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Semigroups for one-dimensional Schrödinger operators with multiplicative Gaussian noise*

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Abstract

Let $H := -\frac{1}{2}\Delta + V$ be a one-dimensional continuum Schrödinger operator. Consider $\hat{H} := H + \xi$, where ξ is a translation invariant Gaussian noise. Under some assumptions on ξ , we prove that if V is locally integrable, bounded below, and grows faster than \log at infinity, then the semigroup $e^{-t\hat{H}}$ is trace class and admits a probabilistic representation via a Feynman-Kac formula. Our result applies to operators acting on the whole line \mathbb{R} , the half line $(0, \infty)$, or a bounded interval $(0, b)$, with a variety of boundary conditions. Our method of proof consists of a comprehensive generalization of techniques recently developed in the random matrix theory literature to tackle this problem in the special case where \hat{H} is the stochastic Airy operator.

Keywords: random Schrödinger operators; Gaussian noise; Schrödinger semigroups; Feynman-Kac formula.

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1 Introduction

Let $I \subset \mathbb{R}$ be an open interval (possibly unbounded) and $V : I \rightarrow \mathbb{R}$ be a function. Let $H := -\frac{1}{2}\Delta + V$ denote a Schrödinger operator with potential V acting on functions $f : I \rightarrow \mathbb{R}$ with prescribed boundary conditions when I has a boundary. In this paper, we are interested in random operators of the form

$$\hat{H} := H + \xi, \tag{1.1}$$

where ξ is a stationary Gaussian noise on \mathbb{R} . Informally, we think of ξ as a centered Gaussian process on \mathbb{R} with a covariance of the form $E[\xi(x)\xi(y)] = \gamma(x - y)$, where γ is an even almost-everywhere-defined function or Schwartz distribution. In many cases that we consider, γ is not an actual function, and thus ξ cannot be defined as a random function on \mathbb{R} ; in such cases ξ can be defined rigorously as a random Schwarz

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distribution, i.e., a centered Gaussian process on an appropriate function space with covariance

$$\mathbf{E}[\xi(f)\xi(g)] = \int_{\mathbb{R}} f(x)\gamma(x-y)g(y) \, dx dy, \quad f, g : \mathbb{R} \rightarrow \mathbb{R}.$$

Among the most powerful tools used to study Schrödinger operators are their semigroups (e.g., [41]); we recall that the semigroup generated by H is the family of operators formally defined as e^{-tH} for $t > 0$. Provided the potentials under consideration are sufficiently well behaved, there is a remarkable connection between Schrödinger semigroups and the theory of stochastic processes that can be expressed in the form of the Feynman-Kac formula (e.g., [41, Theorem A.2.7]): Assuming $I = \mathbb{R}$ for simplicity, for every $f \in L^2(\mathbb{R})$, $t > 0$, and $x \in \mathbb{R}$, one has

$$e^{-tH}f(x) = \mathbf{E}^x \left[\exp \left(- \int_0^t V(B(s)) \, ds \right) f(B(t)) \right] \quad (1.2)$$

where B is a Brownian motion and \mathbf{E}^x signifies that we are taking the expected value with respect to B conditioned on the starting point $B(0) = x$. Apart from the obvious benefit of making Schrödinger semigroups amenable to probabilistic methods, we note that the Feynman-Kac formula can in fact form the basis of the definition of H itself, as done, for instance, in [30].

Our purpose in this paper is to lay out the foundations of a general semigroup theory (or Feynman-Kac formulas) for random Schrödinger operators of the form (1.1). We note that, since we consider very irregular noises (i.e., in general ξ is not a proper function that can be evaluated at points in \mathbb{R}), this undertaking is not a direct application or a trivial extension of the classical theory; see Section 1.1 for more details. As a first step in this program, we show that a variety of tools recently developed in the random matrix theory literature (e.g., [3, 20, 22, 28, 32, 36]) to tackle special cases of this problem can be suitably extended to a rather general setting. The main restriction of our assumptions is that we consider cases where the semigroup $e^{-t\hat{H}}$ is trace class, which implies in particular that \hat{H} must have a purely discrete spectrum.

This paper is organized as follows. In the remainder of this introduction, we present a brief outline of our main results and discuss some motivations and applications. In Section 2, we give a precise statement of our results (our main result is **Theorem 2.24**, and our second main result is **Proposition 2.10**). In Section 3, we provide an outline of the proof of our main results. Finally, in Sections 4 and 5, we go over the technical details of the proof of our results.

1.1 Overview of results

As mentioned earlier in this introduction, much of the challenge involved in our program comes from the fact that, in general, Gaussian noises are Schwartz distributions. This creates two main technical obstacles.

The first obstacle is that it is not immediately obvious how to define the operator \hat{H} . Indeed, if we interpret ξ as being part of the potential of \hat{H} , then the action

$$\hat{H}f = -\frac{1}{2}f'' + (V + \xi)f$$

of the operator on a function f includes the “pointwise product” ξf , which is not well defined if ξ cannot be evaluated at single points in \mathbb{R} . The second obstacle comes from the definition of $e^{-t\hat{H}}$. Arguably, the most natural guess for this semigroup would be to add ξ to the potential in the usual Feynman-Kac formula (1.2), which yields

$$e^{-t\hat{H}}f(x) = \mathbf{E}^x \left[\exp \left(- \int_0^t V(B(s)) + \xi(B(s)) \, ds \right) f(B(t)) \right]. \quad (1.3)$$

However, this again requires the ability to evaluate ξ at every point.

The key to overcoming these obstacles is to interpret ξ as the distributional derivative of an actual Gaussian process. More precisely, let Ξ be the Gaussian process on \mathbb{R} defined as

$$\Xi(x) := \begin{cases} \xi(\mathbf{1}_{[0,x]}), & x \geq 0 \\ \xi(-\mathbf{1}_{[x,0]}), & x \leq 0. \end{cases} \quad (1.4)$$

Assuming Ξ has a version with measurable sample paths (and we neglect boundary values for simplicity), a formal integration by parts yields

$$\xi(f) = \langle f, \Xi' \rangle := -\langle f', \Xi \rangle.$$

Following this line of thought, we may then settle on a “weak” definition of \hat{H} through the form

$$\langle f, \hat{H}g \rangle := \langle f, Hg \rangle + \xi(fg) = \langle f, Hg \rangle - \langle f'g + fg', \Xi \rangle. \quad (1.5)$$

We note that this type of definition for \hat{H} has previously appeared in the literature (e.g., [3, 17, 32, 36]) for various potentials V on the half line $I = (0, \infty)$ as well as $V = 0$ on a bounded interval $I = (0, L)$ ($L > 0$). We also note an alternative approach outlined by Bloemendaal in [2, Appendix A] that allows one (in principle) to recast \hat{H} as the classical Sturm-Liouville operator

$$Sf = -w^{-1}(\frac{w}{2}f')' + (V - 2\Xi^2)f, \quad \text{where } w(x) := \exp\left(4 \int_0^x \Xi(y) dy\right) \quad (1.6)$$

through a suitable Hilbert space isomorphism. Our first result (namely, **Proposition 2.10**) is an extension of these statements: We provide a very succinct proof of the fact that, under fairly general conditions on Ξ and V , the form (1.5) corresponds to a unique self-adjoint operator with compact resolvent, including when I is the whole real line or a bounded interval with a nonzero potential.

The interpretation $\xi = \Xi'$ also leads to a natural candidate for the semigroup generated by \hat{H} : Let $L_t^a(B)$ ($a \in \mathbb{R}, t \geq 0$) be the local time process of the Brownian motion B so that for any measurable function f , we have

$$\int_0^t f(B(s)) ds = \int_{\mathbb{R}} L_t^a(B)f(a) da.$$

Assuming a stochastic integral with respect to Ξ can meaningfully be defined, we may then interpret the problematic term in $e^{-t\hat{H}}$ ’s intuitive derivation (1.3) thusly:

$$\int_0^t V(B(s)) + \xi(B(s)) ds := \int_{\mathbb{R}} L_t^a(B) dQ(a),$$

where Q is the process $dQ(x) = V(x)dx + d\Xi(x)$, which we assume to be independent of B . In the case where $I = \mathbb{R}$, for example, this suggests that

$$e^{-t\hat{H}} f(x) = \mathbf{E}^x \left[\exp \left(- \int_{\mathbb{R}} L_t^a(B) dQ(a) \right) f(B(t)) \right], \quad (1.7)$$

where \mathbf{E}^x now denotes the conditional expectation of $(B|B(0) = x)$ given Ξ . This type of random semigroup has appeared in [20, 22] in the special case where I is the positive half line $(0, \infty)$, $V(x) = x$, and Ξ is a Brownian motion (so that ξ is a Gaussian white noise; see Example 2.28 for more details). Our second and main result (namely, **Theorem**

2.24) provides general sufficient conditions under which a Feynman-Kac formula of the form (1.7) holds (we refer to (2.14) for a statement of our Feynman-Kac formula when I is the half line or a bounded interval). This result can be seen as a comprehensive generalization of [20, Proposition 1.8 (a)] and [22, Corollary 2.2]. We refer to Section 3.2 for a detailed exposition of our method of proof.

One interesting consequence of Theorem 2.24 is the following connection between the random functional (1.7) and the spectrum of \hat{H} : Let $\lambda_1(\hat{H}) \leq \lambda_2(\hat{H}) \leq \dots$ be the eigenvalues of \hat{H} and $\psi_1(\hat{H}), \psi_2(\hat{H}), \dots$ be the associated eigenfunctions, which are defined by the variational principle (i.e., Courant-Fischer) associated with the form (1.5). By Theorem 2.24, in many cases the spectral expansion

$$e^{-t\hat{H}} f = \sum_{k=1}^{\infty} e^{-t\lambda_k(\hat{H})} \langle \psi_k(\hat{H}), f \rangle \psi_k(\hat{H}), \quad f \in L^2(\mathbb{R})$$

admits an explicit probabilistic representation of the form (1.7). We expect this connection to be fruitful in two directions.

On the one hand, a good understanding of \hat{H} 's spectrum could be used to study the geometric properties of the function $u(t, x) := e^{-t\hat{H}} f(x)$, which we may interpret as the solution of the SPDE with multiplicative noise

$$\partial_t u = -(Hu + \xi u), \quad u(0, x) = f(x).$$

We refer to Section 1.2.1 below for more motivation in this direction.

On the other hand, the Feynman-Kac formula can be used to study the properties of the eigenvalues and eigenfunctions of \hat{H} (we refer to [41] for classical examples of this involving the deterministic operator H). In particular, our Feynman-Kac formula provides a means of computing the “Laplace transforms”

$$\mathbf{E} \left[\prod_{i=1}^{\ell} \sum_{k=1}^{\infty} e^{-t_i \lambda_k(\hat{H})} \right] = \mathbf{E} \left[\prod_{i=1}^{\ell} \text{Tr}[e^{-t_i \hat{H}}] \right], \quad t_1, \dots, t_{\ell} > 0, \quad (1.8)$$

which characterize the distribution of \hat{H} 's eigenvalues. In Sections 1.2.2 and 1.2.3, we discuss how the ability to compute (1.8) has led to applications in the study of operator limits of random matrices and the occurrence of number rigidity in the spectrum of general random Schrödinger operators.

1.2 Motivating examples and applications

1.2.1 The Anderson Hamiltonian and parabolic Anderson model

The earliest occurrences of an operator of the form (1.1) in the literature appear to be [16, 24]. The operator that is considered therein is the Anderson Hamiltonian, defined as $A := -\Delta + \xi$, where ξ is a Gaussian white noise. The first mathematically rigorous study of this object appeared in [17]. Following this, there have been several investigations of A 's spectral properties [5, 6, 31], culminating in a recent article of Dumaz and Labb   [13], which provides a comprehensive description of eigenfunction localization and eigenvalue Poisson statistics in the case where A acts on $I = (0, L)$ for large L .

In this context, the Feynman-Kac formula proved in this paper in the case $\hat{H} = A$ creates a rigorous connection between the study of localization in the Anderson Hamiltonian and the study of intermittency for large times in the parabolic Anderson model with continuous noise (c.f., [13, (5) and (6)] and [27, Sections 2.2.3–2.2.4]). We recall that the parabolic Anderson model is the SPDE

$$\partial_t u(t, x) = \Delta u(t, x) + \xi u(t, x), \quad u(0, x) = u_0(x)$$

or, equivalently, $u(t, x) := e^{t(\Delta+\xi)} u_0(x)$. Although several previous works have featured Feynman-Kac-type formulas for the continuum Anderson Hamiltonian or the parabolic Anderson model in one dimension (e.g., [9, Sections 3–4 and Lemma A.1] or [25, Section 3]), ours appears to be the first to make an explicit connection between A 's full spectrum and a Feynman-Kac functional of the form (1.7).

1.2.2 Operator limits of random matrices

One of the most widely studied example of an operator of the form (1.1) is the stochastic Airy operator:

$$A_\beta := -\Delta + x + \xi_\beta, \quad \beta > 0, \quad (1.9)$$

where ξ_β is a Gaussian white noise with variance $4/\beta$, and A_β acts on $I = (0, \infty)$ with Dirichlet or Robin boundary condition at the origin. The interest of studying this operator comes from the fact that its spectrum captures the asymptotic edge fluctuations of a large class of random matrices and β -ensembles. This was first observed by Edelman and Sutton in [15] and is based on the tridiagonal models of Dumitriu and Edelman [14]. The connection was later rigorously established by Ramírez, Rider, and Virág [36], and these developments gave rise to a now very extensive literature concerning operator limits of random matrices, in which general operators of the form (1.1) arise as the limits of a large class of random tridiagonal matrices. We refer to [46] and references therein for a somewhat recent survey.

In [22], Gorin and Shkolnikov introduced an alternative method of studying operator limits of random matrices by proving that large powers of generalized Gaussian β -ensembles admit an operator limit of the form (1.7) (see [22, (2.4)]). These results were later extended to rank 1 additive perturbations of Gaussian β -ensembles in [20]. Since the Gaussian β -ensembles converge to the stochastic Airy operator, these results imply a Feynman-Kac formula of the form (1.7) for $e^{-tA_\beta/2}$. This new Feynman-Kac formula was then used to study the eigenvalues of A_β (see [22, Corollary 2.3 and Proposition 2.6] and [20, Theorem 1.11 and Corollary 1.13]).

In this context, our paper can be viewed as providing a streamlined and unified treatment of trace class semigroups generated by general operators of the form (1.1). In [18], this more general setting is used to extend the operator limit results in [20, 22] to much more general random tridiagonal matrices, including some non-symmetric matrices that could not be treated by any previous method.

1.2.3 Number rigidity in random Schrödinger operators

A point process is number rigid if the number of points inside any bounded set is determined by the configuration of points outside that set. The earliest proof of number rigidity appears to be the work of Aizenman and Martin in [1]. More recently, there has been a notable increase of interest in this property stemming from the work of Ghosh and Peres [21]. Therein, it is proved that the zero set of the planar Gaussian analytic function and the Ginibre process are number rigid. Since then, number rigidity has been shown to be connected to several other interesting properties of point processes (see, e.g., [19, Section 1.2] and references therein).

Due to their ubiquity in mathematical physics, there is a strong incentive to understand any structure that appears in the eigenvalues of random Schrödinger operators, including number rigidity. Up until recently, the only random Schrödinger operator whose eigenvalue point process was known to be number rigid was the stochastic Airy operator A_β in (1.9) with $\beta = 2$ [4], thanks to the special algebraic structure present

in the eigenvalues of this particular object (i.e., A_2 's eigenvalues generate the determinantal Airy-2 point process). In [19], we use the Feynman-Kac formula proved in this paper to show that number rigidity occurs in the spectrum of \hat{H} under very general assumptions on the domain I on which the operator is defined, the boundary conditions on that domain, the regularity of the potential V , and the type of noise; thus providing the first method capable of proving rigidity for general random Schrödinger operators.

2 Main results

In this section, we provide detailed statements of our main results. Throughout this paper, we make the following assumption regarding the interval I on which the operator is defined and its boundary conditions.

Assumption 2.1. *We consider three different types of domains: The full space $I = \mathbb{R}$ (**Case 1**), the positive half line $I = (0, \infty)$ (**Case 2**), and the bounded interval $I = (0, b)$ for some $b > 0$ (**Case 3**).*

In Case 2, we consider Dirichlet and Robin boundary conditions at the origin:

$$\begin{cases} f(0) = 0 & \text{(Case 2-D)} \\ f'(0) + \alpha f(0) = 0 & \text{(Case 2-R)} \end{cases} \quad (2.1)$$

where $\alpha \in \mathbb{R}$ is fixed.

In Case 3, we consider the Dirichlet, Robin, and mixed boundary conditions at the endpoints 0 and b :

$$\begin{cases} f(0) = f(b) = 0 & \text{(Case 3-D)} \\ f'(0) + \alpha f(0) = -f'(b) + \beta f(b) = 0 & \text{(Case 3-R)} \\ f'(0) + \alpha f(0) = f(b) = 0 & \text{(Case 3-M)} \end{cases} \quad (2.2)$$

where $\alpha, \beta \in \mathbb{R}$ are fixed.

Remark 2.2. Case 3-M should technically also include the mixed boundary conditions of the form $f(0) = -f'(b) + \beta f(b) = 0$. However, the latter can easily be obtained from Case 3-M by considering the transformation $x \mapsto f(b - x)$.

Throughout the paper, we make the following assumption on the potential V .

Assumption 2.3. *Suppose that $V : I \mapsto \mathbb{R}$ is nonnegative and locally integrable on I 's closure. If I is unbounded, then we also assume that*

$$\liminf_{x \rightarrow \pm\infty} \frac{V(x)}{\log|x|} = \infty. \quad (2.3)$$

Remark 2.4. As is usual in the theory of Schrödinger operators and semigroups, the assumption that $V \geq 0$ is made for technical ease, and all of our results also apply in the case where V is merely bounded from below on I .

2.1 Self-adjoint operator

Our first result concerns the realization of \hat{H} as a self-adjoint operator. As explained in the passage following equation (1.5), this is done through a sesquilinear form. We begin by introducing the sesquilinear form associated with H :

Definition 2.5. *Let $L^2 = L^2(I)$ denote the set of square integrable functions (equivalence classes up to measure zero) on I , with its usual inner product and norm*

$$\langle f, g \rangle := \int_I f(x)g(x) dx, \quad \|f\|_2 := \sqrt{\langle f, f \rangle}.$$

Let $\text{AC} = \text{AC}(I)$ denote the set of functions that are locally absolutely continuous on I 's closure, and let

$$H_V^1 = H_V^1(I) := \{f \in \text{AC} : \|f\|_2, \|f'\|_2, \|V^{1/2}f\|_2 < \infty\}.$$

We define the following inner product and norm on H_V^1 :

$$\langle f, g \rangle_* := \langle f', g' \rangle + \langle fg, V + 1 \rangle, \quad \|f\|_*^2 := \|f'\|_2^2 + \|V^{1/2}f\|_2^2 + \|f\|_2^2.$$

We define H 's sesquilinear form \mathcal{E} as well as its form domain $D(\mathcal{E}) \subset H_V^1$ for every case in Assumption 2.1 as follows:

$$\begin{aligned} \textbf{Case 1: } & \begin{cases} D(\mathcal{E}) := H_V^1 \\ \mathcal{E}(f, g) := \frac{1}{2}\langle f', g' \rangle + \langle fg, V \rangle \end{cases} \\ \textbf{Case 2-D: } & \begin{cases} D(\mathcal{E}) := \{f \in H_V^1 : f(0) = 0\} \\ \mathcal{E}(f, g) := \frac{1}{2}\langle f', g' \rangle + \langle fg, V \rangle \end{cases} \\ \textbf{Case 2-R: } & \begin{cases} D(\mathcal{E}) := H_V^1 \\ \mathcal{E}(f, g) := \frac{1}{2}\langle f', g' \rangle - \frac{\alpha}{2}f(0)g(0) + \langle fg, V \rangle \end{cases} \\ \textbf{Case 3-D: } & \begin{cases} D(\mathcal{E}) := \{f \in H_V^1 : f(0) = f(b) = 0\} \\ \mathcal{E}(f, g) := \frac{1}{2}\langle f', g' \rangle + \langle fg, V \rangle \end{cases} \\ \textbf{Case 3-R: } & \begin{cases} D(\mathcal{E}) := H_V^1 \\ \mathcal{E}(f, g) := \frac{1}{2}\langle f', g' \rangle - \frac{\alpha}{2}f(0)g(0) - \frac{\beta}{2}f(b)g(b) + \langle fg, V \rangle \end{cases} \\ \textbf{Case 3-M: } & \begin{cases} D(\mathcal{E}) := \{f \in H_V^1 : f(b) = 0\} \\ \mathcal{E}(f, g) := \frac{1}{2}\langle f', g' \rangle - \frac{\alpha}{2}f(0)g(0) + \langle fg, V \rangle \end{cases} \end{aligned}$$

Remark 2.6. As noted by Bloemendal and Virág in [3, Remark 2.5 and (2.11)], the Dirichlet boundary conditions can be specified in the form domain $D(\mathcal{E})$, but the Robin conditions must be enforced by the form itself, since the derivative of an absolutely continuous function is only defined almost everywhere. Taking Case 3-R as an example, by a formal integration by parts we have

$$-\int_0^b f(x)g''(x) dx = f(b)(-g'(b)) + f(0)g'(0) + \langle f', g' \rangle.$$

Substituting $g'(0) = -\alpha g(0)$ and $-g'(b) = -\beta g(b)$ then yields $\mathcal{E}(f, g)$.

We now define the form associated with \hat{H} as a random perturbation of \mathcal{E} coming from the noise. We assume that the Gaussian process Ξ driving the noise is as follows:

Assumption 2.7. $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ is a centered Gaussian process such that:

1. Almost surely, $\Xi(0) = 0$ and Ξ has continuous sample paths.
2. Ξ has stationary increments, that is, for every $x_1, \dots, x_\ell, y_1, \dots, y_\ell \in \mathbb{R}$ ($\ell \in \mathbb{N}$) such that $x_i \leq y_i$ for all $1 \leq i \leq \ell$ and $h \in \mathbb{R}$, the increments

$$\Xi(y_1) - \Xi(x_1), \Xi(y_2) - \Xi(x_2), \dots, \Xi(y_\ell) - \Xi(x_\ell)$$

have the same joint distribution as the shifted increments

$$\Xi(y_1 + h) - \Xi(x_1 + h), \Xi(y_2 + h) - \Xi(x_2 + h), \dots, \Xi(y_\ell + h) - \Xi(x_\ell + h).$$

We may now define ξ as the distributional derivative of Ξ :

Definition 2.8. Let $C_0^\infty = C_0^\infty(I)$ denote the set of functions that are smooth and compactly supported on I 's closure. For every $f \in C_0^\infty$, we define $\xi(f) = \langle f, \Xi' \rangle$ by a formal integration by parts:

$$\xi(f) := \begin{cases} -\langle f', \Xi \rangle & (\text{Cases 1 and 2}) \\ f(b)\Xi(b) - \langle f', \Xi \rangle & (\text{Case 3}) \end{cases} \quad (2.4)$$

(note that we omit the boundary term $-f(0)\Xi(0)$ in the formal integration by parts for Cases 2 and 3, since we assume that $\Xi(0) = 0$).

Our first result is that the sesquilinear form $(f, g) \mapsto \xi(fg)$ for $f, g \in C_0^\infty$ can be continuously extended to H_V^1 , and thus can be added to the form \mathcal{E} as hinted at in (1.5):

Proposition 2.9. Suppose that Assumptions 2.1, 2.3, and 2.7 hold. There exists a finite random variable $c > 0$ such that, almost surely,

$$|\xi(f^2)| \leq c\|f\|_*^2, \quad f \in C_0^\infty. \quad (2.5)$$

Hence $f \mapsto \xi(f^2)$ extends uniquely to a continuous quadratic form on H_V^1 that satisfies (2.5) for all $f \in H_V^1$, which we can then also extend to a sesquilinear form by the polarization identity:

$$\xi(fg) := \frac{\xi((f+g)^2) - \xi((f-g)^2)}{4}, \quad f, g \in H_V^1.$$

In particular, almost surely, we can define the sesquilinear form

$$\hat{\mathcal{E}}(f, g) := \mathcal{E}(f, g) + \xi(fg) \quad (2.6)$$

on the same form domain as \mathcal{E} , that is, for all for all $f, g \in D(\mathcal{E})$.

We may now state our main result regarding our definition of \hat{H} as the self-adjoint operator associated with the form (2.6) on the form domain $D(\mathcal{E})$:

Proposition 2.10. Suppose that Assumptions 2.1, 2.3, and 2.7 hold. Almost surely, there exists a unique self-adjoint operator \hat{H} with dense domain $D(\hat{H}) \subset L^2$ such that

1. $D(\hat{H}) \subset D(\mathcal{E})$;
2. For every $f, g \in D(\hat{H})$, one has $\langle f, \hat{H}g \rangle = \hat{\mathcal{E}}(f, g)$; and
3. \hat{H} has compact resolvent.

Remark 2.11. In Case 1, the statement of Proposition 2.10 is to the best of our knowledge completely new. In Case 2, the closest results are [32, Theorem 2], which assumes that $I = (0, \infty)$ with Dirichlet boundary condition, that V is continuous, and that Ξ is a fractional Brownian motion. In Case 3, the closest result seems to be [17, §2], which only considers the case $V = 0$ with Dirichlet boundary conditions and Ξ a Brownian motion. Proposition 2.9 is a generalization of similar results in [3, Lemma 2.3], [32, Proposition 1-(i)], and [36, Proposition 2.4].

An immediate corollary of Proposition 2.10 is the ability to study the spectrum of \hat{H} using the variational characterization coming from the form $\hat{\mathcal{E}}$:

Definition 2.12. Let A be a semi-bounded self-adjoint operator with discrete spectrum. We use $\lambda_1(A) \leq \lambda_2(A) \leq \dots$ to denote the eigenvalues of A in increasing order, and we use $\psi_1(A), \psi_2(A), \dots$ to denote the associated eigenfunctions.

Corollary 2.13. Under the assumptions of Proposition 2.10, almost surely,

1. $-\infty < \lambda_1(\hat{H}) \leq \lambda_2(\hat{H}) \leq \dots \nearrow +\infty$;

2. the $\psi_k(\hat{H})$ form an orthonormal basis of L^2 ; and
3. for every $k \in \mathbb{N}$,

$$\lambda_k(\hat{H}) = \inf_{\psi \in D(\mathcal{E}), \psi \perp \psi_1(\hat{H}), \dots, \psi_{k-1}(\hat{H})} \frac{\hat{\mathcal{E}}(\psi, \psi)}{\|\psi\|_2^2},$$

with $\psi_k(\hat{H})$ being the minimizer of the above infimum with unit L^2 norm.

2.2 Semigroup

We now state our main result regarding the Feynman-Kac formula for the semigroup generated by \hat{H} . Thanks to Proposition 2.10 and Corollary 2.13, we know that under Assumptions 2.1, 2.3, and 2.7, the semigroup of \hat{H} is the family of bounded self-adjoint operators with spectral expansions

$$e^{-t\hat{H}} f = \sum_{k=1}^{\infty} e^{-t\lambda_k(\hat{H})} \langle \psi_k(\hat{H}), f \rangle \psi_k(\hat{H}), \quad t > 0, f \in L^2. \quad (2.7)$$

In order to state our Feynman-Kac formula for $e^{-t\hat{H}}$, we introduce some notations and further assumptions.

2.2.1 Preliminary definitions

We begin with some preliminary definitions regarding the covariance of the noise ξ and the stochastic processes required to define our Feynman-Kac kernels.

Definition 2.14 (Covariance). Let us denote by $PC_c = PC_c(I)$ the set of functions $f : I \mapsto \mathbb{R}$ that are càdlàg and compactly supported on I 's closure. We say that $f \in PC_c$ is a step function if it can be written as

$$f = \sum_{i=1}^k c_i \mathbf{1}_{[x_i, x_{i+1})} \quad c_i \in \mathbb{R}, \quad -\infty < x_1 < x_2 < \dots < x_{k+1} < \infty. \quad (2.8)$$

To simplify forthcoming definitions and statements, we often extend the domain of $f \in PC_c$ to \mathbb{R} , with the convention that $f(x) = 0$ for all x outside of I 's closure (noting, however, that f 's extension need not be càdlàg on all of \mathbb{R}).

Let $\gamma : PC_c \rightarrow \mathbb{R}$ be an even almost-everywhere-defined function or Schwartz distribution (even in the sense that $\langle f, \gamma \rangle = \langle \tau f, \gamma \rangle$ for every f , where $\tau f(x) = f(-x)$ denotes the reflection map), such that the bilinear map

$$\langle f, g \rangle_\gamma := \int_{\mathbb{R}^2} f(x) \gamma(x-y) g(y) dx dy, \quad f, g \in PC_c \quad (2.9)$$

is a semi-inner-product. We denote the seminorm induced by (2.9) as

$$\|f\|_\gamma := \sqrt{\langle f, f \rangle_\gamma}, \quad f \in PC_c.$$

Remark 2.15. If γ is not an almost-everywhere-defined function, then the integral over $\gamma(x-y)$ in (2.9) may not be well defined. In such cases, we rigorously interpret (2.9) as $\langle f * \tau g, \gamma \rangle = \langle \tau f * g, \gamma \rangle$.

Definition 2.16 (Stochastic Processes, etc.). We use B to denote a standard Brownian motion on \mathbb{R} , X to denote a reflected standard Brownian motion on $(0, \infty)$, and Y to denote a reflected standard Brownian motion on $(0, b)$.

Let $Z = B$, X , or Y . For every $t > 0$ and $x, y \in I$, we define the conditioned processes

$$Z^x := (Z | Z(0) = x) \quad \text{and} \quad Z_t^{x,y} := (Z | Z(0) = x \text{ and } Z(t) = y),$$

and we use \mathbf{E}^x and $\mathbf{E}_t^{x,y}$ to denote the expected value with respect to the law of Z^x and $Z_t^{x,y}$, respectively.

We denote the Gaussian kernel by

$$\mathcal{G}_t(x) := \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}, \quad t > 0, x \in \mathbb{R}.$$

We denote the transition kernels of B , X , and Y as Π_B , Π_X , and Π_Y , respectively. That is, for every $t > 0$,

$$\begin{aligned} \Pi_B(t; x, y) &:= \mathcal{G}_t(x - y) & x, y \in \mathbb{R}, \\ \Pi_X(t; x, y) &:= \mathcal{G}_t(x - y) + \mathcal{G}_t(x + y) & x, y \in (0, \infty), \\ \Pi_Y(t; x, y) &:= \sum_{z \in 2b\mathbb{Z} \pm y} \mathcal{G}_t(x - z) & x, y \in (0, b). \end{aligned}$$

Let $Z = B$, X , or Y . For any time interval $[u, v] \subset [0, \infty)$, we let $a \mapsto L_{[u,v]}^a(Z)$ ($a \in I$) denote the continuous version of the local time of Z (or its conditioned versions) on $[u, v]$, that is,

$$\int_u^v f(Z(s)) \, ds = \int_I L_{[u,v]}^a(Z) f(a) \, da = \langle L_{[u,v]}(Z), f \rangle \quad (2.10)$$

for any measurable function $f : I \rightarrow \mathbb{R}$. In the special case where $u = 0$ and $v = t$, we use the shorthand $L_t(Z) := L_{[0,t]}(Z)$. When there may be ambiguity regarding which conditioning of Z is under consideration, we use $L_{[u,v]}(Z^x)$ and $L_{[u,v]}(Z_t^{x,y})$.

As a matter of convention, if $Z = X$ or Y , then we distinguish the boundary local time from the above, which we define as

$$\mathfrak{L}_{[u,v]}^c(Z) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_u^v \mathbf{1}_{\{c-\varepsilon < Z(s) < c+\varepsilon\}} \, ds$$

for $c \in \partial I$ (i.e., $c = 0$ if $Z = X$ or $c \in \{0, b\}$ if $Z = Y$), with $\mathfrak{L}_t^c(Z) := \mathfrak{L}_{[0,t]}^c(Z)$.

Remark 2.17. Since we use the continuous version of local time, $a \mapsto L_{[u,v]}^a(Z)$ is continuous and compactly supported on I 's closure; in particular, $L_{[u,v]}(Z) \in \text{PC}_c$.

2.2.2 Noise

We now articulate the assumptions that the noise ξ must satisfy for our Feynman-Kac formula to hold. We recall from the introduction that we think of ξ as a centered Gaussian process with covariance $\mathbf{E}[\xi(x)\xi(y)] = \gamma(x - y)$, with γ as in Definition 2.14. Interpreting $\xi(f) = \int_{\mathbb{R}} f(x)\xi(x) \, dx$ for a function f , this suggests that, as a random Schwartz distribution, ξ is a centered Gaussian process with covariance $\mathbf{E}[\xi(f)\xi(g)] = \langle f, g \rangle_{\gamma}$. In similar fashion to Assumption 2.7, we want to interpret ξ as the distributional derivative of some continuous process Ξ , that is, corresponding to (1.4). If ξ 's covariance is given by the semi-inner-product $\langle \cdot, \cdot \rangle_{\gamma}$, then this suggests that Ξ 's covariance is equal to

$$\mathbf{E}[\Xi(x)\Xi(y)] = \begin{cases} \langle \mathbf{1}_{[0,x]}, \mathbf{1}_{[0,y]} \rangle_{\gamma} & \text{if } 0 \leq x, y \\ \langle \mathbf{1}_{[0,x]}, -\mathbf{1}_{[y,0]} \rangle_{\gamma} & \text{if } y \leq 0 \leq x \\ \langle -\mathbf{1}_{[x,0]}, \mathbf{1}_{[0,y]} \rangle_{\gamma} & \text{if } x \leq 0 \leq y \\ \langle \mathbf{1}_{[x,0]}, \mathbf{1}_{[y,0]} \rangle_{\gamma} & \text{if } x, y \leq 0. \end{cases} \quad (2.11)$$

This leads us to the following Assumption:

Assumption 2.18. *The centered Gaussian process $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies Assumption 2.7. Moreover, there exists a $\gamma : \text{PC}_c \rightarrow \mathbb{R}$ as in Definition 2.14 that satisfies the following conditions.*

1. Ξ 's covariance is given by (2.11).
2. There exists $c_\gamma > 0$ and $1 \leq q_1, \dots, q_\ell \leq 2$ (for some $\ell \in \mathbb{N}$) such that

$$\|f\|_\gamma^2 \leq c_\gamma (\|f\|_{q_1}^2 + \dots + \|f\|_{q_\ell}^2), \quad f \in \text{PC}_c, \quad (2.12)$$

where $\|f\|_q := (\int_{\mathbb{R}} |f(x)|^q dx)^{1/q}$ denotes the usual L^q norm.

Then, for every $f \in \text{PC}_c$, we define

$$\xi(f) := \int_{\mathbb{R}} f(x) d\Xi(x), \quad (2.13)$$

where $d\Xi$ denotes stochastic integration with respect to Ξ interpreted in the pathwise sense of Karandikar [26] (see Section 3.2.1 for the details of this construction).

Remark 2.19. Though this is not immediately obvious from the above definition, the pathwise stochastic integral (2.13) actually coincides with (2.4) for every $f \in C_0^\infty$. We note, however, that the extension of ξ to PC_c need not be linear on all of PC_c , and thus may not be a Schwartz distribution in the proper sense on that larger domain. Our interest in defining the stochastic integral in a pathwise sense is that it allows to construct ξ as a random map from PC_c to \mathbb{R} that satisfies the following properties.

1. We can consider the conditional distribution of $\xi(L_t(Z))$ given a fixed realization of Ξ , assuming independence between Z and Ξ .
2. $f \mapsto \xi(f)$ is a centered Gaussian process on PC_c with covariance $\langle \cdot, \cdot \rangle_\gamma$.

In fact, any other pathwise stochastic integral that is an extension of (2.4) and satisfies these two properties leads to the same statement in Theorem 2.24 below. We point to Section 3.2.1 and Appendix A for the details of the proof that ξ has these two properties, and to Section 3.2.2 for an explanation of why any stochastic integral having these two properties gives rise to our main result.

Remark 2.20. The requirement that Ξ be a continuous process with stationary increments in Assumption 2.18 is redundant: Firstly, the covariance (2.11) implies that $\Xi(x) - \Xi(y)$ corresponds to $\xi(\mathbf{1}_{[x,y]})$, which is stationary since the semi-inner-product $\langle \cdot, \cdot \rangle_\gamma$ is translation invariant. Secondly, if we construct Ξ using abstract existence theorems for Gaussian processes (which is possible since $\langle \cdot, \cdot \rangle_\gamma$ is a semi-inner-product), then the assumption (2.12) implies that Ξ has a continuous version by Kolmogorov's theorem for path continuity (see Section 3.3 for details). We nevertheless state these properties as assumptions for clarity.

2.2.3 Feynman-Kac kernels

We now introduce the Feynman-Kac kernels that describe \hat{H} 's semigroup.

Definition 2.21. In Cases 2 and 3, let us define the quantities

$$\bar{\alpha} := \begin{cases} -\infty & (\text{Case 2-D}) \\ \alpha & (\text{Case 2-R}) \end{cases} \quad (\bar{\alpha}, \bar{\beta}) := \begin{cases} (-\infty, -\infty) & (\text{Case 3-D}) \\ (\alpha, \beta) & (\text{Case 3-R}) \\ (\alpha, -\infty) & (\text{Case 3-M}) \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ are as in (2.1) and (2.2). For every $t > 0$, we define the (random) kernel $\hat{K}(t) : I^2 \rightarrow \mathbb{R}$ as

$$\hat{K}(t; x, y) := \begin{cases} \Pi_B(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(B), V \rangle - \xi(L_t(B))} \right] & (\text{Case 1}) \\ \Pi_X(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(X), V \rangle - \xi(L_t(X)) + \bar{\alpha} \mathfrak{L}_t^0(X)} \right] & (\text{Case 2}) \\ \Pi_Y(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(Y), V \rangle - \xi(L_t(Y)) + \bar{\alpha} \mathfrak{L}_t^0(Y) + \bar{\beta} \mathfrak{L}_t^b(Y)} \right] & (\text{Case 3}) \end{cases} \quad (2.14)$$

where we assume that Ξ is independent of B , X , or Y , and $\mathbf{E}_t^{x,y}$ denotes the expected value conditional on Ξ .

Remark 2.22. Let $Z = X$ or Y . In the above definition, we use the convention

$$-\infty \cdot \mathfrak{L}_t^c(Z) = \begin{cases} 0 & \text{if } \mathfrak{L}_t^c(Z) = 0 \\ -\infty & \text{if } \mathfrak{L}_t^c(Z) > 0 \end{cases}$$

for any $c \in \partial I$ as well as $e^{-\infty} = 0$. Thus, if we let $\tau_c(Z) := \inf\{t \geq 0 : Z(t) = c\}$ denote the first hitting time of c , then we can interpret $e^{-\infty \cdot \mathfrak{L}_t^c(Z)} = \mathbf{1}_{\{\tau_c(Z) > t\}}$. In particular, if we remove the term $\xi(L_t(Z))$ from the kernel (2.14), then we recover the classical Feynman-Kac formula for the semigroup of H . See Section 5.1 for more details.

Notation 2.23. Given a Kernel $J : I^2 \rightarrow \mathbb{R}$ (such as $\hat{K}(t)$), we also use J to denote the integral operator induced by the kernel, that is,

$$Jf(x) := \int_I J(x, y) f(y) dy.$$

We say that J is Hilbert-Schmidt if $\|J\|_2 < \infty$, and trace class if $\text{Tr}[|J|] < \infty$.

2.2.4 Main result

Our main result is as follows.

Theorem 2.24 (Feynman-Kac Formula). Suppose that Assumptions 2.1, 2.3, and 2.18 hold. Almost surely, $e^{-t\hat{H}}$ is a Hilbert-Schmidt/trace class integral operator for every $t > 0$. Moreover, for every $t > 0$, the following holds with probability one.

1. $e^{-t\hat{H}} = \hat{K}(t)$.
2. $\text{Tr}[e^{-t\hat{H}}] = \int_I \hat{K}(t; x, x) dx < \infty$.

Remark 2.25. We point to Section 3.2.1 and Appendix A for a justification of the well-posedness of the conditional expectation in (2.14) and that the kernel $\hat{K}(t)$ is Borel measurable, thus making quantities such as

$$\int_I \hat{K}(t; x, y) f(y) dy, \quad \int_{I^2} \hat{K}(t; x, y)^2 dx dy, \quad \text{and} \quad \int_I \hat{K}(t; x, x) dx$$

(where $f \in L^2$) well defined.

Remark 2.26. The closest analogs of Theorem 2.24 in the literature are [20, Proposition 1.8 (a)] and [22, Corollary 2.2], which concern Case 2 in the special case where $V(x) = x$ and Ξ is a Brownian motion. All other cases are new.

Remark 2.27. Though this direction is not explored in this paper, we expect that one could prove (in similar fashion to, e.g., [25, Theorem 4.12]) that the kernels $\hat{K}(t; x, y)$ admit continuous modifications in t , x , and y .

2.3 Optimality and examples

We finish Section 2 by discussing the optimality of the growth condition (2.3) in our results and by providing examples of covariance functions/distributions γ that satisfy Assumption 2.18.

2.3.1 Optimality of potential growth

On the one hand, one of the key aspects of our proof of Proposition 2.10 for unbounded domains I is to show that the growth rate of the squared increment process

$$x \mapsto (\Xi(x+1) - \Xi(x))^2$$

is dominated by V as $|x| \rightarrow \infty$ (see (4.2), (4.3), and the passage that follows). Given that the growth rate of stationary Gaussian processes (such as $\Xi(x+1) - \Xi(x)$) is at most of order $\sqrt{\log|x|}$ (e.g., Corollary B.2), and that in many cases there is also a matching lower bound (e.g., Remark B.4), the growth condition (2.3) appears to be the best one can hope for with the method we use to prove Proposition 2.10. It would be interesting to see if this condition is necessary for \hat{H} to have compact resolvent (perhaps by using the Sturm-Liouville interpretation (1.6)). That being said, for the deterministic operator $H = -\frac{1}{2}\Delta + V$ on $I = (0, \infty)$, it is well known that having a spectrum of discrete eigenvalues that are bounded below is equivalent to $\int_x^{x+\delta} V(y) dy \rightarrow \infty$ as $x \rightarrow \infty$ for all $\delta > 0$; hence it is natural to expect that V must have some kind logarithmic growth to balance the Gaussian potential.

On the other hand, condition (2.3) is necessary to have that $\mathbf{E}[\|\hat{K}(t)\|_2^2] < \infty$ for $t > 0$ close to zero, which is crucial in our proof of Theorem 2.24. Given that the deterministic semigroup e^{-tH} is not trace class for small $t > 0$ when (2.3) does not hold, we do not expect it is possible to improve Theorem 2.24 in that regard. We refer to Remark 5.22 for more details.

2.3.2 Examples

Given the simplicity of Assumption 2.7, it is straightforward to come up with examples of Gaussian noises to which Proposition 2.10 can be applied. In contrast, Assumption 2.18 is a bit more involved. In what follows, we provide examples of covariance functions/distributions γ that satisfy Assumption 2.18.

Example 2.28. Let $\gamma : \text{PC}_c \rightarrow \mathbb{R}$ be an even almost-everywhere-defined function or Schwartz distribution.

1. **(Bounded)** If $\gamma \in L^\infty(\mathbb{R})$, then we call ξ a bounded noise. Depending on the regularity of γ , in many such cases ξ can actually be realized as a continuous Gaussian process on \mathbb{R} with covariance $\mathbf{E}[\xi(x)\xi(y)] = \gamma(x-y)$.
2. **(White)** If $\gamma = \sigma^2\delta_0$ for some $\sigma > 0$, where δ_0 denotes the delta Dirac distribution, then ξ is a Gaussian white noise with variance σ^2 . This corresponds to stochastic integration with respect to a two-sided Brownian motion W with variance σ^2 :

$$\xi(f) = \int_{\mathbb{R}} f(x) dW(x).$$

3. **(Fractional)** If $\gamma(x) := \sigma^2\mathfrak{H}(2\mathfrak{H}-1)|x|^{2\mathfrak{H}-2}$ for $\sigma > 0$ and $\mathfrak{H} \in (1/2, 1)$, then ξ is a fractional noise with variance σ^2 and Hurst parameter \mathfrak{H} . This noise corresponds to stochastic integration with respect to a two-sided fractional Brownian motion $W^{\mathfrak{H}}$ with variance σ^2 and Hurst parameter \mathfrak{H} :

$$\xi(f) = \int_{\mathbb{R}} f(x) dW^{\mathfrak{H}}(x).$$

4. (**L^p -Singular**) Let $\ell \in \mathbb{N}$ and $1 \leq p_1, \dots, p_\ell < \infty$. As a generalization of bounded and fractional noise, we say that ξ is an L^p -singular noise if

$$\gamma = \gamma_1 + \dots + \gamma_\ell + \gamma_\infty,$$

where $\gamma_i \in L^{p_i}(\mathbb{R})$ for $1 \leq i \leq \ell$ and $\gamma_\infty \in L^\infty(\mathbb{R})$. Indeed, the γ_i may have one or several p_i -integrable point singularities, such as $\gamma_i(x) \sim |x|^{-\epsilon}$ as $x \rightarrow 0$ for some $\epsilon \in (0, 1/p_i)$, or $\gamma_i(x) \sim (-\log|x|)^\epsilon$ as $x \rightarrow 0$ for $\epsilon > 0$.

Our last result in this Section is the following.

Proposition 2.29. *For every covariance γ in Example 2.28, there exists a centered Gaussian process Ξ that satisfies Assumption 2.18.*

3 Proof outline

In this section, we provide an outline of the proofs of our main results. Most of the more technical results, which we state here as a string of propositions, are accounted for in Sections 4 and 5. Throughout Section 3, we assume that Assumptions 2.1 and 2.3 are met.

3.1 Outline for Propositions 2.9 and 2.10

In this outline, we assume that Assumption 2.7 holds. Let $\text{FC} \subset C_0^\infty$ be the set of real-valued smooth functions $\varphi : I \rightarrow \mathbb{R}$ such that

1. $\text{supp}(\varphi)$ is a compact subset of I in Cases 1, 2-D, and 3-D;
2. $\text{supp}(\varphi)$ is a compact subset of I 's closure in Cases 2-R and 3-R; and
3. $\text{supp}(\varphi)$ is a compact subset of $[0, b)$ in Case 3-M.

We begin with two classical results in the theory of Schrödinger operators. (For definitions of the functional analysis terminology used in this section, we refer to [39, Section VIII.6], [42, Section 7.5], or [44, Section 2.3].)

Lemma 3.1. *For every $\kappa > 0$, there exists $c = c(\kappa) > 0$ such that for every $f \in \text{AC} \cap L^2$, one has $f(x)^2 \leq \kappa \|f'\|_2^2 + c \|f\|_2^2$ for all $x \in I$.*

Proposition 3.2. *\mathcal{E} is closed and semibounded on $D(\mathcal{E})$, and FC is a form core for \mathcal{E} . H is the unique self-adjoint operator on L^2 whose sesquilinear form is \mathcal{E} , and H has compact resolvent. Lastly, $\|\cdot\|_*$ is equivalent to the “+1 norm” induced by the form \mathcal{E} , where we recall that the latter is defined as*

$$\|f\|_{+1}^2 := \mathcal{E}(f, f) + (c + 1) \|f\|_2^2, \quad f \in D(\mathcal{E}),$$

with $c > 0$ being a constant large enough so that $\mathcal{E}(f, f) + c \|f\|^2 \geq 0$ for every $f \in D(\mathcal{E})$.

Although Lemma 3.1 and Proposition 3.2 can be proved using standard functional-analytic arguments, we were not able to locate an exact statement in the literature that covers every case considered in this paper. For the sake of completeness, we provide a proof and references in Appendix C.

Remark 3.3. Since $\|\cdot\|_*$ and $\|\cdot\|_{+1}$ are equivalent, the claim that \mathcal{E} is closed on $D(\mathcal{E})$ and that FC is a form core is equivalent to the claim that $(D(\mathcal{E}), \langle \cdot, \cdot \rangle_*)$ is a Hilbert space in which FC is dense.

The following proposition, which we prove in Section 4, is a generalization of a result that first appeared in [36], and also uses Lemma 3.1 as a crucial input:

Proposition 3.4. *The inequality (2.5) holds almost surely, and thus $f \mapsto \xi(f^2)$ extends uniquely to a continuous quadratic form on H_V^1 that satisfies (2.5) for all $f \in H_V^1$. Moreover, almost surely, for every $\theta > 0$, there exists $c = c(\theta) > 0$ such that*

$$|\xi(f^2)| \leq \theta \mathcal{E}(f, f) + c \|f\|_2^2, \quad f \in D(\mathcal{E}). \quad (3.1)$$

Thanks to (3.1), almost surely, ξ is an infinitesimally form-bounded perturbation of \mathcal{E} . Therefore, according to the KLMN theorem (e.g., [37, Theorem X.17] or [42, Theorem 7.5.7]), $\hat{\mathcal{E}} = \mathcal{E} + \xi$ is closed and semibounded on $D(\mathcal{E})$, and FC is a form core for $\hat{\mathcal{E}}$. Thus, by [39, Theorem VIII.15], there exists a unique self-adjoint operator \hat{H} satisfying conditions (1) and (2) in the statement of Proposition 2.10. Since H has compact resolvent and \hat{H} is infinitesimally form-bounded by H , the fact that \hat{H} has compact resolvent follows from standard variational estimates (e.g., [38, Theorem XIII.68]).

3.2 Outline for Theorem 2.24

We now go over the outline of the proof of our main result. Throughout, we assume that Assumption 2.18 holds. The outline presented here is separated into five steps. In the first step we provide details on the construction of the pathwise stochastic integral (2.13). In the second step, we introduce smooth-noise approximations of \hat{H} and $\hat{K}(t)$ that serve as the basis of our proof of Theorem 2.24. Then, in the last three steps we prove Theorem 2.24 using these smooth approximations.

3.2.1 Step 1. Stochastic integral

If $f \in PC_c$ is a step function of the form (2.8), then we can define a pathwise stochastic integral in the usual way:

$$\xi(f) = \int_{\mathbb{R}} f(x) d\Xi(x) := \sum_{i=1}^k c_i (\Xi(x_{i+1}) - \Xi(x_i)).$$

Thanks to (2.11), straightforward computations reveal that for such f we have the isometry $\mathbf{E}[\xi(f)^2] = \|f\|_{\gamma}^2$. According to (2.12), step functions are dense in PC_c with respect to $\|f\|_{\gamma}^2$, and thus we may then uniquely define a stochastic integral $\xi^*(f)$ for arbitrary $f \in PC_c$ as the $L^2(\Omega)$ limit of $\xi(f_n)$, where f_n is a sequence of step functions that converges to f in $\|\cdot\|_{\gamma}$ and $L^2(\Omega)$ denotes the space of square-integrable random variables on the same probability space on which Ξ is defined.

We now discuss how $\xi(f)$ for general $f \in PC_c$ can be defined in a pathwise sense as per Karandikar [26]. Given $f \in PC_c$, for every $n \in \mathbb{N}$, define $k(n)$ and $-\infty < \tau_1^{(n)} \leq \tau_2^{(n)} \leq \dots \leq \tau_{k(n)+1}^{(n)} < \infty$ as the quantities

$$\tau_1^{(n)} := \inf \{x \in \mathbb{R} : f(x) \neq 0\}, \quad \tau_{k(n)+1}^{(n)} := \sup \{x \in \mathbb{R} : f(x) \neq 0\}$$

and

$$\tau_k^{(n)} := \inf \{x \geq \tau_{k-1}^{(n)} : |f(x) - f(\tau_{k-1}^{(n)})| \geq 2^{-n}\}, \quad 1 < k \leq k(n).$$

Then, we define the approximate step function

$$f^{(n)} := \sum_{k=1}^{k(n)} f(\tau_k^{(n)}) \mathbf{1}_{[\tau_k^{(n)}, \tau_{k+1}^{(n)})}$$

as well as the pathwise stochastic integral

$$\xi(f) = \int_{\mathbb{R}} f(x) d\Xi(x) := \begin{cases} \lim_{n \rightarrow \infty} \xi(f^{(n)}) & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

On the one hand, as argued in Appendix A (see also [26, Section 1]), the pathwise definition of $f \mapsto \xi(f)$ in (3.2) enables $\hat{K}(t)$'s definition as a conditional expectation of $\xi(L_t(Z))$ given Ξ . On the other hand, $\xi(f)$ retains its meaning as a stochastic integral, since for every $f \in \text{PC}_c$, it holds that $\xi(f) = \xi^*(f)$ almost surely. Indeed, by combining the $L^2(\Omega)$ - $\|\cdot\|_\gamma$ isometry of ξ^* , the definition of $\tau_k^{(n)}$, and (2.12), we get that

$$\begin{aligned} \mathbf{E}\left[\left(\xi(f^{(n)}) - \xi^*(f)\right)^2\right] &= \|f^{(n)} - f\|_\gamma^2 \\ &\leq c_\gamma \left(\sum_{i=1}^{\ell} \|f^{(n)} - f\|_{q_i}^2 \right) \leq c_\gamma 2^{-2n} \left(\sum_{i=1}^{\ell} |\text{supp}(f)|^{2/q_i} \right); \end{aligned}$$

since this is summable in n we conclude that $\xi(f^{(n)}) \rightarrow \xi^*(f)$ almost surely, as desired.

Remark 3.5. Let $f \in C_0^\infty(I)$, and suppose that we restrict our attention to the almost-sure event on which Ξ is continuous. By a summation by parts, we note that

$$\begin{aligned} \xi(f^{(n)}) &= \sum_{k=1}^{k(n)} f(\tau_k^{(n)}) (\Xi(\tau_{k+1}^{(n)}) - \Xi(\tau_k^{(n)})) \\ &= f(\tau_{k(n)}^{(n)}) \Xi(\tau_{k(n)+1}^{(n)}) - f(\tau_1^{(n)}) \Xi(\tau_1^{(n)}) - \sum_{k=2}^{k(n)} \Xi(\tau_k^{(n)}) (f(\tau_k^{(n)}) - f(\tau_{k-1}^{(n)})) \end{aligned}$$

for all $n \in \mathbb{N}$. On the one hand, in Cases 1 and 2, we invariably have that

$$f(\tau_{k(n)}^{(n)}) \Xi(\tau_{k(n)+1}^{(n)}) - f(\tau_1^{(n)}) \Xi(\tau_1^{(n)}) = 0$$

since $\Xi(0) = 0$ and f is compactly supported on I 's closure; similarly, in Case 3,

$$f(\tau_{k(n)}^{(n)}) \Xi(\tau_{k(n)+1}^{(n)}) - f(\tau_1^{(n)}) \Xi(\tau_1^{(n)}) = f(b) \Xi(b).$$

On the other hand, since f is of bounded variation, we have convergence to the usual Riemann-Stieltjes integral:

$$\lim_{n \rightarrow \infty} \sum_{k=2}^{k(n)} \Xi(\tau_k^{(n)}) (f(\tau_k^{(n)}) - f(\tau_{k-1}^{(n)})) = - \int_I \Xi(x) df(x) = -\langle f', \Xi \rangle.$$

In particular, the pathwise stochastic integral defined in (3.2) can be seen as an extension of the Schwartz distribution Ξ' as defined in Definition 2.8 to all of PC_c . However, as noted in an earlier remark, ξ need not preserve its linearity on all of PC_c .

3.2.2 Step 2. Smooth approximations

A key ingredient in the proof of Theorem 2.24 consists of using smooth approximations of Ξ' for which the classical Feynman-Kac formula can be applied, thus creating a connection between \hat{H} as defined via a quadratic form and the kernels $\hat{K}(t)$.

Definition 3.6. Let $\varrho : \mathbb{R} \rightarrow \mathbb{R}$ be a mollifier, that is,

1. ϱ is smooth, compactly supported, nonnegative, even (i.e., $\varrho(x) = \varrho(-x)$), and such that $\int \varrho(x) dx = 1$; and
2. if we define $\varrho_\varepsilon(x) := \varepsilon^{-1} \varrho(x/\varepsilon)$ for every $\varepsilon > 0$, then $\varrho_\varepsilon \rightarrow \delta_0$ as $\varepsilon \rightarrow 0$ in the space of Schwartz distributions, where δ_0 denotes the delta Dirac distribution.

For every $\varepsilon > 0$, we define the stochastic process $\Xi_\varepsilon := \Xi * \varrho_\varepsilon(x)$, where $*$ denotes the convolution.

Remark 3.7. Since ϱ_ε is smooth, the process $\Xi'_\varepsilon = (\Xi * \varrho_\varepsilon)' = \Xi * \varrho'_\varepsilon$ has continuous sample paths. Thanks to (2.11), straightforward computations reveal that Ξ'_ε is a stationary Gaussian process with mean zero and covariance

$$\begin{aligned}\mathbf{E}[\Xi'_\varepsilon(x)\Xi'_\varepsilon(y)] &= \mathbf{E}[(\Xi * \varrho'_\varepsilon)(x)(\Xi * \varrho'_\varepsilon)(y)] \\ &= \int_{\mathbb{R}^2} \mathbf{E}[\Xi(a)\Xi(b)] \varrho'_\varepsilon(a-x) \varrho'_\varepsilon(b-y) \, da db = (\gamma * \varrho_\varepsilon^{*2})(x-y) \quad (3.3)\end{aligned}$$

for every $x, y \in \mathbb{R}$, where the last equality follows from integration by parts.

Moreover, following-up on Remark 3.5, we note that the pathwise stochastic integral ξ is coupled to the random Schwartz distribution

$$f \mapsto \int_{\mathbb{R}} f(x) \Xi'_\varepsilon(x) \, dx, \quad f \in \text{PC}_c$$

in the following way: For every $f \in \text{PC}_c$, the function $f * \varrho_\varepsilon$ is smooth and compactly supported on $I + \text{supp}(\varrho_\varepsilon) \subset \mathbb{R}$, and thus by Remark 3.5 we have that

$$\begin{aligned}\int_I f(x) \Xi'_\varepsilon(x) \, dx &= \int_I f(x) (\Xi * \varrho_\varepsilon)'(x) \, dx = \int_I f(x) (\Xi * \varrho'_\varepsilon)(x) \, dx \\ &= - \int_{\mathbb{R}} (f * \varrho'_\varepsilon)(x) \Xi(x) \, dx = - \int_{\mathbb{R}} (f * \varrho_\varepsilon)'(x) \Xi(x) \, dx = \xi(f * \varrho_\varepsilon). \quad (3.4)\end{aligned}$$

Definition 3.8. For every $\varepsilon > 0$, let us define the sesquilinear form

$$\hat{\mathcal{E}}_\varepsilon(f, g) := \mathcal{E}(f, g) + \langle fg, \Xi'_\varepsilon \rangle$$

on the form domain $D(\mathcal{E})$, and the random kernel

$$\hat{K}_\varepsilon(t; x, y) := \begin{cases} \Pi_B(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(B), V + \Xi'_\varepsilon \rangle} \right] & (\text{Case 1}) \\ \Pi_X(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(X), V + \Xi'_\varepsilon \rangle + \bar{\alpha} \mathfrak{L}_t^0(X)} \right] & (\text{Case 2}) \\ \Pi_Y(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(Y), V + \Xi'_\varepsilon \rangle + \bar{\alpha} \mathfrak{L}_t^0(Y) + \bar{\beta} \mathfrak{L}_t^b(Y)} \right] & (\text{Case 3}) \end{cases}$$

Since Ξ'_ε has regular sample paths, applying classical operator theory to \hat{H}_ε yields the following result:

Proposition 3.9. For every $\varepsilon > 0$, the following holds almost surely: There exists a unique self-adjoint operator \hat{H}_ε with dense domain $D(\hat{H}_\varepsilon) \subset L^2$ such that

1. $D(\hat{H}_\varepsilon) \subset D(\mathcal{E})$;
2. For every $f, g \in D(\hat{H}_\varepsilon)$, one has $\langle f, \hat{H}_\varepsilon g \rangle = \hat{\mathcal{E}}_\varepsilon(f, g)$; and
3. \hat{H}_ε has compact resolvent.

For every $t > 0$, $e^{-t\hat{H}_\varepsilon}$ is a self-adjoint Hilbert-Schmidt/trace class operator, and we have the Feynman-Kac formula $e^{-t\hat{H}_\varepsilon} = \hat{K}_\varepsilon(t)$. In particular,

$$\hat{K}_\varepsilon(t; x, y) = \hat{K}_\varepsilon(t; y, x), \quad t > 0, x, y \in I; \quad (3.5)$$

$$\int_I \hat{K}_\varepsilon(t; x, z) \hat{K}_\varepsilon(\bar{t}; z, y) \, dz = \hat{K}_\varepsilon(t + \bar{t}; x, y), \quad t, \bar{t} > 0, x, y \in I; \quad (3.6)$$

$$\hat{K}_\varepsilon(t)f = \sum_{i=1}^k e^{-t\lambda_k(\hat{H}_\varepsilon)} \langle \psi_k(\hat{H}_\varepsilon), f \rangle \psi_k(\hat{H}_\varepsilon), \quad f \in L^2. \quad (3.7)$$

Moreover, as a direct consequence of the coupling (3.4) and the fact that ξ is a Gaussian process with covariance $\langle \cdot, \cdot \rangle_\gamma$, we can show that the objects introduced in Definition 3.8 serve as good approximations of \hat{H} and $\hat{K}(t)$ in the following sense:

Proposition 3.10. *Almost surely, every vanishing sequence in $(0, 1]$ has a further subsequence $(\varepsilon_n)_{n \in \mathbb{N}}$ along which*

$$\lim_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) = \lambda_k(\hat{H}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\psi_k(\hat{H}_{\varepsilon_n}) - \psi_k(\hat{H})\|_2 = 0 \quad (3.8)$$

for all $k \in \mathbb{N}$, up to possibly relabeling the eigenfunctions of \hat{H} if it has repeated eigenvalues.

Proposition 3.11. *For every $t > 0$, it holds that*

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|_2^2] = 0 \quad (3.9)$$

and

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}\left[\left(\int_I \hat{K}_\varepsilon(t; x, x) - \hat{K}(t; x, x) \, dx\right)^2\right] = 0. \quad (3.10)$$

3.2.3 Step 3. Feynman-Kac formula

We are now in a position to prove Theorem 2.24. We begin by proving that for every $t > 0$, $e^{-t\hat{H}} = \hat{K}(t)$ almost surely. Let us fix some $t > 0$. By Propositions 3.9–3.11, almost surely, there exists a vanishing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that (3.5)–(3.7) holds for every ε_n , and along which the limits (3.8) and

$$\lim_{n \rightarrow \infty} \|\hat{K}_{\varepsilon_n}(t) - \hat{K}(t)\|_2 = 0 \quad (3.11)$$

hold. For the remainder of this step, we assume that we are working with an outcome in this probability-one event.

Since the space $L^2(I \times I)$ of Hilbert-Schmidt integral operators on L^2 is complete, (3.11) means that $\|\hat{K}(t)\|_2 < \infty$. In particular, $\hat{K}(t)$ is compact. Furthermore, given that convergence in Hilbert-Schmidt norm implies weak operator convergence and every $\hat{K}_{\varepsilon_n}(t) = e^{-t\hat{H}_{\varepsilon_n}}$ is nonnegative and symmetric, this implies that $\hat{K}(t)$ is nonnegative and symmetric, hence self-adjoint (e.g., [47, Theorems 4.28 and 6.11]). By the spectral theorem for compact self-adjoint operators (e.g., [45, Theorems 5.4 and 5.6]), we then know that there exists an orthonormal basis $(\Psi_k)_{k \in \mathbb{N}} \subset L^2$ and nonnegative numbers $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \dots \geq 0$ such that $\hat{K}(t)$ satisfies

$$\hat{K}(t)f = \sum_{k=1}^{\infty} \Lambda_k \langle \Psi_k, f \rangle \Psi_k, \quad f \in L^2.$$

Consequently, to prove that $e^{-t\hat{H}} = \hat{K}(t)$, we need only show that $\hat{K}(t)$'s spectral expansion is equivalent to (2.7).

On the one hand, since the Hilbert-Schmidt norm dominates the operator norm, it follows from (3.11) that $\|\hat{K}_{\varepsilon_n}(t) - \hat{K}(t)\|_{\text{op}} \rightarrow 0$; hence $e^{-t\lambda_k(\hat{H}_{\varepsilon_n})} \rightarrow \Lambda_k$ for all $k \in \mathbb{N}$ by (3.7). Given that $\lambda_k(\hat{H}_{\varepsilon_n}) \rightarrow \lambda_k(\hat{H})$ by (3.8), we conclude that $\Lambda_k = e^{-t\lambda_k(\hat{H})}$ for all $k \in \mathbb{N}$. On the other hand, we note that

$$\begin{aligned} & \|\hat{K}_{\varepsilon_n}(t)\psi_k(\hat{H}_{\varepsilon_n}) - \hat{K}(t)\psi_k(\hat{H})\|_2 \\ & \leq \|\hat{K}_{\varepsilon_n}(t)\psi_k(\hat{H}_{\varepsilon_n}) - \hat{K}_{\varepsilon_n}(t)\psi_k(\hat{H})\|_2 + \|\hat{K}_{\varepsilon_n}(t)\psi_k(\hat{H}) - \hat{K}(t)\psi_k(\hat{H})\|_2 \\ & \leq \|\hat{K}_{\varepsilon_n}(t)\|_{\text{op}} \|\psi_k(\hat{H}_{\varepsilon_n}) - \psi_k(\hat{H})\|_2 + \|\hat{K}_{\varepsilon_n}(t) - \hat{K}(t)\|_{\text{op}}. \end{aligned}$$

This vanishes as $n \rightarrow \infty$ for all $k \in \mathbb{N}$. Moreover, the spectral expansion (3.7) and the limit (3.8) imply that

$$\lim_{n \rightarrow \infty} \hat{K}_{\varepsilon_n}(t) \psi_k(\hat{H}_{\varepsilon_n}) = \lim_{n \rightarrow \infty} e^{-t\lambda_k(\hat{H}_{\varepsilon_n})} \psi_k(\hat{H}_{\varepsilon_n}) = e^{-t\lambda_k(\hat{H})} \psi_k(\hat{H})$$

in L^2 ; hence $\hat{K}(t)\psi_k(\hat{H}) = e^{-t\lambda_k(\hat{H})}\psi_k(\hat{H})$. Thus $(e^{-t\lambda_k(\hat{H})}, \psi_k(\hat{H}))_{k \in \mathbb{N}}$ can be taken as the eigenvalue-eigenfunction pairs for $\hat{K}(t)$, concluding the proof that $\hat{K}(t) = e^{-t\hat{H}}$.

3.2.4 Step 4. Trace formula

Next, we prove Theorem 2.24 (2), that is, for every $t > 0$, $\text{Tr}[e^{-t\hat{H}}] = \int_I \hat{K}(t; x, x) dx < \infty$ almost surely. Let $t > 0$ be fixed. By Propositions 3.9 and 3.11, we can find a vanishing sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that (3.5)–(3.7) hold for all ε_n and along which

$$\lim_{n \rightarrow \infty} \|\hat{K}_{\varepsilon_n}(t/2) - \hat{K}(t/2)\|_2 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \left| \int_I \hat{K}_{\varepsilon_n}(t; x, x) - \hat{K}(t; x, x) dx \right| = 0 \quad (3.12)$$

almost surely. Since $e^{-t\hat{H}}$ is by definition a semigroup, we have that

$$\text{Tr}[e^{-t\hat{H}}] = \sum_{k=1}^{\infty} \left(e^{-(t/2)\lambda_k(\hat{H})} \right)^2 = \|e^{-(t/2)\hat{H}}\|_2^2. \quad (3.13)$$

Then, by combining the symmetry and semigroup properties (3.5) and (3.6), the almost sure convergences (3.12), and the almost sure equality $\hat{K}(t/2) = e^{-(t/2)\hat{H}}$ established in the previous step of this proof, we obtain that

$$\begin{aligned} \|e^{-(t/2)\hat{H}}\|_2^2 &= \|\hat{K}(t/2)\|_2^2 = \lim_{n \rightarrow \infty} \|\hat{K}_{\varepsilon_n}(t/2)\|_2^2 \\ &= \lim_{n \rightarrow \infty} \int_{I^2} \hat{K}_{\varepsilon_n}(t/2; x, y)^2 dy dx = \lim_{n \rightarrow \infty} \int_I \left(\int_I \hat{K}_{\varepsilon_n}(t/2; x, y) \hat{K}_{\varepsilon_n}(t/2; y, x) dy \right) dx \\ &= \lim_{n \rightarrow \infty} \int_I \hat{K}_{\varepsilon_n}(t; x, x) dx = \int_I \hat{K}(t; x, x) dx \end{aligned}$$

almost surely. Since we know that $\|\hat{K}(t/2)\|_2 < \infty$ almost surely from the previous step, this concludes the proof of Theorem 2.24 (2).

3.2.5 Step 5. Last properties

We now conclude the proof of Theorem 2.24 by showing that, almost surely, $e^{-t\hat{H}}$ is a Hilbert-Schmidt/trace class integral operator for every $t > 0$. By combining (3.13) with the fact that every Hilbert-Schmidt operator on L^2 has an integral kernel in $L^2(I \times I)$ (e.g., [47, Theorem 6.11]), we need only prove that, almost surely, $e^{-t\hat{H}}$ is trace class for all $t > 0$.

In the previous step of this proof, we have already shown the weaker statement that, for every $t > 0$, $\text{Tr}[e^{-t\hat{H}}] < \infty$ almost surely. By a countable intersection we can extend this to the statement that there exists a probability-one event on which $\text{Tr}[e^{-t\hat{H}}] < \infty$ for every $t \in \mathbb{Q} \cap (0, \infty)$. Since $\lambda_k(\hat{H}) \rightarrow \infty$ as $k \rightarrow \infty$, there exists some $k_0 \in \mathbb{N}$ such that $\lambda_k(\hat{H}) > 0$ for every $k > k_0$. Since $\sum_{k=1}^{k_0} e^{-t\lambda_k(\hat{H})}$ is finite for every t and $\sum_{k=k_0+1}^{\infty} e^{-t\lambda_k(\hat{H})}$ is monotone decreasing in t , the fact that $\text{Tr}[e^{-t\hat{H}}] < \infty$ holds for $t \in \mathbb{Q} \cap (0, \infty)$ implies that it holds for all $t > 0$, concluding the proof of Theorem 2.24.

Remark 3.12. In contrast to the proofs of [20, Proposition 1.8 (a)] and [22, Corollary 2.2] (which we recall apply to Case 2 with $V(x) = x$), the argument presented here uses

smooth approximations of $\hat{K}(t)$ rather than random matrix approximations. Since the present paper does not deal with convergence of random matrices, this choice is natural, and it allows to sidestep several technical difficulties involved with discrete models. With this said, the proof of (3.8) is inspired by the convergence result for the spectrum of random matrices in [3, Section 2] and [36, Section 5]. We refer to Section 5 for the details.

3.3 Proof of Proposition 2.29

The main technical result in the proof of Proposition 2.29 is the following estimate, which is a direct consequence of [19, Lemma 4.2] (as shown in [19], (3.14) is a straightforward consequence of Young's convolution inequality).

Proposition 3.13. *Using the notations of Example 2.28, there exists a constant $c_\gamma > 0$ such that for every $f \in PC_c$, it holds that*

$$\|f\|_\gamma^2 \leq \begin{cases} c_\gamma \|f\|_1^2 & (\text{bounded noise}) \\ c_\gamma \|f\|_2^2 & (\text{white noise}) \\ c_\gamma (\|f\|_2^2 + \|f\|_1^2) & (\text{fractional noise with } \mathfrak{H} \in (\frac{1}{2}, 1)) \\ c_\gamma \left(\sum_{i=1}^\ell \|f\|_{1/(1-1/2p_i)}^2 + \|f\|_1^2 \right) & (L^p\text{-singular noise with } p_i \geq 1). \end{cases} \quad (3.14)$$

Whenever γ is such that $\langle \cdot, \cdot \rangle_\gamma$ is a semi-inner-product, we know from standard existence theorems that there exists a Gaussian process Ξ on \mathbb{R} with covariance (2.11). As argued in Remark 2.20, such a process must have stationary increments. To see that such Ξ have continuous versions, we note that for any $1 \leq q \leq 2$ and $x < y$ such that $y - x \leq 1$, one has $\|\mathbf{1}_{[x,y]}\|_q^4 = (y - x)^{4/q}$ with $4/q > 1$. Thus, given that $1/(1-1/2p) \in (1, 2]$ for every $p \geq 1$, it follows from Proposition 3.13 that there exists some constants $c, r > 0$ such that

$$\mathbb{E}[(\Xi(x) - \Xi(y))^4] = 3! \|\mathbf{1}_{[x,y]}\|_\gamma^4 \leq c|x - y|^{1+r}$$

for every $x < y \in \mathbb{R}$. The existence of a continuous version then follows from the classical Kolmogorov criterion (e.g., [29, Section 14.1]).

4 Proof of Propositions 2.9 and 2.10

In this section, we complete the proof of Propositions 2.9 and 2.10. Following-up on the outline in Section 3.1, it only remains to prove Proposition 3.4.

4.1 Step 1. Reduction to a simple inequality

We begin by showing that Proposition 3.4 can be entirely reduced to the following claim: Almost surely, for every $\theta > 0$, there exists $c = c(\theta) > 0$ such that

$$|\xi(f^2)| \leq \theta \left(\frac{1}{2} \|f'\|_2^2 + \|V^{1/2} f\|_2^2 \right) + c \|f\|_*^2 \quad (4.1)$$

for every $f \in C_0^\infty$.

This is easiest to see in Cases 1, 2-D, and 3-D: On the one hand, in those cases (4.1) directly implies (3.1) for all $f \in FC$, which we can then extend to every $f \in D(\mathcal{E})$ since FC is a form core for \mathcal{E} . On the other hand, (4.1) implies that $|\xi(f^2)| \leq \max\{\theta, c\} \|f\|_*^2$, which yields (2.5). With (2.5) established, the unique continuous extension of $\xi(f^2)$ to H_V^1 then follows from the fact that C_0^∞ is dense in the Hilbert space $(H_V^1, \langle \cdot, \cdot \rangle_*)$.

To see how (4.1) implies the desired estimate in other cases, let us consider for example Case 2-R: By (4.1), almost surely, for every $\bar{\theta} > 0$ there exists $\bar{c} > 0$ such that

$$|\xi(f^2)| \leq \bar{\theta} \left(\frac{1}{2} \|f'\|_2^2 + \|V^{1/2} f\|_2^2 \right) + \bar{c} \|f\|_2^2 = \bar{\theta} \mathcal{E}(f, f) + \frac{\bar{\theta} \alpha}{2} f(0)^2 + \bar{c} \|f\|_2^2.$$

At this point, controlling $f(0)^2$ with Lemma 3.1 yields the desired estimate (with the straightforward substitution $\theta := \bar{\theta}(1 + \frac{\alpha\kappa}{2})$). Cases 3-R and 3-M can be dealt with in the same way.

4.2 Step 2. Proof of (4.1)

We now complete the proof of Proposition 3.4 by proving (4.1). We begin with Cases 1 and 2. Following [32, 36], we define the integrated process

$$\tilde{\Xi}(x) := \int_x^{x+1} \Xi(y) dy, \quad x \in \mathbb{R}$$

so that we can write $\Xi(x) = \tilde{\Xi}(x) + (\Xi(x) - \tilde{\Xi}(x))$; hence for every $f \in C_0^\infty$, one has

$$\xi(f^2) = -\langle 2f'f, \tilde{\Xi} \rangle - \langle 2f'f, \Xi - \tilde{\Xi} \rangle = f^2(0)\tilde{\Xi}(0) + \langle f^2, \tilde{\Xi}' \rangle + 2\langle f'f, \tilde{\Xi} - \Xi \rangle$$

by Definition 2.8 and an integration by parts. By applying Lemma 3.1 to the term $f^2(0)\tilde{\Xi}(0)$ (since $|\tilde{\Xi}(0)| < \infty$ whenever Ξ 's path is continuous), it suffices to prove that almost surely, for every $\theta > 0$, there exists $c > 0$ such that

$$|\langle f^2, \tilde{\Xi}' \rangle| + 2|\langle f'f, \tilde{\Xi} - \Xi \rangle| \leq \theta(\frac{1}{2}\|f'\|_2^2 + \|V^{1/2}f\|_2^2) + c\|f\|_2^2$$

for all $f \in C_0^\infty$. Thanks to Assumption 2.7, the processes $x \mapsto \tilde{\Xi}'(x)$ and $x \mapsto \tilde{\Xi}(x) - \Xi(x)$ are continuous stationary centered Gaussian processes on \mathbb{R} , and thus it follows from standard Gaussian suprema estimates (e.g., Corollary B.2) that there exists a finite random variable $C > 0$ such that, almost surely,

$$|\tilde{\Xi}'(x)|, (\Xi(x) - \tilde{\Xi}(x))^2 \leq C \log(2 + |x|) \quad (4.2)$$

for all $x \in I$. Since $V(x) \gg \log|x|$ as $|x| \rightarrow \infty$, for every $\theta > 0$, there exists $\tilde{c}_1, \tilde{c}_2 > 0$ depending on θ such that

$$C \log(2 + |x|) \leq \frac{\theta}{2}(\tilde{c}_1 + V(x)), \quad \sqrt{C \log(2 + |x|)} \leq \frac{\theta}{2}\sqrt{\tilde{c}_2 + V(x)} \quad (4.3)$$

for all $x \in I$. On the one hand, (4.2) and the above inequality imply that

$$\int_I f(x)^2 |\tilde{\Xi}'(x)| dx \leq \frac{\theta}{2} \|V^{1/2}f\|_2^2 + \frac{\theta\tilde{c}_1}{2} \|f\|_2^2.$$

On the other hand, the same inequalities and $|z\bar{z}| \leq \frac{1}{2}(z^2 + \bar{z}^2)$ imply

$$\begin{aligned} \int_I |f'(x)f(x)| |\tilde{\Xi}(x) - \Xi(x)| dx &\leq \frac{\theta}{2} \int_I |f'(x)f(x)| \sqrt{\tilde{c}_2 + V(x)} dx \\ &\leq \frac{\theta}{2} \left(\int_I f'(x)^2 dx + \int_I f(x)^2 (\tilde{c}_2 + V(x)) dx \right) \leq \frac{\theta}{2} (\|f'\|_2^2 + \|V^{1/2}f\|_2^2) + \frac{\theta\tilde{c}_2}{2} \|f\|_2^2, \end{aligned}$$

concluding the proof.

Suppose then that we are in Case 3. Since Ξ is almost surely continuous by Assumption 2.7, the random variable $C := \sup_{0 \leq x \leq b} |\Xi(x)|$ is finite, and thus

$$|\xi(f^2)| \leq 2C \int_0^b |f'(x)| |f(x)| dx + |\Xi(b)| f(b)^2.$$

An application of the bound $|f'| |f| \leq \frac{\kappa}{2}(f')^2 + \frac{1}{2\kappa}f^2$ for arbitrary $\kappa > 0$ followed by Lemma 3.1 to $f(b)^2$ then yield an upper bound of the form $|\xi(f^2)| \leq \frac{\theta}{2}\|f'\|_2^2 + c\|f\|_2^2$, which is better than (4.1).

5 Proof of Theorem 2.24

In this section, we complete the outline for the proof of Theorem 2.24 provided in Section 3.2 by proving Propositions 3.9–3.11. This is done in Sections 5.6–5.9 below. Before we do this, however, we need several technical results regarding the deterministic semigroup e^{-tH} and the behaviour of the local times $L_t(Z)$ and $\mathcal{L}_t(Z)$. This is done in Sections 5.1–5.5.

5.1 Feynman-Kac formula for deterministic operators

We begin by recording some standard results in semigroup theory. By the Feynman-Kac formula, we expect that $e^{-tH} = K(t)$ for the kernels $K(t)$ defined as follows:

Definition 5.1. *With the same notations as in Definitions 2.16 and 2.21, for every $t > 0$, we define the kernel $K(t) : I^2 \rightarrow \mathbb{R}$ as*

$$K(t; x, y) := \begin{cases} \Pi_B(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(B), V \rangle} \right] & (\text{Case 1}) \\ \Pi_X(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(X), V \rangle + \bar{\alpha} \mathfrak{L}_t^0(X)} \right] & (\text{Case 2}) \\ \Pi_Y(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(Y), V \rangle + \bar{\alpha} \mathfrak{L}_t^0(Y) + \bar{\beta} \mathfrak{L}_t^b(Y)} \right] & (\text{Case 3}) \end{cases} \quad (5.1)$$

To prove this, we begin with a reminder regarding the Kato class of potentials.

Definition 5.2. *We define the Kato class, which we denote by $\mathcal{K} = \mathcal{K}(I)$, as the collection of nonnegative functions $f : I \rightarrow \mathbb{R}$ such that*

$$\sup_{x \in I} \int_{\{y \in I : |x-y| \leq 1\}} f(y) dy < \infty. \quad (5.2)$$

We use $\mathcal{K}_{\text{loc}} = \mathcal{K}_{\text{loc}}(I)$ to denote the class of f 's such that $f \mathbf{1}_K \in \mathcal{K}$ for every compact subset K of I 's closure.

Remark 5.3. There is a large diversity of equivalent definitions of the Kato class, some of which are probabilistic. See, for instance, [41, Section A.2].

Theorem 5.4. *If $V \in \mathcal{K}_{\text{loc}}$, then $e^{-tH} = K(t)$ for all $t > 0$. Moreover,*

$$K(t; x, y) = K(t; y, x), \quad t > 0, x, y \in I; \quad (5.3)$$

$$\int_I K(t; x, z) K(\bar{t}; z, y) dz = K(t + \bar{t}; x, y), \quad t, \bar{t} > 0, x, y \in I. \quad (5.4)$$

While Theorem 5.4 follows from standard functional-analytic methods (e.g., [11]), we were not able to locate an exact statement in the literature that covers Cases 2-R and 3-M. We provide a full proof and references in Appendix D.

It is easy to see from (5.2) that locally integrable functions are in \mathcal{K}_{loc} so that, by Assumption 2.3, $V \in \mathcal{K}_{\text{loc}}$. Therefore, we have the following immediate consequence of Theorem 5.4:

Corollary 5.5. *Theorem 5.4 holds under Assumptions 2.1 and 2.3.*

5.2 Reflected Brownian motion couplings

The local time process of the Brownian motion B is much more well studied than that of its reflected versions X or Y . Thus, it is convenient to reduce statements regarding the local times of the latter into statements concerning the local time of B . In order to achieve this, we use the following couplings of B with X and Y .

5.2.1 Half-line

For any $x > 0$, we can couple B and X in such a way that $X^x(t) = |B^x(t)|$ for every $t \geq 0$. In particular, for any functional F of Brownian paths, one has

$$\mathbf{E}^x[F(X)] = \mathbf{E}^x[F(|B|)]. \quad (5.5)$$

Under the same coupling, we observe that for every positive x, y , and t , one has

$$X_t^{x,y} \stackrel{\text{d}}{=} (|B^x| \mid B^x(t) \in \{-y, y\}).$$

Note that

$$\mathbf{P}[B^x(t) = y \mid B^x(t) \in \{-y, y\}] = \frac{\mathcal{G}_t(x-y)}{\mathcal{G}_t(x-y) + \mathcal{G}_t(x+y)} = \frac{\Pi_B(t; x, y)}{\Pi_X(t; x, y)},$$

and similarly,

$$\mathbf{P}[B^x(t) = -y \mid B^x(t) \in \{-y, y\}] = \frac{\Pi_B(t; x, -y)}{\Pi_X(t; x, y)}.$$

Therefore, for any path functional F , it holds that

$$\begin{aligned} \Pi_X(t; x, y) \mathbf{E}_t^{x,y}[F(X)] &= \Pi_X(t; x, y) \mathbf{E}[F(|B^x|) \mid B^x(t) \in \{-y, y\}] \\ &= \Pi_B(t; x, y) \mathbf{E}_t^{x,y}[F(|B|)] + \Pi_B(t; x, -y) \mathbf{E}_t^{x,-y}[F(|B|)]. \end{aligned} \quad (5.6)$$

According to the strong Markov property and the symmetry about 0 of Brownian motion, we note the equivalence of conditionings

$$(|B^x| \mid B^x(t) = -y) \stackrel{\text{d}}{=} (|B^x| \mid \tau_0(B^x) < t \text{ and } B^x(t) = y), \quad (5.7)$$

where we define the hitting time τ_0 as in Remark 2.22. Indeed, we can obtain the left-hand side of (5.7) from the right-hand side by reflecting $(B^x \mid B^x(t) = -y)$ after it first hits zero and then taking an absolute value (see Figure 1 below for an illustration). Since

$$\mathbf{P}[\tau_0(B^x) < t \mid B^x(t) = y]^{-1} \Pi_B(t; x, -y) = e^{2xy/t} \Pi_B(t; x, -y) = \Pi_B(t; x, y)$$

(this is easily computed from the joint density of the running maximum and current value of a Brownian motion [40, Chapter III, Exercise 3.14]), we see that

$$\Pi_B(t; x, -y) \mathbf{E}_t^{x,-y}[F(|B|)] = \Pi_B(t; x, y) \mathbf{E}_t^{x,y}[1_{\{\tau_0(B) < t\}} F(|B|)].$$

Thus (5.6) becomes

$$\Pi_X(t; x, y) \mathbf{E}_t^{x,y}[F(X)] = \Pi_B(t; x, y) \mathbf{E}_t^{x,y}[(1 + 1_{\{\tau_0(B) < t\}}) F(|B|)]. \quad (5.8)$$

Finally, given that $\Pi_B(t; x, y)/\Pi_X(t; x, y) \leq 1$, if $F \geq 0$, then (5.8) yields the inequality

$$\mathbf{E}_t^{x,y}[F(X)] \leq 2 \mathbf{E}_t^{x,y}[F(|B|)]. \quad (5.9)$$

5.2.2 Bounded interval

For any $x \in (0, b)$, we can couple Y^x and B^x by reflecting the path of the latter on the boundary of $(0, b)$, that is,

$$Y^x(t) := \begin{cases} B^x(t) - 2kb & \text{if } B^x(t) \in [2kb, (2k+1)b], \quad k \in \mathbb{Z}, \\ |B^x(t) - 2kb| & \text{if } B^x(t) \in [(2k-1)b, 2kb], \quad k \in \mathbb{Z}. \end{cases} \quad (5.10)$$

(See Figure 2 below for an illustration of this coupling.) Under this coupling, it is clear that for any $z \in (0, b)$, we have

$$L_t^z(Y^x) = \sum_{a \in 2b\mathbb{Z} \pm z} L_t^a(B^x). \quad (5.11)$$

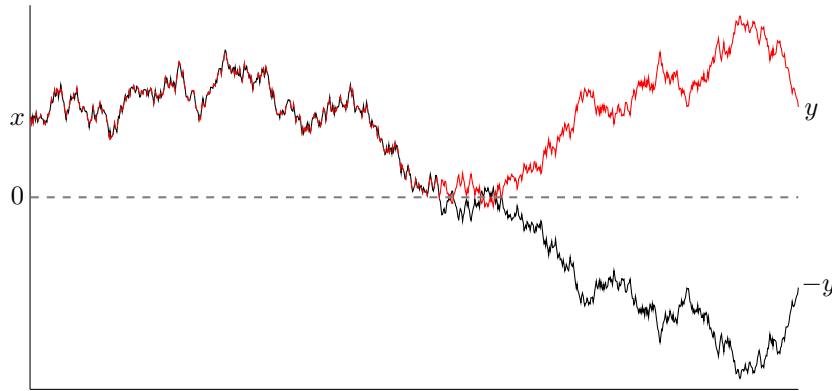


Figure 1: Reflection Principle: The path of $B_t^{x,-y}$ (black) and its reflection after the first passage to zero (red).

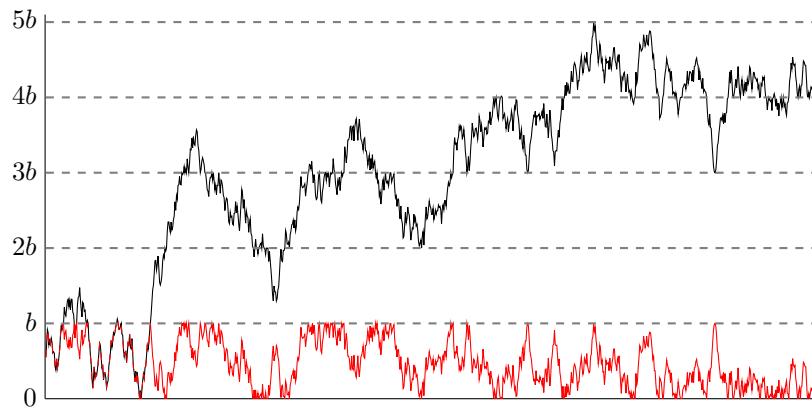


Figure 2: Path of B^x (black) and its reflection on the boundary of $(0, b)$ (red).

5.3 Boundary local time

In this section, we control the exponential moments of the boundary local time of the reflected paths X and Y .

Lemma 5.6. *For every $\theta, t > 0$ and $c \in \{0, b\}$, it holds that*

$$\sup_{x \in (0, \infty)} \mathbf{E}^x [e^{\theta \mathfrak{L}_t^0(X)}], \sup_{x \in (0, b)} \mathbf{E}^x [e^{\theta \mathfrak{L}_t^c(Y)}] < \infty. \quad (5.12)$$

Proof. We begin by proving (5.12) in Case 2 (i.e., the process X). By (5.5) it suffices to prove that

$$\sup_{x \in (0, \infty)} \mathbf{E}^x [e^{\theta \mathfrak{L}_t^0(B)}] < \infty$$

for every $\theta, t > 0$, where

$$\mathfrak{L}_t^0(B) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{-\varepsilon < B(s) < \varepsilon\}} ds.$$

On the one hand, by Brownian scaling, we have the equality in law

$$\mathfrak{L}_t^0(B_t^x) \stackrel{d}{=} t^{1/2} \mathfrak{L}_1^0(B_1^{t^{-1/2}x}). \quad (5.13)$$

On the other hand, according to [35, (1)], for every $x, y \in \mathbb{R}$ and $\ell > 0$, one has

$$\mathbf{P}[\mathfrak{L}_1^0(B^x) \in d\ell, B^x(1) \in dy] = \frac{(|x| + |y| + \ell)e^{-(|x|+|y|+\ell)^2/2}}{\sqrt{2\pi}} d\ell dy;$$

integrating out the y variable then yields

$$\mathbf{P}[\mathfrak{L}_1^0(B^x) \in d\ell] = \frac{2e^{-(|x|+\ell)^2/2}}{\sqrt{2\pi}}. \quad (5.14)$$

Thanks to (5.13) and (5.14), we see that

$$\sup_{x \in (0, \infty)} \mathbf{E}_t^x[e^{\theta \mathfrak{L}_t^0(B)}] \leq \mathbf{E}_1^0[e^{\theta t^{1/2} \mathfrak{L}_1^0(B)}] < \infty$$

for every $\theta, t > 0$; hence (5.12) holds in Case 2.

The proof of (5.12) for Case 3 (i.e., the process Y) follows directly from [34, (2.18) and (3.11')], which states that there exists constants $K, K' > 0$ (depending on θ) such that $\mathbf{E}^x[e^{\theta \mathfrak{L}_t^c(Y)}] \leq K' e^{Kt}$ for all $t > 0$ and $x \in (0, b)$. \square

Next, we aim to extend the result of Lemma 5.6 to the local time of the bridge processes $Z_t^{x,x}$. Before we can do this, we need the following estimate on Π_Z .

Lemma 5.7. *For every $t > 0$, it holds that*

$$\mathfrak{s}_t(Z) := \sup_{x, y \in I} \frac{\Pi_Z(t/2; x, y)}{\Pi_Z(t; x, x)} < \infty. \quad (5.15)$$

Proof. In all three cases, $\Pi_Z(t; x, x) \geq 1/\sqrt{2\pi t}$, and thus it suffices to prove that

$$\sup_{(x, y) \in I^2} \Pi_Z(t; x, y) < \infty. \quad (5.16)$$

In Cases 1 & 2, this is trivial. In Case 3, we recall that, by definition,

$$\Pi_Y(t; x, y) := \sum_{z \in 2b\mathbb{Z} \pm y} \mathcal{G}_t(x - z) = \frac{1}{\sqrt{2\pi t}} \left(\sum_{k \in \mathbb{Z}} e^{-(x+y-2bk)^2/2t} + e^{-(x-y-2bk)^2/2t} \right).$$

According to the integral test for series convergence, we note that for every $b, t > 0$ and $z \in \mathbb{R}$, it holds that

$$\sum_{k=\lceil -z/2b \rceil}^{\infty} \frac{e^{-(z+2bk)^2/2t}}{\sqrt{2\pi t}} \leq \frac{e^{-(z+2b\lceil -z/2b \rceil)^2/2t}}{\sqrt{2\pi t}} + \int_{\lceil -z/2b \rceil}^{\infty} \frac{e^{-(z+2bu)^2/2t}}{\sqrt{2\pi t}} du \leq \frac{1}{\sqrt{2\pi t}} + \frac{1}{b},$$

and similarly for the sum from $k = -\infty$ to $\lfloor -z/2b \rfloor$; hence (5.16) holds. \square

We finish this section with the following.

Lemma 5.8. *For every $\theta, t > 0$ and $c \in \{0, b\}$, it holds that*

$$\sup_{x \in (0, \infty)} \mathbf{E}_t^{x,x}[e^{\theta \mathfrak{L}_t^0(X)}], \sup_{x \in (0, b)} \mathbf{E}_t^{x,x}[e^{\theta \mathfrak{L}_t^c(Y)}] < \infty. \quad (5.17)$$

Proof. As it turns out, (5.17) follows from Lemma 5.6. The trick that we use to prove this makes several other appearances in this paper: Since the exponential function is nonnegative, for every $\theta > 0$, an application of the tower property and the Doob h -transform yields

$$\begin{aligned} \mathbf{E}_t^{x,x}[e^{\theta \mathfrak{L}_t^c(Z)}] &= \mathbf{E}\left[\mathbf{E}_t^{x,x}[e^{\theta \mathfrak{L}_t^c(Z)} | Z_t^{x,x}(t/2)]\right] \\ &= \int_I \mathbf{E}_t^{x,x}[e^{\theta \mathfrak{L}_t^c(Z)} | Z_t^{x,x}(t/2) = y] \frac{\Pi_Z(t/2; x, y) \Pi_Z(t/2; y, x)}{\Pi_Z(t; x, x)} dy. \end{aligned} \quad (5.18)$$

If we condition on $Z_t^{x,x}(t/2) = y$, then the path segments

$$(Z_t^{x,x}(s) : 0 \leq s \leq t/2) \quad \text{and} \quad (Z_t^{x,x}(t/2 + s) : 0 \leq s \leq t/2)$$

are independent of each other and have respective distributions $Z_{t/2}^{x,y}$ and $Z_{t/2}^{y,x}$. Since $\Pi_Z(t/2; \cdot, \cdot)$ is symmetric for every $t > 0$, the time-reversed process $s \mapsto Z_{t/2}^{y,x}(t/2 - s)$ (with $0 \leq s \leq t/2$) is equal in distribution to $Z_{t/2}^{x,y}$. Thus,

$$\begin{aligned} \mathbf{E}_t^{x,x} \left[e^{\theta \mathfrak{L}_t^c(Z)} \middle| Z_t^{x,x}(t/2) = y \right] &= \mathbf{E}_t^{x,x} \left[e^{\theta(\mathfrak{L}_{[0,t/2]}^c(Z) + \mathfrak{L}_{[t/2,t]}^c(Z))} \middle| Z_t^{x,x}(t/2) = y \right] \\ &= \mathbf{E}_{t/2}^{x,y} \left[e^{\theta \mathfrak{L}_{t/2}^c(Z)} \right]^2 \leq \mathbf{E}_{t/2}^{x,y} \left[e^{2\theta \mathfrak{L}_{t/2}^c(Z)} \right], \end{aligned} \quad (5.19)$$

where the equality in (5.19) follows from independence and the fact that local time is invariant with respect to time reversal, and the last term in (5.19) follows from Jensen's inequality.

Let us define the constant $\mathfrak{s}_t(Z) < \infty$ as in (5.15). According to (5.18) and (5.19), we then have that for $t > 0$,

$$\begin{aligned} \mathbf{E}_t^{x,x} \left[e^{\theta \mathfrak{L}_t^c(Z)} \right] &\leq \mathfrak{s}_t(Z) \int_I \mathbf{E}_{t/2}^{x,y} \left[e^{2\theta \mathfrak{L}_{t/2}^c(Z)} \right] \Pi_Z(t/2; x, y) dy \\ &= \mathfrak{s}_t(Z) \mathbf{E}_{t/2}^x \left[e^{2\theta \mathfrak{L}_{t/2}^c(Z)} \right]. \end{aligned} \quad (5.20)$$

Hence the present result is a direct consequence of Lemma 5.6. \square

5.4 L^q norms of local time

In this section, we obtain bounds on the exponential moments of the L^q norms of the local times of B , X , and Y . Such results for B^x are well known (see, for instance, [8, Section 4.2]). For X and Y and the bridge processes, we rely on the couplings introduced in Section 5.2 and the midpoint sampling trick used in the proof Lemma 5.8, respectively. Before we state our result, we need the following.

Lemma 5.9. *For every $\theta, u, v > 0$ and $q \geq 1$, it holds that*

$$\sup_{x \in I} \mathbf{E}^x \left[e^{\theta \|L_{u+v}(Z)\|_q^2} \right] \leq \left(\sup_{x \in I} \mathbf{E}^x \left[e^{2\theta \|L_u(Z)\|_q^2} \right] \right) \left(\sup_{x \in I} \mathbf{E}^x \left[e^{2\theta \|L_v(Z)\|_q^2} \right] \right)$$

Proof. Let $x, y \in I$ be fixed. Conditional on $Z^x(u) = y$, the path segments

$$(Z^x(s) : 0 \leq s \leq u) \quad \text{and} \quad (Z^x(u+t) : 0 \leq t \leq \infty)$$

are independent of each other and have respective distributions $Z_u^{x,y}$ and Z^y . Therefore, by the tower property, we have that

$$\begin{aligned} \mathbf{E}^x \left[e^{\theta \|L_{u+v}(Z)\|_q^2} \right] &= \int_I \mathbf{E}^x \left[e^{\theta \|L_{u+v}(Z)\|_q^2} \middle| Z^x(u) = y \right] \Pi_Z(u; x, y) dy \\ &\leq \int_I \mathbf{E}^x \left[e^{2\theta \|L_u(Z)\|_q^2 + 2\theta \|L_{[u,u+v]}(Z)\|_q^2} \middle| Z^x(u) = y \right] \Pi_Z(u; x, y) dy \\ &= \int_I \mathbf{E}_u^{x,y} \left[e^{2\theta \|L_u(Z)\|_q^2} \right] \mathbf{E}^y \left[e^{2\theta \|L_v(Z)\|_q^2} \right] \Pi_Z(u; x, y) dy \\ &\leq \left(\sup_{y \in I} \mathbf{E}^y \left[e^{2\theta \|L_v(Z)\|_q^2} \right] \right) \int_I \mathbf{E}_u^{x,y} \left[e^{2\theta \|L_u(Z)\|_q^2} \right] \Pi_Z(u; x, y) dy \\ &= \left(\sup_{y \in I} \mathbf{E}^y \left[e^{2\theta \|L_v(Z)\|_q^2} \right] \right) \mathbf{E}^x \left[e^{2\theta \|L_u(Z)\|_q^2} \right], \end{aligned}$$

where the second line follows from Minkowski's inequality and $(z + \bar{z})^2 \leq 2(z^2 + \bar{z}^2)$, and the third line follows from conditional independence of the path segments. The result then follows by taking a supremum over $x \in I$. \square

Lemma 5.10. *Let $1 \leq q \leq 2$. For every $\theta, t > 0$, one has*

$$\sup_{x \in I} \mathbf{E}^x \left[e^{\theta \|L_t(Z)\|_q^2} \right] < \infty. \quad (5.21)$$

Proof. We begin by noting that $\|L_t(Z)\|_1 = t$ by (2.10), and thus the result is trivial if $q = 1$. To prove the result for $1 < q \leq 2$, we claim that it suffices to show that there exists nonnegative random variables $R_1, R_2 \geq 0$ with finite exponential moments in some neighbourhood of zero, as well as constants $\kappa_1, \kappa_2 > 1$ such that

$$\sup_{x \in I} \mathbf{E}^x \left[e^{\theta \|L_t(Z)\|_q^2} \right] \leq \mathbf{E} \left[e^{\theta t^{\kappa_1} R_1} \right] \quad (5.22)$$

or

$$\sup_{x \in I} \mathbf{E}^x \left[e^{\theta \|L_t(Z)\|_q^2} \right] \leq \mathbf{E} \left[e^{\theta t^{\kappa_1} R_1} \right]^{1/2} \mathbf{E} \left[e^{\theta t^{\kappa_2} R_2} \right]^{1/2} \quad (5.23)$$

for all $t > 0$. To see this, suppose (5.22) holds, and let $\theta_0 > 0$ be such that $\mathbf{E}[e^{\theta R_1}] < \infty$ for all $\theta < \theta_0$. Then, for any fixed $\theta > 0$,

$$\sup_{x \in I} \mathbf{E}^x \left[e^{\theta \|L_t(Z)\|_q^2} \right] \leq \mathbf{E} \left[e^{\theta t^{\kappa_1} R_1} \right] < \infty$$

for every $t < (\theta_0/\theta)^{1/\kappa_1}$. In particular, if $u, v \leq (\theta_0/2\theta)^{1/\kappa_1}$, we get from Lemma 5.9 that

$$\sup_{x \in I} \mathbf{E}^x \left[e^{\theta \|L_{u+v}(Z)\|_q^2} \right] \leq \left(\sup_{x \in I} \mathbf{E}^x \left[e^{2\theta \|L_u(Z)\|_q^2} \right] \right) \left(\sup_{x \in I} \mathbf{E}^x \left[e^{2\theta \|L_v(Z)\|_q^2} \right] \right) < \infty.$$

Thus, (5.21) now holds for $t < 2(\theta_0/2\theta)^{1/\kappa_1} = 2^{1-1/\kappa_1}(\theta_0/\theta)^{1/\kappa_1}$. Since $\kappa_1 > 1$, $2^{1-1/\kappa_1} > 1$, and thus by repeating this procedure infinitely often, we obtain by induction that (5.21) holds for all $t > 0$, as desired. Essentially the same argument gives the result if we instead have (5.23).

We then prove (5.22)/(5.23). We argue on a case-by-case basis. Let us begin with Case 1. If we couple $B^x = x + B^0$ for all x , then changes of variables with a Brownian scaling imply that

$$\|L_t(B^x)\|_q^2 = \|L_t(B^0)\|_q^2 \stackrel{d}{=} t \left(\int_{\mathbb{R}} L_1^{t^{-1/2}a}(B^0)^q da \right)^{2/q} = t^{1+1/q} \|L_1(B^0)\|_q^2$$

for every $q > 1$. Thanks to the large deviation result [8, Theorem 4.2.1], we know that for every $q > 1$, there exists some $c_q > 0$ such that

$$\mathbf{P} [\|L_1(B^0)\|_q^2 > u] = e^{-c_q u^{q/(q-1)}(1+o(1))}, \quad u \rightarrow \infty.$$

Thus, in Case 1 (5.22) holds with $R_1 = \|L_1(B^0)\|_q^2$ and $\kappa_1 = 1 + 1/q$.

Consider now Case 2. By coupling $X^x(t) = |B^x(t)|$ for all $t > 0$, we note that for every $a > 0$, one has $L_t^a(X^x) = L_t^a(|B^x|) = L_t^a(B^x) + L_t^{-a}(B^x)$. Therefore,

$$\begin{aligned} \|L_t(X^x)\|_q^2 &= \left(\int_0^\infty L_t^a(X^x)^q da \right)^{2/q} \\ &\leq 2^{2(q-1)/q} \left(\int_0^\infty L_t^a(B^x)^q + L_t^{-a}(B^x)^q da \right)^{2/q} = 2^{2(q-1)/2} \|L_t(B^x)\|_q^2. \end{aligned}$$

Thus, the proof in Case 2 follows from Case 1.

Finally, consider Case 3. Recall the coupling of Y^x and B^x in (5.10), which yields the local time identity (5.11). The argument that follows is inspired from the proof of [10, Lemma 2.1]: Under the coupling (5.11),

$$\begin{aligned} \left(\int_0^b L_t^z (Y^x)^q dz \right)^{1/q} &= \left(\int_0^b \left(\sum_{k \in 2b\mathbb{Z}} L_t^{k+z}(B^x) + L_t^{k-z}(B^x) \right)^q dz \right)^{1/q} \\ &\leq 2^{(q-1)/q} \sum_{k \in 2b\mathbb{Z}} \left(\int_{-b}^b L_t^{k+z}(B^x)^q dz \right)^{1/q}. \end{aligned}$$

Let us denote the maximum and minimum of B^x as

$$M^x(t) := \sup_{s \in [0,t]} B^x(s) \quad \text{and} \quad m^x(t) := \inf_{s \in [0,t]} B^x(s).$$

In order for the integral $\int_{-b}^b L_t^{k+z}(B^x)^2 dz$ to be different from zero, it is necessary that $M^x(t) \geq k - b$ and $m^x(t) \leq k + b$, that is, $M^x(t) + b \geq k \geq m^x(t) - b$. Consequently, for every $q > 1$, one has

$$\begin{aligned} &\sum_{k \in 2b\mathbb{Z}} \left(\int_{-b}^b L_t^{k+z}(B^x)^q dz \right)^{1/q} \\ &= \sum_{k \in 2b\mathbb{Z}} \left(\int_{-b}^b L_t^{k+z}(B^x)^q dz \right)^{1/q} \mathbf{1}_{\{M^x(t)+b \geq k \geq m^x(t)-b\}} \\ &\leq \left(\sum_{k \in 2b\mathbb{Z}} \int_{-b}^b L_t^{k+z}(B^x)^q dz \right)^{1/q} \left(\sum_{k \in 2b\mathbb{Z}} \mathbf{1}_{\{M^x(t)+b \geq k \geq m^x(t)-b\}} \right)^{\frac{q-1}{q}} \\ &= \left(\int_{\mathbb{R}} L_t^a(B^x)^q da \right)^{1/q} \left(\sum_{k \in 2b\mathbb{Z}} \mathbf{1}_{\{M^x(t)+b \geq k \geq m^x(t)-b\}} \right)^{\frac{q-1}{q}} \\ &\leq c_1 t^{1/q} \left(\sup_{a \in \mathbb{R}} L_t^a(B^x) \right)^{\frac{q-1}{q}} (M^x(t) - m^x(t) + c_2)^{\frac{q-1}{q}} \\ &\leq c_1 t^{1/q} \left(c_2^{\frac{q-1}{q}} \left(\sup_{a \in \mathbb{R}} L_t^a(B^x) \right)^{\frac{q-1}{q}} + \left(\sup_{a \in \mathbb{R}} L_t^a(B^x) \cdot (M^x(t) - m^x(t)) \right)^{\frac{q-1}{q}} \right) \end{aligned}$$

where $c_1, c_2 > 0$ only depend on b and q : The inequality on the third line follows from Hölder's inequality; the equality on the fourth line follows from the fact that $\sum_{k \in 2b\mathbb{Z}} \int_{-b}^b L_t^a(B^x)^q da$ is equal to $\int_{\mathbb{R}} L_t^a(B^x)^q da$; the inequality on the fifth line follows from the fact that $\int_{\mathbb{R}} L_t^a(B^x)^q da$ is bounded by $(\sup_{a \in \mathbb{R}} L_t^a(B^x))^{q-1} \|L_t(B^x)\|_1$, where $\|L_t(B^x)\|_1 = t$; and the inequality on the last line follows from the fact that

$$(M^x(t) - m^x(t) + c_2)^{\frac{q-1}{q}} \leq (M^x(t) - m^x(t))^{\frac{q-1}{q}} + c_2^{\frac{q-1}{q}}.$$

By Brownian scaling and translation invariance, we have that

$$t^{1/q} \left(\sup_{a \in \mathbb{R}} L_t^a(B^x) \right)^{\frac{q-1}{q}} \stackrel{d}{=} t^{1/2+1/2q} \left(\sup_{a \in \mathbb{R}} L_1^a(B^0) \right)^{\frac{q-1}{q}}$$

and

$$t^{1/q} \left(\sup_{a \in \mathbb{R}} L_t^a(B^x) \cdot (M^x(t) - m^x(t)) \right)^{\frac{q-1}{q}} \stackrel{d}{=} t \left(\sup_{a \in \mathbb{R}} L_1^a(B^0) \cdot (M^0(1) - m^0(1)) \right)^{\frac{q-1}{q}}.$$

Given that $4(\frac{q-1}{q}) \leq 2$ for all $q \in (1, 2]$ and that there exists $\theta_0 > 0$ small enough so that

$$\mathbf{E} \left[\exp \left(\theta_0 \sup_{a \in \mathbb{R}} L_1^a(B^0)^2 \right) \right], \mathbf{E} \left[e^{\theta_0(M^0(1) - m^0(1))^2} \right] < \infty,$$

(e.g., the proof of [10, Lemma 2.1] and references therein) we finally conclude by Hölder's inequality that (5.23) holds in Case 3 with

$$R_1 = 4c_1^2 \left(c_2 \sup_{a \in \mathbb{R}} L_1^a(B^0) \right)^{2\frac{q-1}{q}}, \quad R_2 = 4 \left(\sup_{a \in \mathbb{R}} L_1^a(B^0) \cdot (M^0(1) - m^0(1)) \right)^{2\frac{q-1}{q}},$$

and $\kappa_1 = 1 + 1/q$ and $\kappa_2 = 2$. \square

Lemma 5.11. *Let $1 \leq q \leq 2$. For every $\theta, t > 0$, one has*

$$\sup_{x \in I} \mathbf{E}_t^{x,x} \left[e^{\theta \|L_t(Z)\|_q^2} \right] < \infty.$$

Proof. Once again, the present result follows from Lemma 5.10. To see this, we use the same trick employed in the proof of Lemma 5.8: For every $\theta > 0$, the tower property and the Doob h -transform yields

$$\mathbf{E}_t^{x,x} \left[e^{\theta \|L_t(Z)\|_q^2} \right] = \int_I \mathbf{E}_t^{x,x} \left[e^{\theta \|L_t(Z)\|_q^2} \middle| Z_t^{x,x}(t/2) = y \right] \frac{\Pi_Z(t/2; x, y) \Pi_Z(t/2; y, x)}{\Pi_Z(t; x, x)} dy.$$

Arguing as in the passage following (5.18),

$$\begin{aligned} \mathbf{E}_t^{x,x} \left[e^{\theta \|L_t(Z)\|_q^2} \middle| Z_t^{x,x}(t/2) = y \right] &= \mathbf{E}_t^{x,x} \left[e^{\theta \|L_{t/2}(Z) + L_{[t/2,t]}(Z)\|_q^2} \middle| Z_t^{x,x}(t/2) = y \right] \\ &\leq \mathbf{E}_t^{x,x} \left[e^{2\theta (\|L_{t/2}(Z)\|_q^2 + \|L_{[t/2,t]}(Z)\|_q^2)} \middle| Z_t^{x,x}(t/2) = y \right] \\ &= \mathbf{E}_{t/2}^{x,y} \left[e^{2\theta \|L_{t/2}(Z)\|_q^2} \right]^2 \leq \mathbf{E}_{t/2}^{x,y} \left[e^{4\theta \|L_{t/2}(Z)\|_q^2} \right], \end{aligned}$$

where the inequality on the second line follows from a combination the triangle inequality and $(z + \bar{z})^2 \leq 2(z^2 + \bar{z}^2)$, the equality on the third line follows from independence and invariance of local time under time reversal, and the inequality on the third line follows from Jensen's inequality.

With $\mathfrak{s}_t(Z)$ as in (5.15), similarly to (5.20) we then have the upper bound

$$\mathbf{E}_t^{x,x} \left[e^{\theta \|L_t(Z)\|_q^2} \right] \leq \mathfrak{s}_t(Z) \mathbf{E}^x \left[e^{4\theta \|L_{t/2}(Z)\|_q^2} \right]$$

for every $t > 0$; whence the present result readily follows from Lemma 5.10. \square

5.5 Compactness properties of deterministic kernels

We now conclude the proofs of our technical results with some estimates regarding the integrability/compactness of the deterministic kernels (5.1). In this section and several others, to alleviate notation, we introduce the following shorthand.

Notation 5.12. *For every $t > 0$, we define the path functional*

$$\mathfrak{A}_t(Z) := \begin{cases} -\langle L_t(B), V \rangle & \text{(Case 1)} \\ -\langle L_t(X), V \rangle + \bar{\alpha} \mathfrak{L}_t^0(X) & \text{(Case 2)} \\ -\langle L_t(Y), V \rangle + \bar{\alpha} \mathfrak{L}_t^0(Y) + \bar{\beta} \mathfrak{L}_t^b(Y) & \text{(Case 3)} \end{cases} \quad (5.24)$$

Lemma 5.13. *For every $p \geq 1$ and $t > 0$,*

$$\int_I \Pi_Z(t; x, x) \mathbf{E}_t^{x,x} \left[e^{p \mathfrak{A}_t(Z)} \right]^{1/p} dx < \infty.$$

Proof. Let us begin with Case 1. By Assumption 2.3, for every $c_1 > 0$, there exists $c_2 > 0$ large enough so that $V(x) \geq c_1 \log(1 + |x|) - c_2$ for every $x \in \mathbb{R}$. Therefore, we have

$$\begin{aligned} \Pi_Z(t; x, x) \mathbf{E}_t^{x,x} \left[e^{p\mathfrak{A}_t(B)} \right]^{1/p} &\leq \frac{e^{c_2 t}}{\sqrt{2\pi t}} \mathbf{E}_t^{x,x} \left[\exp \left(-pc_1 \int_0^t \log(1 + |B(s)|) ds \right) \right]^{1/p} \\ &= \frac{e^{c_2 t}}{\sqrt{2\pi t}} \mathbf{E}_t^{0,0} \left[\exp \left(-pc_1 \int_0^t \log(1 + |x + B(s)|) ds \right) \right]^{1/p}. \end{aligned}$$

By using the inequalities

$$\log(1 + |x + z|) \geq \log(1 + |x|) - \log(1 + |z|) \geq \log(1 + |x|) - |z|, \quad (5.25)$$

which are valid for all $z \in \mathbb{R}$, we get the further upper bound

$$\frac{e^{c_2 t - c_1 t \log(1+|x|)}}{\sqrt{2\pi t}} \mathbf{E}_t^{0,0} \left[\exp \left(pc_1 \int_0^t |B(s)| ds \right) \right]^{1/p}.$$

On the one hand, a Brownian scaling implies that

$$\begin{aligned} \mathbf{E}_t^{0,0} \left[\exp \left(pc_1 \int_0^t |B(s)| ds \right) \right] \\ = \mathbf{E}_1^{0,0} \left[\exp \left(t^{3/2} pc_1 \int_0^1 |B(s)| ds \right) \right] \leq \mathbf{E} \left[\exp \left(t^{3/2} pc_1 \mathcal{S} \right) \right], \quad (5.26) \end{aligned}$$

where $\mathcal{S} = \sup_{s \in [0,1]} |B_1^{0,0}(s)|$. Note that $s \mapsto |B_1^{0,0}(s)|$ is a Bessel bridge of dimension one (see, for instance, [40, Chapter XI]). Consequently, we know that (5.26) is finite for any $t, p, c_1 > 0$ thanks to the tail asymptotic for \mathcal{S} in [23, Remark 3.1] (the Bessel bridge is denoted by ϱ in that paper). On the other hand, for any $t > 0$, we can choose $c_1 > 0$ large enough so that

$$\int_{\mathbb{R}} e^{-c_1 t \log(1+|x|)} dx = \int_{\mathbb{R}} (1 + |x|)^{-c_1 t} dx < \infty,$$

concluding the proof in Case 1.

For Case 2, by Hölder's inequality, we have that

$$\begin{aligned} \Pi_X(t; x, x) \mathbf{E}_t^{x,x} \left[e^{p\mathfrak{A}_t(X)} \right]^{1/p} \\ \leq \Pi_X(t; x, x) \mathbf{E}_t^{x,x} \left[e^{-\langle L_t(X), 2pV \rangle} \right]^{1/2p} \sup_{x \in (0, \infty)} \mathbf{E}_t^{x,x} \left[e^{2p\bar{\alpha}\mathfrak{L}_t^0(X)} \right]^{1/2p}. \end{aligned}$$

The supremum of exponential moments of local time can be bounded by a direct application of Lemma 5.8. Then, by (5.9), we have that

$$\int_0^\infty \Pi_X(t; x, x) \mathbf{E}_t^{x,x} \left[e^{-\langle L_t(X), 2pV \rangle} \right]^{1/2p} dx \leq \frac{2\sqrt{2}}{\sqrt{\pi t}} \int_0^\infty \mathbf{E}_t^{x,x} \left[e^{-\langle L_t(|B|), 2pV \rangle} \right]^{1/2p} dx.$$

This term can be controlled in the same way as Case 1.

For Case 3, since $I = (0, b)$ is finite and $V \geq 0$ (hence $e^{-\langle L_t(Y), pV \rangle} \leq 1$),

$$\begin{aligned} \int_0^b \Pi_Y(t; x, x) \mathbf{E}_t^{x,x} \left[e^{-\langle L_t(Y), pV \rangle + p\bar{\alpha}\mathfrak{L}_t^0(Y) + p\bar{\beta}\mathfrak{L}_t^b(Y)} \right]^{1/p} dx \\ \leq b \left(\sup_{x \in (0,b)} \Pi_Y(t; x, x) \right) \left(\sup_{x \in (0,b)} \mathbf{E}_t^{x,x} \left[e^{p\bar{\alpha}\mathfrak{L}_t^0(Y) + p\bar{\beta}\mathfrak{L}_t^b(Y)} \right]^{1/p} \right). \end{aligned}$$

This is finite by Lemmas 5.7 and 5.8. \square

5.6 Proof of Proposition 3.9

Suppose we can prove that for every $\varepsilon > 0$, the potential $V + \Xi'_\varepsilon$ satisfies Assumption 2.3 with probability one (up to a random additive constant, making it nonnegative). Then, by Proposition 3.2, the \hat{H}_ε are self-adjoint with compact resolvent. Moreover, $\hat{K}_\varepsilon(t) = e^{-t\hat{H}_\varepsilon}$ and the properties (3.5)–(3.7) then follow from Corollary 5.5, and the fact that $e^{-t\hat{H}}$ is trace class follows from Lemma 5.13 in the case $p = 1$. Thus, it only remains to prove the following:

Lemma 5.14. *For every $\varepsilon > 0$, there exists a random $c = c(\varepsilon) \geq 0$ such that the potential $V + \Xi'_\varepsilon + c$ satisfies Assumption 2.3 with probability one.*

Proof. Since Ξ'_ε is continuous, $V + \Xi'_\varepsilon$ is locally integrable on I 's closure. Moreover, if we prove that $|\Xi'_\varepsilon(x)| \ll \log|x|$ as $x \rightarrow \pm\infty$, then the continuity of Ξ'_ε also implies that $V + \Xi'_\varepsilon$ is bounded below and is such that

$$\lim_{x \rightarrow \pm\infty} \frac{V(x) + \Xi'_\varepsilon(x)}{\log|x|} = \infty;$$

hence we can take

$$c(\varepsilon) := \max \left\{ 0, -\inf_{x \in I} (V(x) + \Xi'_\varepsilon(x)) \right\} < \infty.$$

The fact that $|\Xi'_\varepsilon(x)| \ll \log|x|$ follows from Corollary B.2, since Ξ'_ε is stationary. \square

5.7 Proof of Proposition 3.10

Our proof of this result is similar to [3, Section 2] and [36, Section 5], save for the fact that we use smooth approximations instead of discrete ones. We provide the argument in full. Following [3, Fact 2.2] and [36, Fact 2.2], we begin by recording some compactness properties of $\|\cdot\|_*$.

Lemma 5.15. *If $(f_n)_{n \in \mathbb{N}} \subset D(\mathcal{E})$ is such that $\sup_n \|f_n\|_* < \infty$, then there exists $f \in D(\mathcal{E})$ and a subsequence $(n_i)_{i \in \mathbb{N}}$ along which*

1. $\lim_{i \rightarrow \infty} \|f_{n_i} - f\|_2 = 0$;
2. $\lim_{i \rightarrow \infty} \langle g, f'_{n_i} \rangle = \langle g, f' \rangle$ for every $g \in L^2$;
3. $\lim_{i \rightarrow \infty} f_{n_i} = f$ uniformly on compact sets; and
4. $\lim_{i \rightarrow \infty} \langle g, f_{n_i} \rangle_* = \langle g, f \rangle_*$ for every $g \in D(\mathcal{E})$.

Proof. (2) and (4) follow from the Banach-Alaoglu theorem. Next, by combining Lemma 3.1 with the estimate

$$|f_n(x) - f_n(y)| \leq \int_x^y |f'_n(x)| \, dx \leq \|f'_n\|_2 \sqrt{|y - x|}, \quad (5.27)$$

we may extract a further subsequence along which (3) holds by the Arzelà-Ascoli theorem. Finally, in Case 3, (3) immediately implies (1), whereas in Cases 1 and 2, by combining (3) with the Vitali convergence theorem, to prove (1) it suffices to show that for every $\varepsilon > 0$, there exists $K > 0$ large enough and $\delta > 0$ small enough so that

$$\int_{I \setminus [-K, K]} f_{n_i}(x)^2 \, dx \leq \varepsilon^2 \quad \text{and} \quad \sup_{x \in I} \int_{I \cap [x-\delta, x+\delta]} f_{n_i}(x)^2 \, dx \leq \varepsilon^2.$$

The first of these conditions follows from the fact that $\sup_n \|V^{1/2} f_n\|_2 < \infty$ and that $V(x) \gg \log|x|$; the second follows from the uniform bound in Lemma 3.1. \square

Remark 5.16. It is easy to see by definition of $\langle \cdot, \cdot \rangle_*$ that if $f_n \rightarrow f$ in the sense of Lemma 5.15 (1)–(4), then for every $g \in \text{FC}$, one has

$$\lim_{n \rightarrow \infty} \mathcal{E}(g, f_n) = \mathcal{E}(g, f).$$

We can reformulate Proposition 3.4 in terms of $\|\cdot\|_*$ thusly:

Lemma 5.17. *There exist finite random variables $c_1, c_2, c_3 > 0$ such that*

$$c_1 \|f\|_*^2 - c_2 \|f\|_2^2 \leq \hat{\mathcal{E}}(f, f) \leq c_3 \|f\|_*^2, \quad f \in D(\mathcal{E}).$$

We also have the following finite ε variant:

Lemma 5.18. *There exist finite random variables $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ such that for every $\varepsilon \in (0, 1]$,*

$$\tilde{c}_1 \|f\|_*^2 - \tilde{c}_2 \|f\|_2^2 \leq \hat{\mathcal{E}}_\varepsilon(f, f) \leq \tilde{c}_3 \|f\|_*^2, \quad f \in D(\mathcal{E}).$$

Proof. By repeating the proof of Proposition 3.4, we only need to prove that for every $\theta > 0$, there exists $c > 0$ large enough so that

$$|\langle f^2, \Xi'_\varepsilon \rangle| \leq \theta \left(\frac{1}{2} \|f'\|_2^2 + \|V^{1/2} f\|_2^2 \right) + c \|f\|_2^2$$

for every $\varepsilon \in (0, 1]$ and $f \in C_0^\infty$. Let us define

$$\tilde{\Xi}_\varepsilon(x) := \int_x^{x+1} \Xi_\varepsilon(y) dy.$$

Arguing as in the proof of Proposition 3.4, it suffices to show that

$$\sup_{\varepsilon \in (0, 1]} \sup_{0 \leq x \leq b} |\tilde{\Xi}_\varepsilon(x)| < \infty$$

almost surely and that there exist finite random variables $C > 0$ and $u > 1$ independent of $\varepsilon \in (0, 1]$ such that for every $x \in \mathbb{R}$,

$$\sup_{y \in [0, 1]} |\tilde{\Xi}_\varepsilon(x+y) - \tilde{\Xi}_\varepsilon(x)| \leq C \sqrt{\log(u + |x|)}.$$

Let $K > 0$ be such that $\text{supp}(\varrho) \subset [-K, K]$ so that $\text{supp}(\varrho_\varepsilon) \subset [-K, K]$ for all $\varepsilon \in (0, 1]$. On the one hand, since the ϱ_ε integrate to one,

$$\sup_{\varepsilon \in (0, 1]} \sup_{0 \leq x \leq b} \left| \int_{\mathbb{R}} \Xi(x-y) \varrho_\varepsilon(y) dy \right| \leq \sup_{-K \leq x \leq b+K} |\Xi(x)| < \infty.$$

On the other hand, by Corollary B.2 and Remark B.3, for every $x \in I$ and $\varepsilon \in (0, 1]$, one has

$$\begin{aligned} \sup_{y \in [0, 1]} & \left| \int_{\mathbb{R}} (\Xi(x+y-z) - \Xi(x-z)) \varrho_\varepsilon(z) dz \right| \\ & \leq \sup_{w \in [x-K, x+K]} \sup_{y \in [0, 1]} |\Xi(w+y) - \Xi(w)| \leq \sup_{w \in [x-K, x+K]} C \sqrt{\log(2 + |w|)}, \end{aligned}$$

which yields the desired estimate. \square

Remark 5.19. We see from Lemma 5.18 that the forms $(f, g) \mapsto \langle fg, \Xi'_\varepsilon \rangle$ are uniformly form-bounded in $\varepsilon \in (0, 1]$ by \mathcal{E} , in the sense that there exists a $0 < \theta < 1$ and a random $c > 0$ independent of ε such that

$$|\langle f^2, \Xi'_\varepsilon \rangle| \leq \theta \mathcal{E}(f, f) + c \|f\|_2^2, \quad f \in D(\mathcal{E}), \varepsilon \in (0, 1].$$

Among other things, this implies by the variational principle (see, for example, the estimate in [38, Theorem XIII.68]) that for every $k \in \mathbb{N}$ and $\varepsilon \in (0, 1]$, one has

$$(1 - \theta) \lambda_k(H) - c \leq \lambda_k(\hat{H}_\varepsilon) \leq (1 + \theta) \lambda_k(H) + c. \quad (5.28)$$

Finally, we need the following convergence result.

Lemma 5.20. *Almost surely, for every $f, g \in \text{FC}$, it holds that*

$$\lim_{\varepsilon \rightarrow 0} \langle fg, \Xi'_\varepsilon \rangle = \xi(fg). \quad (5.29)$$

Moreover, if $(\varepsilon_n)_{n \in \mathbb{N}} \subset (0, 1]$ converges to zero, $\sup_n \|f_n\|_* < \infty$, and $f_n \rightarrow f$ in the sense of Lemma 5.15 (1)–(4), then almost surely,

$$\lim_{n \rightarrow \infty} \langle f_n g, \Xi'_{\varepsilon_n} \rangle = \xi(fg) \quad (5.30)$$

for every $g \in \text{FC}$.

Proof. Clearly $\Xi_\varepsilon \rightarrow \Xi$ pointwise, hence for (5.29) it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} ((f'g + fg') * \varrho_\varepsilon)(x) \Xi(x) dx = \langle f'g + fg', \Xi \rangle.$$

Since $f'g + fg'$ is compactly supported and Ξ is continuous (hence bounded on compacts), the result follows by dominated convergence.

Let us now prove (5.30). Using again the fact that g and g' are compactly supported, we know that there exists a compact $K \subset \mathbb{R}$ (in Case 3 we may simply take $K = [0, b]$) such that

$$\langle f'_n g + f_n g', \Xi * \varrho_{\varepsilon_n} \rangle = \int_K (f'_n(x)g(x) + f_n(x)g'(x))(\Xi * \varrho_{\varepsilon_n})(x) dx$$

and similarly with f_n replaced by f and $\Xi * \varrho_{\varepsilon_n}$ replaced by Ξ . Given that, as $n \rightarrow \infty$, $\Xi * \varrho_{\varepsilon_n} \mathbf{1}_K \rightarrow \Xi \mathbf{1}_K$ in L^2 , $f'_n g + f_n g' \rightarrow f'g + fg'$ weakly in L^2 , and $\sup_n \|f'_n g + f_n g'\|_2 < \infty$, we conclude that

$$\lim_{n \rightarrow \infty} \langle f'_n g + f_n g', \Xi * \varrho_{\varepsilon_n} \rangle = \langle f'g + fg', \Xi \rangle.$$

Hence (5.30) holds. \square

We finally have all the necessary ingredients to prove the spectral convergence. We first prove that there exists a subsequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that

$$\liminf_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \geq \lambda_k(\hat{H}) \quad (5.31)$$

for every $k \in \mathbb{N}$.

Remark 5.21. For the sake of readability, we henceforth denote any subsequence and further subsequences of $(\varepsilon_n)_{n \in \mathbb{N}}$ as $(\varepsilon_n)_{n \in \mathbb{N}}$ itself.

According to (5.28), the $\lambda_k(\hat{H}_\varepsilon)$ are uniformly bounded, and thus it follows from the Bolzano-Weierstrass theorem that, along a subsequence ε_n , the limits

$$\lim_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) =: l_k$$

exist and are finite for every $k \in \mathbb{N}$, where $-\infty < l_1 \leq l_2 \leq \dots$. Since the eigenvalues are bounded, it follows from Lemma 5.18 that the eigenfunctions $\psi_k(\hat{H}_\varepsilon)$ are bounded in $\|\cdot\|_*$ -norm uniformly in $\varepsilon \in [0, 1)$, and thus there exist functions f_1, f_2, \dots and a further subsequence along which $\psi_k(\hat{H}_{\varepsilon_n}) \rightarrow f_k$ for every k in the sense of Lemma 5.15 (1)–(4). By combining Remark 5.16 and (5.30), this means that

$$l_k \langle g, f_k \rangle = \lim_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \langle g, \psi_k(\hat{H}_{\varepsilon_n}) \rangle = \lim_{n \rightarrow \infty} \hat{\mathcal{E}}_{\varepsilon_n}(g, \psi_k(\hat{H}_{\varepsilon_n})) = \hat{\mathcal{E}}(g, f_k)$$

for all $k \in \mathbb{N}$ and $g \in \text{FC}$. That is, $(l_k, f_k)_{k \in \mathbb{N}}$ consists of eigenvalue-eigenfunction pairs of \hat{H} , though these pairs may not exhaust the full spectrum. Since the l_k are arranged in increasing order, this implies that $l_k \geq \lambda_k(\hat{H})$ for every $k \in \mathbb{N}$, which proves (5.31).

We now prove that we can take \hat{H}' s eigenfunctions in such a way that, along a further subsequence,

$$\limsup_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \lambda_k(\hat{H}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\psi_k(\hat{H}_{\varepsilon_n}) - \psi_k(\hat{H})\|_2 = 0 \quad (5.32)$$

for every $k \in \mathbb{N}$. We proceed by induction. Suppose that (5.32) holds up to $k - 1$ (if $k = 1$ then we consider the base case). Let ψ be an eigenfunction of $\lambda_k(\hat{H})$ orthogonal to $\psi_1(\hat{H}), \dots, \psi_{k-1}(\hat{H})$, and for every $\theta > 0$, let $\varphi_\theta \in \text{FC}$ be such that $\|\varphi_\theta - \psi\|_* < \theta$. Let us define the projections

$$\pi_{\varepsilon_n}(\varphi_\theta) := \varphi_\theta - \sum_{\ell=1}^{k-1} \langle \psi_\ell(\hat{H}_{\varepsilon_n}), \varphi_\theta \rangle \psi_\ell(\hat{H}_{\varepsilon_n})$$

of φ_θ onto the orthogonal of $\psi_1(\hat{H}_{\varepsilon_n}), \dots, \psi_{k-1}(\hat{H}_{\varepsilon_n})$ (if $k = 1$, then we simply have $\pi_{\varepsilon_n}(\varphi_\theta) = \varphi_\theta$). Then, by the variational principle, for any $\theta > 0$,

$$\limsup_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \limsup_{n \rightarrow \infty} \frac{\hat{\mathcal{E}}_{\varepsilon_n}(\pi_{\varepsilon_n}(\varphi_\theta), \pi_{\varepsilon_n}(\varphi_\theta))}{\|\pi_{\varepsilon_n}(\varphi_\theta)\|_2^2}. \quad (5.33)$$

Given that $\|\psi_\ell(\hat{H}_{\varepsilon_n}) - \psi_\ell(\hat{H})\|_2 \rightarrow 0$ for every $\ell \leq k - 1$, one has

$$\lim_{\theta \rightarrow 0} \lim_{n \rightarrow \infty} \pi_{\varepsilon_n}(\varphi_\theta) = \psi$$

in L^2 . Moreover, the convergence of the $\lambda_\ell(\hat{H}_{\varepsilon_n})$ and Lemma 5.18 imply that the $(\psi_\ell(\hat{H}_{\varepsilon_n}))_{\ell=1, \dots, k-1}$ are uniformly bounded in $\|\cdot\|_*$ -norm, and thus

$$\lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} \left\| \sum_{\ell=1}^{k-1} \langle \psi_\ell(\hat{H}_{\varepsilon_n}), \varphi_\theta \rangle \psi_\ell(\hat{H}_{\varepsilon_n}) \right\|_* = 0.$$

We recall that, by Lemma 5.18, the maps $f \mapsto \hat{\mathcal{E}}_\varepsilon(f, f)$ are continuous with respect to $\|\cdot\|_*$ uniformly in $\varepsilon \in (0, 1]$. Consequently,

$$\limsup_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \limsup_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\hat{\mathcal{E}}_{\varepsilon_n}(\pi_{\varepsilon_n}(\varphi_\theta), \pi_{\varepsilon_n}(\varphi_\theta))}{\|\pi_{\varepsilon_n}(\varphi_\theta)\|_2^2},$$

since (5.33) holds for any $\theta > 0$. Then, if we use (5.29) to compute the supremum limit in n , followed by Lemma 5.17 for the limit in θ (recall that $\|\varphi_\theta - \psi\|_* \rightarrow 0$ as $\theta \rightarrow 0$), we conclude that

$$\limsup_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \leq \hat{\mathcal{E}}(\psi, \psi) = \lambda_k(\hat{H}).$$

Since $\liminf_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \geq \lambda_k(\hat{H})$ by the previous step, we now know that $\lambda_k(\hat{H}_{\varepsilon_n}) \rightarrow \lambda_k(\hat{H})$ as $n \rightarrow \infty$. Thus, according to Lemma 5.18, the eigenfunctions $(\psi_k(\hat{H}_{\varepsilon_n}))_{n \in \mathbb{N}}$ are uniformly bounded in $\|\cdot\|_*$ -norm. Thus, there exists $\bar{\psi} \in D(\mathcal{E})$ such that $\psi_k(\hat{H}_{\varepsilon_n}) \rightarrow \bar{\psi}$ in the sense of Lemma 5.15 along a further subsequence. Combining this with Remark 5.16 and (5.30), and the fact that $\lambda_k(\hat{H}_{\varepsilon_n}) \rightarrow \lambda_k(\hat{H})$, we then also have

$$\hat{\mathcal{E}}(g, \bar{\psi}) = \lim_{n \rightarrow \infty} \hat{\mathcal{E}}_{\varepsilon_n}(g, \psi_k(\hat{H}_{\varepsilon_n})) = \lim_{n \rightarrow \infty} \lambda_k(\hat{H}_{\varepsilon_n}) \langle g, \psi_k(\hat{H}_{\varepsilon_n}) \rangle = \lambda_k(\hat{H}) \langle g, \bar{\psi} \rangle$$

for all $g \in \text{FC}$. In particular, $\bar{\psi}$ must be an eigenfunction for $\lambda_k(\hat{H})$, which is orthogonal to $\psi_1(\hat{H}), \dots, \psi_{k-1}(\hat{H})$. Thus we may take $\psi_k(\hat{H}) := \bar{\psi}$, concluding the proof of the proposition since Lemma 5.15 includes L^2 convergence.

5.8 Proof of Proposition 3.11 Part 1

We begin by proving (3.9).

5.8.1 Step 1. Computation of expected L^2 norm

Our first step in the proof of (3.9) is to obtain a formula for $\mathbf{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|_2^2]$ that is amenable to analysis, namely:

$$\begin{aligned} & \mathbf{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|_2^2] \\ &= \int_I \Pi_Z(2t; x, x) \mathbf{E}_{2t}^{x,x} \left[e^{\mathfrak{A}_{2t}(Z)} \left(e^{\frac{1}{2}\|L_{2t}(Z)*\varrho_\varepsilon\|_\gamma^2} - 2e^{\frac{1}{2}\|L_t(Z)*\varrho_\varepsilon+L_{[t,2t]}(Z)\|_\gamma^2} + e^{\frac{1}{2}\|L_{2t}(Z)\|_\gamma^2} \right) \right] dx, \end{aligned} \quad (5.34)$$

where we recall the notation of $\mathfrak{A}_t(Z)$ from (5.24). We now prove (5.34).

By Fubini's theorem,

$$\begin{aligned} & \mathbf{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|_2^2] \\ &= \int_{I^2} \mathbf{E}[\hat{K}_\varepsilon(t; x, y)^2] - 2\mathbf{E}[\hat{K}_\varepsilon(t; x, y)\hat{K}(t; x, y)] + \mathbf{E}[\hat{K}(t; x, y)^2] dydx \\ &= \int_{I^2} \Pi_Z(t; x, y)^2 \mathbf{E} \left[e^{\mathfrak{A}_t(Z_t^{1;x,y})+\mathfrak{A}_t(Z_t^{2;x,y})} \left(\mathbf{E}_\Xi \left[e^{-\langle L_t(Z_t^{1;x,y})+L_t(Z_t^{2;x,y}), \Xi'_\varepsilon \rangle} \right] \right. \right. \\ & \quad \left. \left. - 2\mathbf{E}_\Xi \left[e^{-\langle L_t(Z_t^{1;x,y}), \Xi'_\varepsilon \rangle - \xi(L_t(Z_t^{2;x,y}))} \right] + \mathbf{E}_\Xi \left[e^{-\xi(L_t(Z_t^{1;x,y})) - \xi(L_t(Z_t^{2;x,y}))} \right] \right) \right] dydx, \end{aligned}$$

where $Z_t^{i;x,y}$ ($i = 1, 2$) are i.i.d. copies of $Z_t^{x,y}$ that are independent of Ξ , and \mathbf{E}_Ξ denotes the expected value with respect to Ξ conditional on $Z_t^{i;x,y}$. For every $f_1, f_2 \in \text{PC}_c$, the sum $\xi(f_1) + \xi(f_2)$ is Gaussian with mean zero and variance $\sum_{i,j=1}^2 \langle f_i, f_j \rangle_\gamma = \|f_1 + f_2\|_\gamma^2$. Thanks to (3.4), a straightforward Gaussian moment generating function computation then yields that $\mathbf{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|_2^2]$ is equal to

$$\begin{aligned} & \int_{I^2} \Pi_Z(t; x, y)^2 \mathbf{E} \left[e^{\mathfrak{A}_t(Z_t^{1;x,y})+\mathfrak{A}_t(Z_t^{2;x,y})} \left(e^{\frac{1}{2}\|L_t(Z_t^{1;x,y})*\varrho_\varepsilon+L_t(Z_t^{2;x,y})*\varrho_\varepsilon\|_\gamma^2} \right. \right. \\ & \quad \left. \left. - 2e^{\frac{1}{2}\|L_t(Z_t^{1;x,y})*\varrho_\varepsilon+L_t(Z_t^{2;x,y})\|_\gamma^2} + e^{\frac{1}{2}\|L_t(Z_t^{1;x,y})+L_t(Z_t^{2;x,y})\|_\gamma^2} \right) \right] dydx. \end{aligned}$$

By symmetry of $\Pi_Z(t)$, the time-reversed process $s \mapsto Z_t^{2;x,y}(t-s)$ is equal in distribution to $Z_t^{2;y,x}$; since local time is invariant under time-reversal, $\mathbf{E}[\|\hat{K}_\varepsilon(t) - \hat{K}(t)\|_2^2]$ now equals

$$\begin{aligned} & \int_{I^2} \Pi_Z(t; x, y) \Pi_Z(t; y, x) \mathbf{E} \left[e^{\mathfrak{A}_t(Z_t^{1;x,y})+\mathfrak{A}_t(Z_t^{2;y,x})} \left(e^{\frac{1}{2}\|L_t(Z_t^{1;x,y})*\varrho_\varepsilon+L_t(Z_t^{2;y,x})*\varrho_\varepsilon\|_\gamma^2} \right. \right. \\ & \quad \left. \left. - 2e^{\frac{1}{2}\|L_t(Z_t^{1;x,y})*\varrho_\varepsilon+L_t(Z_t^{2;y,x})\|_\gamma^2} + e^{\frac{1}{2}\|L_t(Z_t^{1;x,y})+L_t(Z_t^{2;y,x})\|_\gamma^2} \right) \right] dydx. \quad (5.35) \end{aligned}$$

As noted in the proof of Lemma 5.8, for every $x, y \in I$ and $t > 0$, if we condition the path $Z_{2t}^{x,x}$ on $Z_{2t}^{x,x}(t) = y$, then the path segments

$$(Z_{2t}^{x,x}(s) : 0 \leq s \leq t) \quad \text{and} \quad (Z_{2t}^{x,x}(t+s) : 0 \leq s \leq t)$$

have the same joint distribution as $Z_t^{1;x,y}$ and $Z_t^{2;y,x}$. Moreover, $Z_{2t}^{x,x}(t)$ has density

$$y \mapsto \frac{\Pi_Z(t; x, y) \Pi_Z(t; y, x)}{\Pi_Z(2t; x, x)}$$

by the Doob h -transform. Given that the functions $f \mapsto \langle f, V \rangle$, $f \mapsto \bar{\alpha}f$ and $f \mapsto \bar{\beta}f$ are all linear in f , and that local time is additive in the sense that

$$L_{[u,v]}(Z) + L_{[v,w]}(Z) = L_{[u,w]}(Z) \quad \text{and} \quad \mathfrak{L}_{[u,v]}^c(Z) + \mathfrak{L}_{[v,w]}^c(Z) = \mathfrak{L}_{[u,w]}^c(Z)$$

for all $0 < u < v < w$, (5.34) is then a consequence of applying Fubini's theorem to (5.35) with the rearrangement

$$\Pi_Z(t; x, y)\Pi_Z(t; y, x) = \Pi_Z(2t; x, x) \frac{\Pi_Z(t; x, y)\Pi_Z(t; y, x)}{\Pi_Z(2t; x, x)}.$$

Indeed, we note for instance that

$$\begin{aligned} & \int_I \Pi_Z(2t; x, x) \left(\int_I \frac{\Pi_Z(t; x, y)\Pi_Z(t; y, x)}{\Pi_Z(2t; x, x)} \right. \\ & \quad \cdot \mathbf{E} \left[e^{\mathfrak{A}_t(Z_t^{1;x,y}) + \mathfrak{A}_t(Z_t^{2;y,x}) + \frac{1}{2} \|L_t(Z_t^{1;x,y}) * \varrho_\varepsilon + L_t(Z_t^{2;y,x}) * \varrho_\varepsilon\|_\gamma^2} \right] dy \Big) dx \\ &= \int_I \Pi_Z(2t; x, x) \left(\int_I \mathbf{E} \left[e^{\mathfrak{A}_{2t}(Z_{2t}^{x,x}) + \frac{1}{2} \|L_{2t}(Z_{2t}^{x,x}) * \varrho_\varepsilon\|_\gamma^2} \middle| Z_{2t}^{x,x}(t) = y \right] \mathbf{P}[Z_{2t}^{x,x}(t) \in dy] dy \right) dx \\ &= \int_I \Pi_Z(2t; x, x) \mathbf{E} \left[e^{\mathfrak{A}_{2t}(Z_{2t}^{x,x}) + \frac{1}{2} \|L_{2t}(Z_{2t}^{x,x}) * \varrho_\varepsilon\|_\gamma^2} \right] dx; \end{aligned}$$

a similar argument applied to the terms on the second line of (5.35) then yields (5.34).

5.8.2 Step 2. Convergence inside expectation

With (5.34) in hand, our second step to prove (3.9) is to show that, for every $x \in I$, we have the almost sure limit

$$\lim_{\varepsilon \rightarrow 0} e^{\frac{1}{2} \|L_t(Z_{2t}^{x,x}) * \varrho_\varepsilon\|_\gamma^2} - 2e^{\frac{1}{2} \|L_t(Z_{2t}^{x,x}) * \varrho_\varepsilon + L_{[t,2t]}(Z_{2t}^{x,x})\|_\gamma^2} + e^{\frac{1}{2} \|L_{2t}(Z_{2t}^{x,x})\|_\gamma^2} = 0. \quad (5.36)$$

This is a simple consequence of (2.12) coupled with the fact that if $f \in L^q$ for some $q \geq 1$, then $\|f * \varrho_\varepsilon - f\|_q \rightarrow 0$ as $\varepsilon \rightarrow 0$.

5.8.3 Step 3. Convergence inside integral

Our next step is to prove that for every $x \in I$, we have the limit in expectation

$$\lim_{\varepsilon \rightarrow 0} \mathbf{E}_{2t}^{x,x} \left[e^{\mathfrak{A}_{2t}(Z)} \left(e^{\frac{1}{2} \|L_{2t}(Z) * \varrho_\varepsilon\|_\gamma^2} - 2e^{\frac{1}{2} \|L_t(Z) * \varrho_\varepsilon + L_{[t,2t]}(Z)\|_\gamma^2} + e^{\frac{1}{2} \|L_{2t}(Z)\|_\gamma^2} \right) \right] = 0. \quad (5.37)$$

Thanks to (5.36), for this it suffices to prove that the prelimit variables in (5.37) are uniformly integrable in $\varepsilon > 0$, which itself can be reduced to the claim that

$$\sup_{\varepsilon > 0} \mathbf{E}_{2t}^{x,x} \left[e^{2\mathfrak{A}_{2t}(Z)} \left(e^{\frac{1}{2} \|L_{2t}(Z) * \varrho_\varepsilon\|_\gamma^2} - 2e^{\frac{1}{2} \|L_t(Z) * \varrho_\varepsilon + L_{[t,2t]}(Z)\|_\gamma^2} + e^{\frac{1}{2} \|L_{2t}(Z)\|_\gamma^2} \right)^2 \right] < \infty.$$

By combining Hölder's inequality with $(z - 2\bar{z} + \tilde{z})^2 \leq 16(z^2 + \bar{z}^2 + \tilde{z}^2)$, it is enough to prove that

$$\mathbf{E}_{2t}^{x,x} \left[e^{4\mathfrak{A}_{2t}(Z)} \right] < \infty \quad (5.38)$$

and

$$\begin{aligned} \sup_{\varepsilon > 0} \mathbf{E}_{2t}^{x,x} \left[e^{\|L_{2t}(Z) * \varrho_\varepsilon\|_\gamma^2} \right], \quad \sup_{\varepsilon > 0} \mathbf{E}_{2t}^{x,x} \left[e^{\|L_t(Z) * \varrho_\varepsilon + L_{[t,2t]}(Z)\|_\gamma^2} \right], \\ \mathbf{E}_{2t}^{x,x} \left[e^{\|L_{2t}(Z)\|_\gamma^2} \right] < \infty. \end{aligned} \quad (5.39)$$

By combining the assumption $V \geq 0$ (hence $e^{-4\langle L_{2t}(Z), V \rangle} \leq 1$) and Lemma 5.8, we immediately obtain (5.38). Next, it follows from (2.12) that

$$\mathbf{E}_{2t}^{x,x} \left[e^{\|L_{2t}(Z)\|_\gamma^2} \right] \leq \mathbf{E}_{2t}^{x,x} \left[e^{c_\gamma \sum_{i=1}^\ell \|L_{2t}(Z)\|_{q_i}^2} \right].$$

This is finite by Lemma 5.11 since $1 \leq q_i \leq 2$ for all $1 \leq i \leq \ell$. According to Young's convolution inequality, the fact that the ϱ_ε integrate to one implies that $\|f * \varrho_\varepsilon\|_q \leq \|f\|_q \|\varrho_\varepsilon\|_1 \leq \|f\|_q$. Thus, it follows from (2.12) that

$$\sup_{\varepsilon > 0} \mathbf{E}_{2t}^{x,x} \left[e^{\|L_{2t}(Z) * \varrho_\varepsilon\|_\gamma^2} \right] \leq \mathbf{E}_{2t}^{x,x} \left[e^{c_\gamma \sum_{i=1}^\ell \|L_{2t}(Z)\|_{q_i}^2} \right] < \infty.$$

Since $\|\cdot\|_\gamma$ is a seminorm, it satisfies the triangle inequality, and thus

$$\|L_t(Z_{2t}^{x,x}) * \varrho_\varepsilon + L_{[t,2t]}(Z_{2t}^{x,x})\|_\gamma^2 \leq 2\|L_t(Z_{2t}^{x,x}) * \varrho_\varepsilon\|_\gamma^2 + 2\|L_{[t,2t]}(Z_{2t}^{x,x})\|_\gamma^2.$$

Given that $L_t(Z_{2t}^{x,x})$ and $L_{[t,2t]}(Z_{2t}^{x,x})$ are both smaller than $L_{2t}(Z_{2t}^{x,x})$, applying once again (2.12) and Young's inequality yields

$$\sup_{\varepsilon > 0} \mathbf{E}_{2t}^{x,x} \left[e^{\|L_t(Z) * \varrho_\varepsilon + L_{[t,2t]}(Z)\|_\gamma^2} \right] \leq \mathbf{E}_{2t}^{x,x} \left[e^{4c_\gamma \sum_{i=1}^\ell \|L_{2t}(Z)\|_{q_i}^2} \right],$$

which is finite by Lemma 5.11. We therefore conclude that (5.39) holds, and thus (5.37) as well.

5.8.4 Step 4. Convergence of the integral

Our final step in the proof of (3.9) is to show that (5.34) converges to zero. Given (5.37), by applying the dominated convergence theorem, it suffices to find an integrable function that dominates

$$\Pi_Z(2t; x, x) \mathbf{E}_{2t}^{x,x} \left[e^{\mathfrak{A}_{2t}(Z)} \left(e^{\frac{1}{2}\|L_{2t}(Z) * \varrho_\varepsilon\|_\gamma^2} - 2e^{\frac{1}{2}\|L_t(Z) * \varrho_\varepsilon + L_{[t,2t]}(Z)\|_\gamma^2} + e^{\frac{1}{2}\|L_{2t}(Z)\|_\gamma^2} \right) \right] \quad (5.40)$$

for every $\varepsilon > 0$. By Holder's inequality, this is bounded by

$$\begin{aligned} & \Pi_Z(2t; x, x) \mathbf{E}_{2t}^{x,x} \left[e^{2\mathfrak{A}_{2t}(Z)} \right]^{1/2} \\ & \cdot \sup_{\varepsilon > 0, x \in I} \mathbf{E}_{2t}^{x,x} \left[\left(e^{\frac{1}{2}\|L_{2t}(Z) * \varrho_\varepsilon\|_\gamma^2} - 2e^{\frac{1}{2}\|L_t(Z) * \varrho_\varepsilon + L_{[t,2t]}(Z)\|_\gamma^2} + e^{\frac{1}{2}\|L_{2t}(Z)\|_\gamma^2} \right)^2 \right]^{1/2}. \end{aligned} \quad (5.41)$$

uniformly in $\varepsilon > 0$. Thanks to Lemma 5.13 with $p = 2$, the first line of (5.41) is integrable. Then, by arguing in exactly the same way as in Section 5.8.3 (i.e., Young's convolution inequality, (2.12), etc.), the term on the second line of (5.41) is bounded by

$$\sup_{x \in I} C \mathbf{E}_{2t}^{x,x} \left[e^{\theta c_\gamma \sum_{i=1}^\ell \|L_{2t}(Z)\|_{q_i}^2} \right]^{1/2}$$

for some constants $C, \theta > 0$. This is finite by Lemma 5.11. We therefore conclude that (5.40) is dominated by an integrable function for all $\varepsilon > 0$; hence (3.9) holds.

Remark 5.22. Arguing as in (5.34), we have the formula

$$\mathbf{E}[\|\hat{K}(t)\|_2^2] = \int_I \Pi_Z(2t; x, x) \mathbf{E}_{2t}^{x,x} \left[e^{\mathfrak{A}_{2t}(Z) + \frac{1}{2} \|L_{2t}(Z)\|_\gamma^2} \right] dx.$$

Considering Case 1 for simplicity, it follows from (the reverse) Hölder's inequality that for every $p > 1$, the above is bounded below by

$$\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \mathbf{E}_{2t}^{x,x} \left[e^{-\langle L_{2t}(B), V/p \rangle} \right]^p \mathbf{E}_{2t}^{0,0} \left[e^{-\frac{1}{2(p-1)} \|L_{2t}^a(B)\|_\gamma^2} \right]^{-(p-1)} dx$$

for every $x \in \mathbb{R}$. If $V(x) \leq c_1 \log(1 + |x|) + c_2$ for some $c_1 > 0$ and large enough $c_2 > 0$, then an argument similar to the proof of Lemma 5.13 (using the bound $\log(1 + |z + \bar{z}|) \leq \log(1 + |z|) + |\bar{z}|$ instead of (5.25)) yields the further lower bound

$$\zeta_t \int_{\mathbb{R}} (1 + |x|)^{-c_1 t} dx$$

for some finite $\zeta_t > 0$ that only depends on t ; this blows up whenever $t \leq 1/c_1$. Thus, if we do not assume (2.3), then there is always some $t_0 > 0$ such that $\mathbf{E}[\|\hat{K}(t)\|_2^2] = \infty$ for all $t \leq t_0$. Essentially the same argument implies that $\|e^{-tH}\|_2 = \infty$ for all $t \leq t_0$ for the deterministic operator H as well.

5.9 Proof of Proposition 3.11 Part 2

We now prove (3.10). By Fubini,

$$\begin{aligned} \mathbf{E} \left[\left(\int_I \hat{K}_\varepsilon(t; x, x) - \hat{K}(t; x, x) dx \right)^2 \right] &= \int_{I^2} \mathbf{E}[\hat{K}_\varepsilon(t; x, x) \hat{K}_\varepsilon(t; y, y)] \\ &\quad - 2\mathbf{E}[\hat{K}_\varepsilon(t; x, x) \hat{K}(t; y, y)] + \mathbf{E}[\hat{K}(t; x, x) \hat{K}(t; y, y)] dxdy. \end{aligned}$$

Arguing as in the previous section, the above is seen to be equal to

$$\begin{aligned} \int_{I^2} \Pi_Z(t; x, x) \Pi(t; y, y) \mathbf{E} \left[e^{\mathfrak{A}_t(Z_t^{1;x,x}) + \mathfrak{A}_t(Z_t^{2;y,y})} \left(e^{\frac{1}{2} \|L_t(Z_t^{1;x,x}) * \varrho_\varepsilon + L_t(Z_t^{2;y,y}) * \varrho_\varepsilon\|_\gamma^2} \right. \right. \\ \left. \left. - 2e^{\frac{1}{2} \|L_t(Z_t^{1;x,x}) * \varrho_\varepsilon + L_t(Z_t^{2;y,y})\|_\gamma^2} + e^{\frac{1}{2} \|L_t(Z_t^{1;x,x}) + L_t(Z_t^{2;y,y})\|_\gamma^2} \right) \right] dxdy, \end{aligned}$$

where $Z_t^{1;x,x}$ and $Z_t^{2;y,y}$ are independent processes with respective distributions $Z_t^{x,x}$ and $Z_t^{y,y}$. At this point, essentially the same argument that we used to prove (3.9) in the previous section yields (3.10).

A Measurability of kernel

We begin by proving that, in Case 1, for every realization of Ξ as a continuous function, $(x, y) \mapsto \hat{K}(t; x, y)$ can be made a Borel measurable function on \mathbb{R}^2 .

Notation A.1. Let $C_{[0,t]}$ be the set of continuous functions $f : [0, t] \rightarrow \mathbb{R}$, which we equip with the uniform topology. Let $C = C(\mathbb{R})$ be the space of continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$, equipped with the uniform-on-compacts topology; and let $C_0 = C_0(\mathbb{R})$ be the space of continuous and compactly supported functions $f : \mathbb{R} \rightarrow \mathbb{R}$, equipped with the uniform topology. We use $P_t^{0,0}$ to denote the probability measure of the Brownian bridge on $C_{[0,t]}$, and assume that C is equipped with the probability measure of Ξ .

By Fubini's theorem, it suffices to prove that there exists a measurable map

$$F : \mathbb{R}^2 \otimes C_{[0,t]} \otimes C \rightarrow \mathbb{R}$$

such that for every $(x, y) \in \mathbb{R}^2$, $\omega \in C_{[0,t]}$, and $\bar{\omega} \in C$, we can interpret

$$e^{-\langle L_t(B_t^{x,y}), V \rangle - \xi(L_t(B_t^{x,y}))} = F((x, y), \omega, \bar{\omega}) \quad (\text{A.1})$$

(here, $\bar{\omega} \in C$ corresponds to a realization of Ξ , and $((x, y), \omega) \in \mathbb{R}^2 \otimes C_{[0,t]}$ corresponds to a realization of the Brownian bridge $B_t^{x,y}$ with deterministic endpoints x and y and random dynamics given by the Brownian path $B_t^{0,0}$). Indeed, if this holds, then for every realization of the noise $\bar{\omega} \in C$, we can define the Borel measurable function

$$\hat{K}(t; x, y) := \int_{C_{[0,t]}} \Pi_B(t; x, y) F((x, y), \omega, \bar{\omega}) d\Pi_t^{0,0}(\omega), \quad x, y \in \mathbb{R},$$

which corresponds to the expected value of $\Pi_B(t; x, y) e^{-\langle L_t(B_t^{x,y}), V \rangle - \xi(L_t(B_t^{x,y}))}$ given Ξ .

Given a realization $\omega \in C_{[0,t]}$ of $B_t^{0,0}$ and $x, y \in \mathbb{R}$, we can construct a realization of $B_t^{x,y}$ by using the measurable map $F_1 := \mathbb{R}^2 \otimes C_{[0,t]} \rightarrow C_{[0,t]}$ defined as

$$F_1((x, y), \omega) := \left(\omega(s) + \frac{(t-s)x}{t} + \frac{sy}{t} : 0 \leq s \leq t \right).$$

Next, we let $F_2 : C_{[0,t]} \rightarrow C_0$ be the measurable function that maps ω to its (continuous) local time. More precisely, let $E \subset C_{[0,t]}$ be the event on which the limit

$$L_t^a(\omega) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{a \leq \omega(s) < a + \varepsilon\}} ds$$

exists and is finite for all $a \in \mathbb{R}$, and the resulting function $a \mapsto L_t^a(\omega)$ is an element of C_0 . (We know from [40, Chapter VI, Corollaries 1.8 and 1.9] that E has probability one under the law of $B_t^{x,y}$.) Then, for every $\omega \in C_{[0,t]}$, we define the function $F_2(\omega) \in C_0$ as

$$(F_2(\omega))(a) := L_t^a(\omega) \mathbf{1}_{\{\omega \in E\}}, \quad a \in \mathbb{R}.$$

(To see that this is measurable, note that $\omega \mapsto L_t^a(\omega) \mathbf{1}_{\{\omega \in E\}}$ is measurable for every fixed $a \in \mathbb{R}$, and that the Borel σ -algebra on C_0 is generated by evaluation maps.)

Finally, let

$$F_3(f, \bar{\omega}) = \int_{\mathbb{R}} f(x) d\bar{\omega}(x) := \begin{cases} \lim_{n \rightarrow \infty} F_3^{(n)}(f, \bar{\omega}) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

be the limit of the measurable maps $F_3^{(n)} : C_0 \otimes C \rightarrow \mathbb{R}$ defined as

$$F_3^{(n)}(f, \bar{\omega}) := \sum_{k=1}^{k(n)} f(\tau_k^{(n)}) \left(\bar{\omega}(\tau_{k+1}^{(n)}) - \bar{\omega}(\tau_k^{(n)}) \right),$$

as per Section 3.2.1/Karandikar [26]. We may then define (A.1) using the compositions of measurable maps

$$F((x, y), \omega, \bar{\omega}) := e^{-\langle F_2 \circ F_1((x, y), \omega), V \rangle - F_3 \circ F_2 \circ F_1((x, y), \omega, \bar{\omega})}.$$

In order to prove that the diagonal $x \mapsto K(t; x, x)$ is Borel measurable, we apply the same argument, except that $x = y$. Then, in order to prove the measurability in Cases 2 and 3, we can use the same argument, except that we add a few additional steps to construct the conditioned processes

$$(B^x \mid B^x(t) \in \{y, -y\}) \quad \text{or} \quad (B^x \mid B^x(t) \in 2b\mathbb{Z} \pm y),$$

and then use the couplings discussed in Section 5.2 to construct $X_t^{x,y}$ and $Y_t^{x,y}$ and their local times from the latter in the space of continuous and compactly supported functions on $[0, \infty)$ and $[0, b]$, respectively.

B Tails of Gaussian suprema

Throughout this section, we assume that $(\mathfrak{X}(\mathfrak{x}))_{\mathfrak{x} \in \mathfrak{T}}$ is a continuous centered Gaussian process on some index space \mathfrak{T} . We have the following result regarding the behaviour of the tails of \mathfrak{X} 's supremum.

Theorem B.1 ([29, (5.151)]). *Let us define*

$$v^2 := \sup_{\mathfrak{x} \in \mathfrak{T}} \mathbf{E}[\mathfrak{X}(\mathfrak{x})^2] \quad \text{and} \quad m := \mathbf{Med} \left[\sup_{\mathfrak{x} \in \mathfrak{T}} \mathfrak{X}(\mathfrak{x}) \right],$$

where \mathbf{Med} denotes the median. It holds that

$$\mathbf{P} \left[\sup_{\mathfrak{x} \in \mathfrak{T}} \mathfrak{X}(\mathfrak{x}) \geq t \right] \leq 1 - \Phi((t - m)/v) \leq e^{-(t-m)^2/2v^2}, \quad t \geq 0,$$

where Φ denotes the standard Gaussian CDF.

Using this Gaussian tails result, we can control the asymptotic growth of functions involving Gaussian Suprema.

Corollary B.2. *Let $\mathfrak{T} = \mathbb{R}$, and suppose that \mathfrak{X} is stationary. There exists a finite random variable $C > 0$ such that, almost surely,*

$$|\mathfrak{X}(x)| \leq C \sqrt{\log(2 + |x|)}, \quad x \in \mathbb{R}.$$

Proof. For every $n \in \mathbb{Z} \setminus \{0\}$ and $c > 0$, define the events

$$\begin{aligned} E_n^{(c)} &:= \left\{ \sup_{x \in [n, n+1]} |\mathfrak{X}(x)| \geq c \sqrt{\log |n|} \right\} \\ &= \left\{ \sup_{x \in [n, n+1]} \mathfrak{X}(x) \geq c \sqrt{\log |n|} \right\} \cup \left\{ \sup_{x \in [n, n+1]} -\mathfrak{X}(x) \geq c \sqrt{\log |n|} \right\}. \end{aligned}$$

By the Borel-Cantelli lemma, it suffices to prove that $\sum_n \mathbf{P}[E_n^{(c)}] < \infty$ for a large enough $c > 0$. Since \mathfrak{X} is stationary, for every n , it holds that

$$\sup_{x \in [n, n+1]} \mathbf{E}[\mathfrak{X}(x)^2] = \mathbf{E}[\mathfrak{X}(0)^2] =: \sigma^2$$

and

$$\mathbf{Med} \left[\sup_{x \in [n, n+1]} \mathfrak{X}(x) \right] = \mathbf{Med} \left[\sup_{x \in [0, 1]} \mathfrak{X}(x) \right] =: \mu;$$

the same holds true for $-\mathfrak{X}$. Thus, by applying Theorem B.1 to the suprema of \mathfrak{X} and $-\mathfrak{X}$ on $[n, n+1]$ and a union bound, $\mathbf{P}[E_n^{(c)}] \leq 2 \exp((c \sqrt{\log |n|} - \mu)^2 / 2\sigma^2)$. Since this is summable in n for large enough $c > 0$, the result is proved. \square

Remark B.3. By examining the proof of Corollary B.2, we note that we can easily also prove the stronger statement that, almost surely,

$$\sup_{y \in [x, x+1]} |\mathfrak{X}(y)| \leq C \sqrt{\log(2 + |x|)},$$

since

$$\sup_{y \in [x, x+1]} |\mathfrak{X}(y)| \leq \sup_{y \in [\lfloor x \rfloor, \lfloor x \rfloor + 1]} |\mathfrak{X}(y)| + \sup_{y \in [\lfloor x \rfloor + 1, \lfloor x \rfloor + 2]} |\mathfrak{X}(y)|.$$

Remark B.4. In the setting of Corollary B.2, if we also assume that

$$\lim_{|x| \rightarrow \infty} \mathbf{E}[\mathfrak{X}(0)\mathfrak{X}(x)] = 0,$$

then we can prove that the upper bound in Corollary B.2 is optimal, in the sense that we also have a matching lower bound of the form

$$\sup_{|y| \leq x} |\mathfrak{X}(y)| \geq \tilde{C} \sqrt{\log(2 + |x|)}, \quad x \in \mathbb{R}$$

for some $0 < \tilde{C} < C$ (see, e.g., [7, Section 2.1] and references therein).

C Schrödinger operator theory

C.1 Proof of Lemma 3.1

Consider first Cases 1 and 2. Since f is continuous and square-integrable, $f(x) \rightarrow 0$ as $x \rightarrow \infty$, and thus

$$f(x)^2 \leq \int_x^\infty |(f(y)^2)'| dy \leq 2 \int_I |f(y)| |f'(y)| dy.$$

The result then follows from the fact that for every $\kappa > 0$, we have the inequality $|z\bar{z}| \leq \frac{\kappa}{2} z^2 + \frac{1}{2\kappa} \bar{z}^2$. Suppose then that we are in Case 3. Define the function

$$h(x) := \begin{cases} 1 & \text{if } x \in [0, b/2], \\ 1 - \frac{x-b/2}{b/2} & \text{if } x \in (b/2, b]. \end{cases}$$

Then, for every $x \in [0, b/2]$, one has

$$\begin{aligned} f(x)^2 = f(x)^2 h(x) &\leq \int_x^b |(f(y)^2 h(y))'| dy \\ &\leq 2 \int_I |f(y) f'(y)| h(y) dy + \int_I f(y)^2 |h'(y)| dy. \end{aligned}$$

Since $h \leq 1$ and $|h'| \leq 2/b$, the same inequality used in Cases 1 and 2 yields the result. To prove the bound for $x \in (b/2, b]$, we apply the same method with the function

$$h(x) := \begin{cases} \frac{2x}{b} & \text{if } x \in [0, b/2], \\ 1 & \text{if } x \in (b/2, b]. \end{cases}$$

C.2 Proof of Proposition 3.2

C.2.1 Step 1. Norm equivalence and \mathcal{E} is semibounded

We begin by proving that $\|\cdot\|_{+1}$ and $\|\cdot\|_*$ are equivalent and that \mathcal{E} is semibounded. In Cases 1, 2-D, and 3-D, it suffices to observe that, because $V \geq 0$, one has

$$\mathcal{E}(f, f) = \frac{1}{2} \|f'\|_2^2 + \|V^{1/2} f\|_2^2 \geq 0.$$

In the other cases, where $\mathcal{E}(f, f)$ contains boundary terms of the form $-\alpha f(0)^2$ and $-\beta f(b)^2$, we get that \mathcal{E} is semibounded and the equivalence of norms from Lemma 3.1.

C.2.2 Step 2. \mathcal{E} is closed

Knowing that $\|\cdot\|_{+1}$ and $\|\cdot\|_*$ are equivalent, to prove that \mathcal{E} is closed, it suffices to show that $(D(\mathcal{E}), \langle \cdot, \cdot \rangle_*)$ is a Hilbert space. This follows from the fact that Sobolev spaces and the L^2 space with measure $V(x)dx$ are complete, noting further that $\|g_n' - g'\|_2 \rightarrow 0$ as $n \rightarrow \infty$ for a sequence $(g_n)_{n \in \mathbb{N}} \subset D(\mathcal{E})$ and $g \in H_V^1$ implies by the fundamental theorem of calculus that $g_n \rightarrow g$ pointwise. Hence in cases 2-D, 3-D, and 3-M, boundary conditions of the form $g_n(0) = 0$ and $g_n(b) = 0$ are preserved in the limit g .

C.2.3 Step 3. Form core

By equivalence of $\|\cdot\|_{+1}$ and $\|\cdot\|_*$, to prove that FC is a form core for \mathcal{E} , it suffices to show that FC is dense in $(D(\mathcal{E}), \langle \cdot, \cdot \rangle_*)$.

We begin by noting that it suffices to prove the result in Cases 1, 2-D, and 3-D. To illustrate this, consider Case 2-R: Let $\tilde{I} := (-1, \infty)$, and define $\tilde{V} : \tilde{I} \rightarrow [0, \infty)$ as $\tilde{V}(x) = 0$ for $x \in (-1, 0]$ and $\tilde{V}(x) = V(x)$ for $x \in (0, \infty)$. If the result is proved in Case 2-D, then we know that for every locally absolutely continuous $\tilde{f} : \tilde{I} \rightarrow \mathbb{R}$ such that

$$\tilde{f}(-1) = 0 \quad \text{and} \quad \int_{-1}^{\infty} \tilde{f}'(x)^2 + (\tilde{V}(x) + 1)\tilde{f}(x)^2 dx < \infty, \quad (\text{C.1})$$

there exists a sequence $(\tilde{\varphi}_n)_{n \in \mathbb{N}}$ of smooth functions compactly supported in \tilde{I} such that

$$\lim_{n \rightarrow \infty} \int_{-1}^{\infty} (\tilde{f}'(x) - \tilde{\varphi}'_n(x))^2 + (\tilde{V}(x) + 1)(\tilde{f}(x) - \tilde{\varphi}_n(x))^2 dx = 0.$$

We then get the result for Case 2-R by noting that the restriction of $\tilde{\varphi}_n$ to $(0, \infty)$ is an element of FC, and that every function $f \in D(\mathcal{E})$ can be extended to an \tilde{f} of the form (C.1). A similar extension argument can be used in Cases 3-R and 3-M.

Next, we argue that it suffices to prove the result in Case 3-D. We illustrate this in Case 2-D: Let ψ be a smooth cutoff function such that $\psi(x) = 1$ for $x \in (0, 1/2]$ and $\psi(x) = 0$ for $x \geq 1$. Then, for every $R > 0$, we let $\psi_R(x) := \psi(x/R)$. Given that $\psi'_R(x)^2 = (1/R)^2 \psi'(x/R)^2 \rightarrow 0$ and $(\psi_R(x) - 1)^2 \rightarrow 0$ pointwise in $x \in (0, \infty)$ as $R \rightarrow \infty$, for every $f \in D(\mathcal{E})$, it is easy to check that $\|\psi_R f - f\|_* \rightarrow 0$ as $R \rightarrow 0$ by dominated convergence. Next, since $\text{supp}(\psi_R f)$ is compact, if the result holds in Case 3-D, then we can find a smooth $\varphi_R : (0, \infty) \rightarrow \mathbb{R}$ with $\text{supp}(\varphi_R) \subset \text{supp}(\psi_R f)$ such that $\|\varphi_R - \psi_R f\|_* < 1/R$. Taking $R \rightarrow \infty$ then yields the result in Case 2-D; a similar cutoff argument holds for Case 1.

It now only remains to prove the result in Case 3-D. By [12, Lemma 7.1.1], we know that, in Case 3-D, for every $f \in D(\mathcal{E})$, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \text{FC}$ such that $\|\varphi'_n - f'\|_2 \rightarrow 0$ and $\|\varphi_n - f\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Since $\sup_n \|\varphi'_n\|_2 < \infty$ and $\sup_n \|\varphi_n\|_2 < \infty$, it follows from Lemma 3.1 and (5.27) that $(\varphi_n)_{n \in \mathbb{N}}$ is uniformly bounded and equicontinuous, hence we get $\|(\varphi_{n_i} - f)V^{1/2}\|_2 \rightarrow 0$ along some subsequence $(n_i)_{i \in \mathbb{N}}$ by Arzelà-Ascoli. Therefore, $\|\varphi_{n_i} - f\|_* \rightarrow 0$ as $i \rightarrow \infty$, concluding the proof.

C.2.4 Step 4. Unique form for H and compact resolvent

Since \mathcal{E} is closed and semibounded on $D(\mathcal{E})$, the fact that H is the unique operator with form \mathcal{E} follows from [39, Theorem VIII.15]. It only remains to prove that H has compact resolvent: In Case 3, this follows from the fact that H is in this case a regular Sturm-Liouville operator, and in Cases 1 and 2, from the fact that $V(x) \gg \log|x|$ as $x \rightarrow \pm\infty$: Indeed, H is in those cases limit point (e.g., [48, Chapter 7.3 and Theorem 7.4.3]), and compactness of the resolvent is given by [38, Theorem XIII.67] or the Molchanov criterion as stated in [48, Page 213].

D Proof of Theorem 5.4

D.1 Proof of (5.3)

On the one hand, since the Gaussian kernel \mathcal{G}_t is even, the transition kernels satisfy $\Pi_Z(t; x, y) = \Pi_Z(t; y, x)$ for every $t > 0$ and $x, y \in I$. On the other hand, given that

$$(Z_t^{x,y}(t-s) : 0 \leq s \leq t) \stackrel{d}{=} (Z_t^{y,x}(s) : 0 \leq s \leq t)$$

$\mathbf{E}_t^{x,y}[F(Z)] = \mathbf{E}_t^{y,x}[F(Z)]$ for any path functional F that is invariant under time reversal. In particular, since local time is invariant under time reversal, we have (5.3).

D.2 Proof of (5.4)

For every $x, y, z \in I$ and $t, \bar{t} > 0$, if we condition the path $Z_{t+\bar{t}}^{x,y}$ on $Z_{t+\bar{t}}^{x,y}(t) = z$, then the path segments

$$(Z_{t+\bar{t}}^{x,y}(s) : 0 \leq s \leq t) \quad \text{and} \quad (Z_{t+\bar{t}}^{x,y}(t+s) : 0 \leq s \leq \bar{t}) \quad (\text{D.1})$$

are independent and have respective distributions $Z_t^{x,z}$ and $Z_{\bar{t}}^{z,y}$. Moreover, by Doob's h -transform, $Z_{t+\bar{t}}^{x,y}(t)$ has density

$$z \mapsto \frac{\Pi_Z(t; x, z)\Pi_Z(\bar{t}; z, y)}{\Pi_Z(t + \bar{t}; x, y)}.$$

(5.4) is then a consequence of Fubini's theorem and additivity of local time: Letting $Z_t^{1;x,z}$ and $Z_{\bar{t}}^{2;z,y}$ denote independent processes with respective distributions $Z_t^{x,z}$ and $Z_{\bar{t}}^{z,y}$ for all z , we have that $\int_I K(t; x, z)K(\bar{t}; z, y) dz$ is equal to (recall the notation \mathfrak{A}_t from (5.24))

$$\begin{aligned} & \Pi_Z(t + \bar{t}; x, y) \int_I \mathbf{E}\left[e^{\mathfrak{A}_t(Z_t^{1;x,z}) + \mathfrak{A}_{\bar{t}}(Z_{\bar{t}}^{2;z,y})}\right] \frac{\Pi_Z(t; x, z)\Pi_Z(\bar{t}; z, y)}{\Pi_Z(t + \bar{t}; x, y)} dz \\ &= \Pi_Z(t + \bar{t}; x, y) \int_I \mathbf{E}\left[e^{\mathfrak{A}_{t+\bar{t}}(Z_{t+\bar{t}}^{x,y})} \middle| Z_{t+\bar{t}}^{x,y}(t) = z\right] \mathbf{P}[Z_{t+\bar{t}}^{x,y}(t) \in dz] dz \\ &= K(t + \bar{t}; x, y), \end{aligned}$$

as desired.

D.3 Feynman-Kac formula

We now complete the proof of Theorem 5.4 by showing that $e^{-tH} = K(t)$ for all $t > 0$. The proof of this in Case 1 can be found in [43, Theorem 4.9]. For Cases 2-D and 3-D, we refer to [11, (34) and Theorem 3.27]. For Case 3-R, we have [33, (3.3') and (3.4), Theorem 3.4 (b), and Lemmas 4.6 and 4.7]. It now only remains to prove the result in cases Cases 3-M and 2-R:

D.3.1 Case 3-M

Let us assume that we are considering Case 3-M, that is, the operator $H = -\frac{1}{2}\Delta + V$ is acting on $(0, b)$ with mixed boundary conditions (as in Assumption 2.1) and

$$K(t; x, y) = \Pi_Y(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(Y), V \rangle + \alpha \mathfrak{L}_t^0(Y) - \infty \cdot \mathfrak{L}_t^b(Y)} \right].$$

As argued in [33, Pages 62 and 63], it can be shown that

1. $K(t)$ is a strongly continuous semigroup on L^2 ; and
2. if, for every $n \in \mathbb{N}$, we define

$$K_n(t; x, y) = \Pi_Y(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(Y), V \rangle + \alpha \mathfrak{L}_t^0(Y) - n \mathfrak{L}_t^b(Y)} \right],$$

then for every $t > 0$, $\|K_n(t) - K(t)\|_{\text{op}} \rightarrow 0$ as $n \rightarrow \infty$.

Item (1) above implies that $K(t)$ has a generator, so it only remains to prove that this generator is in fact H . By Lemma 5.13 in the case $p = 1$, we know that the $K_n(t)$ and

$K(t)$ are compact. Therefore, if we let H_n be the operator $-\frac{1}{2}\Delta + V$ on $(0, b)$ with Robin boundary

$$f'(0) + \alpha f(0) = -f'(b) - nf(0) = 0,$$

then by repeating the argument in Section 3.2.3, we need only prove that $H_n \rightarrow H$ in the sense of convergence of eigenvalues and L^2 -convergence of eigenfunctions.

If we define the matrices

$$A := \begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and the vector function $F(x) := [f'(x), f(x)]^\top$, then we can represent H 's boundary conditions in matrix form as $AF(0) + BF(b) = 0$. Similarly, if we let

$$C_n := \begin{bmatrix} 0 & 0 \\ 1/n & 1 \end{bmatrix},$$

then H_n 's boundary conditions are represented as $AF(0) + C_n F(b) = 0$. Given that $\|B - C_n\| \rightarrow 0$ as $n \rightarrow \infty$, it follows from [48, Theorems 3.5.1 and 3.5.2] that for every $k \in \mathbb{N}$, $\lambda_k(H_n) \rightarrow \lambda_k(H)$ and $\psi_k(H_n) \rightarrow \psi_k(H)$ uniformly on compacts. Since $(0, b)$ is bounded, this implies L^2 -convergence of the eigenfunctions, concluding the proof.

D.3.2 Case 2-R

Let us now assume that H acts on $(0, \infty)$ with Robin boundary at the origin and that

$$K(t; x, y) = \Pi_X(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(X), V \rangle + \alpha \mathfrak{L}_t^0(X)} \right].$$

The same arguments used in [33, Theorem 3.4 (b)] imply that this semigroup is strongly continuous on L^2 , and we know it is compact by Lemma 5.13.

For every $n \in \mathbb{N}$, let $H_n = -\frac{1}{2}\Delta + V$, acting on $(0, n)$ with mixed boundary conditions

$$f(0) + \alpha f'(0) = f(n) = 0.$$

By the previous section, the semigroup generated by this operator is given by

$$K_n(t; x, y) = \Pi_{Y_n}(t; x, y) \mathbf{E}_t^{x,y} \left[e^{-\langle L_t(Y_n), V \rangle + \alpha \mathfrak{L}_t^0(Y_n) - \infty \cdot \mathfrak{L}_t^n(Y_n)} \right],$$

where Y_n is a reflected Brownian motion on $(0, n)$. Arguing as in the previous section, it suffices to prove that $K_n(t) \rightarrow K(t)$ in operator norm and $H_n \rightarrow H$ in the sense of eigenvalues and eigenfunctions.

We begin with the semigroup convergence. We first note that $\|K_n(t) - K(t)\|_{\text{op}}$ is ambiguous, since $K_n(t)$ and $K(t)$ do not act on the same space. However, by using an argument similar to (5.6), we can extend the kernel $K_n(t)$ to $(0, \infty)^2$ by defining

$$\tilde{K}_n(t; x, y) = \Pi_X(t; x, y) \mathbf{E}_t^{x,y} \left[\mathbf{1}_{\{\tau_{[n,\infty)}(X) > t\}} e^{-\langle L_t(X), V \rangle + \alpha \mathfrak{L}_t^0(X)} \right],$$

where $\tau_{[n,\infty)}$ is the first hitting time of $[n, \infty)$. This transformation does not affect the eigenvalues, and the eigenfunctions are similarly extended from functions on $(0, n)$ vanishing on the boundary to functions on $(0, \infty)$ that are supported on $(0, n)$. One has

$$\|\tilde{K}_n(t) - K(t)\|_2^2 = \int_0^\infty \tilde{K}_n(2t; x, x) - 2\tilde{K}_{n,0}(2t; x, x) + K(2t; x, x) \, dx,$$

where

$$\tilde{K}_{n,0}(2t; x, x) = \Pi_X(2t; x, x) \mathbf{E}_{2t}^{x,x} \left[\mathbf{1}_{\{\tau_{[n,\infty)}(X) > t\}} e^{-\langle L_{2t}(X), V \rangle + \alpha \mathfrak{L}_{2t}^0(X)} \right].$$

Thus it suffices to prove that

$$\lim_{n \rightarrow \infty} \int_0^\infty \tilde{K}_{n,0}(2t; x, x) dx, \lim_{n \rightarrow \infty} \int_0^\infty \tilde{K}_n(2t; x, x) dx = \int_0^\infty K(2t; x, x) dx.$$

Since $X_{2t}^{x,x}$ is almost surely continuous, hence bounded, the result is a straightforward application of monotone convergence (both with $\mathbf{E}^{x,x}$ and the dx integral).

We now prove convergence of eigenvalues and eigenvectors. Let \mathcal{E} denote the form of H and $D(\mathcal{E})$ its domain, as defined in Definition 2.5 for Case 2-R. We note that we can think of H_n as the operator with the same form \mathcal{E} but acting on the smaller domain

$$D_n := \{f \in H_V^1((0, \infty)) : f(x) = 0 \text{ for every } x \geq n\} \subset D(\mathcal{E}).$$

These domains are increasing, in that $D_1 \subset D_2 \subset \dots \subset D(\mathcal{E})$. A straightforward modification of the convergence argument presented in Section 5.6 gives the desired result (at least through a subsequence).

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