Some rigorous results on the phase transition of finitary random interlacements*

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Abstract

In this paper, we show several rigorous results on the phase transition of Finitary Random Interlacements (FRI). For the high intensity regime, we show the existence of a critical fiber length, and find its exact asymptotic as intensity goes to infinity. At the same time, our result for the low intensity regime proves the global existence of a non-trivial phase transition with respect to the system intensity.

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1   Introduction

The model of finitary random interlacements (FRI) was first introduced by Bowen [1] to solve a special case of the Gaboriau-Lyons problem. Intuitively speaking, FRI (denoted by $\mathcal{FI}^{u,T}$) can be seen as a "cloud of twisted yarn" composed of finite "fibers" on $\mathbb{Z}^d$. The fibers within this system each forms a geometrically killed simple random walk with an expected length T. Meanwhile, the starting point of which is sampled according to a Poisson point process with intensity measure proportional to system intensity u and inversely proportional to $T + 1$ [1, 2, 3, 8]. See Section 2 for precise definitions.

In [1], Bowen proved that as $T \to +\infty$, $\mathcal{FI}^{u,T}$ converges under weak $\ast-$ topology to the celebrated random interlacements (RI) $\mathcal{I}^u$ introduced by Sznitman [10]. At the same time, FRI can also be seen as a finitary version of RI, as it can be defined as massive interlacements on a graph equipped with killing measure (see Chapter V of [7]).

On the other hand, unlike the classic system of random interlacements, whose trajectories always form an a.s. connected network [4, 10], the collection of edges traversed by FRI on $\mathbb{Z}^d$, $d \geq 3$ has exhibited a non-trivial percolative phase transition [8]. And since there are now two parameters to play with, one may characterize such transition on phase space $\{(u, T) :, u > 0, T > 0\}$ from either of the following aspects:

(1) One may first fix the intensity u and examine the evolution of FRI with respect to T. This is also the problem Bowen studied in his first paper [1]. In addition to the local convergence as $T \to \infty$, Bowen also proved that the FRI on non-amenable graphs will a.s. have infinite connected cluster(s) for all sufficiently large T. He proposed a

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parallel question for FRI on $\mathbb{Z}^d$, which was affirmatively answered in [8], where the existence of the following phase transition with respect to T was proved: For any $d \geq 3$ and $u > 0$, there are $T_0, T_1 \in (0, \infty)$, such that $\mathcal{FI}^{u,T}$ a.s. has no infinite cluster for all $T < T_0$, and a.s. has a unique infinite cluster for all $T > T_1$ (see Theorem 1, 2 in [8] for details). The geometric properties of this infinite cluster, such as local uniqueness and order of chemical distance, for sufficiently large T, was later obtained in [2]. However, as mentioned in Section 1.1 of [8], the uniqueness of phase transition and the existence/uniqueness of a critical fiber length T_c remain open. Unlike the case of RI, this turns out to be a highly non-trivial problem, since it was recently proved that there is no global stochastic monotonicity with respect to T for $\mathcal{FI}^{u,T}$. See Theorem 1, [3] for details. On the other hand, numerical simulations in Section 5 of [3] provided evidences on the existence and uniqueness of T_c. It was further conjectured in Conjecture 5 [3] that T_c is asymptotically an inverse linear function with respect to u.

(2) As in [7], one may also fix T and examine the evolution of FRI with respect to its intensity u. Note that FRI is by definition monotone with respect to u. So the question of interest here is whether there is always a non-trivial phase transition. I.e., for a fixed T, we want to show FRI does not percolate for all sufficiently small u and percolate for all sufficiently large u. [7] proved this for all T small enough, and conjectured it can be extended to all $T \in (0, \infty)$.

In this paper, we prove that, despite lacking global monotonicity, FRI is stochastically increasing with respect to T for all $T \in (0, 1)$, which implies the existence and uniqueness of T_c for all sufficiently large u. Meanwhile, we also show that the upper bound of T_c found by [3, 7] in the high intensity regime is actually sharp in the limit, and give an exact asymptotic of T_c as $u \to \infty$. Moreover, for the low intensity regime, we prove a polynomial lower bound for the phase diagram, which at the same time proves the conjecture on the global existence of a non-trivial phase transition with respect to u. Our proofs are largely based on the "decoupling" methods first invented in [11].

This paper is organized as follows: In Section 2 we introduce precise definitions of FRI together with necessary notations and preliminaries. We state our main results in Section 3. Local monotonicity is shown in Section 4. And at last we estimate the asymptotic of critical values in Section 5.

2   Notations and preliminaries

Some basic notations: In this paper, we denote the l^∞ distance and Euclidean distance by $|\cdot|$ and $|\cdot|_2$ respectively. The undirected edge set of $\mathbb{Z}^d$ is denoted by $\mathbb{L}^d$ (i.e. $\mathbb{L}^d := \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y|_2 = 1\}$). For any subsets $A, B \subset \mathbb{Z}^d$, the l^∞ distance between them is defined as $d(A, B) := \min\{|x - y| : x \in A, y \in B\}$. For finite subset $D \subset \mathbb{Z}^d$, let $\partial D := \{x \in D : \exists y \in \mathbb{Z}^d \setminus D \text{ such that } \{x, y\} \in \mathbb{L}^d\}$ be its inner boundary and $|D|$ be the cardinality of D, without causing further confusion.

Connection between two sets: For sets $A, B \subset \mathbb{Z}^d$ and a collection of edges in $\mathbb{L}^d$ denoted by E, we say A and B are connected by E (written by $A \overset{E}{\leftrightarrow} B$) iff there exists a sequence of vertices $(x_0, ..., x_n)$ such that $x_0 \in A$, $x_n \in B$ and that for any $0 \leq i \leq n - 1$, $\{x_i, x_{i+1}\} \in E$.

Statements on constants: we will use $c, c_1, c_2, ...$ as local constants ("local" means their values may vary according to contexts) and $C, C_1, C_2, ...$ as global constants ("global" means constants will keep their values throughout the whole paper).

Random walks and relative stopping times: We denote the law of simple random walks starting from x on $\mathbb{Z}^d$ by P_x and the law of geometrically killed simple random walks starting from x with killing rate $\frac{1}{T+1}$ at each step by $P_x^{(T)}$.

For a random walk $\{X_i\}_{n=0}^\infty$ and $A \subset \mathbb{Z}^d$, define the corresponding hitting time and

entrance time as $H_A = \min\{k \geq 0 : X_k \in A\}$ and $\bar{H}_A = \min\{k \geq 1 : X_k \in A\}$. Here we use the convention that $\min \emptyset = \infty$.

Capacity with killing measure: For any $K \subset \mathbb{Z}^d$, the escaping probability is defined as $Es_K^{(T)}(x) := P_x^{(T)}(\bar{H}_K = \infty)$. Then the capacity of K with killing measure $\frac{1}{T+1}$ is defined by

$$cap^{(T)}(K) := 2d * \sum_{x \in K} Es_K^{(T)}(x).$$

In particular, we note the following trivial upper bound

$$cap^{(T)}(K) \leq 2d * |K|. \tag{2.1}$$

Definitions of FRI: According to [8], FRI has two equivalent definitions. Denote the set of all finite nearest-neighbor paths on $\mathbb{Z}^d$ by $W^{[0,\infty)}$. Then $v^{(T)} := \sum_{x \in \mathbb{Z}^d} \frac{2d}{T+1} P_x^{(T)}$ is a $\sigma-$finite measure on $W^{[0,\infty)}$.

Definition 2.1. *For $0 < u, T < \infty$, finitary random interlacements $\mathcal{FI}^{u,T}$ is the Poisson point process with intensity measure $u * v^{(T)}$. We denote the law of $\mathcal{FI}^{u,T}$ by $P^{u,T}$.*

Definition 2.2. *Let $\{N_x\}_{x \in \mathbb{Z}^d} \overset{i.i.d.}{\sim} Pois(\frac{2du}{T+1})$. For each site $x \in \mathbb{Z}^d$, start N_x independent geometrically killed simple random walks with law $P_x^{(T)}$. Let $\mathcal{FI}^{u,T}$ be the point measure on $W^{[0,\infty)}$ composed of all the trajectories above starting from all $x \in \mathbb{Z}^d$.*

With a slight abuse of notations, for $\eta \in W^{[0,\infty)}$, we may write $\eta \in \mathcal{FI}^{u,T}$ if $\mathcal{FI}^{u,T}(\eta) = 1$. In this paper, $\mathcal{FI}^{u,T}$ is also regarded as a bond percolation model on $\mathbb{L}^d$. I.e. we say an edge $e \in \mathbb{L}^d$ is open in the FRI iff there exists a path in $\mathcal{FI}^{u,T}$ containing e, denoted by $\mathcal{FI}^{u,T}(e) = 1$ (otherwise, $\mathcal{FI}^{u,T}(e) = 0$). For the simplicity of notations, for any $A, B \subset \mathbb{Z}^d$, we denote

$$A \xleftrightarrow{\mathcal{FI}^{u,T}} B := A \xleftrightarrow{\{e \in \mathbb{L}^d : \mathcal{FI}^{u,T}(e) = 1\}} B. \tag{2.2}$$

We say $\mathcal{FI}^{u,T}$ percolates iff there exists an infinite connected cluster composed of open edges. It has been proved in Theorem 2, [3] that $\mathcal{FI}^{u,T}$ contains at most one open infinite connected cluster.

FRI on a finite set: Let K be a finite subset of $\mathbb{Z}^d$. For each path η_i in $\mathcal{FI}^{u,T}$, we denote the part of η_i after intersecting K by $\hat{\eta}_i^K$. Precisely, for path $\eta_i = (\eta_i(0), ..., \eta_i(l_i))$, define $\hat{\eta}_i^K := (\eta_i(H_K), ..., \eta_i(l_i))$ if $H_K < \infty$ and $\hat{\eta}_i^K := \emptyset$ if $H_K = \infty$. By Lemma 2.1 of [8], $\sum_i \delta_{\hat{\eta}_i^K}$ has the same law as a Poisson point process with intensity measure $u * \sum_{x \in K} Es_K^{(T)}(x) P_x^{(T)}$. As a direct corollary, the number of paths intersecting K in $\mathcal{FI}^{u,T}$ is a Poisson random variable with parameter $u * cap^{(T)}(K)$.

Independence in FRI: For disjoint sets $A_1, A_2, ..., A_m \subset \mathbb{Z}^d$, consider an arbitrary sequence of events $E_1, E_2, ..., E_m$ satisfying that for all $1 \leq i \leq m$, E_i depends only on the paths in $\mathcal{FI}^{u,T}$ starting from A_i. By Definition 2.2, $E_1, E_2, ..., E_m$ are mutually independent.

Critical values of FRI:

Definition 2.3. *For $u > 0$, $d \geq 3$, we define that*

$$T_c^-(u,d) := \sup\{T_0 > 0 : \forall 0 < T < T_0, \mathcal{FI}^{u,T} \text{ does not percolate}\}$$

and

$$T_c^+(u,d) := \inf\{T_0 > 0 : \forall T > T_0, \mathcal{FI}^{u,T} \text{ percolates}\}.$$

Remark 2.4. *By Theorem 1 and 2 of [8], it has been proved that for all $d \geq 3$ and $u > 0$, $0 < T_c^-(u,d) \leq T_c^+(u,d) < \infty$.*

3 Main results

In this paper, we first show that for all $d \geq 3$ and $u > 0$, $\mathcal{FI}^{u,T}$ is stochastically increasing with respect to T for all $T \in (0,1]$. It is worth noting that $\mathcal{FI}^{u,T}$ has been proved not enjoying monotonicity for larger T's (see Theorem 1, [3]).

Theorem 3.1. *For any $u > 0$, $0 < T_1 < T_2 < \infty$ such that $T_1 * T_2 \leq 1$, then $\mathcal{FI}^{u,T_2}$ stochasitcally dominates $\mathcal{FI}^{u,T_1}$. I.e. there is a coupling between $\mathcal{FI}^{u,T_1}$ and $\mathcal{FI}^{u,T_2}$ such that almost surely for any edge $e \in \mathbb{L}^d$, $\mathcal{FI}^{u,T_1}(e) \leq \mathcal{FI}^{u,T_2}(e)$.*

By Theorem 3.1, we know that for any $d \geq 3$ and $u > 0$, if $0 < T_c^- \leq T_c^+ < 1$, then there must exist a unique critical fiber length between 0 and 1. In fact, according to the upper bound of T_c^+ and the lower bound of T_c^- given in Theorem 3 (iv) and Proposition 2, [3], $0 < T_c^- \leq T_c^+ < 1$ holds for all sufficiently large u. Therefore, we have the following corollary:

Corollary 3.2. *For all $d \geq 3$, there is a $U_d < \infty$ such that for all $u \geq U_d$,*

$$T_c^+(u,d) = T_c^-(u,d) := T_c(u,d).$$

I.e., $\mathcal{FI}^{u,T}$ percolates a.s. for all $T > T_c$, and does not percolates a.s. for all $T < T_c$.

Our next result provides the exact asymptotic of T_c as $u \to \infty$, which gives an affirmative answer to Part 2, Conjecture 5, [3].

Theorem 3.3. *For all $d \geq 3$,*

$$\lim_{u \to \infty} u * T_c(u,d) = \frac{-\log(1 - p_d^c)}{2}, \tag{3.1}$$

where p_d^c is the critical value of Bernoulli bond percolation on $\mathbb{L}^d$.

Remark 3.4. In Theorem 4.2 [3] and on Page 263 [7], it has been shown that

$$\limsup_{u \to \infty} u * T_c^+ \leq \frac{-\log(1 - p_d^c)}{2}. \tag{3.2}$$

So here we only need to prove an asymptotically sharp lower bound for the subcritical phase (a partial result on the asymptotic order was given in Proposition 2, [3]).

For the low intensity regime (i.e. $u \ll 1$), we obtain the following estimate on the subcritical phase:

Theorem 3.5. *For any $d \geq 3$ and $\delta > 0$, there exist constants $0 < U_0(d,\delta) < 1$ and $C_1(d,\delta) > 0$ such that for any $0 < u \leq U_0$,*

$$T_c^- \geq C_1 u^{-\frac{1}{d-1}+\delta}. \tag{3.3}$$

Combining Theorem 3.5 and the supercritical estimates obtained in (v) of Theorem 3, [3],

Corollary 3.6. *When $d = 3$,*

$$0.5 \leq \liminf_{u \to 0} \frac{\log(T_c^-)}{-\log(u)} \leq \limsup_{u \to 0} \frac{\log(T_c^+)}{-\log(u)} \leq 2; \tag{3.4}$$

When $d \geq 4$,

$$\frac{1}{d-1} \leq \liminf_{u \to 0} \frac{\log(T_c^-)}{-\log(u)} \leq \limsup_{u \to 0} \frac{\log(T_c^+)}{-\log(u)} \leq 1. \tag{3.5}$$

At the same time, Theorem 3.3 together with Theorem 3.5 also give an affirmative answer to the conjecture proposed on Page 263, [7] which predicted the global existence of a non-trivial phase transition with respect to u:

Theorem 3.7. *For all $d \geq 3$ and $T \in (0,\infty)$, there is a $u_c = u_c(d,T) \in (0,\infty)$ such that $\mathcal{FI}^{u,T}$ percolates a.s. for all $u \in (u_c,\infty)$, and does not percolate a.s. for all $u \in (0, u_c)$.*

4 Local stochastic monotoncity

In order to prove Theorem 3.1, we introduce a new mechanism to construct $\mathcal{FI}^{u,T}$. Denote the collection of all **directed** nearest-neighbor edges by $\hat{\mathbb{L}}^d := \{x \to y : \{x, y\} \in \mathbb{L}^d\}$. Let $\{N_{x \to y}\}_{x \to y \in \hat{\mathbb{L}}^d} \overset{i.i.d.}{\sim} Pois\left(\frac{uT}{(T+1)^2}\right)$, $\{Y_k\}_{k=1}^{\infty} \overset{i.i.d.}{\sim} Geo\left(\frac{T}{T+1}\right)$ and $\left\{X^{y,k}_\cdot\right\}_{k \in \mathbb{N}^+, y \in \mathbb{Z}^d}$ be a sequence of independent random walks with law $P_0^{(T)}$. For $u \geq 0$, we have the following equivalence in distribution:

Lemma 4.1.

$$\bigcup_{x \to y \in \hat{\mathbb{L}}^d} \bigcup_{1 \leq k \leq N_{x \to y}, Y_k \geq 1} \left\{\{x,y\}, \{X_i^{y,k}, X_{i+1}^{y,k}\}, 0 \leq i \leq Y_k - 1\right\} \overset{d}{=} \mathcal{FI}^{u,T}.$$

Proof. For any $x \in \mathbb{Z}^d$ and $x \to y \in \hat{\mathbb{L}}^d$, by Definition 2.2 and property of Poisson point process, we note that the number of paths starting from x with length ≥ 1 and the first step $x \to y$ is a Poisson variable with parameter $\frac{T}{T+1} * \frac{1}{2d} * \frac{2du}{T+1} = \frac{uT}{(T+1)^2}$. In addition, by the memoryless property of geometric distribution, the remaining part of each path after the first step removed is still a geometrically killed random walk with killing rate $\frac{1}{T+1}$. Then Lemma 4.1 follows. $\square$

Now Theorem 3.1 can be proved as follows:

Proof of Theorem 3.1: When $T_1 < T_2$, $0 < T_1 * T_2 < 1$, we define three independent sequences of random variables:

$$\left\{N_{x \to y}^{(1)}\right\} \overset{i.i.d.}{\sim} Pois\left(\frac{uT_1}{(T_1+1)(T_2+1)}\right),$$

$$\left\{N_{x \to y}^{(2)}\right\} \overset{i.i.d.}{\sim} Pois\left(\frac{uT_1(T_2-T_1)}{(T_1+1)^2(T_2+1)}\right)$$

and

$$\left\{N_{x \to y}^{(3)}\right\} \overset{i.i.d.}{\sim} Pois\left(\frac{u(T_2-T_1)(1-T_1T_2)}{(T_1+1)^2(T_2+1)^2}\right).$$

Meanwhile, one may construct

$$\left\{\left(Y_k^{(T_1)}, Y_k^{(T_2)}\right)\right\}_{k=1}^{\infty} \overset{i.i.d.}{\sim} \left(Geo\left(\frac{T_1}{T_1+1}\right), Geo\left(\frac{T_2}{T_2+1}\right)\right)$$

such that for any $k \geq 1$, $P\left(Y_k^{(T_1)} \leq Y_k^{(T_2)}\right) = 1$.

Note that for any $x \to y$, $N_{x \to y}^{(1)} + N_{x \to y}^{(2)} \sim Pois\left(\frac{uT_1}{(T_1+1)^2}\right)$ and $N_{x \to y}^{(1)} + N_{x \to y}^{(2)} + N_{x \to y}^{(3)} \sim Pois\left(\frac{uT_2}{(T_2+1)^2}\right)$. By Lemma 4.1, we have

$$\bigcup_{x \to y \in \hat{\mathbb{L}}^d} \bigcup_{1 \leq k \leq N_{x \to y}^{(1)} + N_{x \to y}^{(2)}, Y_k^{(T_1)} \geq 1} \left\{\{x,y\}, \{X_i^{y,k}, X_{i+1}^{y,k}\}, 0 \leq i \leq Y_k^{(T_1)} - 1\right\} \overset{d}{=} \mathcal{FI}^{u,T_1}. \quad (4.1)$$

Note that for any $k \geq 1$, if $Y_k^{(T_1)} \geq 1$, then $Y_k^{(T_2)} \geq 1$. Use Lemma 4.1 again,

$$\bigcup_{x \to y \in \hat{\mathbb{L}}^d} \bigcup_{1 \leq k \leq N_{x \to y}^{(1)} + N_{x \to y}^{(2)} + N_{x \to y}^{(3)}, Y_k^{(T_2)} \geq 1} \left\{\{x,y\}, \{X_i^{y,k}, X_{i+1}^{y,k}\}, 0 \leq i \leq Y_k^{(T_2)} - 1\right\} \overset{d}{=} \mathcal{FI}^{u,T_2}.$$

$$(4.2)$$

Finally, by comparing the LHS's of (4.1) and (4.2), we get the stochastic domination in Theorem 3.1. $\square$

5 Asymptotic of critical values

In this section, we prove Theorem 3.3 and Theorem 3.5. We begin the proof with inducing notations according to [9] necessary for renormalization argument.

1. Let L_0 and l_0 be positive integers to be determined later. For $n \geq 1$, let $L_n = l_0^n * L_0$ and $\mathbb{L}_n = L_n * \mathbb{Z}^d$.

2. Set $B_{n,x} := x + ([0, L_n) \cap \mathbb{Z})^d$ and $\widetilde{B}_{n,x} = \bigcup_{y \in \mathbb{L}_n : d(B_{n,y}, B_{n,x}) \leq 1} B_{n,y}$.

3. Let $\mathcal{I}_n = \{n\} \times \mathbb{L}_n$. For any $(n, x) \in \mathcal{I}_n$, write

$$\mathcal{H}_1(n, x) = \left\{ (n-1, y) \in \mathcal{I}_{n-1} : B_{n-1,y} \subset B_{n,x}, B_{n-1,y} \cap \partial B_{n,x} \neq \emptyset \right\},$$

$$\mathcal{H}_2(n, x) = \left\{ (n-1, y) \in \mathcal{I}_{n-1} : B_{n-1,y} \cap \{z \in \mathbb{Z}^d : d(z, B_{n,x}) = \lfloor \frac{L_n}{2} \rfloor\} \neq \emptyset \right\}.$$

Recall that $d(\cdot, \cdot)$ is the metric on $\mathbb{Z}^d$ (see Section 2 for precise definition).

4. For $x \in \mathbb{L}^d$, $n \geq 0$, let

$$\Lambda_{n,x} = \{\mathcal{T} \subset \bigcup_{k=0}^n \mathcal{I}_k : \mathcal{T} \cap \mathcal{I}_n = (n, x) \ and \ \forall (k, y) \in \mathcal{T} \cap \mathcal{I}_k, 0 < k \leq n, has \ two$$

$$descendants \ (k-1, y_i(k, y)) \in \mathcal{H}_i(k, y), i = 1, 2 \ such \ that$$

$$\mathcal{T} \cap \mathcal{I}_{k-1} = \bigcup_{(k,y) \in \mathcal{T} \cap \mathcal{I}_{k-1}} \{(k-1, y_1(k, y)), (k-1, y_2(k, y))\}\}.$$

$$(5.1)$$

By (2.8) of [9], one has

$$|\Lambda_{n,x}| \leq \left(c_0(d) * l_0^{2(d-1)} \right)^{2^n}. \tag{5.2}$$

Based on the settings above, we can decompose events of interests and estimate their probabilities by choosing proper values of L_0 and l_0. Roughly speaking, we need to select L_0 to control the $0-$level event and select l_0 to guarantee the "almost independence" between trajectories in different boxes, according to u and T. See Figure 1 for an illustration of this renormalization scheme:

Now we give the proof of Theorem 3.3.

5.1 Proof of Theorem 3.3

Recall that in Theorem 3.(iii) of [3] and on Page 263 of [7], it has been shown that

$$\limsup_{u \to \infty} u * T_c^+ \leq \frac{-\log(1 - p_d^c)}{2}. \tag{5.3}$$

Thus it is now sufficient to prove that: for any $\epsilon > 0$, there exists $U'(d, \epsilon) > 0$ such that for any $u > U'$ and $T > 0$ satisfying

$$u * T \leq \frac{-\log(1 - p_d^c)}{2} - \epsilon, \tag{5.4}$$

$\mathcal{FI}^{u,T}$ does not percolate. I.e.,

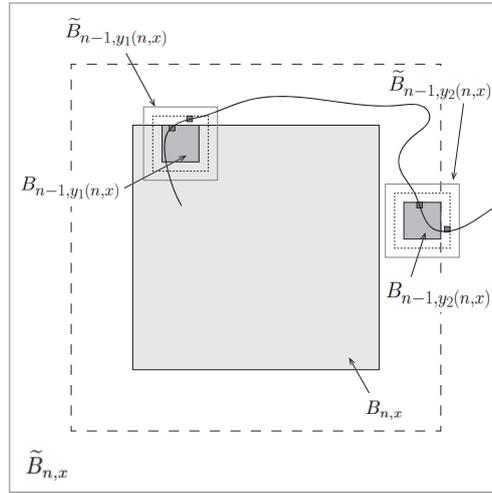$$\liminf_{u \to \infty} u * T_c^- \geq \frac{-\log(1 - p_d^c)}{2}. \tag{5.5}$$

Figure 1: An illustration of the renormalization scheme

Assume that u is sufficiently large while T satisfies (5.4). Take $L_0 = \lfloor u^{\frac{1}{2d}} \rfloor > 10$ and $l_0 = 10$. Let $B_{0,0}^* = \{y \in \mathbb{Z}^d : d(\{y\}, \widetilde{B}_{0,0}) \le 1\}$ and recall the notation $\hat{\eta}^K$ from Section 2. By (5.4), we have

$$P^{u,T}\left(\exists \eta \in \mathcal{FI}^{u,T} \ such \ that \ \{\eta(i), \eta(i+1)\} \subset \widetilde{B}_{0,0} \ for \ some \ i \ge 1\right)$$

$$\le P^{u,T}\left(\exists \eta \in \mathcal{FI}^{u,T} \ such \ that \ the \ length \ of \ \hat{\eta}^{B_{0,0}^*} \ \ge 2\right)$$

$$= 1 - \exp(-u * cap^{(T)}(B_{0,0}^*) * \left(\frac{T}{T+1}\right)^2) \tag{5.6}$$

$$\le u * cap^{(T)}(B_{0,0}^*) * \left(\frac{T}{T+1}\right)^2$$

$$\le c * u^{1.5} * T^2 \le c' * u^{-0.5}.$$

Let $\hat{\mathcal{FI}}^{u,T} := \{e \in \mathbb{L}^d : e = \{\eta(0), \eta(1)\}, \eta \in \mathcal{FI}^{u,T}\}$. By Definition 2.2, one may see that $\hat{\mathcal{FI}}^{u,T}$ has the same distribution as the collection of open edges of a Bernoulli bond percolation with parmeter $1 - \exp(-\frac{uT}{(T+1)^2}) < p_d^c$.

By Theorem 6.1 of [5], we have

$$P^{u,T}\left(B_{0,0} \xleftrightarrow{\hat{\mathcal{FI}}^{u,T}} \partial \widetilde{B}_{0,0}\right) \le (2L_0+1)^d * P^{u,T}\left(0 \xleftrightarrow{\hat{\mathcal{FI}}^{u,T}} \partial B_{0,0}\right) \le (2L_0+1)^d * e^{-cL_0}. \tag{5.7}$$

Note that when the event in the LHS of (5.6) does not occur, $\left\{B_{0,0} \xleftrightarrow{\mathcal{FI}^{u,T}} \partial \widetilde{B}_{0,0}\right\}$ is equivalent to $\left\{B_{0,0} \xleftrightarrow{\hat{\mathcal{FI}}^{u,T}} \partial \widetilde{B}_{0,0}\right\}$. By (5.6) and (5.7),

$$P^{u,T}\left(B_{0,0} \xleftrightarrow{\mathcal{FI}^{u,T}} \partial \widetilde{B}_{0,0}\right) \le c' * u^{-0.5} + (2L_0 + 1)^d * e^{-cL_0}. \tag{5.8}$$

For any $n \geq 0$ and $x \in \mathbb{L}_n$, we write that

$$A_{n,x} := \left\{ B_{n,x} \xleftrightarrow{\mathcal{FI}^{u,T}} \partial \widetilde{B}_{n,x} \right\}. \tag{5.9}$$

Recall the notation $\Lambda_{n,x}$ in (5.1). For any $\mathcal{T} \in \Lambda_{n,x}$, like Equation (2.13) of [9], we write

$$A_{\mathcal{T}} := \bigcap_{(0,y) \in \mathcal{T} \cap \mathcal{I}_0} A_{0,y}. \tag{5.10}$$

Similar to (2.14) of [9], we have

$$A_{n,x} \subset \bigcup_{\mathcal{T} \in \Lambda_{n,x}} A_{\mathcal{T}}. \tag{5.11}$$

Here we need a decoupling inequality, which is parallel to Lemma 5.4, [2]. Let $\hat{B}_{n,x} = \{y : d(\{y\}, \widetilde{B}_{n,x}) \leq L_{n+1}\}$ and $F_{n,x} := \bigcap_{\eta \in \mathcal{FI}^{u,T}, \eta(0) \in (\hat{B}_{n,x})^c} \{\eta \cap \widetilde{B}_{n,x} = \emptyset\}$.

Given event $F_{n,x}^c$, there must exist a path $\eta \in \mathcal{FI}^{u,T}$ such that $\eta \cap \partial \hat{B}_{n,x} \neq \emptyset$ and $\eta \cap \partial \widetilde{B}_{n,x} \neq \emptyset$. Recalling the construction of FRI on a finite set in Section 2 (here we take $K = \partial \hat{B}_{n,x} \cup \partial \widetilde{B}_{n,x}$), the number of such paths is a Poisson random variable with parameter

$$u * \left[\sum_{y \in \partial \hat{B}_{n,x}} Es_K^{(T)}(y) * P_y^{(T)} \left(H_{\widetilde{B}_{n,x}} < \infty \right) + \sum_{y \in \partial \widetilde{B}_{n,x}} Es_K^{(T)}(y) * P_y^{(T)} \left(H_{\partial \hat{B}_{n,x}} < \infty \right) \right]$$

$$\leq c * u * (L_{n+1})^d * \left(1 - \frac{1}{T+1} \right)^{L_{n+1}} \leq c_1 u^{1.5} 10^{nd} e^{-c_2 L_{n+1}}. \tag{5.12}$$

Recalling that $l_0 = 10$ and (5.12), when u is sufficiently large, we have: for any $n \geq 0$,

$$P^{u,T} (F_{n,x}) \geq \exp(-c_1 u^{1.5} 10^{nd} e^{-c_2 L_{n+1}}) \geq 1 - e^{-c_3 * L_0 * 2^n}. \tag{5.13}$$

Suppose that $\mathcal{T} \cap \mathcal{I}_{n-1} = \{(n-1,y_1), (n-1,y_2)\}$, where $y_1, y_2 \in \mathbb{L}_{n-1}$. Since $L_0 > 10$, $\hat{B}_{n-1,y_1} \cap \hat{B}_{n-1,y_2} = \emptyset$. We denote that $\mathcal{T}_i = \{(m,y) \in \mathcal{T} : y \in \widetilde{B}_{n-1,y_i}\}$, $i \in \{1,2\}$, and that

$$A_{\mathcal{T}_i} = \bigcap_{(0,y) \in \mathcal{T}_i \cap \mathcal{I}_0} A_{0,y} \; and \; \hat{A}_{\mathcal{T}_i} = \bigcap_{(0,y) \in \mathcal{T}_i \cap \mathcal{I}_0} \left\{ B_{0,y} \xleftrightarrow{\{\eta \in \mathcal{FI}^{u,T} : \eta(0) \in \hat{B}_{n-1,y_i}\}} \partial \widetilde{B}_{0,y} \right\}. \tag{5.14}$$

Note that $\hat{A}_{\mathcal{T}_1}$ and $\hat{A}_{\mathcal{T}_2}$ are independent and for $i \in \{1,2\}$, $\hat{A}_{\mathcal{T}_i} \subset A_{\mathcal{T}_i}$. In addition, if F_{n-1,y_1} and F_{n-1,y_2} both happen, the events $\hat{A}_{\mathcal{T}}$ and $\hat{A}_{\mathcal{T}_1} \cap \hat{A}_{\mathcal{T}_2}$ will then be equivalent.

By induction and (5.13), and similar to (5.26) of [2], we have

$$\begin{aligned} P^{u,T} (A_{\mathcal{T}}) + 2e^{-c_3 * L_0 * 2^n} &\leq P^{u,T} \left(A_{\mathcal{T}} \cap F_{n-1,y_1} \cap F_{n-1,y_2} \right) + 4e^{-c_3 * L_0 * 2^n} \\ &\leq P^{u,T} \left(\hat{A}_{\mathcal{T}_1} \cap \hat{A}_{\mathcal{T}_2} \right) + 4e^{-c_3 * L_0 * 2^n} \\ &\leq \left(P^{u,T} (A_{\mathcal{T}_1}) + 2e^{-c_3 * L_0 * 2^{n-1}} \right) * \left(P^{u,T} (A_{\mathcal{T}_2}) + 2e^{-c_3 * L_0 * 2^{n-1}} \right) \\ &\leq ... \leq \left(P^{u,T} (A_{0,0}) + 2e^{-c_3 * L_0} \right)^{2^n}. \end{aligned} \tag{5.15}$$

Combining (5.2), (5.8), (5.11) and (5.15),

$$P^{u,T} (A_{n,0}) \leq \left(c_0 * l_0^{2(d-1)} \right)^{2^n} * \left(c' * u^{-0.5} + (2L_0 + 1)^d * e^{-cL_0} + 2e^{-c_3 * L_0} \right)^{2^n}. \tag{5.16}$$

Since $L_0 = \lfloor u^{\frac{1}{2d}} \rfloor$ and $l_0 = 10$, the sufficiently large u will also guarantee that

$$c_0 * l_0^{2(d-1)} * \left(c' * u^{-0.5} + (2L_0 + 1)^d * e^{-cL_0} + 2e^{-c_3 * L_0} \right) < 0.5. \tag{5.17}$$

Therefore, as $n \to \infty$, $P^{u,T}(A_{n,0})$ converges to 0 and so does $P^{u,T}\left(0 \xleftrightarrow{\mathcal{FI}^{u,T}} \partial \widetilde{B}_{n,0} \right)$ since $P^{u,T}\left(0 \xleftrightarrow{\mathcal{FI}^{u,T}} \partial \widetilde{B}_{n,0} \right) \leq P^{u,T}(A_{n,0})$. So $\mathcal{FI}^{u,T}$ fails to percolate.

In conclusion,

$$\frac{-\log(1 - p_d^c)}{2} \leq \liminf_{u \to \infty} u * T_c^- \leq \limsup_{u \to \infty} u * T_c^+ \leq \frac{-\log(1 - p_d^c)}{2}. \tag{5.18}$$

By (5.18) and Corollary 3.2, we conclude the proof of Theorem 3.3 $\qquad\square$

5.2 Proof of Theorem 3.5

Before we give the proof of Theorem 3.5, we need an estimate on the diameter of the range of geometrically killed random walks.

Lemma 5.1. *Let $\{X_n^{(T)}\}$ be a geometrically killed random walk with law $P_0^{(T)}$. Then there exist $c_1, c_2 > 0$ such that for all $L > 0$,*

$$P_0^{(T)}\left[\max_{0 \leq i \leq \infty} |X_i^{(T)}| \geq L \right] \leq c_1 e^{-c_2 * T^{-\frac{1}{3}} * L^{\frac{2}{3}}}. \tag{5.19}$$

Proof. By Theorem 1.5.1 of [6], for any positive integer m, we have

$$\begin{aligned}
P_0^{(T)}\left[\max_{0 \leq i \leq \infty} |X_i^{(T)}| \geq L \right] &\leq (1 - \frac{1}{T+1})^m + P_0\left[\max_{0 \leq i \leq m} |X_i| \geq L \right] \\
&\leq \exp(-c * T^{-1} m) + c' * \exp(-\frac{L}{\sqrt{m}}).
\end{aligned} \tag{5.20}$$

Taking $m = \lfloor (TL)^{\frac{2}{3}} \rfloor$ in (5.20) finishes the proof. $\qquad\square$

By Theorem 3.1, for any fixed $u > 0$, $\mathcal{FI}^{u,T}$ is stochastically increasing on $T \in (0, 1]$. Thus, it's sufficient to show that for $d \geq 3$ and $\epsilon > 0$, there exist $\bar{c}(d, \epsilon) > 0$ and $U''(d, \epsilon) > 0$ such that for any $u < U''$ and $T \geq 1$ satisfying

$$u \leq \bar{c}(T * (\log(T+1))^{3+\epsilon})^{-(d-1)}, \tag{5.21}$$

$\mathcal{FI}^{u,T}$ does not percolate. In fact, the result proved here is sightly stronger than the statement of Theorem 3.5 and the RHS of (5.21) can be replaced by a polynomial of T to make the proof a little shorter.

We use the same approach as in the proof of Theorem 3.3 but with different choices of L_0 and l_0. To be precise, we set $L_0 = 10$, $l_0 = \lfloor (c'T * (\log(T+1))^{3+\epsilon})^{0.5} \rfloor$, where c' is to be determined later. Note that the number of paths intersecting $B_{0,0}$ in $\mathcal{FI}^{u,T}$ is a Poisson random variable with parameter $u * cap^{(T)}(B_{0,0})$. Therefore,

$$\begin{aligned}
P^{u,T}\left(0 \xleftrightarrow{\mathcal{FI}^{u,T}} \partial B_{0,0} \right) &\leq P^{u,T}(\text{there exists at least one path intersecting } B_{0,0}) \\
&\leq 1 - \exp(-u * cap^{(T)}(B_{0,0})) \leq c'' * u.
\end{aligned} \tag{5.22}$$

For any $k \in \mathbb{N}^+$, $0 \leq r < L_{n+1}$ and $L = k * L_{n+1} + r$, by Lemma 5.1, we have

$$P_0^{(T)}\left(\max_{0 \leq i \leq \infty} |X_i^{(T)}| \geq L \right) \leq c_1 \exp(-c_2'(d) k^{\frac{2}{3}} (c')^{\frac{1}{3}(n+1)} T^{\frac{1}{3}n} (\log(T+1))^{\frac{1}{3}(3+\epsilon)(n+1)}). \tag{5.23}$$

Recall the notation $F_{n,x}$ in the proof of Theorem 3.3. By (5.23), when the constant $c'(d, \epsilon)$ is sufficiently large, for $u > 0$ and $T \geq 1$ satisfying (5.21),

$$
\begin{aligned}
P^{u,T}\left((F_{n,x})^c\right) &\leq \sum_{y \in (\hat{B}_{n,0})^c} P^{u,T}\left(there\ is\ no\ path\ starting\ from\ y\ and\ intersecting\ \widetilde{B}_{n,x}\right) \\
&\leq \sum_{L \geq L_{n+1}} c * L^{d-1} * \left[1 - \exp\left(-\frac{2du}{T+1} * P_0^{(T)}\left(\max_{0 \leq i \leq \infty} |X_i^{(T)}| \geq L)\right)\right)\right] \\
&\leq \sum_{k=1}^{\infty} c\left[(k+1)L_{n+1}\right]^{d-1} * \frac{2du}{T+1} * L_{n+1} \\
&\quad * \exp\left(-c_2' k^{\frac{2}{3}} (c')^{\frac{1}{3}(n+1)} T^{\frac{1}{3}n} \left(\log(T+1)\right)^{\frac{1}{3}(3+\epsilon)(n+1)}\right) \\
&\leq c * u * (L_{n+1})^d * \exp\left(-c_2''(d) * (c')^{n+1} * T^{\frac{1}{3}n} * \left(\log(T+1)\right)^{\frac{1}{3}(3+\epsilon)(n+1)}\right) \\
&\leq \exp\left(-c_2'''(d) * c' * \left(\log(T+1)\right)^{\frac{1}{3}(3+\epsilon)} * 2^n\right).
\end{aligned}
\tag{5.24}
$$

Similar to (5.16), by (5.22) and (5.24), we have

$$
P^{u,T}\left(0 \xleftrightarrow{\mathcal{FI}^{u,T}} \partial\widetilde{B}_{n,0}\right) \leq \left[c_0 l_0^{2(d-1)} * \left(2\exp(-c_2''' * c' * \left(\log(T+1)\right)^{\frac{1}{3}(3+\epsilon)}) + c'' * u\right)\right]^{2^n}.
\tag{5.25}
$$

Taking $\bar{c} = (4c_0 c'')^{-1}$ and $c'(d, \epsilon) > 0$ large enough such that for any $T \geq 1$,

$$
c_0 l_0^{2(d-1)} * 2\exp(-c_2''' * c' * \left(\log(T+1)\right)^{\frac{1}{3}(3+\epsilon)}) < 0.25.
\tag{5.26}
$$

Therefore, when u is sufficiently small and $T \geq 1$ satisfies (5.21),

$$
\lim_{n \to \infty} P^{u,T}\left(0 \xleftrightarrow{\mathcal{FI}^{u,T}} \partial\widetilde{B}_{n,0}\right) = 0
$$

and thus $\mathcal{FI}^{u,T}$ does not percolate. $\qquad\square$

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