in the stronger

- L^1 sense if $\mathbb{E}[Z_1 \log^+ Z_1] < \infty$,
- almost sure sense if (A5) holds.

Proof. Define

$$\phi(t) := e^{2\alpha t} \sum_{k=1}^N e^{-2\alpha X_k} \mathbb{1}_{[0, X_k)}(t) f(t-X_k), \quad t \in \mathbb{R}$$

and notice that, for any $t \geq 0$, with $\|f\|_\infty := \sup_{x \in \mathbb{R}} f(x)$,

$$\mathbb{E} \left[\sup_{s \leq t} \phi(s) \right] \leq e^{2\alpha t} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \right] \|f\|_\infty = e^{2\alpha t} m(2\alpha) \|f\|_\infty < \infty, \quad (3.8)$$

that is, Condition (3.2) in [22] holds. Further, for $t \geq 0$,

$$\mathbb{E}[e^{-\alpha t} \phi(t)] = e^{\alpha t} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \mathbb{1}_{[0, X_k)}(t) f(t-X_k) \right] = e^{\alpha t} \int_{(t, \infty)} e^{-2\alpha x} f(t-x) \mu(dx)$$

is càdlàg as a function of t by the dominated convergence theorem. On the other hand, the inequality

$$\begin{aligned} \mathbb{E}[e^{-\alpha t} \phi(t)] &= e^{\alpha t} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \mathbb{1}_{\{X_k > t\}} f(t-X_k) \right] \\ &\leq \|f\|_\infty \mathbb{E} \left[\sum_{k=1}^N e^{-\alpha X_k} \mathbb{1}_{\{X_k > t\}} \right] = \|f\|_\infty \int_{(t, \infty)} e^{-\alpha x} \mu(dx) \end{aligned}$$

together with (2.4) shows that the nonnegative càdlàg function $t \mapsto \mathbb{E}[e^{-\alpha t} \phi(t)]$ is bounded from above by a directly Riemann integrable function on $[0, \infty)$, hence it is directly Riemann integrable on $[0, \infty)$. Consequently, the assumptions of Theorem 3.1 in [22] are satisfied. The cited theorem gives that, as $t \rightarrow \infty$,

$$\begin{aligned} e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} f(t-S(u)) &= e^{-\alpha t} \sum_{u \in \mathcal{G}} [\phi]_u(t-S(u)) = e^{-\alpha t} \mathcal{Z}_t^\phi \\ &\xrightarrow{\mathbb{P}} \frac{W}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} f(x-X_k) dx \right]. \quad (3.9) \end{aligned}$$

In the special case $f = 1$, we find

$$e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \xrightarrow{\mathbb{P}} \frac{W}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} dx \right] = \frac{1-m(2\alpha)}{-\alpha m'(\alpha)} W. \quad (3.10)$$

As for the L^1 -convergence, use Corollary 3.3 in [22].

Finally, we pass to the a.s. convergence. Since ϕ is càdlàg and

$$e^{-\alpha t} \phi(t) = e^{\alpha t} \sum_{k=1}^N e^{-2\alpha X_k} \mathbb{1}_{[0, X_k)}(t) f(t-X_k) \leq \|f\|_\infty \sum_{k=1}^N e^{-\alpha X_k} \mathbb{1}_{[0, X_k)}(t),$$

condition (2.6), which is Condition 5.1 from [22], entails

$$\mathbb{E} \left[\sup_{t \geq 0} \frac{e^{-\alpha t} \phi(t)}{g(t)} \right] < \infty.$$

This is Condition 5.2 from [22] with the particular ϕ and $h = g$. Thus, the assumptions of Theorem 5.4 in [22] are satisfied. According to this theorem, the convergence in (3.6) holds also in the almost sure sense. \square

The lemma has the following corollary, which we use in the proofs of Theorems 2.2 and 2.3, but not in the proof of Theorem 2.1.

Corollary 3.2. *Suppose that (A1) through (A4) hold. Then, for any fixed $\delta > 0$, as $t \rightarrow \infty$,*

$$e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} v_{\delta+t-S(u)} \rightarrow c_\delta W \quad \text{in } L^1 \tag{3.11}$$

where

$$c_\delta := \frac{1}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} v_{\delta+x-X_k} dx \right]. \tag{3.12}$$

The convergence in (3.11) holds in the stronger almost sure sense if (A5) holds. Further, c_δ is nondecreasing as a function of δ with $c_\delta > 0$ for every $\delta > 0$. The limits of c_δ as $\delta \rightarrow 0$ and $\delta \rightarrow \infty$ are given by

$$\lim_{\delta \downarrow 0} c_\delta = 0 \quad \text{and} \quad c_\infty := \lim_{\delta \uparrow \infty} c_\delta = \frac{\sigma^2}{-\alpha m'(\alpha)}. \tag{3.13}$$

Proof. The validity of (A4) implies that the function $t \mapsto v_t$ is bounded by (3.4). From the discussion preceding (3.4), we infer that $t \mapsto v_t$ is nondecreasing and right-continuous, hence càdlàg. Thus, for any fixed $\delta > 0$, (3.11) follows from Lemma 3.1 with $f(t) = v_{\delta+t}$, $t \in \mathbb{R}$. We infer $c_\delta > 0$ from $0 < \sigma^2 = v_0 \leq v_t$ for $t \geq 0$ and the representation

$$\int_0^{X_k} e^{\alpha x} v_{\delta+x-X_k} dx = \mathbb{1}_{\{X_k \leq \delta\}} \int_0^{X_k} e^{\alpha x} v_{\delta+x-X_k} dx + \mathbb{1}_{\{X_k > \delta\}} \int_{X_k-\delta}^{X_k} e^{\alpha x} v_{\delta+x-X_k} dx > 0.$$

Since $t \mapsto v_t$ is a nondecreasing function, so is $\delta \mapsto c_\delta$. Hence, $\lim_{\delta \downarrow 0} c_\delta$ and $c_\infty := \lim_{\delta \uparrow \infty} c_\delta$ exist. Since $v_t = 0$ for $t < 0$ and $\lim_{t \rightarrow \infty} v_t = \sigma^2/(1 - m(2\alpha))$, (3.13) follows with the help of the monotone convergence theorem. \square

Proof of Theorem 2.1. For $t \geq 0$, we infer from (3.3)

$$e^{\alpha t/2} (W - W_t) = e^{\alpha t/2} \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} ([W]_u - 1) \quad \text{a.s.},$$

which given \mathcal{H}_t is a weighted sum of independent, centered and square-integrable random variables (Theorem 4.14 in [18]). We show that the distribution of this sum given \mathcal{H}_t converges in probability to the distribution of a centered normal random variable. For simplicity, we write $\mathbb{E}_t[\cdot]$ to denote the conditional expectation given \mathcal{H}_t . Then, by (2.10),

$$\begin{aligned} \mathbb{E}_t \left[\left(e^{\alpha t/2} \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} ([W]_u - 1) \right)^2 \right] &= e^{\alpha t} \mathbb{E}_t \left[\sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} ([W]_u - 1)^2 \right] \\ &= \frac{\sigma^2}{1 - m(2\alpha)} e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)}. \end{aligned} \tag{3.14}$$

Observe that $\mathbb{E}[Z_1^2] < \infty$ entails $\mathbb{E}[Z_1 \log^+ Z_1] < \infty$ whence, by Lemma 3.1, (3.6) holds in L^1 and thereupon

$$\mathbb{E}_t \left[\left(e^{\alpha t/2} \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} ([W]_u - 1) \right)^2 \right] \rightarrow \frac{\sigma^2}{-\alpha m'(\alpha)} W \quad \text{in } L^1 \quad (3.15)$$

as $t \rightarrow \infty$. Further, for $x \geq 0$, we define

$$\sigma_W^2(x) := \mathbb{E}[(W - 1)^2 \mathbb{1}_{\{|W-1|>x\}}]$$

and notice that $\lim_{x \rightarrow \infty} \sigma_W^2(x) = 0$ by (3.4) and the dominated convergence theorem. Consequently, for all $\varepsilon > 0$,

$$\begin{aligned} & \sum_{u \in \mathcal{I}_t} \mathbb{E}_t \left[\left(e^{\alpha t/2} e^{-\alpha S(u)} ([W]_u - 1) \right)^2 \mathbb{1}_{\{|e^{\alpha t/2} e^{-\alpha S(u)} ([W]_u - 1)| > \varepsilon\}} \right] \\ &= e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \sigma_W^2(\varepsilon e^{-\alpha t/2} e^{\alpha S(u)}) \leq \sigma_W^2(\varepsilon e^{\alpha t/2}) e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \xrightarrow{P} 0 \end{aligned}$$

as $t \rightarrow \infty$ by (3.6). Thus, (2.5) through (2.7) in [9] hold, and we conclude (2.12). \square

For the proof of the functional central limit theorem, Theorem 2.2, we need some preparatory lemmas.

Lemma 3.3. *Suppose that (A1) through (A4) hold. Then the family $((W - W_t)^2)_{t \in \mathbb{R}}$ is uniformly integrable. In other words, the function $\sigma_t^2(x) := \mathbb{E}[|W - W_t|^2 \mathbb{1}_{\{|W - W_t| > x\}}]$, $t \in \mathbb{R}$, $x \geq 0$ is a bounded on $\mathbb{R} \times [0, \infty)$ with*

$$\sup_{t \in \mathbb{R}} \sigma_t^2(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (3.16)$$

Proof. From the discussion preceding (3.4), we know that $W_t \rightarrow W$ a. s. and in L^2 , and that $W_t = \mathbb{E}[W | \mathcal{H}_t]$ a. s. Thus, the family $(W_t^2)_{t \geq 0}$ is uniformly integrable, hence so is the family $((W - W_t)^2)_{t \geq 0}$, which implies (3.16). \square

Lemma 3.4. *Suppose that (A1) through (A4) hold and let $0 \leq r < s < \infty$. Then, as $t \rightarrow \infty$ and with $\text{Cov}_t[\cdot, \cdot]$ and $\text{Var}_t[\cdot]$ denoting conditional covariance and variance given \mathcal{H}_t , respectively,*

$$\begin{aligned} \text{Cov}_t[e^{\alpha t/2}(W - W_{t+r}), e^{\alpha t/2}(W - W_{t+s})] &= \text{Var}_t[e^{\alpha t/2}(W - W_{t+s})] \\ &\rightarrow \frac{e^{-\alpha s} \sigma^2}{-\alpha m'(\alpha)} W \quad \text{in } L^1. \end{aligned} \quad (3.17)$$

Notice that the expression on the right-hand side of (3.17) is exactly the (conditional) covariance of the limiting process in (2.13).

Proof. For any $t \geq 0$, we have

$$\begin{aligned} & \text{Cov}_t[e^{\alpha t/2}(W - W_{t+r}), e^{\alpha t/2}(W - W_{t+s})] \\ &= \text{Cov}_t[e^{\alpha t/2}(W - W_{t+s} + W_{t+s} - W_{t+r}), e^{\alpha t/2}(W - W_{t+s})] \\ &= \text{Var}_t[e^{\alpha t/2}(W - W_{t+s})] + \text{Cov}_t[e^{\alpha t/2}(W_{t+s} - W_{t+r}), e^{\alpha t/2}(W - W_{t+s})] \\ &= \text{Var}_t[e^{\alpha t/2}(W - W_{t+s})]. \end{aligned}$$

since increments of square-integrable martingales are (conditionally) uncorrelated. It thus remains to investigate $\text{Var}_t[e^{\alpha t/2}(W - W_{t+s})]$ as $t \rightarrow \infty$. Here, arguing as in the proof of Theorem 2.1 and using Lemma 3.1, we infer, with $\sigma_t^2 = \mathbb{E}[(W - W_t)^2]$ for $t \in \mathbb{R}$,

$$\begin{aligned} \text{Var}_t[e^{\alpha t/2}(W - W_{t+s})] &= e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \sigma_{t+s-S(u)}^2 \\ &\rightarrow \frac{W}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} \sigma_{s+x-X_k}^2 dx \right] =: d_s W \end{aligned}$$

as $t \rightarrow \infty$ in L^1 , where $d_s \geq 0$ is a constant. We now calculate the constant d_s , but avoid evaluating it directly. Instead, notice that

$$\begin{aligned} \text{Var}_t [e^{\alpha t/2}(W - W_{t+s})] &= \mathbb{E}_t [(e^{\alpha t/2}(W - W_{t+s}))^2] \\ &= \mathbb{E}_t [\mathbb{E}_{t+s} [(e^{\alpha t/2}(W - W_{t+s}))^2]]. \end{aligned}$$

By (3.15), $\mathbb{E}_{t+s} [(e^{\alpha t/2}(W - W_{t+s}))^2]$ converges in L^1 as $t \rightarrow \infty$, in particular, the family $(\mathbb{E}_{t+s} [(e^{\alpha t/2}(W - W_{t+s}))^2])_{t \geq 0}$ is uniformly integrable. Hence so is $(\mathbb{E}_t [(e^{\alpha t/2}(W - W_{t+s}))^2])_{t \geq 0}$. Consequently,

$$\mathbb{E} [\mathbb{E}_t [(e^{\alpha t/2}(W - W_{t+s}))^2]] \rightarrow d_s \quad \text{as } t \rightarrow \infty.$$

On the other hand,

$$\mathbb{E} [\mathbb{E}_t [(e^{\alpha t/2}(W - W_{t+s}))^2]] = e^{-\alpha s} \mathbb{E} [\mathbb{E}_t [(e^{\alpha(t+s)/2}(W - W_{t+s}))^2]] \rightarrow \frac{e^{-\alpha s} \sigma^2}{-\alpha m'(\alpha)}$$

by (3.15). Hence, $d_s = \frac{e^{-\alpha s} \sigma^2}{-\alpha m'(\alpha)}$. □

Proof of Theorem 2.2. We first prove weak convergence of the finite-dimensional distributions on $[0, \infty)$. To this end, fix $n \in \mathbb{N}$ and $0 \leq s_1 < s_2 \dots < s_n$. Abbreviate $t + s_k$ by t_k , $k = 1, \dots, n$ and define $t_{n+1} := \infty$ and $W_{t_{n+1}} := W$. We use the Cramér-Wold device which reduces the problem to studying the convergence in law of the following linear combinations:

$$\sum_{k=1}^n \gamma_k e^{\alpha t/2} (W - W_{t_k})$$

for fixed $\gamma_1, \dots, \gamma_n \in \mathbb{R}$. Recall that Var_t and Cov_t denote the conditional variance and covariance given \mathcal{H}_t , respectively. Lemma 3.4 yields

$$\begin{aligned} \text{Var}_t \left[\sum_{k=1}^n \gamma_k e^{\alpha t/2} (W - W_{t_k}) \right] &= \sum_{k=1}^n \gamma_k^2 e^{\alpha t} \text{Var}_t [W - W_{t_k}] + 2 \sum_{1 \leq j < k \leq n} \gamma_j \gamma_k \text{Cov}_t [W - W_{t_j}, W - W_{t_k}] \\ &\rightarrow \left(\sum_{k=1}^n \gamma_k^2 e^{-\alpha s_k} + 2 \sum_{1 \leq j < k \leq n} \gamma_j \gamma_k e^{-\alpha s_k} \right) \frac{\sigma^2}{-\alpha m'(\alpha)} W \quad \text{in } L^1 \end{aligned} \tag{3.18}$$

as $t \rightarrow \infty$. Note that the expression in the last line is exactly the conditional variance of the linear combinations corresponding to the limiting process in (2.13) since

$$\begin{aligned} \text{Var} \left[\sum_{k=1}^n \gamma_k \left(\frac{\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} B_{e^{-\alpha s_k}} \mid W \right] &= \text{Var} \left[\sum_{k=1}^n \gamma_k B_{e^{-\alpha s_k}} \right] \frac{\sigma^2}{-\alpha m'(\alpha)} W \\ &= \left(\sum_{k=1}^n \gamma_k^2 e^{-\alpha s_k} + 2 \sum_{1 \leq j < k \leq n} \gamma_j \gamma_k e^{-\alpha s_k} \right) \frac{\sigma^2}{-\alpha m'(\alpha)} W \quad \text{a. s.} \end{aligned}$$

Next, we check the Lindeberg-Feller condition. Using the decomposition

$$\sum_{k=1}^n \gamma_k e^{\alpha t/2} (W - W_{t_k}) = \sum_{k=1}^n \gamma_k e^{\alpha t/2} \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} ([W]_u - [W_{t_k-}]_u \circ (S(u))),$$

we argue as in the proof of Theorem 2.1. Recalling the notation $\sigma_r^2(x) = \mathbb{E}[|W - W_r|^2 \mathbb{1}_{\{|W - W_r| > x\}}]$, we infer, for all $\varepsilon > 0$,

$$\begin{aligned} & \sum_{u \in \mathcal{I}_t} \mathbb{E}_t \left[\left(e^{\alpha(t/2 - S(u))} ([W]_u - [W_{t_k-}]_u \circ (S(u))) \right)^2 \right. \\ & \quad \left. \cdot \mathbb{1}_{\{|e^{\alpha(t/2 - S(u))} ([W]_u - [W_{t_k-}]_u \circ (S(u)))| > \varepsilon\}} \right] \\ & = e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \sigma_{t_k - S(u)}^2 (e^{-\alpha t/2} e^{\alpha S(u)} \varepsilon) \xrightarrow{\mathbb{P}} 0 \end{aligned}$$

by (3.6) and (3.16).

This implies that the finite-dimensional distributions of $(e^{\alpha t/2} (W - W_{t+s}))_{s \geq 0}$ given \mathcal{H}_t converge weakly to those of $((\sigma^2 / (-\alpha m'(\alpha)) W)^{1/2} B_{e^{-\alpha s}})_{s \geq 0}$ given \mathcal{F}_∞ . From this, we conclude that

$$(e^{\alpha t/2} (W - W_{t+s}))_{s \geq 0} \xrightarrow{\text{fdd}} \left(\left(\frac{\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} B_{e^{-\alpha s}} \right)_{s \geq 0} \text{ as } t \rightarrow \infty \tag{3.19}$$

where $\xrightarrow{\text{fdd}}$ denotes weak convergence of finite-dimensional distributions.

The next step is to check that the distributions of the family

$$(e^{\alpha t/2} (W_t - W_{t+s}))_{s \geq 0}, t \geq 0 \text{ are tight in } D([0, \infty)). \tag{3.20}$$

We use Aldous's tightness criterion, see e.g. Theorem 16.10 on p. 178 of [5]. To this end, we first check condition (16.22) in the cited source, i.e.,

$$\lim_{x \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P} \left(\sup_{0 \leq s \leq b} |e^{\alpha t/2} (W_t - W_{t+s})| \geq x \right) = 0 \text{ for all } b > 0. \tag{3.21}$$

For any fixed $b, t, x > 0$, by Chebyshev's inequality and Doob's maximal L^p -inequality (with $p = 2$),

$$\mathbb{P} \left(\sup_{0 \leq s \leq b} |e^{\alpha t/2} (W_t - W_{t+s})| \geq x \right) \leq \frac{4e^{\alpha t}}{x^2} \mathbb{E}[|W_{t+b} - W_t|^2] \text{ for all } b > 0. \tag{3.22}$$

Here, an application of (3.11) yields

$$\begin{aligned} e^{\alpha t} \mathbb{E}[|W_{t+b} - W_t|^2] & = e^{\alpha t} \mathbb{E} \left[\left(\sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} ([W_{t+b-}]_u \circ S(u) - 1) \right)^2 \right] \\ & = e^{\alpha t} \mathbb{E} \left[\sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} v_{t+b-S(u)} \right] \rightarrow c_b \text{ as } t \rightarrow \infty. \end{aligned}$$

Using this after taking the limsup as $t \rightarrow \infty$ in (3.22), and then letting $x \rightarrow \infty$ gives (3.21).

We now turn to the second condition of Aldous's criterion, namely, for all $\varepsilon, b > 0$

$$\lim_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \sup_{\tau} \mathbb{P}(e^{\alpha t/2} |(W_t - W_{t+\tau+\delta}) - (W_t - W_{t+\tau})| \geq \varepsilon) = 0 \tag{3.23}$$

where \sup_{τ} is the supremum over all discrete stopping times $0 \leq \tau \leq b$ with respect to the filtration $(\mathcal{H}_{t+s})_{s \geq 0}$. Here, 'discrete' means that τ takes only finitely many values. To

prove (3.23), fix $\varepsilon, b > 0$ and let $\tau \leq b$ be a discrete stopping time. Then $\tau_t := t + \tau$ is a stopping time with respect to $(\mathcal{H}_s)_{s \geq 0}$. We claim that \mathcal{I}_{τ_t} is an optional line. Clearly, it is a random line. Further, if $t \leq t_1 \leq \dots \leq t_m \leq t + b$ are the values τ takes, then, for any deterministic line $L \subseteq \mathcal{I}$,

$$\{\mathcal{I}_{\tau_t} \preceq L\} = \bigcup_{k=1}^m (\{\tau_t = t_k\} \cap \{\mathcal{I}_{t_k} \preceq L\}) \in \mathcal{F}_L$$

since τ_t is a stopping time with respect to $(\mathcal{H}_s)_{s \geq 0}$, thus $\{\tau_t = t_k\} \in \mathcal{H}_{t_k} = \mathcal{F}_{\mathcal{I}_{t_k}}$ and, consequently, $\{\tau_t = t_k\} \cap \{\mathcal{I}_{t_k} \preceq L\} \in \mathcal{F}_L$ by the definition of $\mathcal{F}_{\mathcal{I}_{t_k}}$. We may now use the strong Markov branching property (Theorem 4.14 in [18]) to conclude that

$$\begin{aligned} \mathbb{P}(e^{\alpha t/2} |(W_t - W_{t+\tau+\delta}) - (W_t - W_{t+\tau})| \geq \varepsilon) &= \mathbb{P}(e^{\alpha t/2} |(W_{t+\tau+\delta}) - W_{t+\tau}| \geq \varepsilon) \\ &\leq \frac{e^{\alpha t}}{\varepsilon^2} \mathbb{E}[|(W_{t+\tau+\delta}) - W_{t+\tau}|^2] \\ &= \frac{e^{\alpha t}}{\varepsilon^2} \mathbb{E} \left[\mathbb{E} \left[\left(\sum_{u \in \mathcal{I}_{\tau_t}} e^{-\alpha S(u)} ([W_{\tau_t+\delta-\cdot}]_u \circ S(u) - 1) \right)^2 \middle| \mathcal{F}_{\mathcal{I}_{\tau_t}} \right] \right] \\ &= \frac{e^{\alpha t}}{\varepsilon^2} \mathbb{E} \left[\sum_{u \in \mathcal{I}_{t+\tau}} e^{-2\alpha S(u)} v_{t+\tau+\delta-S(u)} \right]. \end{aligned}$$

Here, if $0 \leq s_1 \leq \dots \leq s_m \leq b$ denote the values of τ , then

$$\begin{aligned} e^{\alpha t} \sum_{u \in \mathcal{I}_{t+\tau}} e^{-2\alpha S(u)} v_{t+\tau+\delta-S(u)} &= \sum_{k=1}^m e^{\alpha s_k} \mathbb{1}_{\{\tau=s_k\}} e^{-\alpha(t+s_k)} \sum_{u \in \mathcal{I}_{t+s_k}} e^{-2\alpha S(u)} v_{t+\tau+\delta-S(u)} \\ &\rightarrow \sum_{k=1}^m e^{\alpha s_k} \mathbb{1}_{\{\tau=s_k\}} c_\delta W = e^{\alpha \tau} c_\delta W \quad \text{in } L^1 \text{ as } t \rightarrow \infty \end{aligned}$$

by Corollary 3.2. Hence,

$$\limsup_{t \rightarrow \infty} \sup_{\tau} \mathbb{P}(e^{\alpha t/2} |(W_t - W_{t+\tau+\delta}) - (W_t - W_{t+\tau})| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} e^{\alpha b} c_\delta.$$

Thus, (3.23) follows from (3.13). Combining (3.21) and (3.23) with Theorem 16.10 on p. 178 of [5] yields (3.20). Since the increments of the processes $(e^{\alpha t/2}(W_t - W_{t+s}))_{s \geq 0}$ and $(e^{\alpha t/2}(W - W_{t+s}))_{s \geq 0}$ are the same, Theorem 16.5 in [5] in combination with the fact that $e^{\alpha t/2}(W - W_t)$ converges in distribution implies that also the distributions of the family $(e^{\alpha t/2}(W - W_{t+s}))_{s \geq 0}, t \geq 0$ are tight in $D([0, \infty))$. Together with the convergence of the finite-dimensional distributions, we obtain

$$(e^{\alpha t/2}(W - W_{t+s}))_{s \geq 0} \Rightarrow \left(\left(\frac{\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} B_{e^{-\alpha s}} \right)_{s \geq 0} \text{ as } t \rightarrow \infty \text{ on } D([0, \infty)). \quad (3.24)$$

For any $r > 0$ the shift operator $\theta_r : f(\cdot) \mapsto f(\cdot + r)$ is an isometry between $D([0, b])$ and $D([-r, b - r])$. Hence, θ_r is a continuous mapping from $D([0, \infty))$ to $D([-r, \infty))$. In particular, for any $r > 0$, by the scaling invariance of Brownian motion, we infer

$$\begin{aligned} (e^{\alpha t/2}(W - W_{t+s}))_{s \geq -r} &= \theta_r \left((e^{\alpha t/2}(W - W_{t-r+s})) \right)_{s \geq -r} \\ &\Rightarrow \theta_r \left(e^{\alpha r/2} \left(\frac{\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} B_{e^{-\alpha s}} \right)_{s \geq -r} \\ &\stackrel{\text{law}}{=} \left(\left(\frac{\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} B_{e^{-\alpha s}} \right)_{s \geq 0}, \end{aligned}$$

as $t \rightarrow \infty$ on $D([-r, \infty))$. Here, $\stackrel{\text{law}}{=}$ denotes equality of distributions. Since this holds for every $r > 0$, we arrive at (2.13). \square

3.3 Law of the iterated logarithm: proofs

The key tool in our proof of the law of the iterated logarithm for Nerman’s martingale is Proposition 7.2 on p. 436 in [2].

Lemma 3.5. *Let $(\mathcal{R}_n)_{n \in \mathbb{N}_0}$ be an increasing sequence of σ -fields and $(T_n)_{n \in \mathbb{N}_0}$ be a sequence of random variables such that*

$$\sum_{n \geq 0} \sup_{y \in \mathbb{R}} |\mathbb{P}(T_n \leq y | \mathcal{R}_n) - \Phi(y)| < \infty \quad \text{a. s.} \tag{3.25}$$

where $\Phi(y) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx$, $y \in \mathbb{R}$. Then

$$\limsup_{n \rightarrow \infty} \frac{T_n}{\sqrt{\log n}} \leq \sqrt{2} \quad \text{a. s.}$$

If there is a $k \in \mathbb{N}$ such that T_n is \mathcal{R}_{n+k} -measurable for each $n \in \mathbb{N}_0$, then

$$\limsup_{n \rightarrow \infty} \frac{T_n}{\sqrt{\log n}} = \sqrt{2} \quad \text{a. s.}$$

The next result is Lemma A.2 in [13], an infinite version of the Berry-Esseen inequality for independent, centered random variables. It is likely that this fact is also given in other sources.

Lemma 3.6. *Let Y_1, Y_2, \dots be independent random variables with $\mathbb{E}[Y_i] = 0$, $\sigma_{Y_i}^2 := \text{Var}[Y_i] < \infty$ and $\rho_{Y_i} := \mathbb{E}[|Y_i|^3]$, $i \in \mathbb{N}$. If $\sum_{i \geq 1} \sigma_{Y_i}^2 < \infty$, then, for an absolute constant C ,*

$$\sup_{y \in \mathbb{R}} \left| \mathbb{P} \left(\frac{\sum_{i \geq 1} Y_i}{(\sum_{i \geq 1} \sigma_{Y_i}^2)^{1/2}} \leq y \right) - \Phi(y) \right| \leq C \frac{\sum_{i \geq 1} \rho_{Y_i}}{(\sum_{i \geq 1} \sigma_{Y_i}^2)^{3/2}}. \tag{3.26}$$

Recall that the limits in (2.14) and (2.15) are considered on the survival set \mathcal{S} . We only give a complete proof for the upper limit. Investigating $W_t - W$ rather than $W - W_t$ gives the result for the lower limit at no extra cost. Although the scheme of the proof is similar to that of Theorem 3.4 on p. 130 in [2] in which a Markov branching process was investigated, technical details differ at places. Without loss of generality we assume in what follows that $\mathbb{P}(\mathcal{S}) = 1$ (otherwise we have to use Lemma 3.5 with the probability measure \mathbb{P} replaced with $\mathbb{P}(\cdot | \mathcal{S})$ and write “a. s. on the survival set \mathcal{S} ” rather than “a. s.” throughout). This assumption ensures that W is positive a. s. rather than with positive probability.

For $t, r > 0$, we use the following representations derived from (3.2) and (3.3):

$$W_{t+r} - W_t = \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} ([W_{r+t-\cdot}]_u \circ S(u) - 1)$$

and $W - W_t = \sum_{u \in \mathcal{I}_t} e^{-\alpha S(u)} ([W]_u - 1).$

Recall that the $S(u)$, $u \in \mathcal{I}_t$ are \mathcal{H}_t -measurable, whereas the $[W_{r-x}]_u$, $u \in \mathcal{I}_t$ and the $[W]_u$, $u \in \mathcal{I}_t$ are independent of \mathcal{H}_t , see Theorem 4.14 in [18]. Since we do not assume $\mathbb{E}[|W_t|^3] < \infty$, we start by investigating the sums as above with truncated summands. For $t \geq 0$ and $r \in (0, \infty]$, let

$$W_{t,r}(u) := e^{-\alpha S(u)} ([W_{r+t-\cdot}]_u \circ S(u) - 1) \mathbb{1}_{\{e^{\alpha t/2} e^{-\alpha S(u)} |[W_{r+t-\cdot}]_u \circ S(u) - 1| \leq 1\}}$$

and

$$V_{t,r} = \sum_{u \in \mathcal{I}_t} (W_{t,r}(u) - \mathbb{E}_t[W_{t,r}(u)]). \tag{3.27}$$

Lemma 3.7. For $r \in (0, \infty]$, with c_r as defined in (3.12), we have

$$\lim_{t \rightarrow \infty} e^{\alpha t} \text{Var}_t[V_{t,r}] = c_r W \quad \text{a. s.} \tag{3.28}$$

Proof. Conditionally on \mathcal{H}_t , the random variables $W_{t,r}(u)$, $u \in \mathcal{I}_t$ are independent (but not identically distributed). By definition of $V_{t,r}$, we have

$$\text{Var}_t[V_{t,r}] = \sum_{u \in \mathcal{I}_t} \mathbb{E}_t[W_{t,r}(u)^2] - \sum_{u \in \mathcal{I}_t} (\mathbb{E}_t[W_{t,r}(u)])^2 =: G'_{t,r} - G''_{t,r}.$$

It is sufficient to show that

$$\lim_{t \rightarrow \infty} e^{\alpha t} G'_{t,r} = c_r W \quad \text{a. s.} \tag{3.29}$$

$$\text{and} \quad \lim_{t \rightarrow \infty} e^{\alpha t} G''_{t,r} = 0 \quad \text{a. s.} \tag{3.30}$$

To this end, for $t \in \mathbb{R}$, let F_t and G_t denote the distribution functions of $|W_t - 1|$ and $\sup_{0 \leq s \leq t} |W_s - 1|$, respectively. For instance, $F_t(x) = \mathbb{P}(|W_t - 1| \leq x)$ for $x \in \mathbb{R}$.

Proof of (3.29). We have

$$G'_{t,r} = \sum_{u \in \mathcal{I}_t} \mathbb{E}_t[W_{t,r}(u)^2] = \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \int_{[0, e^{-\alpha t/2} e^{\alpha S(u)}]} x^2 dF_{r+t-S(u)}(x).$$

For $u \in \mathcal{I}_t$, $S(u) > t$ and therefore, for every $c > 0$, $e^{-\alpha t/2} e^{\alpha S(u)} \geq e^{\alpha t/2} \geq c$ for all sufficiently large t . Consequently

$$\sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} v_{r+t-S(u)}(c) \leq G'_{t,r} \leq \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} v_{r+t-S(u)}$$

where $v_s(c) := \int_{[0, c]} x^2 dF_s(x)$. Corollary 3.2 yields

$$\begin{aligned} e^{\alpha t} G'_{t,r} &\leq e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} v_{r+t-S(u)} \\ &\rightarrow \frac{W}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} v_{r+x-X_k} dx \right] = c_r W \end{aligned} \tag{3.31}$$

a. s. and in L^1 as $t \rightarrow \infty$ where the definition of c_r should be recalled from (3.12). Analogously, for any $c > 0$,

$$\begin{aligned} e^{\alpha t} G'_{t,r} &\geq e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} v_{r+t-S(u)}(c) \\ &\rightarrow \frac{W}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} v_{r+x-X_k}(c) dx \right] \end{aligned} \tag{3.32}$$

a. s. and in L^1 . Since $v_s(c) \uparrow v_s$ as $c \uparrow \infty$, (3.29) follows from the monotone convergence theorem.

Proof of (3.30). Since $\mathbb{E}[[W_s]_u - 1] = 0$ for all $s \in \mathbb{R}$,

$$\begin{aligned} G''_{t,r} &= \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \left(\mathbb{E} \left(([W_{r+t-\cdot}]_u \circ S(u) - 1) \mathbb{1}_{\{e^{\alpha t/2} e^{-\alpha S(u)} |[W_{r+t-\cdot}]_u \circ S(u) - 1| \leq 1\}} \right) \right)^2 \\ &= \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \left(\mathbb{E} \left(([W_{r+t-\cdot}]_u \circ S(u) - 1) \mathbb{1}_{\{e^{\alpha t/2} e^{-\alpha S(u)} |[W_{r+t-\cdot}]_u \circ S(u) - 1| > 1\}} \right) \right)^2. \end{aligned}$$

In view of $[W_s]_u - 1 \leq |[W_s]_u - 1|$ and $S(u) > t$ for $u \in \mathcal{I}_t$, we have

$$\begin{aligned} G''_{t,r} &\leq \sum_{u \in \mathcal{I}_t} \left(e^{-2\alpha S(u)} \left(\int_{(e^{-\alpha t/2} e^{\alpha S(u)}, \infty)} x \, dF_{r+t-S(u)}(x) \right)^2 \right) \\ &\leq \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \left(\int_{(c, \infty)} x \, dF_{r+t-S(u)}(x) \right)^2 \\ &=: \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \bar{v}_{r+t-S(u)}(c) \end{aligned}$$

for all sufficiently large t where $\bar{v}_x(c) := (\mathbb{E}[|W_x - 1| \mathbf{1}_{\{|W_x - 1| \geq c\}}])^2$ for $x \geq 0$. Since $((W_t - 1)^2)_{t \geq 0}$ is a submartingale,

$$0 \leq \bar{v}_t(c) \leq \mathbb{E}[(W_t - 1)^2] \leq \mathbb{E}[(W - 1)^2] = \sigma_W^2 < \infty,$$

i.e., $t \mapsto \bar{v}_t(c)$ is a nonnegative and bounded function. Doob’s maximal inequality enables us to apply the dominated convergence theorem to show that the function $t \mapsto \bar{v}_t(c)$ is càdlàg. Consequently, we may apply (3.7) to conclude that

$$\begin{aligned} e^{\alpha t} G''_{t,r} &\leq \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} \bar{v}_{r+t-S(u)}(c) \\ &\rightarrow \frac{W}{-m'(\alpha)} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} \bar{v}_{r+x-X_k}(c) \, dx \right] \quad \text{a. s. and in } L^1. \end{aligned}$$

Further,

$$\|\bar{v}(\cdot)(c)\|_\infty := \sup_{t \in \mathbb{R}} v_t(c) = \sup_{t \in \mathbb{R}} \mathbb{E}[|W_t - 1| \mathbf{1}_{\{|W_t - 1| \geq c\}}] \rightarrow 0 \quad \text{as } c \rightarrow \infty.$$

Therefore, as $c \rightarrow \infty$,

$$\begin{aligned} \mathbb{E} \left[\sum_{k=1}^N e^{-2\alpha X_k} \int_0^{X_k} e^{\alpha x} \bar{v}_{r+x-X_k}(c) \, dx \right] &= \int_{[0, \infty)} e^{-2\alpha t} \int_0^t e^{\alpha x} \bar{v}_{r+x-t}(c) \, dx \, \mu(dt) \\ &\leq \frac{\|\bar{v}(\cdot)(c)\|_\infty}{\alpha} \int_{[0, \infty)} e^{-2\alpha t} (e^{\alpha t} - 1) \, \mu(dt) \rightarrow 0. \end{aligned}$$

This proves (3.30). □

Our proof of (2.14) consists of two parts. In the first part, Lemma 3.8, we obtain (2.14) with the limit $t \rightarrow \infty$ taken along the points of a lattice δn , $n \in \mathbb{N}$ where $\delta > 0$ is fixed but arbitrary. In the second part, we extend the convergence in (2.14) along lattice sequences to arbitrary sequences $t \rightarrow \infty$.

Lemma 3.8. *For every $\delta > 0$, we have*

$$\limsup_{n \rightarrow \infty} \frac{e^{\alpha n \delta / 2}}{\sqrt{\log(n \delta)}} (W - W_{n \delta}) = \left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} \quad \text{a. s.} \tag{3.33}$$

Proof. Fix an arbitrary $\delta > 0$ and $r \in \delta \mathbb{N} \cup \{\infty\}$. We claim that (3.25) holds for the random variables

$$T_n := V_{\delta n, r} / \sqrt{\text{Var}_{\delta n}[V_{\delta n, r}]}, \quad n \in \mathbb{N}_0$$

and $\mathcal{R}_n = \mathcal{H}_{\delta n}$. Conditionally given $\mathcal{H}_{\delta n}$, $V_{\delta n,r}$ is a weighted sum of independent, centered random variables to which Lemma 3.6 applies. In particular, (3.26) yields

$$\begin{aligned} & \sup_{y \in \mathbb{R}} \left| \mathbb{P} \left(\frac{V_{\delta n,r}}{\sqrt{\text{Var}_{\delta n}[V_{\delta n,r}]}} \leq y \mid \mathcal{H}_{\delta n} \right) - \Phi(y) \right| \\ & \leq C \frac{\sum_{u \in \mathcal{I}_{\delta n}} \mathbb{E}_{\delta n} [|W_{\delta n,r}(u) - \mathbb{E}_{\delta n}[W_{\delta n,r}(u)]|^3]}{(\text{Var}_{\delta n}[V_{\delta n,r}])^{3/2}} \\ & \leq 8C \frac{\sum_{u \in \mathcal{I}_{\delta n}} \mathbb{E}_{\delta n} [|W_{\delta n,r}(u)|^3]}{(\text{Var}_{\delta n}[V_{\delta n,r}])^{3/2}}, \end{aligned}$$

where $C > 0$ is a finite absolute constant. In view of (3.28), the condition (3.25) will follow from the almost sure finiteness of

$$I := \sum_{n \geq 0} e^{3\alpha\delta n/2} \sum_{u \in \mathcal{I}_{\delta n}} \mathbb{E}_{\delta n} [|W_{\delta n,r}(u)|^3]. \tag{3.34}$$

Here, $\mathbb{E}_{\delta n} [|W_{\delta n,r}(u)|^3] = e^{-3\alpha S(u)} \int_{[0,\infty)} x^3 \mathbb{1}_{\{e^{-\alpha\delta n/2} e^{\alpha S(u)} \geq x\}} dF_{r+\delta n-S(u)}(x)$. If $S(u) > \delta n + r$, then $F_{r+\delta n-S(u)}$ is the distribution function of the Dirac measure at 0 and hence the integral vanishes. Otherwise, since $F_t \geq G_t \geq G_\infty$ pointwise for any $t > 0$, we may estimate

$$\begin{aligned} & \int_{[0,\infty)} e^{3\alpha\delta n/2-3\alpha S(u)} x^3 \mathbb{1}_{\{e^{-\alpha\delta n/2} e^{\alpha S(u)} \geq x\}} dF_{r+\delta n-S(u)}(x) \\ & \leq \int_{[0,\infty)} \min\{e^{3\alpha\delta n/2-3\alpha S(u)} x^3, 1\} dF_{r+\delta n-S(u)}(x) \\ & \leq \int_{[0,\infty)} \min\{e^{3\alpha\delta n/2-3\alpha S(u)} x^3, 1\} dG_\infty(x). \end{aligned}$$

As in Section 2.3, denote by $(S_n)_{n \in \mathbb{N}_0}$ a random walk with i. i. d. increments $S_n - S_{n-1}$, $n \in \mathbb{N}$ and increment law $\mathbb{P}(S_n - S_{n-1} \in dx) := e^{-\alpha x} \mu(dx)$. By the many-to-one lemma [25, Section 1.3],

$$\begin{aligned} \mathbb{E}[I] & \leq \int_{[0,\infty)} \mathbb{E} \left[\sum_{n \geq 0} \sum_{u \in \mathcal{I}_{\delta n}} \min\{e^{3\alpha\delta n/2-3\alpha S(u)} x^3, 1\} \right] dG_\infty(x) \\ & = \int_{[0,\infty)} \mathbb{E} \left[\sum_{n \geq 0} \min\{e^{3\alpha\delta n/2-3\alpha S_{\tau_{\delta n}}} x^3, 1\} e^{\alpha S_{\tau_{\delta n}}} \right] dG_\infty(x), \end{aligned}$$

where $\tau_{\delta n} = \inf\{k \in \mathbb{N}_0 : S_k > \delta n\}$ is the first-passage time of the level δn of the random walk $(S_n)_{n \in \mathbb{N}_0}$ for $n \in \mathbb{N}_0$. The integrand above can be estimated as follows

$$\begin{aligned} \sum_{n \geq 0} \min\{e^{3\alpha\delta n/2-3\alpha S_{\tau_{\delta n}}} x^3, 1\} e^{\alpha S_{\tau_{\delta n}}} & \leq \sum_{n \geq 0} \min\{e^{-3\alpha S_{\tau_{\delta n}}/2} x^3, 1\} e^{\alpha S_{\tau_{\delta n}}} \\ & = x^3 \sum_{n: S_{\tau_{\delta n}} \geq \frac{2}{\alpha} \log x} e^{-\alpha S_{\tau_{\delta n}}/2} + \sum_{n: S_{\tau_{\delta n}} \leq \frac{2}{\alpha} \log x} e^{\alpha S_{\tau_{\delta n}}}. \end{aligned}$$

Let X be a copy of S_1 independent of $(S_n)_{n \in \mathbb{N}_0}$. Since for any $i \in \mathbb{N}$ the set $\{n \in \mathbb{N}_0 : \tau_{\delta n} = i\}$ has cardinality at most $\delta^{-1}(S_i - S_{i-1}) + 1$, for any real y and $\beta > 0$, we may estimate

$$\begin{aligned} \mathbb{E} \left[\sum_{n: S_{\tau_{\delta n}} \geq y} e^{-\beta S_{\tau_{\delta n}}} \right] & \leq \mathbb{E} \left[\sum_{i \geq 1} \mathbb{1}_{\{S_i \geq y\}} e^{-\beta S_i} (\delta^{-1}(S_i - S_{i-1}) + 1) \right] \\ & = e^{-\beta y} \mathbb{E} \left[\sum_{i \geq 0} \mathbb{1}_{\{S_i + X - y \geq 0\}} e^{-\beta(S_i + X - y)} (\delta^{-1}X + 1) \right]. \end{aligned}$$

Now let $f(x) := e^{\beta x} \mathbb{1}_{(-\infty, 0]}(x)$ for $x \in \mathbb{R}$, and let U denote the renewal measure associated with $(S_n)_{n \in \mathbb{N}_0}$, i.e., $U(B) = \sum_{n \in \mathbb{N}_0} \mathbb{P}(S_n \in B)$ for Borel sets $B \subseteq \mathbb{R}$. The function f is directly Riemann integrable, hence $f * U(x) := \int f(x-u) U(du)$ is bounded by some finite constant $C > 0$ (that may depend on β) by the key renewal theorem. Consequently,

$$\begin{aligned} & e^{-\beta y} \mathbb{E} \left[\sum_{i \geq 0} \mathbb{1}_{\{S_i + X - y \geq 0\}} e^{-\beta(S_i + X - y)} (\delta^{-1} X + 1) \right] \\ &= e^{-\beta y} \mathbb{E} [f * U(y-x) (\delta^{-1} X + 1)] \\ &\leq C e^{-\beta y} \mathbb{E} [\delta^{-1} X + 1] = C(-\delta^{-1} m'(\alpha) + 1) e^{-\beta y}. \end{aligned}$$

Similarly, by increasing the value of $C > 0$ if necessary,

$$\begin{aligned} \mathbb{E} \left[\sum_{n: S_{\tau_{\delta n}} \leq y} e^{\beta S_{\tau_{\delta n}}} \right] &\leq \mathbb{E} \left[\sum_{i \geq 1} \mathbb{1}_{\{S_i \leq y\}} e^{\beta S_i} (\delta^{-1} (S_i - S_{i-1}) + 1) \right] \\ &= e^{\beta y} \mathbb{E} \left[\sum_{i \geq 0} \mathbb{1}_{\{S_i + X - y \leq 0\}} e^{\beta(S_i + X - y)} (\delta^{-1} X + 1) \right] \\ &\leq C e^{\beta y} \mathbb{E} [\delta^{-1} X + 1] = C(-\delta^{-1} m'(\alpha) + 1) e^{\beta y}. \end{aligned}$$

Hence, for some constant $C' > 0$,

$$\mathbb{E} \left[\sum_{n \geq 0} \min\{e^{3\alpha\delta n/2 - 3\alpha S_{\tau_{\delta n}}} x^3, 1\} e^{\alpha S_{\tau_{\delta n}}} \right] \leq C x^2,$$

which, in turn gives,

$$\mathbb{E}[I] \leq C \int_{[0, \infty)} x^2 dG_\infty(x) = C \mathbb{E}[\sup_{s \geq 0} (W_s - 1)^2] < \infty$$

in view of (3.5). In particular, $I < \infty$ almost surely and the condition (3.25) is fulfilled.

An appeal to Lemma 3.5 with $T_n = V_{\delta n, r} / \sqrt{\text{Var}_{\delta n}[V_{\delta n, r}]}$ in combination with (3.28) gives, for fixed $r \in \delta\mathbb{N}$,

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha\delta n}}{\log(\delta n)}} V_{\delta n, r} = \sqrt{2c_r W} \quad \text{a. s.} \tag{3.35}$$

because $V_{\delta n, r}$ is $\mathcal{H}_{\delta n+r}$ -measurable; whereas

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha\delta n}}{\log(\delta n)}} V_{\delta n, \infty} \leq \sqrt{2c_\infty W} = \left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} \quad \text{a. s.} \tag{3.36}$$

Next, we shall prove that (3.35) and (3.36) entail

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha\delta n}}{\log(\delta n)}} (W_{\delta n+r} - W_{\delta n}) = \sqrt{2c_r W} \quad \text{a. s.} \tag{3.37}$$

for fixed $r \in \delta\mathbb{N}$ and

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha\delta n}}{\log(\delta n)}} (W - W_{\delta n}) \leq \sqrt{2c_\infty W} = \left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} \quad \text{a. s.} \tag{3.38}$$

To this end, it is enough to check that, for $r \in \delta\mathbb{N} \cup \{\infty\}$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} e^{\alpha\delta n/2} \sum_{u \in \mathcal{I}_{\delta n}} e^{-\alpha S(u)} |[W_{r+\delta n-}]_u \circ S(u) - 1| \\ & \quad \cdot \mathbb{1}_{\{e^{\alpha\delta n/2} e^{-\alpha S(u)} |[W_{r+\delta n-}]_u \circ S(u) - 1| > 1\}} = 0 \quad \text{a. s.} \end{aligned} \tag{3.39}$$

and

$$\lim_{n \rightarrow \infty} e^{\alpha \delta n / 2} \sum_{u \in \mathcal{I}_{\delta n}} |\mathbb{E}_{\delta n}[W_{\delta n, r}(u)]| = 0 \quad \text{a. s.} \tag{3.40}$$

Since, for $u \in \mathcal{I}_{\delta n}$, $\mathbb{E}_{\delta n}[[W_r]_u - 1] = 0$ and $S(u)$ is $\mathcal{H}_{\delta n}$ -measurable, we have

$$\begin{aligned} & |\mathbb{E}_{\delta n}[W_{\delta n, r}(u)]| \\ &= |\mathbb{E}_{\delta n}[e^{-\alpha S(u)}((W_{r+\delta n-} \cdot)_u \circ S(u) - 1) \mathbf{1}_{\{e^{\alpha \delta n / 2} e^{-\alpha S(u)} |(W_{r+\delta n-} \cdot)_u \circ S(u) - 1| \leq 1\}}]| \\ &= |\mathbb{E}_{\delta n}[e^{-\alpha S(u)}((W_{r+\delta n-} \cdot)_u \circ S(u) - 1) \mathbf{1}_{\{e^{\alpha \delta n / 2} e^{-\alpha S(u)} |(W_{r+\delta n-} \cdot)_u \circ S(u) - 1| > 1\}}]| \\ &\leq \mathbb{E}_{\delta n}[e^{-\alpha S(u)} |(W_{r+\delta n-} \cdot)_u \circ S(u) - 1| \mathbf{1}_{\{e^{\alpha \delta n / 2} e^{-\alpha S(u)} |(W_{r+\delta n-} \cdot)_u \circ S(u) - 1| > 1\}}]. \end{aligned}$$

Hence, both relations (3.39) and (3.40) follow if we can show that

$$\mathbb{E} \left[\sum_{n \geq 0} e^{\alpha \delta n / 2} \sum_{u \in \mathcal{I}_{\delta n}} e^{-\alpha S(u)} \int_{(e^{-\alpha \delta n / 2} e^{\alpha S(u)}, \infty)} x \, dF_{r+\delta n-S(u)}(x) \right] < \infty. \tag{3.41}$$

To see this, notice that

$$\begin{aligned} & \mathbb{E} \left[\sum_{n \geq 0} e^{\alpha \delta n / 2} \sum_{u \in \mathcal{I}_{\delta n}} e^{-\alpha S(u)} \int_{(e^{-\alpha \delta n / 2} e^{\alpha S(u)}, \infty)} x \, dF_{r+\delta n-S(u)}(x) \right] \\ & \leq \mathbb{E} \left[\sum_{n \geq 0} e^{\alpha \delta n / 2} \sum_{u \in \mathcal{I}_{\delta n}} e^{-\alpha S(u)} \int_{(e^{\alpha(\delta n / 2 + r)}, \infty)} x \, dG_r(x) \right] \\ & = \sum_{n \geq 0} e^{\alpha \delta n / 2} \int_{(e^{\alpha(\delta n / 2 + r)}, \infty)} x \, dG_r(x) \\ & \leq \int_{(1, \infty)} x \left(\sum_{n=0}^{(2/\delta)(\frac{\log x}{\alpha} - r)} e^{\alpha \delta n / 2} \right) dG_r(x) \\ & \leq \text{const} \cdot \int_{(1, \infty)} x^2 \, dG_r(x) < \infty. \end{aligned}$$

The proof of (3.37) and (3.38) is complete.

It remains to show that “ \leq ” can be replaced by “ $=$ ” in (3.38). As has already been remarked at the beginning of the proof, once we have proved (3.38), we also have

$$\liminf_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W - W_{\delta n}) \geq - \left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} \quad \text{a. s.} \tag{3.42}$$

For any $r \in \delta \mathbb{N}$, the following equality holds

$$\begin{aligned} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W - W_{\delta n}) &= \sqrt{\frac{e^{\alpha(\delta n + r)}}{\log(\delta n + r)}} (W - W_{\delta n + r}) \sqrt{\frac{\log(\delta n + r)}{\log(\delta n)}} e^{-\alpha \delta r / 2} \\ &\quad + \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W_{\delta n + r} - W_{\delta n}). \end{aligned}$$

From (3.37) and (3.42) we infer

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W - W_{\delta n}) \\ & \geq \liminf_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha(\delta n+r)}}{\log(\delta n+r)}} (W - W_{\delta n+r}) \sqrt{\frac{\log(\delta n+r)}{\log(\delta n)}} e^{-\alpha \delta r/2} \\ & \quad + \limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W_{\delta n+r} - W_{\delta n}) \\ & \geq -\left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W\right)^{1/2} e^{-\alpha \delta r/2} + (2c_r W)^{1/2}. \end{aligned}$$

Letting $r \rightarrow \infty$, we arrive at

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W - W_{\delta n}) \geq (2c_\infty W)^{1/2} = \left(\frac{2\sigma^2}{-\alpha m'(\alpha)}\right)^{1/2} \text{ a. s.} \quad \square$$

Proof of Theorem 2.3. We have to show that (2.14) with δn replacing t entails (2.14). Plainly,

$$\limsup_{t \rightarrow \infty} \sqrt{\frac{e^{\alpha t}}{\log t}} (W - W_t) \geq \limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W - W_{\delta n}) = \left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W\right)^{1/2} \text{ a. s.}$$

Thus, it remains to prove the converse inequality

$$\limsup_{t \rightarrow \infty} \sqrt{\frac{e^{\alpha t}}{\log t}} (W - W_t) \leq \left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W\right)^{1/2} \text{ a. s.}, \quad (3.43)$$

To this end, fix arbitrary $\delta, \rho > 0$, let $n \in \mathbb{N}_0$ and notice that the following particular case of (3.37) holds

$$\limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W_{\delta n} - W_{\delta(n+1)}) = \sqrt{2c_\delta W} \text{ a. s.}$$

This in combination with the fact that $W_{\delta n} - W_{\delta(n+1)}$ is $\mathcal{H}_{\delta(n+1)}$ -measurable and the conditional Borel-Cantelli lemma (see, for instance, Theorem 5.3.2 on p. 240 in [7]) implies

$$\sum_{n \geq 1} \mathbb{P}(W_{\delta n} - W_{\delta(n+1)} > \varepsilon_n | \mathcal{H}_{\delta n}) < \infty \text{ a. s.} \quad (3.44)$$

for

$$\varepsilon_n = (1 + \rho) \sqrt{\frac{\log(\delta n)}{e^{\alpha \delta n}}} \sqrt{2c_\delta W_{\delta n}}. \quad (3.45)$$

For $t \in [\delta n, \delta(n+1))$, define

$$A_t := \mathbb{E}_t[(W_{\delta(n+1)} - W_t)^2]$$

and observe that, in view of Lemma 3.1 and the fact that the martingale $(W_t)_{t \geq 0}$ is L^2 -bounded,

$$\sup_{t \geq 0} e^{\alpha t} A_t = \sup_{t \geq 0} e^{\alpha t} \sum_{u \in \mathcal{I}_t} e^{-2\alpha S(u)} v_{\delta(n+1)-S(u)} < \infty \text{ a. s.} \quad (3.46)$$

Now let

$$B_n := \sup_{t \in [\delta n, \delta(n+1))} (W_{\delta n} - W_t - (2A_t)^{1/2})$$

and

$$t_n^* := \inf\{s \geq \delta n : W_{\delta n} - W_s - (2A_s)^{1/2} > \varepsilon_n\}.$$

Then t_n^* , being the hitting time of an open set by an adapted, right-continuous process, is an optional time for $(\mathcal{H}_t)_{t \geq 0}$. Thus,

$$\begin{aligned} \mathbb{P}(W_{\delta n} - W_{\delta(n+1)} > \varepsilon_n | \mathcal{H}_{\delta n}) &\geq \mathbb{P}(W_{\delta n} - W_{\delta(n+1)} > \varepsilon_n, t_n^* < \delta(n+1) | \mathcal{H}_{\delta n}) \\ &\geq \mathbb{P}(W_{t_n^*} - W_{\delta(n+1)} > -(2A_{t_n^*})^{1/2}, t_n^* < \delta(n+1) | \mathcal{H}_{\delta n}) \\ &= \mathbb{E}\left[\mathbb{P}(W_{t_n^*} - W_{\delta(n+1)} > -(2A_{t_n^*})^{1/2} | \mathcal{H}_{t_n^*}) \mathbf{1}_{\{t_n^* < \delta(n+1)\}} \middle| \mathcal{H}_{\delta n}\right] \\ &\geq 2^{-1} \mathbb{P}(t_n^* < \delta(n+1) | \mathcal{H}_{\delta n}) = 2^{-1} \mathbb{P}(B_n > \varepsilon_n | \mathcal{H}_{\delta n}) \end{aligned}$$

having used the definition of t_n^* and the fact that t_n^* is optional for the second inequality, the tower property of the conditional expectations for the first equality and the Markov inequality for the last inequality.

This entails

$$\sum_{n \geq 1} \mathbb{P}(B_n > \varepsilon_n | \mathcal{H}_{\delta n}) < \infty \quad \text{a. s.}$$

and thereupon $B_n \leq \varepsilon_n$ eventually a. s. Thus,

$$\begin{aligned} &\limsup_{t \rightarrow \infty} \sqrt{\frac{e^{\alpha t}}{\log t}} (W - W_t) \\ &\leq e^{\alpha \delta / 2} \left(\limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} (W - W_{\delta n}) + \limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} \sup_{t \in [\delta n, \delta(n+1)]} (W_{\delta n} - W_t) \right) \\ &= e^{\alpha \delta / 2} \left(\left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} + \limsup_{n \rightarrow \infty} \sqrt{\frac{e^{\alpha \delta n}}{\log(\delta n)}} B_n \right) \\ &\leq e^{\alpha \delta / 2} \left(\left(\frac{2\sigma^2}{-\alpha m'(\alpha)} W \right)^{1/2} + (1 + \rho) \sqrt{2c_\delta W} \right), \end{aligned}$$

where the equality is a consequence of (3.46). Recall the definition of c_δ from (3.31) and notice that $\lim_{\delta \rightarrow 0+} c_\delta = 0$ by Corollary 3.2. With this at hand, letting $\delta \downarrow 0$ gives (3.43). \square

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