

# The sharp $K_4$ -percolation threshold on the Erdős–Rényi random graph

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## Abstract

We locate the critical threshold  $p_c \sim 1/\sqrt{3n \log n}$  at which it becomes likely that the complete graph  $K_n$  can be obtained from the Erdős–Rényi graph  $\mathcal{G}_{n,p}$  by iteratively completing copies of  $K_4$  minus an edge. This refines work of Balogh, Bollobás and Morris that bounds the threshold up to multiplicative constants.

**Keywords:** bootstrap percolation; random graph; triadic closure; weak saturation.

**MSC2020 subject classifications:** 05C80; 60K35.

Submitted to EJP on February 22, 2021, final version accepted on September 27, 2021.

## 1 Introduction

Triangles play an important role in networks. For instance, the concept of triadic closure (see, e.g., Simmel [13] and Granovetter [9]) from social network theory is the observation that if there are edges (e.g., representing friendship) between vertices  $x, y$  and  $x, z$ , then the edge between  $y, z$  (if not already present) is likely to be added eventually (e.g., once  $x$  finds an opportunity to introduce  $y$  and  $z$ ). This gives rise to the special case  $H = K_3$  of the process called *H-graph bootstrap percolation* introduced by Bollobás [6] (under the name of *weak saturation*). Let  $\langle G \rangle_H$  denote the graph obtained from  $G$  by iteratively completing copies of  $H$  minus an edge. A graph  $G$  is said to *H-percolate* if all missing edges are eventually added, that is, if  $\langle G \rangle_H$  is the complete graph on the vertices of  $G$ .

Following Balogh, Bollobás and Morris [4], we suppose that the underlying network is the Erdős–Rényi [8] graph, that is, the random subgraph  $\mathcal{G}_{n,p}$  of the complete graph  $K_n$  where edges are included independently with probability  $p$ . The critical threshold, at which  $\mathcal{G}_{n,p}$  is likely to *H-percolate*, is defined formally as

$$p_c(n, H) = \inf \{ p > 0 : \mathbf{P}(\langle \mathcal{G}_{n,p} \rangle_H = K_n) \geq 1/2 \}.$$

A graph  $K_3$ -percolates if and only if it is connected, so this case follows by the classical work [8]. In this article, we focus on the next case,  $H = K_4$ . This is a stricter version

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of triadic closure, where an edge between  $u, v$  is added only if  $u$  and  $v$  are incident to triangles that share an edge (e.g., people become friends if they have mutual friends who are friends). In [4],  $p_c(n, K_4)$  is estimated up to multiplicative constants. Our main result locates the sharp threshold.

**Theorem 1.1.** *The critical  $K_4$ -percolation threshold on the Erdős–Rényi graph  $\mathcal{G}_{n,p}$  satisfies  $p_c(n, K_4) \sim 1/\sqrt{3n \log n}$ .*

### 1.1 Outline

The upper bound is proved in [2], via a connection with classical 2-neighbor bootstrap percolation [12, 7, 11], which we now explain. In this model, vertices are “infected” if they have at least 2 infected neighbors. Suppose that some set  $I$  of vertices in a graph  $G = (V, E)$  are initially infected. Let  $\langle I, G \rangle_2$  denote the set of eventually infected vertices. If all vertices are eventually infected,  $\langle I, G \rangle_2 = V$ , we say that  $I$  is contagious for  $G$ . It is easy to see (by induction) that if some edge in  $G$  is contagious, then  $G$  will  $K_4$ -percolate. Therefore, the upper bound in Theorem 1.1 follows since, as shown in [2],  $1/\sqrt{3n \log n}$  is the sharp threshold for the existence of such a seed edge in  $\mathcal{G}_{n,p}$ .

To prove the lower bound in Theorem 1.1, we essentially show that none of the other ways in which  $\mathcal{G}_{n,p}$  can percolate are more likely. The analysis is somewhat involved, as there are many ways in which percolating subgraphs of  $\mathcal{G}_{n,p}$  can “merge” (see Section 2 below) to form larger percolating subgraphs. Similar issues are involved, for instance, with the pioneering work of Holroyd [10].

The key to overcoming this difficulty is to observe that, for any graph  $G$  (larger than a single edge) that  $K_4$ -percolates, the minimum degree of  $G$  is at least 2 (no isolated vertices or pendant edges) and its subgraph  $C \subset G$  obtained by successively deleting vertices of degree 2 also  $K_4$ -percolates. We call  $C$  the core of  $G$ . The case that  $C$  is a single edge (a seed edge) is described above. Otherwise,  $C$  has minimum degree at least 3, in which case we call  $C$  a 3-core. Hence, a percolating graph  $G$  is either a seed graph, or else it has a 3-core. In either case, the vertex set  $V(C)$  is contagious for  $G$ . See Section 3.1 for more details.

There are two other main ingredients in the proof of the lower bound. First, by a detailed combinatorial analysis, based on the clique process (see Section 2 below) introduced in [4], we (roughly speaking) show that there are at most  $(2/e)^q q! q^q$  percolating 3-cores of size  $q$  (that is, on  $q$  vertices). See Section 3 for the precise estimates. Then, with this at hand, we utilize the following tail estimates [3] (which complement the central limit theorems in [11]).

Let  $P(q, k)$  denote the probability that for a given set  $I \subset [n]$  (independent of  $\mathcal{G}_{n,p}$ ) of size  $q$ , we have that  $|\langle I, \mathcal{G}_{n,p} \rangle_2| \geq k$ .

**Lemma 1.2** ([3]). *Fix  $\alpha > 0$  and put  $p = \sqrt{\alpha/(n \log n)}$ . Let  $\varepsilon \in [0, 1)$  and  $\beta \in [\beta_\varepsilon, 1/\alpha]$ , where  $\beta_\varepsilon = (1 - \sqrt{1 - \varepsilon})/\alpha$ . Put  $k_\alpha = \alpha^{-1} \log n$  and  $q_\alpha = (2\alpha)^{-1} \log n$ . Suppose that  $q/q_\alpha \rightarrow \varepsilon$  and  $k/k_\alpha \rightarrow \alpha\beta$  as  $n \rightarrow \infty$ . Then  $P(q, k) = n^{\xi + o(1)}$ , where*

$$\xi = -\frac{\alpha\beta^2}{2} + \begin{cases} (2\alpha\beta - \varepsilon)(2\alpha)^{-1} \log(e(\alpha\beta)^2/(2\alpha\beta - \varepsilon)) & \beta \in [\beta_\varepsilon, \varepsilon/\alpha] \\ \beta \log(\alpha\beta) - \varepsilon(2\alpha)^{-1} \log(\varepsilon/e) & \beta \in [\varepsilon/\alpha, 1/\alpha]. \end{cases}$$

This follows by the main result in [3], setting  $r = 2$  and replacing the parameters  $\vartheta, \alpha, \beta$  therein with  $k_\alpha, \varepsilon, \alpha\beta$ , respectively.

Using this, together with the upper bound  $(2/e)^q q! q^q$  for percolating 3-cores of size  $q$ , we argue (see Section 4.1) that, when  $p$  is sub-critical, the expected number of percolating subgraphs of  $\mathcal{G}_{n,p}$  of size  $k = \beta \log n$ , for  $\beta \in [\beta_\varepsilon, 1/\alpha]$ , with a core of size  $q \leq (3/2) \log n$  is bounded by  $n^{\mu + o(1)}$ , where

$$\mu(\alpha, \beta) = 3/2 + \beta \log(\alpha\beta) - \alpha\beta^2/2.$$

The almost sure non-existence of percolating 3-cores of size  $q \geq (3/2) \log n$  in  $\mathcal{G}_{n,p}$  is handled separately (see Section 4.2), by showing that such a graph would have to be created through a highly unlikely merging of other graphs of “macroscopic” size. This leads to the following result, yielding the lower bound in Theorem 1.1.

(In this work, the size of a graph is its number of vertices, not its number of edges.)

**Theorem 1.3.** Fix  $\alpha \in (0, 1/3)$  and put  $p = \sqrt{\alpha/(n \log n)}$ . With high probability the largest cliques in  $\langle \mathcal{G}_{n,p} \rangle_{K_4}$  are of size  $(\beta_* + o(1)) \log n$ , where  $\mu(\alpha, \beta_*) = 0$ .

### 1.2 Discussion

The critical window for the connectivity of  $\mathcal{G}_{n,p}$  is well-understood. With high probability  $\mathcal{G}_{n,p}$  is connected (hence  $K_3$ -percolating) if and only if it has no isolated vertices. If  $p = (\log n + \epsilon)/n$ ,  $\mathcal{G}_{n,p}$  will  $K_3$ -percolate with probability  $\exp(-e^{-\epsilon})(1 + o(1))$ , as  $n \rightarrow \infty$ . It would be interesting to obtain similarly detailed information for  $K_4$ -percolation.

For all  $r \geq 5$ ,  $p_c(n, K_r)$  is estimated up to poly-logarithmic factors in [4]. More recently [5], the threshold  $p_c(n, H)$  has been located up to constant factors, for all  $H$  in a certain class of “reasonably balanced” graphs, which includes all such  $K_r$ . Interestingly, the connection with classical bootstrap percolation described above does not lead to the critical threshold when  $r \geq 5$ . Instead, near  $p_c$ ,  $\mathcal{G}_{n,p}$  percolates in some other way, that is not yet fully understood.

Although  $H$ -percolation can, in general, behave quite differently than the present case  $H = K_4$ , we think the ideas in this work will be useful in improving the bounds for  $p_c$  in other cases of interest.

## 2 The clique process

The *clique process*, introduced in [4], describes the  $K_4$ -percolation dynamics in a way that is amenable to analysis. In this section, we recall some basic observations about this process which we will require. See [4] for the proofs.

**Definition 2.1.** Three graphs  $G_i = (V_i, E_i)$  form a triangle if there are distinct vertices  $x, y, z$  such that  $x \in V_1 \cap V_2$ ,  $y \in V_1 \cap V_3$  and  $z \in V_2 \cap V_3$ . If  $|V_i \cap V_j| = 1$  for all  $i \neq j$ , we say that they form exactly one triangle.

In [4] the following observation is made.

**Lemma 2.2.** Suppose that  $G_i = (V_i, E_i)$  percolate.

- (i) If the  $G_i$  form a triangle then  $G_1 \cup G_2 \cup G_3$  percolates.
- (ii) If  $|V_1 \cap V_2| \geq 2$  then  $G_1 \cup G_2$  percolates.

This leads to the following process.

**Definition 2.3.** A clique process for a graph  $G$  is a sequence  $(\mathcal{C}_t)_{t=0}^\tau$  of sets of subgraphs of  $G$  such that:

- (i)  $\mathcal{C}_0 = E(G)$  is the edge set of  $G$ .
- (ii) For each  $t < \tau$ ,  $\mathcal{C}_{t+1}$  is obtained from  $\mathcal{C}_t$  by either (a) merging two subgraphs  $G_1, G_2 \in \mathcal{C}_t$  with at least two common vertices, or (b) merging three subgraphs  $G_1, G_2, G_3 \in \mathcal{C}_t$  that form exactly one triangle.
- (iii)  $\mathcal{C}_\tau$  is such that no further operations as in (ii) are possible.

The reason for the name is that (by induction), for any  $t \leq \tau$  and  $H \in \mathcal{C}_t$ ,  $H$  percolates (on its vertex set  $V(H)$ ), and hence  $\langle H \rangle_{K_4}$  is a clique in  $\langle G \rangle_{K_4}$ . Let us also mention here that this terminology mirrors that of the so-called “rectangle process” used in [10] and other works about bootstrap percolation on grids.

The description above is slightly modified from that presented in [4], as we note that if three percolating graphs form more than one triangle, then they can be merged

by applying Lemma 2.2(ii) twice. Therefore, for convenience, we reserve the use of Lemma 2.2(i) in any clique process for the case that *exactly* one triangle is formed. This simplifies the combinatorial analysis in Lemma 3 below.

Finally, let us record the following observation, see [4].

**Lemma 2.4.** *Let  $G$  be a finite graph and  $(C_t)_{t=0}^\tau$  a clique process for  $G$ . For each  $t \leq \tau$ ,  $C_t$  is a set of edge-disjoint, percolating subgraphs of  $G$ . Furthermore,  $\langle G \rangle_{K_4}$  is the edge-disjoint, triangle-free union of cliques  $\bigcup_{H \in C_\tau} \langle H \rangle_{K_4}$ . Hence  $G$  percolates if and only if  $C_\tau = \{G\}$ . In particular,  $C_\tau = C'_\tau$  for any two clique processes  $(C_t)_{t=0}^\tau$  and  $(C'_t)_{t=0}^\tau$  for  $G$ .*

### 2.1 Consequences

The following consequences of Lemma 2.4, derived in [4] using the clique process, play a crucial role in the current work.

**Lemma 2.5.** *If  $G = (V, E)$  percolates then  $|E| \geq 2|V| - 3$ .*

This result was first proved in [6]. A proof by the clique process is given in [4].

**Definition 2.6.** *We call  $|E| - (2|V| - 3)$  the excess of a graph  $G = (V, E)$ . A graph is edge-minimal if its excess is 0.*

To prove Lemma 2.5, the following observations are made in [4].

**Lemma 2.7.** *Suppose that  $G_i = (V_i, E_i)$  are edge-disjoint, percolating graphs.*

- (i) *If the  $G_i$  form exactly one triangle, then the excess of  $G_1 \cup G_2 \cup G_3$  is the sum of the excesses of the  $G_i$ .*
- (ii) *If  $|V_1 \cap V_2| = m \geq 2$ , then the excess of  $G_1 \cup G_2$  is the sum of the excesses of the  $G_i$ , plus  $2m - 3 > 0$ .*

Hence, if  $G$  is an edge-minimal percolating graph, then every step of any clique process for  $G$  involves merging three subgraphs that form exactly one triangle. The simplest example of this is when two of the  $G_i$  are a single edge sharing a common vertex (leading one to the connection with 2-neighbor bootstrap percolation). If all steps of a clique process for  $G$  are of this form, then  $G$  is a seed graph, as defined in Section 1.1 above.

Finally, since at most three subgraphs are merged in any step of a clique process, a Aizenman–Lebowitz [1] type condition follows, see [4].

**Lemma 2.8.** *Let  $G$  be a graph and  $k \geq 1$ . If  $G$  has no percolating subgraphs of size  $k' \in [k, 3k]$  then  $G$  has no percolating subgraphs larger than  $k$ .*

## 3 Combinatorial bounds

We first address the issue of estimating the number of percolating graphs with various structural properties. Most crucially, we require reasonably sharp estimates for the number of percolating graphs with few vertices of degree 2. The proofs of the main results in this section Lemmas 3.7 and 3.8 are fairly straightforward, but rather laborious. As such, we only sketch the main ideas in the proofs in this section. Detailed proofs appear in Appendices A and B below.

### 3.1 Structure of percolating graphs

We first make some simple, but critical, observations about the structure of percolating graphs. Informally, we note (as discussed in Section 1) that a percolating graph  $G$  has a percolating core  $C$  that is either a single edge or a graph of minimum degree at least 3 (see Lemma 3.3 below). Once  $C$  percolates (by the  $K_4$ -percolation dynamics) the full percolation of  $G$  can be completed by successively adding vertices according to the 2-neighbor dynamics.

We first observe that if a vertex of degree 2 is removed from a percolating graph, the resulting subgraph still percolates.

**Lemma 3.1.** *Suppose that  $G$  percolates and  $v \in V(G)$  is of degree 2. Then the subgraph  $G_v \subset G$  induced by  $V - \{v\}$  percolates.*

Let us remark that in [6] a closely related result is proved. Namely, if  $3 \leq r < 7$ , and  $v$  is of minimum degree  $d = r - 2 + p$  in a  $K_r$ -percolating graph  $G$  with the minimum possible number of edges, then  $G_v$  can be made into a  $K_r$ -percolating graph by adding at most  $p$  edges to it. Indeed, this is how it is shown (by induction) in [6] that  $(r - 2)k - \binom{r-1}{2}$  is the minimum possible number of edges in a  $K_r$ -percolating graph (for these particular values of  $r$ ).

Lemma 3.1 follows easily using the clique process.

*Proof.* Recall that, in each step of the clique process, *edge-disjoint* percolating graphs are merged to form a larger percolating graph. Consider the first time in the process that (at least) one of the edges  $e_1, e_2$  incident to  $v$  is involved in a merge. In this step, in fact, both  $e_1, e_2$  are merged with some other percolating graph  $G'$ , since percolating graphs (larger than a single edge) have minimum degree at least 2. If this is the last step of the clique process, the result is immediate since then  $G' = G_v$ . Otherwise, in the last step of the clique process, one of the (two or three) graphs being merged contains  $e_1, e_2$  and all other graphs involved do not contain  $v$ . Hence the result follows by induction on the size of  $G$ .  $\square$

Recall (see Section 1.1) that  $\langle I, G \rangle_2$  is the set of vertices eventually infected by the 2-neighbor dynamics on  $G$ , when  $I$  is initially infected.

**Definition 3.2.** *Similarly, for a subgraph  $H \subset G$ , we let  $\langle H, G \rangle_2$  denote the subgraph of  $G$  induced by  $\langle V(H), G \rangle_2$ .*

Note that, by Lemma 2.2(i) and induction, if  $H \subset G$  percolates then so does  $\langle H, G \rangle_2$ . Informally speaking, a percolating graph can be grown by adding vertices with (at least) two neighbors in the graph. On the other hand, Lemma 3.1 says that if vertices of degree 2 are removed from a graph then it still percolates. Hence we make the following observation about the structure of percolating graphs.

**Lemma 3.3.** *Let  $G$  be a percolating graph. Then either:*

- (i)  $G = \langle e, G \rangle_2$  for some edge  $e \in E(G)$ , or else,
- (ii)  $G = \langle C, G \rangle_2$  for some percolating  $C \subset G$  of minimum degree at least 3.

*Furthermore:*

- (iii) *the excess of  $G$  is equal to the excess of  $C$ .*

Note that, in the first case,  $G$  is a seed graph and  $e$  is a seed edge. In the latter case,  $C$  is the 3-core of  $G$ . If  $G = C$ , we say that  $G$  is a 3-core.

*Proof.* Recall (see Definition 2.6) that the excess of a graph  $G = (V, E)$  is its number of edges  $|E|$  minus the minimum possible number of edges  $2|V| - 3$  in a percolating graph of size  $|V|$ . Therefore, the excess of  $G$  is clearly equal to the excess of  $G_v$ , for any  $v \in V$  of degree 2. (This is an easy special case of Lemma 2.7(i).) Hence part (iii) follows by a simple induction.

Parts (i) and (ii) follow by Lemma 3.1 and induction, noting that if  $C$  is the subgraph of  $G$  obtained by successively deleting vertices of degree 2, then  $V(G) = \langle V(C), G \rangle_2$  (and hence  $G = \langle C, G \rangle_2$  in the notation of Definition 3.2) by the definition of the 2-neighbor dynamics. That is, if we initially infect all vertices in  $C$ , then all vertices in  $G$  will be infected eventually. To see this, note that we can simply infect all vertices outside of  $C$  one at a time, in the time-reversed order in which they were deleted in obtaining  $C$  from  $G$ .  $\square$

### 3.2 Basic estimates

In this section, we use Lemma 3.3 to obtain upper bounds for the number of percolating graphs of a given size.

**Definition 3.4.** We say that a percolating graph  $G$  is irreducible if removing any edge from  $G$  results in a non-percolating graph.

In other words, an irreducible graph is minimal in the poset of percolating graphs. Recall (see Definition 2.6) that a graph  $G$  of size  $k$  is edge-minimal if it has exactly  $2k - 3$  edges (the smallest possible number). Seed graphs, for instance, are irreducible and edge-minimal. It is easy to see that all irreducible percolating graphs of size  $2 < k \leq 6$  are seed graphs, and so, in particular, have a vertex of degree 2. There are, however, irreducible percolating graphs of size  $k = 7$  (and larger) with no vertices of degree 2, see e.g. Figure 1.

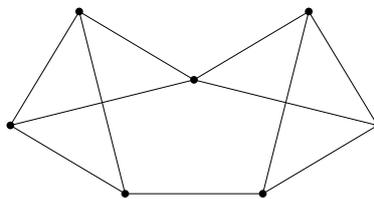


Figure 1: The smallest irreducible percolating 3-core.

Clearly, a graph  $G$  percolates if and only if it has an irreducible percolating subgraph  $G' \subset G$  with  $V(G) = V(G')$ . Therefore, in proving Theorem 1.3, it suffices to restrict our attention to irreducible graphs. Indeed, any percolating subgraph of  $\mathcal{G}_{n,p}$  contains an irreducible percolating subgraph on the same vertex set.

In bounding the possible number of irreducible percolating graphs  $G$  of a given size  $k$ , the relevant quantities are its number  $i$  of vertices in  $G$  of degree 2, the size  $q$  of its core  $C \subset G$ , and its number  $\ell$  of excess edges. (Recall that, in this work, the size of graph is its number of vertices.)

**Definition 3.5.** Let  $I_q^\ell(k, i)$  be the number of labelled, irreducible percolating graphs  $G$  of size  $k$  with an excess of  $\ell$  edges,  $i$  vertices of degree 2, and a core  $C \subset G$  of size  $q$ . For convenience, we say that any such  $G$  “contributes” to  $I_q^\ell(k, i)$ . If  $i = 0$ , and hence  $q = k$ , we simply write  $C^\ell(k) = I_k^\ell(k, 0)$ . If  $\ell = 0$ , we write  $I_q(k, i)$  and  $C(k)$ .

Note that  $I_2(k, i)$  is the number of labelled, irreducible (and edge-minimal) seed graphs of size  $k$  with  $i$  vertices of degree 2.

By Lemma 3.3(iii), if a graph  $G$  contributes to  $I_q^\ell(k, i)$  then its core has excess  $\ell$ . As noted above, there are no irreducible 3-cores on  $q \leq 6$  vertices. Hence  $I_q^\ell(k, i) = 0$  if  $2 < q \leq 6$ .

**Definition 3.6.** We let  $I^\ell(k, i) = \sum_q I_q^\ell(k, i)$  denote the number of labelled, irreducible percolating graphs  $G$  of size  $k$ , with an excess of  $\ell$  edges and  $i$  vertices of degree 2. If  $\ell = 0$ , we write  $I(k, i)$ .

We obtain the following estimates for  $I^\ell(k, i)$ , assuming the excess is  $\ell \leq 3$ . The method of proof could presumably (with additional work) provide bounds for larger  $\ell$ , however, quite fortunately, percolating graphs with a larger excess can be dealt with using less accurate estimates (see Lemma 4.3 below).

**Lemma 3.7.** For all  $k \geq 2$ ,  $\ell \leq 3$  and relevant  $i$ , we have that

$$I^\ell(k, i) \leq (2/e)^k k! k^{k+2\ell+i}.$$

In particular,  $C^\ell(k) \leq (2/e)^k k! k^{k+2\ell}$ .

Note that, for small values of  $i$  (and all large enough  $k$ ), this is much smaller than the total number of seed graphs of size  $k$ , which in [2] is shown to be roughly equal to  $k!k^k$ .

See Appendix A below for the proof. The argument is quite lengthy, as there are several cases (increasing in  $\ell$ ) to consider, depending on the nature of the last step in the clique process. Before moving on, we sketch the main ideas.

First, we note that the cases  $i > 0$  follow by a simple induction, since if  $G$  has  $i$  vertices of degree 2, then removing such a vertex from  $G$  results in a graph with  $j \in \{i, i \pm 1\}$  vertices of degree 2. Consideration of these cases leads to the constant  $2/e$ .

The case of 3-cores ( $i = 0$ ) is the “heart” of the proof. The following observations are the key: If  $G$  is an irreducible percolating 3-core, then in the last step of a clique process, either (i) three graphs  $G_1, G_2, G_3$  are merged that form exactly one triangle on  $T = \{v_1, v_2, v_3\}$ , or else (ii) two graphs  $G_1, G_2$  are merged that share  $m \geq 2$  vertices  $S = \{v_1, v_2, \dots, v_m\}$ . In case (i), if some  $G_j$  has a vertex  $v$  of degree 2, then necessarily  $v \in T$ , as else  $G$  would have a vertex of degree 2. In other words, if a percolating 3-core is formed by merging three graphs with vertices of degree 2, then all such vertices belong to the triangle that they form. On the other hand, in case (ii), it can be seen that  $G_1, G_2$  are 3-cores (no vertices of degree 2). Indeed, if some  $v \in S$  is of degree 2 in some  $G_j$ , then  $(G_j)_v$  percolates by Lemma 3.1, and so by Lemma 2.2 it would follow that  $G$  minus some edge incident to  $v$  in  $G_j$  still percolates, in contradiction to the irreducibility of  $G$ .

The above observations provide enough control over the combinatorics to allow for a fairly simple (albeit lengthy) inductive proof of Lemma 3.7. These observations are also utilized in the proof of Proposition 4.9 below.

### 3.3 Sharper estimates

Next, using Lemma 3.7 as a starting point, we obtain the following upper bounds for  $I_q^\ell(k, i)$ .

**Lemma 3.8.** *Fix  $\varepsilon > 0$ . For some constant  $\vartheta = \vartheta(\varepsilon) \geq 1$ , the following holds. For all  $k \geq 2$ ,  $\ell \leq 3$ , and relevant  $q, i$ , we have that*

$$I_q^\ell(k, i) \leq \vartheta \psi_\varepsilon(q/k)^k k! k^{k+2\ell+i}$$

where

$$\psi_\varepsilon(y) = \max\{3/(2e) + \varepsilon, (e/2)^{1-2y} y^2\}.$$

Note that this lemma improves upon Lemma 3.7 only when  $\varepsilon < 1/(2e)$ , as otherwise  $\psi_\varepsilon(y) \geq 2/e$  for all  $y$ . On the other hand, for any given  $\varepsilon < 1/(2e)$ , we have that  $\psi_\varepsilon(y)$  is non-decreasing and  $\psi_\varepsilon(y) \rightarrow 2/e$  as  $y \uparrow 1$ . Indeed,  $\psi_\varepsilon(y) = 3/(2e) + \varepsilon$  for  $y \leq y_*$  and  $\psi_\varepsilon(y) = (e/2)^{1-2y} y^2$  for  $y > y_*$ , where

$$3/(2e) + \varepsilon = (e/2)^{1-2y_*} y_*^2. \tag{3.1}$$

We define  $y_0 = y_*(0) \approx 0.819$ , and note that  $y_*(\varepsilon) \downarrow y_0$ , as  $\varepsilon \downarrow 0$ .

The main ideas in the proof are as follows: First, we note that the case  $i = k - q$  follows essentially directly by Lemma 3.7. We establish the remaining cases by induction, noting that if a graph  $G$  contributes to  $I_q^\ell(k, i)$  and  $i < k - q$ , then there is a vertex  $v$  in  $G$  of degree 2 with a neighbor that is *not* in the core  $C \subset G$ . It follows that either (1) there is such a vertex  $v$  so that some neighbor of  $v$  is of degree 2 in  $G_v$ , or else (ii) there are vertices  $u \neq w$  of degree 2 in  $G$  with a common neighbor that is not in  $C$ . Beyond these observations, the proof is mostly calculus, see Appendix B below.

### 4 Proof of Theorem 1.3

With our key Lemmas 1.2, 3.7 and 3.8 at hand, we turn to the proof of Theorem 1.3. The argument is divided into two parts (Sections 4.1 and 4.2) where, respectively, percolating subgraphs of  $\mathcal{G}_{n,p}$  with small and large cores are considered.

#### 4.1 Percolating subgraphs with small cores

First, we show that for sub-critical  $p$ , with high probability  $\mathcal{G}_{n,p}$  has no percolating subgraphs significantly larger than  $\beta_* \log n$  with a “small” (smaller than  $(3/2) \log n$ ) core.

**Proposition 4.1.** *Fix  $\alpha \in (0, 1/3)$  and put  $p = \sqrt{\alpha/(n \log n)}$ . Then, for any  $\delta > 0$ , with high probability  $\mathcal{G}_{n,p}$  has no irreducible percolating subgraphs  $G$  of size  $k \geq (\beta_* + \delta) \log n$  with a core  $C \subset G$  of size  $q \leq (3/2) \log n$ .*

First, we note that  $\beta_*$  in Theorem 1.3 is well-defined.

**Lemma 4.2.** *Fix  $\alpha \in (0, 1/3)$ . For  $\beta > 0$ , let*

$$\mu(\alpha, \beta) = 3/2 + \beta \log(\alpha\beta) - \alpha\beta^2/2.$$

*The function  $\mu(\alpha, \beta)$  is decreasing in  $\beta$ , with a unique zero  $\beta_* \in (0, 3)$ .*

*Proof.* Differentiating  $\mu(\alpha, \beta)$  with respect to  $\beta$ , we obtain  $1 + \log(\alpha\beta) - \alpha\beta$ . Since  $\log x < x - 1$  for all positive  $x \neq 1$ , we find that  $\mu(\alpha, \beta)$  is decreasing in  $\beta$ . Moreover, since  $\alpha < 1/3$ , we have that  $\mu(\alpha, 3) < (3/2)(3\alpha - 1) < 0$ . The result follows, noting that  $\mu(\alpha, \beta) \rightarrow 3/2 > 0$  as  $\beta \downarrow 0$ . □

Recall that the bounds in Lemmas 3.7 and 3.8 apply only to graphs with an excess of  $\ell \leq 3$  edges. For graphs with larger excess, we will apply the following result, which follows by an elementary union bound.

**Lemma 4.3.** *Fix  $\alpha \in (0, 1/3)$  and put  $p = \sqrt{\alpha/(n \log n)}$ . Then with high probability  $\mathcal{G}_{n,p}$  contains no subgraphs of size  $k \leq 2 \log n$  and excess  $\ell > 3$ . Similarly, with high probability  $\mathcal{G}_{n,p}$  contains no subgraphs of size  $k \leq 9 \log n$  and excess  $\ell > 27$ .*

*Proof.* The expected number of subgraphs of size  $k = \beta \log n$  in  $\mathcal{G}_{n,p}$  with an excess of  $\ell$  edges is bounded by

$$\binom{n}{k} \binom{k}{2k-3+\ell} p^{2k-3+\ell} \leq \left(\frac{e^3}{16} k n p^2\right)^k \left(\frac{e}{4} k p\right)^{\ell-3} \leq n^\nu \log^\ell n$$

where

$$\nu(\beta, \ell) = -(\ell - 3)/2 + \beta \log(\alpha\beta e^3/16).$$

Note that  $\nu$  is convex in  $\beta$ , and that

$$2 \log(2/3 \cdot e^3/16) \approx -0.356 < 0$$

and

$$9 \log(9/3 \cdot e^3/16) \approx 11.934 < 12.$$

Therefore, since  $\alpha < 1/3$ , it follows that  $\nu(\beta, \ell) < -(\ell - 3)/2$  if  $\beta \leq 2$ , and  $\nu(9, \ell) < -(\ell - 27)/2$  if  $\beta \leq 9$ .

Altogether we find that the expected number of subgraphs of size  $k \leq 2 \log n$  and excess  $\ell > 3$  is bounded by

$$2 \log n \cdot \sum_{\ell>3} n^{-(\ell-3)/2} \log^\ell n \leq \frac{O(\log^5 n)}{\sqrt{n}} \sum_{\ell \geq 0} \left(\frac{\log n}{\sqrt{n}}\right)^\ell \leq \frac{O(\log^5 n)}{\sqrt{n}} \ll 1.$$

Similarly, the expected number of subgraphs of size  $k \leq 9 \log n$  and excess  $\ell > 27$  is bounded by

$$\frac{O(\log^{29} n)}{\sqrt{n}} \ll 1.$$

Therefore, with high probability, no such subgraphs exist. □

**Definition 4.4.** Let  $E^\ell(q, k)$  denote the expected number of irreducible percolating cores  $C \subset \mathcal{G}_{n,p}$  of size  $q$  and excess  $\ell$  such that  $|\langle C, \mathcal{G}_{n,p} \rangle_2| \geq k$ .

Combining Lemmas 1.2 and 3.7, we obtain the following estimate. Recall  $\beta_\varepsilon, k_\alpha, q_\alpha$  as in Lemma 1.2, and  $\mu$  in Lemma 4.2.

**Lemma 4.5.** Fix  $\alpha \in (0, 1/3)$  and put  $p = \sqrt{\alpha/(n \log n)}$ . Let  $\varepsilon \in [0, 3\alpha]$  and  $\beta \in [\beta_\varepsilon, 1/\alpha]$ . Suppose that  $q/q_\alpha \rightarrow \varepsilon$  and  $k/k_\alpha \rightarrow \alpha\beta$  as  $n \rightarrow \infty$ . Then, for any  $\ell \leq 3$ , we have that  $E^\ell(q, k) \leq n^{\mu_\varepsilon + o(1)}$ , where  $\mu_\varepsilon(\alpha, \beta) = \mu(\alpha, \beta)$  for  $\beta \in [\varepsilon/\alpha, 1/\alpha]$ , and

$$\mu_\varepsilon(\alpha, \beta) = \mu(\alpha, \beta) - \beta \log(\alpha\beta) + \frac{\varepsilon}{2\alpha} \log(\varepsilon/e) + \frac{2\alpha\beta - \varepsilon}{2\alpha} \log\left(\frac{e(\alpha\beta)^2}{2\alpha\beta - \varepsilon}\right)$$

for  $\beta \in [\beta_\varepsilon, \varepsilon/\alpha]$ .

*Proof.* Suppose that  $k \sim \beta \log n$  and  $q \sim \varepsilon(2\alpha)^{-1} \log n$ . Then, for any  $\ell \leq 3$ , it follows by Lemmas 1.2 and 3.7 that

$$E^\ell(q, k) \leq \binom{n}{q} C^\ell(q) p^{2q-3+\ell} P(q, k) \leq q^{2\ell} p^{\ell-3} \left(\frac{2}{e} q n p^2\right)^q P(q, k) \leq n^{\nu+o(1)}$$

where

$$\nu = 3/2 + \varepsilon(2\alpha)^{-1} \log(\varepsilon/e) + \xi_\varepsilon(\alpha, \beta) = \mu_\varepsilon(\alpha, \beta). \quad \square$$

Having established Lemma 4.5, we aim to prove Proposition 4.1 by the first moment method. We first show that, for some  $\varepsilon' \in (0, 3\alpha)$ , with high probability there are no irreducible percolating cores in  $\mathcal{G}_{n,p}$  of size  $\varepsilon(2\alpha)^{-1} \log n$ , with  $\varepsilon \in [\varepsilon', 3\alpha]$ . We record a slightly more general result, allowing for  $O(1)$  vertices of degree 2, as this will be required in Section 4.2 below.

**Lemma 4.6.** Fix  $\alpha \in (0, 1/3)$  and put  $p = \sqrt{\alpha/(n \log n)}$ . Fix some  $i_* \geq 0$ . Define  $\varepsilon_* \in (0, 3\alpha)$  implicitly by  $3/2 + \varepsilon_*(2\alpha)^{-1} \log(\varepsilon_*/e) = 0$ . Then, for any  $\eta > 0$ , with high probability  $\mathcal{G}_{n,p}$  has no irreducible percolating subgraphs  $G$  of size

$$\frac{\varepsilon_* + \eta}{2\alpha} \log n \leq k \leq \frac{3}{2} \log n$$

and  $i \leq i_*$  vertices of degree 2.

*Proof.* By Lemma 4.3, it suffices to consider subgraphs  $G$  with excess  $\ell \leq 3$ . By Lemma 3.7, the expected number of such subgraphs of size  $k = \varepsilon(2\alpha)^{-1} \log n$  is bounded by

$$\binom{n}{k} p^{2k-3+\ell} I^\ell(k, i) \leq k^{2\ell+i} p^{\ell-3} \left(\frac{2}{e} k n p^2\right)^k \leq n^{\nu+o(1)}$$

where  $\nu(\varepsilon) = 3/2 + \varepsilon(2\alpha)^{-1} \log(\varepsilon/e)$ . Since  $\nu$  is decreasing in  $\varepsilon < 1$ ,  $\nu \rightarrow 3/2 > 0$  as  $\varepsilon \downarrow 0$ , and  $\nu(3\alpha) = (3/2) \log(3\alpha) < 0$ ,  $\varepsilon_*$  satisfying  $\nu(\varepsilon_*) = 0$  is well-defined. Noting that  $\nu(\varepsilon) \leq \nu(\varepsilon_* + \eta) < 0$  for all  $\varepsilon \in [\varepsilon_* + \eta, 3\alpha]$ , the lemma follows by a simple union bound, summing over all  $O(1)$  relevant values of  $i$  and all  $O(\log n)$  relevant values of  $k$ . □

Next, we use Lemma 4.5 to rule out the remaining cases  $\varepsilon \leq \varepsilon_* + \eta$  (where  $\eta > 0$  is a small constant, to be determined below). In order to apply Lemma 4.5, we first verify that, for such  $\varepsilon$ , we have  $\beta_* \geq \beta_\varepsilon$ .

**Lemma 4.7.** Fix  $\alpha \in (0, 1/3)$ . Let  $\beta_\varepsilon, \beta_*, \varepsilon_*$  be as in Lemmas 1.2, 4.2 and 4.6. Then, for some sufficiently small  $\eta(\alpha) > 0$ , we have that  $\beta_* \geq \beta_\varepsilon$  for all  $\varepsilon \in [0, \varepsilon_* + \eta]$ .

*Proof.* By Lemma 4.2 and the continuity of  $\mu(\alpha, \beta_\varepsilon)$  in  $\varepsilon$ , it suffices to show that  $\mu(\alpha, \beta_\varepsilon) > 0$ , for all  $\varepsilon \in [0, \varepsilon_*]$ . Let  $\delta_\varepsilon = 1 - \sqrt{1 - \varepsilon}$ , so that  $\beta_\varepsilon = \delta_\varepsilon/\alpha$ . Note that

$$\mu(\alpha, \beta_\varepsilon) = 3/2 + (2\alpha)^{-1}(2\delta_\varepsilon \log \delta_\varepsilon - \delta_\varepsilon^2).$$

Therefore, by the bound  $\log x \leq x - 1$ ,

$$\frac{\partial}{\partial \varepsilon} \mu(\alpha, \beta_\varepsilon) = (2\alpha)^{-1}(1 + \log(\delta_\varepsilon)/(1 - \delta_\varepsilon)) \leq 0.$$

It thus suffices to verify that  $\mu(\alpha, \beta_{\varepsilon_*}) > 0$ . To this end note that, by the definition of  $\varepsilon_*$  (see Lemma 4.6),

$$\mu(\alpha, \beta_{\varepsilon_*}) = (2\alpha)^{-1}(2\delta_{\varepsilon_*} \log \delta_{\varepsilon_*} - \delta_{\varepsilon_*}^2 - \varepsilon_* \log(\varepsilon_*/e)).$$

By Lemma 4.6, we have that  $\varepsilon_* = \delta_{\varepsilon_*}(2 - \delta_{\varepsilon_*}) \in (0, 1)$ , and so  $\delta_{\varepsilon_*} \in (0, 1)$ . Hence the lemma follows if we show that  $\nu(\delta) > 0$  for all  $\delta \in (0, 1)$ , where

$$\nu(\delta) = 2\delta \log \delta - \delta^2 - \delta(2 - \delta) \log(\delta(2 - \delta)/e).$$

Note that

$$\nu(\delta)/\delta = \delta \log \delta - (2 - \delta) \log(2 - \delta) + 2(1 - \delta).$$

Differentiating this expression with respect to  $\delta$ , we obtain  $\log(\delta(2 - \delta)) < 0$ , for all  $\delta < 1$ . Noting that  $\nu(1) = 0$ , the lemma follows.  $\square$

It can be seen that, for all sufficiently large  $\varepsilon < \varepsilon_*$ , we have that  $\beta_* < \varepsilon/\alpha$ , where  $\mu_\varepsilon \neq \mu$ . Therefore, we require the following bound.

**Lemma 4.8.** Fix  $\alpha \in (0, 1/3)$ . Let  $\varepsilon \in [0, 1)$  and  $\beta_\varepsilon, \mu_\varepsilon$  be as in Lemmas 1.2 and 4.5. Then  $\mu_\varepsilon(\alpha, \beta) \leq \mu(\alpha, \beta)$ , for all  $\beta \in [\beta_\varepsilon, 1/\alpha]$ .

*Proof.* Since  $\mu(\alpha, \beta) = \mu_\varepsilon(\alpha, \beta)$  for  $\beta \in [\varepsilon/\alpha, 1/\alpha]$ , we may assume that  $\beta < \varepsilon/\alpha$ . Let  $\delta = \alpha\beta$ . Then

$$\alpha(\mu(\alpha, \beta) - \mu_\varepsilon(\alpha, \beta)) = \delta \log \delta - \frac{\varepsilon}{2} \log(\varepsilon/e) - \frac{2\delta - \varepsilon}{2} \log\left(\frac{e\delta^2}{2\delta - \varepsilon}\right).$$

Differentiating this expression with respect to  $\delta$ , we obtain

$$\varepsilon/\delta - 1 - \log(\delta/(2\delta - \varepsilon)) \leq 0,$$

by the inequality  $\log x \geq (x - 1)/x$ . Since  $\mu(\alpha, \varepsilon/\alpha) = \mu_\varepsilon(\alpha, \varepsilon/\alpha)$ , the lemma follows.  $\square$

Finally, we prove the main result of this section.

*Proof of Proposition 4.1.* Let  $\delta > 0$  be given. By Lemma 4.2, we may assume without loss of generality that  $\beta_* + \delta < 1/\alpha$ .

Note that by Lemmas 3.3(iii) and 4.3, it suffices to show that with high probability there are no irreducible percolating subgraphs of size  $k \geq (\beta_* + \delta) \log n$  with a core of size  $q \leq (3/2) \log n$  and excess  $\ell \leq 3$ . Also note that, if such a subgraph exists, then by Lemma 3.1 there is such a subgraph of size  $k = \beta \log n$  for some  $\beta \in [\beta_* + \delta, 1/\alpha]$ .

Select  $\eta > 0$  as in Lemma 4.7. By Lemma 4.6, with high probability there are no percolating 3-cores of size

$$\frac{\varepsilon_* + \eta}{2\alpha} \log n \leq q \leq \frac{3}{2} \log n.$$

On the other hand, by the choice of  $\eta$ , Lemmas 4.5, 4.7 and 4.8 imply that for any  $\beta \in [\beta_*, 1/\alpha]$ ,  $\varepsilon \leq \varepsilon_* + \eta$  and  $\ell \leq 3$ , the expected number of irreducible percolating subgraphs of size  $k = \beta \log n$  with a core of size  $q = \varepsilon(2\alpha)^{-1} \log n$  and excess  $\ell$  is bounded by  $n^{\mu+o(1)}$ , where  $\mu = \mu(\alpha, \beta)$ . Therefore (summing over all  $O(\log^2 n)$  relevant values of  $k, q, \ell$ ) we find, by Lemma 4.2 and a union bound, that with high probability there are no irreducible subgraphs of size

$$(\beta_* + \delta) \log n \leq k \leq \frac{1}{\alpha} \log n$$

with a core of size  $q \leq (\varepsilon_* + \eta)(2\alpha)^{-1} \log n$  and excess  $\ell \leq 3$ .

Altogether, it follows that, with high probability, there are no irreducible percolating subgraphs of size  $k \geq (\beta_* + \delta) \log n$  with a core of size  $q \leq (3/2) \log n$ .  $\square$

#### 4.2 Percolating subgraphs with large cores

To complete the proof of Theorem 1.3, we rule out the existence of “large” (larger than  $(3/2) \log n$ ) percolating 3-cores.

**Proposition 4.9.** *Fix  $\alpha \in (0, 1/3)$  and put  $p = \sqrt{\alpha/(n \log n)}$ . Then with high probability  $\mathcal{G}_{n,p}$  has no irreducible percolating 3-cores  $C$  of size*

$$(3/2) \log n \leq q \leq 9 \log n.$$

Before establishing a proof, we observe that Propositions 4.1 and 4.9 easily imply our main result.

*Proof of Theorem 1.3.* Let  $\delta > 0$  be given. By Lemma 4.2, we may assume without loss of generality that  $\beta_* + \delta < 3$ . Hence, by Lemmas 2.8 and 3.3, if  $\mathcal{G}_{n,p}$  has a percolating subgraph that is larger than  $(\beta_* + \delta) \log n$ , then it has some irreducible percolating subgraph  $G$  of size

$$(\beta_* + \delta) \log n \leq k \leq 9 \log n$$

with a core  $C \subset G$  of size  $q \leq k$ . By Proposition 4.9, with high probability  $q \leq (3/2) \log n$ . However then, by Proposition 4.1, with high probability  $\mathcal{G}_{n,p}$  contains no such subgraphs  $G$ . Therefore, with high probability, all percolating subgraphs of  $\mathcal{G}_{n,p}$  are of size  $k \leq (\beta_* + \delta) \log n$ . On the other hand, as shown in [2],  $\mathcal{G}_{n,p}$  has seed subgraphs larger than  $(\beta_* - \delta) \log n$ , completing the proof.  $\square$

Turning now to the proof of Proposition 4.9, we first observe that  $\mathcal{G}_{n,p}$  has no large percolating subgraphs with small cores and few vertices of degree 2.

**Lemma 4.10.** *Fix  $\alpha \in (0, 1/3)$  and put  $p = \sqrt{\alpha/(n \log n)}$ . Fix some  $i_* \geq 1$ . With high probability  $\mathcal{G}_{n,p}$  has no irreducible percolating subgraph  $G$  of size  $k \geq (3/2) \log n$  with a core  $C \subset G$  of size  $q \leq (3/2) \log n$  and  $i \leq i_*$  vertices of degree 2.*

This is essentially a straightforward consequence of Lemma 3.8.

*Proof.* By Lemma 3.3 and 4.3, we may assume that if  $\mathcal{G}_{n,p}$  has an irreducible percolating subgraph  $G$  of size  $k = \beta \log n$  with a core of size  $q \leq (3/2) \log n$ , then  $G$  has excess  $\ell \leq 3$ . By Proposition 4.1 and Lemma 4.2 and 4.6, we may further assume that  $\beta \in [3/2, 3]$  and  $q = yk$ , where  $y\beta \in [0, 3/2 - \varepsilon]$ , for some  $\varepsilon > 0$ . Without loss of generality, we assume that  $\varepsilon < 1/(2e)$  and  $\log(3/(2e) + \varepsilon) < -1/2$  (which is possible, since  $1 + 2 \log(3/(2e)) \approx -0.189 < 0$ ). By Lemma 3.8 and since  $\alpha < 1/3$ , for some constant  $\vartheta \geq 1$ , the expected number of such subgraphs  $G$  (for a given  $k, q, i$ ) is bounded by

$$\binom{n}{k} p^{2k-3+\ell} I_q^\ell(k, i) \leq \vartheta k^{2\ell+i} p^{\ell-3} (knp^2 \psi_\varepsilon(q/k))^k \ll n^\nu$$

where

$$\nu(\beta, \psi_\varepsilon(y)) = 3/2 + \beta \log(\beta/3) + \beta \log \psi_\varepsilon(y).$$

Here,  $\psi_\varepsilon(y)$  is as defined in Lemma 3.8, that is,

$$\psi_\varepsilon(y) = \max\{3/(2e) + \varepsilon, (e/2)^{1-2y}y^2\}.$$

Recall that  $\psi_\varepsilon(y) = 3/(2e) + \varepsilon$  for  $y \leq y_*$  and  $\psi_\varepsilon(y) = (e/2)^{1-2y}y^2$  for  $y > y_*$ , where  $y_* = y_*(\varepsilon)$  is as defined by (3.1). Moreover,  $y_* \downarrow y_0$  as  $\varepsilon \downarrow 0$ , where  $y_0 \approx 0.819$ .

Finally, observe that, for some  $\delta > 0$ ,  $\nu(\beta, \psi_\varepsilon(y)) < -\delta$  for all relevant all  $\beta, y$ . This follows by basic calculus, see Appendix C below. Hence the lemma follows by a simple union bound, summing over all  $O(\log^2 n)$  relevant values of  $k, q, i$ .  $\square$

Finally, we prove Proposition 4.9. The main idea is as follows: Suppose that  $\mathcal{G}_{n,p}$  has an irreducible percolating 3-core  $C$  of size  $k = \beta \log n$ , for some  $\beta \in [3/2, 9]$ . By Lemma 4.3, we can assume that its excess is  $\ell \leq 27$ . Hence, in the last step of a clique process for  $C$ , either 2 or 3 percolating subgraphs are merged that have few vertices of degree 2 (by the observations following Lemma 3.7 above). Therefore, by Lemma 4.10, each of these subgraphs is either smaller than  $(3/2) \log n$ , or else has a 3-core larger than  $(3/2) \log n$ . Hence, in proving Proposition 4.9, the key is to consider  $C$  in  $\mathcal{G}_{n,p}$  of *minimal* size larger than  $(3/2) \log n$ . By Lemma 4.6, there is some  $\beta_1 < 3/2$  so that with high probability  $\mathcal{G}_{n,p}$  has no percolating subgraphs of size  $\beta \log n$  with few vertices of degree 2, for  $\beta \in [\beta_1, 3/2]$ . Hence such a graph  $C$ , if it exists, is the result of the (unlikely) event that 2 or 3 percolating graphs, all of which are smaller than  $\beta_1 \log n$  and have few vertices of degree 2, are merged to form a percolating 3-core of size at least  $(3/2) \log n$ . Informally, this corresponds to a costly “macroscopic jump” in the clique process.

*Proof of Proposition 4.9.* By Lemma 4.6, there is some  $\beta_1 < 3/2$  so that with high probability  $\mathcal{G}_{n,p}$  has no irreducible percolating subgraphs of size

$$\beta_1 \log n \leq k \leq \frac{3}{2} \log n$$

with  $i \leq 15$  vertices of degree 2.

Suppose that  $\mathcal{G}_{n,p}$  has an irreducible 3-core  $C$  of size  $k = \beta \log n$ , for some  $\beta \in [3/2, 9]$ . By Lemma 4.3, we may assume that its excess is  $\ell \leq 27$ . Assume, moreover, that  $C$  is of minimal size amongst such subgraphs. Then by Lemma 2.7 there are two possibilities for the last step of a clique process for  $C$ :

- (i) Three irreducible percolating subgraphs  $G_j, j \in \{1, 2, 3\}$ , are merged which form exactly one triangle  $T = \{v_1, v_2, v_3\}$ , such that for some  $i_j \leq 2$  and  $k_j, \ell_j \geq 0$  with  $\sum k_j = k + 3$  and  $\sum \ell_j = \ell$ , the  $G_j$  contribute to  $I^{\ell_j}(k_j, i_j)$ . If any  $i_j > 0$ , the  $i_j$  vertices of  $G_j$  of degree 2 belong to  $T$ .
- (ii) For some  $m \leq (\ell + 3)/2 \leq 15$ , two percolating subgraphs  $G_j, j \in \{1, 2\}$ , are merged that share exactly  $m$  vertices  $S = \{v_1, v_2, \dots, v_m\}$ , such that for some  $k_j, \ell_j \geq 0$  with  $\sum k_j = k + m$  and  $\sum \ell_j = \ell - (2m - 3)$ , the  $G_j$  contribute to  $C^{\ell_j}(k_j)$ .

In either case, by the choice of  $C$ , all  $G_j$  have a core smaller than  $(3/2) \log n$ . Hence, by Lemmas 3.3 and 4.3, we may assume that each  $\ell_j \leq 3$ . Also, by Lemma 4.10 and the choice of  $\beta_1$ , we may further assume that all  $G_j$  are smaller than  $\beta_1 \log n$ .

**Case (i).** Let  $k_j, \ell_j$  be as in (i). Let  $k_j - (j - 1) = \varepsilon_j k$ , so that  $\sum \varepsilon_j = 1$ . Without loss of generality we assume that  $k_1 \geq k_2 \geq k_3$ . Hence  $\varepsilon_1, \varepsilon_2$  satisfy  $1/3 \leq \varepsilon_1 \leq \beta_1/\beta < 1$  and  $(1 - \varepsilon_1)/2 \leq \varepsilon_2 \leq \min\{\varepsilon_1, 1 - \varepsilon_1\}$ . The number of 3-cores  $C$  as in (i) for these given values  $k_j, \ell_j$  is bounded by

$$\binom{k}{k_1, k_2 - 1, k_3 - 2} \binom{k_1}{2} 2! \binom{k_2 - 1}{1} \prod_{j=1}^3 \sum_{i=0}^2 \binom{2}{i} \frac{I^{\ell_j}(k_j, i)}{\binom{k_j}{i}}.$$

Applying Lemma 3.7 and the inequality  $k! < ek(k/e)^k$  (and recalling  $\ell_j \leq 3$ ), this is bounded by

$$\binom{k}{k-k_1} \binom{k-k_1}{k_3-2} (4ek^8)^3 \left(\frac{2}{e^2}\right)^{k+3} \prod_{j=1}^3 k_j^{2k_j}.$$

By the inequality  $\binom{n}{k} < (ne/k)^k$ , and noting that

$$k_j^{2k_j} \leq (ek)^{2(j-1)} (k_j - (j-1))^{2(k_j - (j-1))},$$

we see that the above expression is bounded by  $(2e^{-2}\eta(\varepsilon_1, \varepsilon_2)k^2)^k n^{o(1)}$ , where

$$\begin{aligned} \eta(\varepsilon_1, \varepsilon_2) &= \left(\frac{e}{1-\varepsilon_1}\right)^{1-\varepsilon_1} \left(\frac{(1-\varepsilon_1)e}{\varepsilon_3}\right)^{\varepsilon_3} \varepsilon_1^{2\varepsilon_1} \varepsilon_2^{2\varepsilon_2} \varepsilon_3^{2\varepsilon_3} \\ &= \frac{e^{1-\varepsilon_1+\varepsilon_3}}{(1-\varepsilon_1)^{\varepsilon_2}} \varepsilon_1^{2\varepsilon_1} \varepsilon_2^{2\varepsilon_2} \varepsilon_3^{\varepsilon_3}. \end{aligned}$$

Therefore, since  $\alpha < 1/3$ , the expected number of 3-cores  $C$  as in (i) for these given values  $k_j, \ell_j$  is at most

$$\binom{n}{k} p^{2k-3} \left(\frac{2}{e^2}\eta(\varepsilon_1, \varepsilon_2)k^2\right)^k n^{o(1)} = p^{-3} \left(\frac{2}{e}\alpha\beta\eta(\varepsilon_1, \varepsilon_2)\right)^k n^{o(1)} \ll n^\nu$$

where

$$\nu(\beta, \varepsilon_1, \varepsilon_2) = \frac{3}{2} + \beta \log \left(\frac{2}{3e}\beta\eta(\varepsilon_1, \varepsilon_2)\right).$$

In Appendix D it is shown, by basic calculus, that for some  $\delta > 0$ ,  $\nu(\beta, \varepsilon_1, \varepsilon_2) < -\delta$  for all relevant  $\beta, \varepsilon_1, \varepsilon_2$ . Therefore, taking a union bound (summing over all  $O(\log^3 n)$  relevant values of  $k, k_j, \ell_j$ ) we find that with high probability  $\mathcal{G}_{n,p}$  has no subgraphs  $C$  as in (i) above.

The next case is similar. We only sketch the details.

**Case (ii).** Let  $k_j, \ell_j, m$  be as in (ii). Let  $k_1 = \varepsilon_1 k$  and  $k_2 - m = \varepsilon_2 k$ , so that  $\sum \varepsilon_j = 1$ . Without loss of generality we assume that  $k_1 \geq k_2$ . Hence  $\varepsilon_1, \varepsilon_2$  satisfy  $1/2 \leq \varepsilon_1 \leq \beta_1/\beta < 1$  and  $\varepsilon_2 = 1 - \varepsilon_1$ . The number of 3-cores  $C$  as in (ii) for these given values  $k_j, \ell_j, m$  is bounded by

$$\binom{k}{k_2-m} \binom{k_1}{m} m! \prod_{j=1}^2 C^{\ell_j}(k_j, i).$$

Therefore, arguing as in Case (i), we find that the expected number of 3-cores  $C$  as in (ii) for these given values  $k_j, \ell_j, m$  is at most  $\ll n^\nu$ , where  $\nu = \nu(\beta, \varepsilon_1, 1 - \varepsilon_1)$  is as in Case (i). Hence, once again, by taking a union bound (summing over all  $O(\log^2 n)$  relevant values of  $k, k_j, \ell_j, m$ ) we find that with high probability  $\mathcal{G}_{n,p}$  has no subgraphs  $C$  as in (i) above.

The proof is complete. □

## A Basic estimates

*Proof of Lemma 3.7.* It is easily verified that the statement of the lemma holds for  $k \leq 4$ . For  $k > 4$ , we claim moreover that for all  $\ell \leq 3$  and relevant  $i$ ,

$$I^\ell(k, i) \leq A\zeta^k \binom{k}{i} k! k^{k+2\ell} \tag{A.1}$$

where  $\zeta = 2/e$  and  $A = 6/(\zeta^5 5! 5^5)$ . Since  $A < 1$  and  $\binom{k}{i} \leq k^i$ , the lemma follows.

The constant  $A$  (appearing in (A.1) but not in Lemma 3.7) is used as a device in the proof to control the case of 3-cores ( $i = 0$ ), that can be formed by “merging” multiple percolating graphs of “macroscopic” size.

The proof is by induction. By the choice of  $A$ , we note that (A.1) holds for  $k = 5$ . Indeed,  $I^\ell(5, i) \leq \binom{5}{i} \binom{4}{2}$  for  $i \in \{1, 2, 3\}$  and  $\ell = 0$ , and  $I^\ell(5, i) = 0$  otherwise. Assume that for some  $k > 5$ , (A.1) holds for all  $4 < k' < k$ ,  $\ell \leq 3$  and relevant  $i$ .

The case  $i > 0$ , where  $G$  has at least one vertex of degree 2 follows easily, and elucidates the choice of  $\zeta = 2/e$ .

**Case 1** ( $i > 0$ ). Suppose that  $G$  is a graph contributing to  $I^\ell(k, i)$ , where  $i > 0$  and  $\ell \leq 3$ . Let  $v \in V(G)$  be the vertex of degree 2 in  $G$  of minimal index. By considering which two of the  $k - i$  vertices of  $G$  of degree larger than 2 are neighbors of  $v$ , we find that  $I^\ell(k, i)$  is bounded from above by

$$\binom{k}{i} \binom{k-i}{2} \sum_{j=0}^2 \binom{2}{j} \frac{I^\ell(k-1, i-1+j)}{\binom{k-1}{i-1+j}}.$$

In this sum,  $j \in \{0, 1, 2\}$  is the number of neighbors of  $v$  that are of degree 2 in the subgraph of  $G_v$  of  $G$  induced by  $V(G) - \{v\}$ . Applying the inductive hypothesis, we obtain

$$I^\ell(k, i) \leq A\zeta^k \binom{k}{i} k! k^{k+2\ell} \cdot \frac{2}{\zeta} \left(\frac{k-1}{k}\right)^k \leq A\zeta^k \binom{k}{i} k! k^{k+2\ell},$$

as required.

The remaining cases deal with 3-cores  $G$  of size  $k$ , where  $i = 0$ . First, we establish the case  $i = \ell = 0$  of edge-minimal 3-cores. The cases  $i = 0$  and  $\ell \in \{1, 2, 3\}$  are proved by adapting this argument.

**Case 2** ( $i = \ell = 0$ ). Let  $G$  be a graph contributing to  $C(k) = I(k, 0)$ . Then, by Lemma 2.7, in the last step of a clique process for  $G$ , three edge-minimal percolating subgraphs  $G_j$ ,  $j \in \{1, 2, 3\}$ , are merged which form exactly one triangle on some  $T = \{v_1, v_2, v_3\} \subset V(G)$ . Moreover, each  $G_j$  has at most 2 vertices of degree 2, and if some  $G_j$  has such a vertex  $v$  then necessarily  $v \in T$  (as else  $G$  would have a vertex of degree 2). Also if  $k_j = |V(G_j)|$ , with  $k_1 \geq k_2 \geq k_3$ , then (i)  $\sum_{j=1}^3 k_j = k + 3$ , (ii)  $k_1, k_2 \geq 4$  and (iii)  $k_3 = 2$  or  $k_3 \geq 4$  (since if some  $k_j = 3$  or some  $k_j = k_{j'} = 2$ ,  $j \neq j'$ , then  $G$  would have a vertex of degree 2).

Since the inductive hypothesis only holds for graphs with more than 4 vertices, it is convenient to deal with the case  $k_1 = 4$  separately: Note that the only irreducible percolating 3-cores of size  $k$  with  $k_1 = 4$  are of size  $k \in \{7, 9\}$ . These are the graph in Figure 1 and the graph obtained from this graph by replacing the bottom edge with a copy of  $K_4$  minus an edge. It is easy to verify that (A.1) holds if  $k \in \{7, 9\}$ , and so in the arguments below we assume that  $k_1 > 4$  and  $k \geq 8$ .

We take three cases, with respect to whether (i)  $k_2 = 4$ , (ii)  $k_2 > 4$  and  $k_3 \in \{2, 4\}$ , or (iii)  $k_3 > 4$ .

**Case 2(i)** ( $i = \ell = 0$  and  $k_2 = 4$ ). Note that if  $k_2 = 4$  then  $k_3 \in \{2, 4\}$ . The number of graphs  $G$  as above with  $k_3 = 2$  and  $k_2 = 4$  is bounded from above by

$$\binom{k}{k-3} \binom{k-3}{2} 2! \binom{3}{1} \sum_{i=0}^2 \binom{2}{i} \frac{I(k-3, i)}{\binom{k-3}{i}}.$$

Here the first binomial selects the vertices for the subgraph of size  $k_1 = k - 3$ , the next three factors select and order the vertices in the triangle  $T$ , and the rightmost factor bounds the number of possibilities for the subgraph of size  $k_1 = k - 3$  (recalling that it can have at most 2 vertices of degree 2, and if it contains any such vertex  $v$ , then  $v \in T$ ).

Applying the inductive hypothesis (recall that we may assume that  $k_1 > 4$ ), the above expression is bounded by

$$A\zeta^k k!k^k \cdot \frac{(k-3)^{k-1}}{k^k} \frac{2}{\zeta^3} \leq A\zeta^k k!k^k \cdot \frac{1}{k} \frac{2}{\zeta^3 e^3}.$$

Here, and throughout this proof, we use the fact that  $(\frac{k-x}{k})^{k-y} \leq e^{-x}$  provided that  $2y \leq x < k$  and  $x > 0$ . To see this, note that  $(\frac{k-x}{k})^{k-y} \rightarrow e^{-x}$  as  $k \rightarrow \infty$ , and

$$\begin{aligned} \frac{\partial}{\partial k} \left(\frac{k-x}{k}\right)^{k-y} &= \left(\frac{k-x}{k}\right)^{k-y} \left(\log\left(\frac{k-x}{k}\right) + \frac{x(k-y)}{k(k-x)}\right) \\ &\geq \left(\frac{k-x}{k}\right)^{k-y} \frac{x(x-2y)}{2k(k-x)} \geq 0, \end{aligned}$$

by the inequality  $\log u \geq (u^2 - 1)/(2u)$  (which holds for  $u \in (0, 1]$ ).

Similarly, the number of graphs  $G$  as above such that  $k_1 = k_2 = 4$  is bounded by

$$\binom{k}{k-5, 3, 2} \binom{k-5}{2} 2! \binom{3}{1} \sum_{i=0}^2 \binom{2}{i} \frac{I(k-5, i)}{\binom{k-5}{i}}.$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k!k^k \cdot \frac{(k-5)^{k-3}}{k^k} \frac{1}{\zeta^5} \leq A\zeta^k k!k^k \cdot \frac{1}{k^{5/2} \sqrt{k-5}} \frac{1}{\zeta^5 e^5}.$$

Altogether, we find that the number of graphs  $G$  contributing to  $C(k)$  with  $k_2 = 4$ , divided by  $A\zeta^k k!k^k$ , is bounded by

$$\gamma_1 = \frac{1}{8} \frac{2}{\zeta^3 e^3} + \frac{1}{8^{5/2} \sqrt{3}} \frac{1}{\zeta^5 e^5} < 0.04. \tag{A.2}$$

**Case 2(ii)** ( $i = \ell = 0$ ,  $k_2 > 4$  and  $k_3 \in \{2, 4\}$ ). Note that in this case we may further assume that  $k \geq 9$ . For a given  $k_1, k_2 > 4$ , the number of graphs  $G$  as above with  $k_3 = 2$  (in which case  $k_1 + k_2 = k + 1$ ) is bounded by

$$\binom{k}{k_1, k_2 - 1} \binom{k_1}{2} 2! \binom{k_2 - 1}{1} \prod_{j=1}^2 \sum_{i=0}^2 \binom{2}{i} \frac{I(k_j, i)}{\binom{k_j}{i}}.$$

Applying the inductive hypothesis, this is bounded by

$$A\zeta^k k!k^k \cdot 4^2 A\zeta \frac{k_1^{k_1+2} k_2^{k_2+2}}{k^k}.$$

Since  $k_2 = k + 1 - k_1$ , we have that

$$\frac{\partial}{\partial k_1} k_1^{k_1+2} k_2^{k_2+2} = -k_1^{k_1+1} k_2^{k_2+1} (k_1 k_2 \log(k_2/k_1) - 2(k_1 - k_2)).$$

By the bound  $\log x \leq x - 1$ , we see that

$$k_1 k_2 \log(k_2/k_1) - 2(k_1 - k_2) \leq -(k_2 + 2)(k_1 - k_2) \leq 0.$$

Hence, setting  $k_1$  to be the maximum relevant value  $k_1 = k - 4$  (when  $k_2 = 5$ ), we find

$$\frac{k_1^{k_1+2} k_2^{k_2+2}}{k^k} \leq \frac{5^7 (k-4)^{k-2}}{k^k} \leq \frac{1}{k^2} \frac{5^7}{e^4}$$

for all relevant  $k_1, k_2$ . Therefore, summing over the at most  $k/2$  possibilities for  $k_1, k_2$ , we find that at most

$$A\zeta^k k!k^k \cdot \frac{1}{k} \frac{A\zeta^4 2^5 5^7}{2e^4}$$

graphs  $G$  with  $k_3 = 2$  and  $k_2 > 4$  contribute to  $C(k)$ .

The case of  $k_3 = 4$  is very similar. In this case, for a given  $k_1, k_2 > 4$  such that  $k_1 + k_2 = k - 1$ , the number of graphs  $G$  as above is bounded by

$$\binom{k}{k_1, k_2 - 1, 2} \binom{k_1}{2} 2! \binom{k_2 - 1}{1} \prod_{j=1}^2 \sum_{i=0}^2 \binom{2}{i} \frac{I(k_j, i)}{\binom{k_j}{i}},$$

which, by the inductive hypothesis, is bounded by

$$A\zeta^k k!k^k \cdot \frac{4^2}{2} \frac{A}{\zeta} \frac{k_1^{k_1+2} k_2^{k_2+2}}{k^k}.$$

Arguing as in the previous case, we see that the above expression is maximized when  $k_2 = 5$  and  $k_1 = k - 6$ . Hence, summing over the at most  $k/2$  possibilities for  $k_1, k_2$ , there are at most

$$A\zeta^k k!k^k \cdot \frac{1}{(k-6)k^2} \frac{4A5^7}{\zeta e^6}$$

graphs  $G$  that contribute to  $C(k)$  with  $k_3 = 4$  and  $k_2 > 4$ .

We conclude that the number of graphs  $G$  that contribute to  $C(k)$  with  $k_2 > 4$  and  $k_3 \in \{2, 4\}$ , divided by  $A\zeta^k k!k^k$ , is bounded by

$$\gamma_2 = \frac{1}{9} \frac{A\zeta^4 2^5 5^7}{2e^4} + \frac{1}{3 \cdot 9^2} \frac{4A5^7}{\zeta e^6} < 0.07. \tag{A.3}$$

**Case 2(iii)** ( $i = \ell = 0$  and  $k_3 > 4$ ). In this case we may further assume that  $k \geq 12$ . For a given  $k_1, k_2, k_3 > 4$  such that  $k_1 + k_2 + k_3 = k + 3$ , the number of graphs  $G$  as above is bounded by

$$\binom{k}{k_1, k_2 - 1, k_3 - 2} \binom{k_1}{2} 2! \binom{k_2 - 1}{1} \prod_{j=1}^3 \sum_{i=0}^2 \binom{2}{i} \frac{I(k_j, i)}{\binom{k_j}{i}}.$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k!k^k \cdot 4^3 A^2 \zeta^3 \frac{k_1^{k_1+2} k_2^{k_2+2} k_3^{k_3+2}}{k^k}.$$

As in the previous cases considered, the above expression is maximized when  $k_2 = k_3 = 5$  and  $k_1 = k - 7$ . Hence, summing over the at most  $k^2/12$  choices for the  $k_j$ , we find that at most

$$A\zeta^k k!k^k \cdot \frac{1}{((k-7)k)^{3/2}} \frac{A^2 \zeta^3 4^3 5^{14}}{12e^7}$$

graphs  $G$  contribute to  $C(k)$  with  $k_3 > 4$ . Hence, the number of such graphs, divided by  $A\zeta^k k!k^k$ , is bounded by

$$\gamma_3 = \frac{1}{(5 \cdot 12)^{3/2}} \frac{A^2 \zeta^3 4^3 5^{14}}{12e^7} < 0.01. \tag{A.4}$$

Finally, combining (A.2), (A.3) and (A.4), we find that

$$\frac{C(k)}{A\zeta^k k!k^k} \leq \gamma_1 + \gamma_2 + \gamma_3 < 0.12 < 1, \tag{A.5}$$

completing the proof of Case 2.

It remains to consider the cases  $i = 0$  and  $\ell \in \{1, 2, 3\}$ , corresponding to 3-cores  $G$  with non-zero excess. In these cases, it is possible that only 2 subgraphs are merged in the last step of a clique process for  $G$ . We prove the cases  $\ell = 1, 2, 3$  separately, however they all follow by adjusting the proof of Case 2.

First, we note that if two graphs  $G_1, G_2$  with at least 2 vertices in common are merged to form an irreducible percolating 3-core  $G$ , then necessarily each  $G_j$  contains more than 4 vertices. In particular, such a graph  $G$  contains at least 8 vertices. This allow us to apply the inductive hypothesis in these cases (recall that we claim that (A.1) holds only for graphs with more than 4 vertices), without taking additional sub-cases as in the proof of Case 2. Moreover, as discussed below the statement of Lemma 3.7, in this case we also have that the  $G_j$  are 3-cores.

**Case 3** ( $i = 0$  and  $\ell = 1$ ). If  $G$  contributes to  $C^1(k)$ , then by Lemma 2.7, in the last step of a clique process for  $G$ , there are two cases to consider:

- (i) Three percolating subgraphs  $G_j, j \in \{1, 2, 3\}$ , are merged which form exactly one triangle  $T = \{v_1, v_2, v_3\}$ , such that for some  $i_j \leq 2$  and  $k_j, \ell_j \geq 0$  with  $\sum k_j = k + 3$  and  $\sum \ell_j = 1$ , we have that  $G_j$  contributes to  $I^{\ell_j}(k_j, i_j)$ . Moreover, if any  $i_j > 0$ , the  $i_j$  vertices of  $G_j$  of degree 2 belong to  $T$ .
- (ii) Two percolating subgraphs  $G_j, j \in \{1, 2\}$ , are merged that share exactly two vertices  $S = \{v_1, v_2\}$ , such that for some  $k_j$  with  $\sum k_j = k + 2$ , we have that the  $G_j$  contribute to  $C(k_j, i_j)$ .

We claim that, by the arguments in Case 2 leading to (A.5), the number of graphs  $G$  satisfying (i), divided by  $A\zeta^k k! k^{k+2}$ , is bounded by

$$\gamma_1 + 2\gamma_2 + 3\gamma_3 < 0.21. \tag{A.6}$$

To see this, note the only difference between (i) of the present case and Case 2 above is that here one of the  $G_j$  has exactly 1 excess edge. Note that if one of the graphs  $G_j$  has an excess edge, then necessarily  $k_j > 4$ . Recall that graphs  $G$  that contribute to  $C(k)$ , as considered in Cases 2(i),(ii),(iii) above, have exactly 1, 2, 3 subgraphs  $G_j$  with  $k_j > 4$ , respectively. Moreover, recall that the number of such graphs  $G$ , divided by  $A\zeta^k k! k^k$ , is bounded by  $\gamma_1, \gamma_2, \gamma_3$ , respectively, in these cases. Therefore, applying the inductive hypothesis, and noting that if  $G_j$  has exactly  $\ell_j = 1$  excess edge then it contributes an extra factor of  $k_j^2 < k^2$ , it follows that the number of graphs  $G$  as in (i) of the present case, divided by  $A\zeta^k k! k^{k+2}$ , is bounded by  $\sum_{j=1}^3 j\gamma_j$ , as claimed. (By (A.2), (A.3) and (A.4), this sum is bounded by 0.21.)

On the other hand, arguing along the lines as in Case 2, the number of graphs  $G$  satisfying (ii), for a given  $k_1, k_2 > 4$  such that  $k_1 + k_2 = k + 2$ , is bounded by

$$\binom{k}{k_1, k_2 - 2} \binom{k_1}{2} 2! \prod_{j=1}^2 C(k_j, i).$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k! k^k \cdot A\zeta^2 \frac{k_1^{k_1+2} k_2^{k_2+2}}{k^k}.$$

Arguing as in Case 2, we find that this expression is maximized when  $k_2 = 5$  and  $k_1 = k - 3$ . Hence, summing over the at most  $k/2$  choices for  $k_1, k_2$ , the number of graphs  $G$  satisfying (ii), divided by  $A\zeta^k k! k^{k+2}$ , is at most

$$\gamma_4 = \frac{1}{8^2} \frac{A\zeta^2 5^7}{2e^3} < 0.01. \tag{A.7}$$

Altogether, by (A.6) and (A.7), we conclude that

$$\frac{C^1(k)}{A\zeta^k k! k^{k+2}} \leq \gamma_1 + 2\gamma_2 + 3\gamma_3 + \gamma_4 < 0.22 < 1, \tag{A.8}$$

completing the proof of Case 3.

**Case 4** ( $i = 0$  and  $\ell = 2$ ). This case is nearly identical to Case 3. By Lemma 2.7, in the last step of a clique process for a graph  $G$  that contributes to  $C^2(k)$ , either (i) three graphs that form exactly one triangle are merged whose excesses sum to 2, or else (ii) two graphs that share exactly two vertices are merged whose excesses sum to 1. Hence, by the arguments in Case 3 leading to (A.8), we find that

$$\frac{C^2(k)}{A\zeta^k k! k^{k+4}} \leq \gamma_1 + 3\gamma_2 + 6\gamma_3 + 2\gamma_4 < 0.33 < 1, \tag{A.9}$$

as required.

**Case 5** ( $i = 0$  and  $\ell = 3$ ). Since  $\ell = 3$ , it is now possible that in the last step of a clique process for a graph  $G$  contributing to  $C^\ell(k)$ , two graphs are merged that share three vertices. Apart from this difference, the argument is completely analogous to the previous cases.

If  $G$  contributes to  $C^3(k)$ , then by Lemma 2.7, in the last step of a clique process for  $G$ , there are three cases to consider:

- (i) Three percolating subgraphs  $G_j$ ,  $j \in \{1, 2, 3\}$ , are merged which form exactly one triangle  $T = \{v_1, v_2, v_3\}$ , such that for some  $i_j \leq 2$  and  $k_j, \ell_j \geq 0$  with  $\sum k_j = k + 3$  and  $\sum \ell_j = 3$ , we have that  $G_j$  contributes to  $I^{\ell_j}(k_j, i_j)$ . If any  $i_j > 0$ , the corresponding  $i_j$  vertices of  $G_j$  of degree 2 belong to  $T$ .
- (ii) Two percolating subgraphs  $G_j$ ,  $j \in \{1, 2\}$ , are merged that share exactly two vertices  $S = \{v_1, v_2\}$ , such that for some  $k_j$  with  $\sum k_j = k + 2$ , we have that the  $G_j$  contribute to  $C(k_j)$ .
- (iii) Two percolating subgraphs  $G_j$ ,  $j \in \{1, 2\}$ , are merged that share exactly three vertices  $R = \{v_1, v_2, v_3\}$ , such that for some  $k_j$  with  $\sum k_j = k + 3$ , we have that the  $G_j$  contribute to  $C(k_j)$ .

As in Case 4, we find by the arguments in Case 3 leading to (A.8) that the number of graphs  $G$  satisfying (i) or (ii), divided by  $A\zeta^k k! k^{k+6}$ , is bounded by

$$\gamma_1 + 4\gamma_2 + 10\gamma_3 + 3\gamma_4 < 0.45. \tag{A.10}$$

By the arguments in Case 3 leading to (A.7), the number of graphs  $G$  satisfying (iii), for a given  $k_1, k_2 > 4$  such that  $k_1 + k_2 = k + 3$ , is bounded by

$$\binom{k}{k_1, k_2 - 3} \binom{k_1}{3} 3! \prod_{j=1}^2 C(k_j).$$

By the inductive hypothesis, this is bounded by

$$A\zeta^k k! k^k \cdot A\zeta^3 \frac{k_1^{k_1+3} k_2^{k_2+3}}{k^k}.$$

This expression is maximized when  $k_2 = 5$  and  $k_1 = k - 2$ . Hence, summing over the at most  $k/2$  choices for  $k_1, k_2$ , the number of graphs  $G$  satisfying (iii), divided by  $A\zeta^k k! k^{k+6}$ , is at most

$$\gamma_5 = \frac{1}{8^4} \frac{A\zeta^3 5^8}{2e^2} < 0.01. \tag{A.11}$$

Therefore, by (A.10) and (A.11), we have that

$$\frac{C^3(k)}{A\zeta^k k! k^{k+6}} \leq \gamma_1 + 4\gamma_2 + 10\gamma_3 + 3\gamma_4 + \gamma_5 < 0.46 < 1,$$

completing the proof of Case 5.

This last case completes the induction. We conclude that (A.1) holds for all  $k > 4$ ,  $\ell \leq 3$  and relevant  $i$ , and the lemma follows.  $\square$

## B Sharper estimates

*Proof of Lemma 3.8.* Let  $\varepsilon > 0$  be given. We may assume that  $\varepsilon < 1/(2e)$ , as otherwise the statement of lemma follows by Lemma 3.7. We claim that, for some  $\vartheta(\varepsilon) \geq 1$  (to be determined below), and for all  $k \geq 2$ ,  $\ell \leq 3$  and relevant  $q, i$ , we have that

$$I_q^\ell(k, i) \leq \vartheta \binom{k}{i} \psi_\varepsilon(q/k)^k k! k^{k+2\ell}. \tag{B.1}$$

**Case 1** ( $i = k - q$ ). We first observe that Lemma 3.7 implies the case  $i = k - q$ . Indeed, if  $q = k$ , in which case  $i = 0$ , then (B.1) follows immediately by Lemma 3.7, noting that  $I_k^\ell(k, 0) = C^\ell(k)$  and  $\psi(1) = 2/e$ . On the other hand, if  $i = k - q > 0$  then

$$I_q^\ell(k, k - q) = \binom{k}{k - q} \binom{q}{2}^{k - q} C^\ell(q),$$

since all  $k - q$  vertices of degree 2 in a graph that contributes to  $I_q^\ell(k, k - q)$  are neighbors of 2 vertices in its core. We claim that the right hand side is bounded by

$$\binom{k}{k - q} (e/2)^{k - 2q} (q/k)^{2k} k! k^{k + 2\ell}.$$

Since  $(e/2)^{k - 2q} (q/k)^{2k} \leq \psi(q/k)^k$ , (B.1) follows. To see this, note that by Lemma 3.7, we have that

$$\frac{\binom{q}{2}^{k - q} C^\ell(q)}{(e/2)^{k - 2q} (q/k)^{2k} k! k^{k + 2\ell}} \leq \left(\frac{q}{k}\right)^{2\ell} \frac{q!}{(q/e)^q} \frac{(k/e)^k}{k!} \leq \frac{q!}{(q/e)^q} \frac{(k/e)^k}{k!}.$$

By the inequalities  $1 \leq i! / (\sqrt{2\pi i} (i/e)^i) \leq e^{1/(12i)}$ , it is easy to verify that the right hand side above is bounded by 1, for all relevant  $q \leq k$ . Hence (B.1) holds also in the case  $i = k - q > 0$ .

**Case 2** ( $i < k - q$ ). Fix some  $k_\varepsilon \geq 1/(1 - y_*)^2$  (where  $y_*$  is as in (3.1)) such that, for all  $k \geq k_\varepsilon$  and relevant  $q$ , we have that

$$1 + \frac{2}{k - 1} \binom{k - 2}{k - 1}^k \frac{\psi_\varepsilon(q/(k - 2))^{k - 2}}{\psi_\varepsilon(q/(k - 1))^{k - 1}} = 1 + O(1/k) \leq 1 + \delta,$$

where

$$\delta = \min \left\{ 1 - \frac{3/(2e)}{3/(2e) + \varepsilon}, 1 - \frac{3(1 - y_*)}{y_*^2} \right\}.$$

Note that, since  $3(1 - y)/y^2 < 1$  for all  $y > (\sqrt{21} - 3)/2 \approx 0.791$ , and recalling (see (3.1)) that  $y_* > y_0 \approx 0.819$ , it follows that  $\delta > 0$ .

Select  $\vartheta \geq 1$  so that (B.1) holds for all  $k \leq k_\varepsilon$  and relevant  $q, \ell, i$ . By Case 1 and since  $\vartheta \geq 1$ , we have that (B.1) holds for all  $k, q$  in the case that  $i = k - q$ . We establish the remaining cases  $i < k - q$  by induction. Assume that for some  $k > k_\varepsilon$ , (B.1) holds for all  $k' < k$  and relevant  $q, \ell, i$ .

In any graph  $G$  contributing to  $I_q^\ell(k, i)$ , where  $i < k - q$ , there is some vertex of degree 2 with at least one of its two neighbors not in the core of  $G$ . There are two cases to consider: either

- (i) there is a vertex  $v$  of degree 2 such that at least one of its two neighbors is of degree 2 in  $G_v$  (obtained from  $G$  by deleting  $v$ ), or else,
- (ii) there is no such vertex  $v$ , however there are vertices  $u \neq w$  of degree 2 in  $G$  with a common neighbor that is not in the core  $C$  of  $G$ .

Note that, in case (i), removing  $v$  results in a graph with  $j \in \{i, i + 1\}$  vertices of degree 2. On the other hand, in case (ii), removing  $u$  and  $w$  results in a graph with  $j \in \{i - 2, i - 1, i\}$  vertices of degree 2. Hence, for  $i < k - q$ , we find that  $I_q^\ell(k, i) / \binom{k}{i}$  is bounded by

$$\frac{I_q^\ell(k - 1, i + 1)}{\binom{k-1}{i+1}} \binom{k - i - q}{2} + \frac{I_q^\ell(k - 1, i)}{\binom{k-1}{i}} (k - i - q)(k - i) + (k - i - q)(k - i)^2 \sum_{j=0}^2 \frac{I_q^\ell(k - 2, i - 2 + j)}{\binom{k-2}{i-2+j}}.$$

Applying the inductive hypothesis, it follows (after simple, but somewhat tedious simplifications) that

$$\frac{I_q^\ell(k, i)}{\vartheta \binom{k}{i} \psi_\varepsilon(q/k)^k k! k^{k+2\ell}} \leq \Psi_\varepsilon(q, k) \left[ 1 + \frac{2}{k - 1} \left( \frac{k - 2}{k - 1} \right)^k \frac{\psi_\varepsilon(q/(k - 2))^{k-2}}{\psi_\varepsilon(q/(k - 1))^{k-1}} \right]$$

where

$$\Psi_\varepsilon(q, k) = \frac{3}{2} \frac{k - q}{k} \left( \frac{k - 1}{k} \right)^k \frac{\psi_\varepsilon(q/(k - 1))^{k-1}}{\psi_\varepsilon(q/k)^k}.$$

By the choice of  $k_\varepsilon$ , and since  $k \geq k_\varepsilon$ , we have that

$$\frac{I_q^\ell(k, i)}{\vartheta \binom{k}{i} \psi_\varepsilon(q/k)^k k! k^{k+2\ell}} \leq \Psi_\varepsilon(q, k)(1 + \delta). \tag{B.2}$$

Next, we show that  $\Psi_\varepsilon(q, k) < 1 - \delta$ , completing the induction. To this end, we take cases with respect to whether (i)  $q/(k - 1) \leq y_*$ , (ii)  $y_* \leq q/k$ , or (iii)  $q/k < y_* < q/(k - 1)$ .

**Case 2(i)** ( $q/(k - 1) \leq y_*$ ). In this case  $\psi_\varepsilon(q/m) = 3/(2e) + \varepsilon$ , for each  $m \in \{k - 1, k\}$ . It follows, by the choice of  $\delta$ , that

$$\Psi_\varepsilon(q, k) \leq \left( \frac{k - 1}{k} \right)^k \frac{3/2}{3/(2e) + \varepsilon} \leq \frac{3/(2e)}{3/(2e) + \varepsilon} < 1 - \delta,$$

as required.

**Case 2(ii)** ( $y_* \leq q/k$ ). In this case, we have that  $\psi(q/m)^m = (e/2)^{m-2q} (q/m)^{2m}$ , for each  $m \in \{k - 1, k\}$ . Hence

$$\Psi_\varepsilon(q, k) = \frac{3}{e} \left( \frac{k}{k - 1} \right)^{k-1} \frac{(k - q)(k - 1)}{q^2} \leq \frac{3(1 - y)}{y^2},$$

where  $y = q/k$ . Since the right hand side is decreasing in  $y$ , we find, by the choice of  $\delta$ , that

$$\Psi_\varepsilon(q, k) \leq \frac{3(1 - y_*)}{y_*^2} < 1 - \delta.$$

**Case 2(iii)** ( $q/k < y_* < q/(k - 1)$ ). In this case,  $\psi_\varepsilon(q/k) = 3/(2e) + \varepsilon$  and

$$\psi_\varepsilon(q/(k - 1))^{k-1} = (e/2)^{k-1-2q} (q/(k - 1))^{2(k-1)}.$$

Hence

$$\Psi_\varepsilon(q, k) = \frac{3}{e} \left( \frac{k}{k - 1} \right)^{k-1} \frac{(k - q)(k - 1)}{q^2} \frac{(e/2)^{k-2q} (q/k)^{2k}}{(3/(2e) + \varepsilon)^k}.$$

As in the previous case, we consider the quantity  $y = q/k$ . The above expression is bounded by

$$\frac{3(1-y)}{y^2} \left( \frac{(e/2)^{1-2y}y^2}{3/(2e) + \varepsilon} \right)^k.$$

We claim that this expression is increasing in  $y \leq y_*$ . By (3.1) and the choice of  $\delta$ , it follows that

$$\Psi_\varepsilon(q, k) \leq \frac{3(1-y_*)}{y_*^2} < 1 - \delta,$$

as required. To establish the claim, simply note that

$$\begin{aligned} \frac{\partial}{\partial y} \frac{1-y}{y^2} ((2/e)^y y)^{2k} &= \frac{1}{y^3} ((2/e)^y y)^{2k} (2(1-y)(1+y \log(2/e))k + y - 2) \\ &> \frac{2}{y^3} ((2/e)^y y)^{2k} ((1-y)^2 k - 1) \geq 0 \end{aligned}$$

for all  $y \leq y_*$ , since  $k \geq k_\varepsilon \geq 1/(1-y_*)^2$ .

Altogether, we conclude that  $\Psi_\varepsilon(q, k) \leq 1 - \delta$ , for all relevant  $q$ . By (B.2), it follows that

$$\frac{I_q^\ell(k, i)}{\vartheta \binom{k}{i} \psi_\varepsilon(q/k)^k k! k^{k+2\ell}} \leq 1 - \delta^2 < 1$$

completing the induction. We conclude that (B.1) holds for  $k \geq 2$ ,  $\ell \leq 3$  and relevant  $q, i$ . Since  $\binom{k}{i} \leq k^i$ , the lemma follows.  $\square$

### C Details in the proof of Lemma 4.10

In this section, to complete the proof of Lemma 4.10, we verify that, for some  $\delta > 0$ , we have that  $\nu(\beta, \psi_\varepsilon(y)) < -\delta$  for all relevant  $\beta, y$ . Note that  $\nu$  is convex in  $\beta$ . Therefore it suffices to consider the extreme points  $\beta = 3/2$  and  $\beta = \min\{3, 3/(2y)\}$  in the range  $y \in [0, 1 - 2\varepsilon/3]$ .

Since  $\psi_\varepsilon(1) = 2/e$ , we have that  $\nu(3/2, \psi_\varepsilon(1)) = 0$ . Hence, for some  $\delta_1 > 0$ , we have that  $\nu(3/2, \psi_\varepsilon(y)) < -\delta_1$  for all  $y \in [0, 1 - 2\varepsilon/3]$ . Next, for  $\beta = \min\{3, 3/(2y)\}$ , we treat the cases (i)  $y \in [0, 1/2]$  and  $\beta = 3$  and (ii)  $y \in [1/2, 1 - 2\varepsilon/3]$  and  $\beta = 3/(2y)$  separately. If  $y \leq 1/2$ , then  $\psi_\varepsilon(y) = 3/(2e) + \varepsilon$ , in which case, by the choice of  $\varepsilon$ ,

$$\nu(3, \psi_\varepsilon(y)) = \frac{3}{2}(1 + 2 \log(3/(2e) + \varepsilon)) < 0.$$

On the other hand, for  $y \geq 1/2$ , we need to show that

$$\nu(3/(2y), \psi_\varepsilon(y)) = \frac{3}{2} \left( 1 + \frac{1}{y} \log \left( \frac{\psi_\varepsilon(y)}{2y} \right) \right) < 0.$$

To this end, we first note that differentiating  $\nu(3/(2y), 3/(2e) + \varepsilon)$  twice with respect to  $y$ , we obtain

$$\frac{3}{2y^3} \left( 3 + 2 \log \left( \frac{3/(2e) + \varepsilon}{2y} \right) \right) \geq \frac{3}{2} \left( 3 + 2 \log \left( \frac{3}{4e} \right) \right) \approx 0.637 > 0.$$

Therefore it suffices to consider the extreme points  $y = 1/2$  and  $y = 1$ . Noting that, by the choice of  $\varepsilon$ , we have that

$$\nu(3, 3/(2e) + \varepsilon) = \frac{3}{2}(1 + 2 \log(3/(2e) + \varepsilon)) < 0$$

and

$$\begin{aligned} \nu(3/2, 3/(2e) + \varepsilon) &= \frac{3}{2} \left( 1 + \log \left( \frac{3/(2e) + \varepsilon}{2} \right) \right) \\ &< \frac{3}{2} (1 + 2 \log(3/(2e) + \varepsilon)) < 0, \end{aligned}$$

it follows that  $\nu(3/(2y), 3/(2e) + \varepsilon) < 0$  for all  $y \in [1/2, 1]$ . Next, we observe that differentiating  $\nu(3/(2y), (e/2)^{1-2y}y^2)$  with respect to  $y$ , we obtain

$$\frac{3}{2y^2} (1 - \log(e/4)) \geq 3 \log 2 > 0.$$

Therefore, since  $\nu(3/(2y), (e/2)^{1-2y}y^2) \rightarrow \nu(3/2, \psi_\varepsilon(1)) = 0$  as  $y \uparrow 1$ , it follows that  $\nu(3/(2y), (e/2)^{1-2y}y^2) < 0$  for all  $y \in [1/2, 1 - 2\varepsilon/3]$ . Altogether, there is some  $\delta_2 > 0$  so that  $\nu(\min\{3, 3/(2y)\}, \psi_\varepsilon(y)) < -\delta_2$  for all  $y \in [0, 1 - 2\varepsilon/3]$ .

Taking  $\delta = \min\{\delta_1, \delta_2\}$ , it follows that  $\nu(\beta, \psi_\varepsilon(y)) < -\delta$ , for all relevant  $\beta, y$ , as required.

### D Details in the proof of Proposition 4.9

We finish the proof of Proposition 4.9 by showing that, for some  $\delta > 0$ , we have  $\nu(\beta, \varepsilon_1, \varepsilon_2) < -\delta$ , for all relevant  $\beta, \varepsilon_1, \varepsilon_2$ . Since  $\nu$  is convex in  $\beta$ , we can restrict to the extreme points  $\beta = 3/2$  and  $\beta = 3/(2\varepsilon_1) > \beta_1/\varepsilon_1$ . To this end, observe that when  $\beta = 3/2$ , we have that  $\nu < 0$  if and only if  $\eta < 1$ . Similarly, when  $\beta = 3/(2\varepsilon_1)$ ,  $\nu < 0$  if and only if  $\eta < \varepsilon_1 e^{1-\varepsilon_1}$ . Since  $\varepsilon_1 e^{1-\varepsilon_1} \leq 1$  for all relevant  $\varepsilon_1$ , it suffices to establish the latter claim. To this end, we observe that

$$\begin{aligned} \frac{\partial}{\partial \varepsilon_2} \eta(\varepsilon_1, \varepsilon_2) &= \eta(\varepsilon_1, \varepsilon_2) \log \left( \frac{e\varepsilon_2^2}{(1-\varepsilon_1)(1-\varepsilon_1-\varepsilon_2)} \right) \\ &\geq \eta(\varepsilon_1, \varepsilon_2) \log(e/2) > 0 \end{aligned}$$

for all relevant  $\varepsilon_2 \geq (1 - \varepsilon_1)/2$ . Therefore, we need only show that

$$\zeta(\varepsilon_1) = \frac{\eta(\varepsilon_1, \min\{\varepsilon_1, 1 - \varepsilon_1\})}{\varepsilon_1 e^{1-\varepsilon_1}} < 1 - \delta$$

for some  $\delta > 0$  and all relevant  $\varepsilon_1$ . We treat the cases  $\varepsilon_1 \in [1/3, 1/2]$  and  $\varepsilon_1 \in [1/2, 1]$  separately.

For  $\varepsilon_1 \in [1/3, 1/2]$ , we have

$$\zeta(\varepsilon_1) = \frac{\eta(\varepsilon_1, \varepsilon_1)}{\varepsilon_1 e^{1-\varepsilon_1}} = \frac{(e(1-2\varepsilon_1))^{1-2\varepsilon_1} \varepsilon_1^{4\varepsilon_1-1}}{(1-\varepsilon_1)^{\varepsilon_1}}.$$

Hence

$$\frac{\partial}{\partial \varepsilon_1} \zeta(\varepsilon_1) = \zeta(\varepsilon_1) \left( \log \left( \frac{\varepsilon_1^4}{(1-\varepsilon_1)(1-2\varepsilon_1)^2} \right) + \frac{\varepsilon_1^2 + \varepsilon_1 - 1}{\varepsilon_1(1-\varepsilon_1)} \right).$$

The terms  $\varepsilon_1^4/((1-\varepsilon_1)(1-2\varepsilon_1)^2)$  and  $(\varepsilon_1^2 + \varepsilon_1 - 1)/(\varepsilon_1(1-\varepsilon_1))$  are increasing for  $\varepsilon_1 \in [1/3, 1/2]$ , as is easily verified. Hence  $\zeta(\varepsilon_1)$  is decreasing in  $\varepsilon_1$  for  $1/3 \leq \varepsilon_1 \leq x_1 \approx 0.439$  and increasing for  $x_1 \leq \varepsilon_1 \leq 1/2$ . Therefore, since  $\zeta(1/3) = (e/6)^{1/3} < 1$  and  $\zeta(1/2) = 1/\sqrt{2} < 1$ , we have that, for some  $\delta_1 > 0$ ,  $\zeta(\varepsilon_1) < 1 - \delta_1$  for all  $\varepsilon_1 \in [1/3, 1/2]$ .

Similarly, for  $\varepsilon \in [1/2, 1)$ , we have

$$\zeta(\varepsilon_1) = \frac{\eta(\varepsilon_1, 1 - \varepsilon_1)}{\varepsilon_1 e^{1-\varepsilon_1}} = (1 - \varepsilon_1)^{1-\varepsilon_1} \varepsilon_1^{2\varepsilon_1-1}.$$

Hence

$$\frac{\partial}{\partial \varepsilon_1} \zeta(\varepsilon_1) = \zeta(\varepsilon_1) \left( \log \left( \frac{\varepsilon_1^2}{1 - \varepsilon_1} \right) + \frac{\varepsilon_1 - 1}{\varepsilon_1} \right).$$

Since  $\varepsilon_1^2/(1 - \varepsilon_1)$  and  $(\varepsilon_1 - 1)/\varepsilon_1$  are increasing in  $\varepsilon_1 \in [1/2, 1)$ , we find that  $\zeta(\varepsilon_1)$  is decreasing in  $\varepsilon_1$  for  $1/2 \leq \varepsilon_1 \leq x_2 \approx 0.692$  and increasing for  $x_2 \leq \varepsilon_1 < 1$ . Note that  $\zeta(1/2) = 1/\sqrt{2} < 1$  and  $\zeta(1) = 1$ . Hence, for some  $\delta_2 > 0$ ,  $\zeta(\varepsilon_1) < 1 - \delta_2$  for all  $\varepsilon_1 \in [1/2, \beta_1/\beta] \subset [1/2, 1)$ .

Setting  $\delta' = \min\{\delta_1, \delta_2\}$ , we find that  $\zeta(\varepsilon_1) < 1 - \delta'$  for all relevant  $\varepsilon_1$ . It follows that, for some  $\delta > 0$ , we have that  $\nu(\beta, \varepsilon_1, \varepsilon_2) < -\delta$ , for all relevant  $\beta, \varepsilon_1, \varepsilon_2$ .

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**Acknowledgments.** The author gratefully acknowledges the financial support of NSERC of Canada, and thanks Omer Angel for many helpful discussions.

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