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An exit measure construction of the total local time of super-Brownian motion

Jieliang Hong*

Abstract

We use a renormalization of the total mass of the exit measure from the complement of a small ball centered at $x \in \mathbb{R}^d$ for $d \le 3$ to give a new construction of the total local time L^x of super-Brownian motion at x.

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1 Introduction and main results

The local time of super-Brownian motion (SBM) has been well studied by many authors, e.g., Adler and Lewin [1], Barlow, Evans and Perkins [2], Krone [9], Sugitani [14], etc. It may be formally defined as the density function of the occupation measure of super-Brownian motion. Let $M_F = M_F(\mathbb{R}^d)$ be the space of finite measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ equipped with the topology of weak convergence of measures. A super-Brownian motion $X = (X_t, t \ge 0)$ starting at $\mu \in M_F$ is a continuous M_F-valued strong Markov process defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $X_0 = \mu$ a.s. Write $\mu(\phi) = \int \phi(x)\mu(dx)$ for any measure μ. It is well known that super-Brownian motion is the solution to the following *martingale problem* (see [13], II.5): For any $\phi \in C_b^2(\mathbb{R}^d)$,

$$X_t(\phi) = X_0(\phi) + M_t(\phi) + \int_0^t X_s(\frac{\Delta}{2}\phi)ds, \tag{1.1}$$

where $(M_t(\phi))_{t\ge 0}$ is a continuous $\mathcal{F}_t$-martingale such that $M_0(\phi) = 0$ and the quadratic variation of $M(\phi)$ is

$$[M(\phi)]_t = \int_0^t X_s(\phi^2)ds.$$

Here $C_b^2(\mathbb{R}^d)$ is the space of bounded functions which are twice continuously differentiable. The above martingale problem uniquely characterizes the law $\mathbb{P}_{X_0}$ of super-Brownian motion X, starting from $X_0 \in M_F$, on $C([0,\infty), M_F)$, the space of continuous functions from $[0,\infty)$ to M_F furnished with the compact-open topology.

*University of British Columbia, Canada. E-mail: jlhong@math.ubc.ca

For any $0 \le t \le \infty$, the occupation measure of super-Brownian motion X up to time t is the random measure defined by

$$I_t(A) = \int_0^t X_s(A) ds. \tag{1.2}$$

In dimensions $d \le 3$, the occupation measure I_t has a density, L_t^x, called the local time of X, which satisfies

$$I_t(f) = \int_0^t X_s(f)\, ds = \int_{\mathbb{R}^d} f(x) L_t^x \, dx \text{ for all continuous } f \text{ with compact support.} \tag{1.3}$$

Moreover, Theorems 2 and 3 of Sugitani [14] imply that $(t,x) \mapsto L_t^x$ is continuous on $[0,\infty) \times S(X_0)^c$, where $S(\mu) = \text{Supp}(\mu)$ denotes the closed support of a measure μ. The extinction time of X is a.s. finite (see, e.g., Chp II.5 in [13]) and so we set $L^x = L_\infty^x$ to be the (total) local time of X. We define the range, $\mathcal{R}$, of X to be $\mathcal{R} = \text{Supp}(I_\infty)$.

Now consider SBM under the canonical measure $\mathbb{N}_{x_0}$, which is a σ-finite measure on $C([0,\infty), M_F)$. If $\Xi = \sum_{i\in I} \delta_{\nu^i}$ is a Poisson point process on $C([0,\infty), M_F)$ with intensity $\mathbb{N}_{X_0}(d\nu) = \int \mathbb{N}_x(d\nu) X_0(dx)$, then

$$X_t = \sum_{i\in I} \nu_t^i = \int \nu_t \; \Xi(d\nu),\ t > 0, \tag{1.4}$$

has the law, $\mathbb{P}_{X_0}$, of a super-Brownian motion X starting from X_0. We refer the readers to Theorem II.7.3(c) of [13] for more details. The global continuity of the total local time L^x under $\mathbb{N}_{x_0}$ is given in [6] (see, e.g., Theorem 1.2 of the same reference). By (1.4) we may decompose the total local time L^x under $\mathbb{P}_{X_0}$ as

$$L^x = \sum_{i\in I} L^x(\nu^i) = \int L^x(\nu) \Xi(d\nu). \tag{1.5}$$

Intuitively the total local time L^x measures the amount of mass distributed by super-Brownian motion on the singleton x. This mechanism is pretty similar to the exit measure from the complement of a small ball centered at x. To define the exit measure in an appropriate way, we first recall Le Gall's Brownian snake.

Let $\mathcal{W} = \cup_{s\ge 0} C([0,s], \mathbb{R}^d)$ be equipped with the natural metric (see, e.g., Chp. IV.1 of Le Gall [11]). For any $w \in \mathcal{W}$, we write $\zeta(w) = s$ if $w \in C([0,s], \mathbb{R}^d)$. We call $\zeta(w)$ the lifetime of w. The Brownian snake $W = (W_t, t \ge 0)$ is a $\mathcal{W}$-valued continuous strong Markov process. Let $\zeta_t = \zeta(W_t)$ and use $\hat{W}(t) = W_t(\zeta_t)$ to denote the tip of the snake at time t. Recall the canonical measure $\mathbb{N}_x$ of super-Brownian motion from above. By slightly abusing the notation, we let $\mathbb{N}_x$ denote the excursion measure of the snake, on $C([0,\infty), \mathcal{W})$, starting from the trivial path at $x \in \mathbb{R}^d$ with zero lifetime. Then we may use the Brownian snake W to construct a measure-valued process $X(W) = (X_t(W), t \ge 0)$ under $\mathbb{N}_x$ such that the law of $X(W)$ under $\mathbb{N}_x$ is equal to that of a super-Brownian motion under the canonical measure $\mathbb{N}_x$, thus justifying our abusive notation. We use $X_t(W)$ to denote the super-Brownian motion associated with the snake W instead of the integral with respect to X_t. This should be clear if one recalls that W is not a function on $\mathbb{R}^d$ but the snake. The construction of the super-Brownian motion $X(W)$ by the snake W is not important for our discussion here, and so we refer the interested readers to Theorem IV.4 of [11] for more information. If $\Xi = \sum_{j\in J} \delta_{W_j}$ is a Poisson point process on $\mathcal{W}$ with intensity $\mathbb{N}_{X_0}(dW) = \int \mathbb{N}_x(dW) X_0(dx)$, then it follows from (1.4) that

$$X_t = \sum_{j\in J} X_t(W_j) = \int X_t(W) \Xi(dW) \text{ for } t > 0 \tag{1.6}$$

has the law, $\mathbb{P}_{X_0}$, of a super-Brownian motion X starting from X_0. It also follows from (1.5) that the total local time L^x under $\mathbb{P}_{X_0}$ may be decomposed as

$$L^x = \sum_{j \in J} L^x(W_j) = \int L^x(W) \Xi(dW). \tag{1.7}$$

Now we turn to the exit measure. The exit measure from an open set G, under $\mathbb{P}_{X_0}$ or $\mathbb{N}_{X_0}$, is a random finite measure supported on ∂G and is denoted by X_G (see Chp. V of [11] for the construction of the exit measure). Intuitively X_G represents the mass started at X_0 which is stopped at the instant it leaves G. We note [11] also suffices as a reference for the properties of X_G described below. Let $B(x_0, \varepsilon) = B_\varepsilon(x_0) = \{x : |x - x_0| < \varepsilon\}$ denote an open ball centered at $x_0 \in \mathbb{R}^d$ with radius $\varepsilon > 0$. Define the complement of a closed ball centered at x_0 with radius $\varepsilon > 0$ to be

$$G_\varepsilon^{x_0} = G_\varepsilon(x_0) = \{x : |x - x_0| > \varepsilon\} \text{ and let } G_\varepsilon = G_\varepsilon(0). \tag{1.8}$$

For any K_1, K_2 non-empty, set

$$d(K_1, K_2) = \inf\{|x - y| : x \in K_1, y \in K_2\}.$$

We assume that $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ satisfy $d(\overline{B_\varepsilon(x_0)}, S(X_0)) > 0$. In what follows we will only be considering exit measures X_G for $G = G_\varepsilon^{x_0}$ with $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ as above. Under $\mathbb{N}_x$ we have the range $\mathcal{R}$ of super-Brownian motion $X = X(W)$, defined by $\mathcal{R} = S(I_\infty)$ with I_∞ as in (1.2), may also be written as (see, e.g., equation (8) in the proof of Theorem IV.7(iii) of [11])

$$\mathcal{R} = \{\hat{W}(s) : s \in [0, \sigma]\}, \tag{1.9}$$

where $\sigma = \sigma(W) = \inf\{t > 0 : \zeta_t = 0\} > 0$ is the length of the excursion path. For any $x \in G$, under $\mathbb{N}_x$ we may use the definition of exit measure in Chp. V of [11] to get (see also (2.3) of [8])

$$X_G \text{ is a finite random measure supported on } \partial G \cap \mathcal{R} \text{ a.e.} \tag{1.10}$$

The extension of (1.10) to $\mathbb{N}_{X_0}$ is immediate as $\mathbb{N}_{X_0}(dW) = \int \mathbb{N}_x(dW) X_0(dx)$. It also works under $\mathbb{P}_{X_0}$ as we may, equivalently, set (see, e.g., (2.23) of [12])

$$X_G = \sum_{j \in J} X_G(W_j) = \int X_G(W) \Xi(dW), \tag{1.11}$$

where Ξ is a Poisson point process on $\mathcal{W}$ with intensity $\mathbb{N}_{X_0}$.

Let $d(x, K) = \inf\{|x - y| : y \in K\}$. It has been shown in Proposition 6.2(b) of [8] that for any $x \in S(X_0)^c$, under $\mathbb{N}_{X_0}$ or $\mathbb{P}_{X_0}$, the family $\{X_{G^x_{r_0 - r}}(1), 0 \le r < r_0\}$ with $r_0 = d(x, S(X_0))/2$ has a càdlàg version which is a supermartingale if $d = 3$; a martingale if $d = 2$. Throughout the rest of the paper, we will always work with this càdlàg version. For any $\varepsilon > 0$, set

$$\psi_0(\varepsilon) = \begin{cases} \frac{1}{\pi} \log^+(1/\varepsilon), & \text{in } d = 2, \\ \frac{1}{2\pi} \frac{1}{\varepsilon}, & \text{in } d = 3. \end{cases} \tag{1.12}$$

The following result gives a new construction of the total local time L^x in terms of the local asymptotic behavior of the exit measures at x. This result is also useful in the construction of a boundary local time measure whose support is the topological boundary of the range of super-Brownian motion in $d = 2$ and $d = 3$ (see [7]).

Notation. For a collection of random variables $\{\xi_t, t \in T\}$, we say ξ_t converges in measure to ξ_{t_0} under $\mathbb{N}_{X_0}$ as $t \to t_0$ if for any $\eta > 0$, $\mathbb{N}_{X_0}(|\xi_t - \xi_{t_0}| > \eta) \to 0$ as $t \to t_0$. The same definition applies under $\mathbb{P}_{X_0}$.

Theorem 1.1. *Let $d = 2$ or $d = 3$ and $X_0 \in M_F(\mathbb{R}^d)$. For any $x \in S(X_0)^c$, we have*

$$X_{G^x_\varepsilon}(1)\psi_0(\varepsilon) \text{ converges in measure to } L^x \text{ under } \mathbb{N}_{X_0} \text{ or } \mathbb{P}_{X_0} \text{ as } \varepsilon \downarrow 0, \tag{1.13}$$

where ψ_0 is as in (1.12)*. Moreover, in $d = 3$ the convergence holds $\mathbb{N}_{X_0}$-a.e. or $\mathbb{P}_{X_0}$-a.s.*

Remark 1.2. In $d = 3$, the family $\mathcal{A} := \{X_{G^x_{r_0-r}}(1)\psi_0(r_0 - r), 0 \le r < r_0\}$ with $r_0 = d(x, S(X_0))/2$ is indeed a martingale (see the proof of the above theorem in Section 3). This allows us to use martingale convergence to conclude a.s. convergence in $d = 3$. In $d = 2$, we already know from Proposition 6.2(b) of [8] that the family $\{X_{G^x_{r_0-r}}(1), 0 \le r < r_0\}$ is a martingale, and so one can check that $\mathcal{A}$ will be a submartingale in $d = 2$. Whether or not a.s. convergence holds in $d = 2$ remains unresolved.

2 The special Markov property

We will state the special Markov property for the Brownian snake from [10] that plays an essential role in our proof. We first deal with $\mathbb{N}_{X_0}$. Recall that we are working with exit measures X_G for $G = G^{x_0}_\varepsilon$ with $x_0 \in \mathbb{R}^d$ and $\varepsilon > 0$ satisfying $d(\overline{B_\varepsilon(x_0)}, S(X_0)) > 0$. Define

$$\begin{aligned} S_G(W_u) &= \inf\{t \le \zeta_u : W_u(t) \notin G\} \ \ (\inf \emptyset = \infty), \\ \eta^G_s(W) &= \inf\{t : \int_0^t 1(\zeta_u \le S_G(W_u))\, du > s\}, \\ \mathcal{E}_G &= \sigma(W_{\eta^G_s}, s \ge 0) \vee \{\mathbb{N}_{X_0} - \text{null sets}\}, \end{aligned} \tag{2.1}$$

where $s \to W_{\eta^G_s}$ is continuous (see p. 401 of [10]). Intuitively one may think of $\mathcal{E}_G$ as the σ-field generated by the excursions of W inside G. Write the open set $\{u : S_G(W_u) < \zeta_u\}$ as countable union of disjoint open intervals, $\cup_{i\in I}(a_i, b_i)$. Then for all $u \in [a_i, b_i]$, one notices $S_G(W_u) = S^i_G < \infty$ where $S^i_G = S_G(W_{a_i}) > 0$, and we may define

$$W^i_s(t) = W_{(a_i+s)\wedge b_i}(S^i_G + t) \text{ for } 0 \le t \le \zeta_{(a_i+s)\wedge b_i} - S^i_G.$$

In this way, we have W^i are the excursions of W outside G for each $i \in I$. Proposition 2.3 of [10] implies that X_G is $\mathcal{E}_G$-measurable and Corollary 2.8 of the same reference gives the following *special Markov property*:

$$\begin{cases} \text{Conditional on } \mathcal{E}_G, \text{ the point measure } \sum_{i\in I} \delta_{W^i} \text{ is a Poisson} \\ \text{point measure with intensity } \mathbb{N}_{X_G}. \end{cases} \tag{2.2}$$

Here $\mathbb{N}_{X_G}(dW) = \int \mathbb{N}_x(dW)X_G(dx)$ is a (random) intensity measure on the space of the snake, i.e. $C([0,\infty), \mathcal{W})$. Consider $G = G^x_{\varepsilon_1}$ and $D = G^x_{\varepsilon_2}$ with $\varepsilon_1 > \varepsilon_2 > 0$. We can define the exit measure $X_D(W^i)$ for each W^i following the construction of exit measure in Chapter V.1 of [11]. As in (2.6) of [8], one may conclude

$$X_D = \sum_{i\in I} X_D(W^i). \tag{2.3}$$

If U is an open subset of $S(X_0)^c$, then L_U, the restriction of the total local time L^x to U, is in $C(U, \mathbb{R})$ which is the set of continuous functions on U. Here are some consequences of (2.2) that are already proved in Proposition 2.2(a) of [8].

Proposition 2.1. *For any $X_0 \in M_F(\mathbb{R}^d)$, fix some $x \in S(X_0)^c$. Define $G_1 = G^x_{\varepsilon_1}$ and $G_2 = G^x_{\varepsilon_2}$ with $0 < \varepsilon_2 < \varepsilon_1 < d(x, S(X_0))$.*
(i) If $\psi_1 : C(\overline{G_1}^c, \mathbb{R}) \to [0,\infty)$ is Borel measurable, then

$$\mathbb{N}_{X_0}(\psi_1(L_{\overline{G_1}^c})|\mathcal{E}_{G_1}) = \mathbb{E}_{X_{G_1}}(\psi_1(L_{\overline{G_1}^c})).$$

(ii) If $\psi_2 : M_F(\mathbb{R}^d) \to [0,\infty)$ is Borel measurable, then

$$\mathbb{N}_{X_0}(\psi_2(X_{G_2})|\mathcal{E}_{G_1}) = \mathbb{E}_{X_{G_1}}(\psi_2(X_{G_2})).$$

The σ-finiteness of $\mathbb{N}_{X_0}$ is not an issue here as we may define the above conditional expectation by, e.g., using Radon-Nikodym derivative.

We will need a version of the above under $\mathbb{P}_{X_0}$ as well, which follows immediately from Proposition 2.3 of [8].

Proposition 2.2. *For any $X_0 \in M_F(\mathbb{R}^d)$, fix some $x \in S(X_0)^c$. Define $G_1 = G^x_{\varepsilon_1}$ and $G_2 = G^x_{\varepsilon_2}$ with $0 < \varepsilon_2 < \varepsilon_1 < d(x, S(X_0))$.*
(i) If $\phi_1 : C(\overline{G_1}^c, \mathbb{R}) \to [0,\infty)$ is Borel measurable, then

$$\mathbb{E}_{X_0}(\phi_1(L_{\overline{G_1}^c})) = \mathbb{E}_{X_0}\Big(\mathbb{E}_{X_{G_1}}(\phi_1(L_{\overline{G_1}^c}))\Big).$$

(ii) If $\phi_2 : M_F(\mathbb{R}^d) \to [0,\infty)$ is Borel measurable, then

$$\mathbb{E}_{X_0}(\phi_2(X_{G_2})) = \mathbb{E}_{X_0}\Big(\mathbb{E}_{X_{G_1}}(\phi_2(X_{G_2}))\Big).$$

3 Construction of the total local time by exit measure

In this section we will give the proof of Theorem 1.1. We assume throughout this section that $d = 2$ or $d = 3$. The Laplace transform of L^x derived in Lemma 2.2 of [12] is given by

$$\mathbb{E}_{X_0}\Big(\exp(-\lambda L^x)\Big) = \exp\left(-\int_{\mathbb{R}^d} V^\lambda(x-y) X_0(dy)\right), \tag{3.1}$$

where V^λ is the unique solution to

$$\frac{\Delta V^\lambda}{2} = \frac{(V^\lambda)^2}{2} - \lambda\delta_0, \quad V^\lambda > 0 \text{ on } \mathbb{R}^d. \tag{3.2}$$

Here δ_0 is the Dirac delta function and the above differential equation is interpreted in a distributional sense. One can check that V^λ is radially symmetric and we may write $V^\lambda(|x|)$ for $V^\lambda(x)$. Recall ψ_0 from (1.12). It is known that (see, e.g., p. 187 of [4]) V^λ is smooth in $\mathbb{R}^d\backslash\{0\}$, and near the origin, Lemma 8 of [3] gives that

$$\lim_{x\to 0} \frac{V^\lambda(x)}{\psi_0(|x|)} = \lambda. \tag{3.3}$$

Proof of Theorem 1.1. The outline for the proof is as follows: First we get some L^2 convergence, associated with $X_{G^x_\varepsilon}$ and L^x, using the Laplace transforms. Then we show that this implies the convergence in measure. When $d = 3$, we prove there is an a.s. limit by the martingale arguments. It is then immediate that L^x, as the limit of convergence in measure, is in fact the a.s. limit, thus completing the proof.

We first consider the $\mathbb{N}_{X_0}$ case. Fix any $x \in S(X_0)^c$ and let $\delta := d(x, S(X_0)) > 0$. For any $\lambda > 0$ and $0 < \varepsilon < \delta/2$, we have

$$\begin{aligned} I :=&\mathbb{N}_{X_0}\Big(\Big(\exp(-\lambda X_{G^x_\varepsilon}(1)\psi_0(\varepsilon)) - \exp(-\lambda L^x)\Big)^2\Big)\\ =&\mathbb{N}_{X_0}\Big(\exp(-2\lambda X_{G^x_\varepsilon}(1)\psi_0(\varepsilon)) + \exp(-2\lambda L^x) - 2\exp(-\lambda X_{G^x_\varepsilon}(1)\psi_0(\varepsilon))\exp(-\lambda L^x)\Big)\\ =&\mathbb{N}_{X_0}\Big(\exp(-2\lambda X_{G^x_\varepsilon}(1)\psi_0(\varepsilon)) + \mathbb{E}_{X_{G^x_\varepsilon}}\Big(\exp(-2\lambda L^x)\Big)\\ &\qquad - 2\exp(-\lambda X_{G^x_\varepsilon}(1)\psi_0(\varepsilon))\mathbb{E}_{X_{G^x_\varepsilon}}\Big(\exp(-\lambda L^x)\Big)\Big), \end{aligned} \tag{3.4}$$

where we have used Proposition 2.1 (i) in the last equality. Apply (3.1) with $X_0 = X_{G_\varepsilon^x}$ to get

$$\begin{aligned}\mathbb{E}_{X_{G_\varepsilon^x}}\Big(\exp(-\lambda L^x)\Big) &= \exp\Big(-\int_{\mathbb{R}^d} V^\lambda(x-y)X_{G_\varepsilon^x}(dy)\Big)\\ &= \exp\Big(-\int_{\mathbb{R}^d} V^\lambda(\varepsilon)X_{G_\varepsilon^x}(dy)\Big) = \exp(-X_{G_\varepsilon^x}(1)V^\lambda(\varepsilon)).\end{aligned} \tag{3.5}$$

In the second equality we have used the fact that the exit measure $X_{G_\varepsilon^x}$ is supported on ∂G_ε^x by (1.10) and then apply the radial symmetry of V^λ to get $V^\lambda(x-y) = V^\lambda(|x-y|) = V^\lambda(\varepsilon)$ for any $y \in \partial G_\varepsilon^x$. The above still holds true if we replace λ with 2λ in (3.5). Use the above in (3.4) to arrive at

$$\begin{aligned} I =&\mathbb{N}_{X_0}\Big(\exp(-2\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)) + \exp(-X_{G_\varepsilon^x}(1)V^{2\lambda}(\varepsilon))\\ &\qquad - 2\exp(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon))\exp(-X_{G_\varepsilon^x}(1)V^\lambda(\varepsilon))\Big) \qquad (3.6)\\ =&\mathbb{N}_{X_0}\Big(\exp(-2\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)) - \exp(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon))\exp(-X_{G_\varepsilon^x}(1)V^\lambda(\varepsilon))\Big)\\ &+ \mathbb{N}_{X_0}\Big(\exp(-X_{G_\varepsilon^x}(1)V^{2\lambda}(\varepsilon)) - \exp(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon))\exp(-X_{G_\varepsilon^x}(1)V^\lambda(\varepsilon))\Big)\\ :=&I_1 + I_2.\end{aligned}$$

We first deal with I_1.

$$\begin{aligned}|I_1| \le&\mathbb{N}_{X_0}\Big(\Big|\exp\Big(-2\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big) - \exp\Big(-(\lambda + \frac{V^\lambda(\varepsilon)}{\psi_0(\varepsilon)})X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big|\Big)\\ =&\mathbb{N}_{X_0}\Big(\Big|X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\lambda'(\varepsilon)X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big(2\lambda - (\lambda + \frac{V^\lambda(\varepsilon)}{\psi_0(\varepsilon)})\Big)\Big|\Big)\\ \le&\Big|\lambda - \frac{V^\lambda(\varepsilon)}{\psi_0(\varepsilon)}\Big| \cdot \mathbb{N}_{X_0}\Big(X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\lambda'(\varepsilon)X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big),\end{aligned} \tag{3.7}$$

where the second line is by the mean value theorem with $\lambda'(\varepsilon)(\omega)$ chosen between 2λ and $\lambda + V^\lambda(\varepsilon)/\psi_0(\varepsilon)$. When $\varepsilon > 0$ is small, (3.3) implies $V^\lambda(\varepsilon)/\psi_0(\varepsilon) > \lambda/2$, and so $\mathbb{N}_{X_0}$-a.e. we have $\lambda'(\varepsilon) \ge \min\{2\lambda, \lambda + V^\lambda(\varepsilon)/\psi_0(\varepsilon)\} > 3\lambda/2 > \lambda$. Hence (3.7) becomes

$$|I_1| \le \Big|\lambda - \frac{V^\lambda(\varepsilon)}{\psi_0(\varepsilon)}\Big| \cdot \mathbb{N}_{X_0}\Big(X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big). \tag{3.8}$$

Recall $\delta = d(x, S(X_0))$. Define $S(X_0)^{>\delta/4} = \{y : d(y, S(X_0)) > \delta/4\}$ so that for any $0 < \varepsilon < \delta/2$, we have $\partial G_\varepsilon^x \subset S(X_0)^{>\delta/4}$. Recall $\mathcal{R}$ from (1.9). Apply (1.10) to see for all $0 < \varepsilon < \delta/2$, we have

$$\mathcal{R} \cap S(X_0)^{>\delta/4} = \emptyset \text{ implies } X_{G_\varepsilon^x}(1) = 0, \quad \mathbb{N}_{X_0}\text{-a.e.} \tag{3.9}$$

Use the above to get

$$\begin{aligned}&\mathbb{N}_{X_0}\Big(X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big)\\ =&\mathbb{N}_{X_0}\Big(X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)1(\mathcal{R} \cap S(X_0)^{>\delta/4} \neq \emptyset)\Big)\\ \le&\lambda^{-1}e^{-1}\mathbb{N}_{X_0}(\mathcal{R} \cap S(X_0)^{>\delta/4} \neq \emptyset) := \lambda^{-1}e^{-1}C(X_0, \delta) < \infty,\end{aligned} \tag{3.10}$$

where the first inequality is by $xe^{-\lambda x} \le \lambda^{-1}e^{-1}, \forall x \ge 0$. The finiteness of $C(X_0, \delta)$ follows from Proposition VI.2 of [11]. Hence (3.8) becomes

$$|I_1| \le \Big|\lambda - \frac{V^\lambda(\varepsilon)}{\psi_0(\varepsilon)}\Big| \cdot \lambda^{-1}e^{-1}C(X_0, \delta) \to 0 \text{ as } \varepsilon \downarrow 0, \tag{3.11}$$

where the convergence to 0 follows from (3.3).

Turning to I_2, we have

$$\begin{aligned}|I_2| \le & \mathbb{N}_{X_0}\Big(\Big|\exp\Big(-\frac{V^{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)-\exp\Big(-(\lambda+\frac{V^{\lambda}(\varepsilon)}{\psi_0(\varepsilon)})X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big|\Big)\\ = & \mathbb{N}_{X_0}\Big(\Big|X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\hat\lambda(\varepsilon)X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big(\frac{V^{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}-(\lambda+\frac{V^{\lambda}(\varepsilon)}{\psi_0(\varepsilon)})\Big)\Big|\Big)\\ \le & \Big|\frac{V^{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}-\lambda-\frac{V^{\lambda}(\varepsilon)}{\psi_0(\varepsilon)}\Big|\cdot\mathbb{N}_{X_0}\Big(X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\hat\lambda(\varepsilon)X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big),\end{aligned} \tag{3.12}$$

where in the second line we have used the mean value theorem with $\hat\lambda(\varepsilon)(\omega)$ chosen between $V^{2\lambda}(\varepsilon)/\psi_0(\varepsilon)$ and $\lambda + V^{\lambda}(\varepsilon)/\psi_0(\varepsilon)$. When $\varepsilon > 0$ is small, (3.3) implies $V^{2\lambda}(\varepsilon)/\psi_0(\varepsilon) > 3\lambda/2$ and $V^{\lambda}(\varepsilon)/\psi_0(\varepsilon) > \lambda/2$. So $\mathbb{N}_{X_0}$-a.e. we have

$$\hat\lambda(\varepsilon) \ge \min\Big\{\frac{V^{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}, \lambda+\frac{V^{\lambda}(\varepsilon)}{\psi_0(\varepsilon)}\Big\} > \frac{3\lambda}{2} > \lambda. \tag{3.13}$$

Use the above to see that (3.12) becomes

$$|I_2| \le \Big|\frac{V^{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}-\lambda-\frac{V^{\lambda}(\varepsilon)}{\psi_0(\varepsilon)}\Big|\cdot\mathbb{N}_{X_0}\Big(X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\exp\Big(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)\Big)\Big).$$

Apply (3.10) to see that

$$\begin{aligned}|I_2| \le & \Big|\frac{V^{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}-\lambda-\frac{V^{\lambda}(\varepsilon)}{\psi_0(\varepsilon)}\Big|\cdot\lambda^{-1}e^{-1}C(X_0,\delta)\\ \le & \Big(\Big|\frac{V^{2\lambda}(\varepsilon)}{\psi_0(\varepsilon)}-2\lambda\Big|+\Big|\lambda-\frac{V^{\lambda}(\varepsilon)}{\psi_0(\varepsilon)}\Big|\Big)\cdot\lambda^{-1}e^{-1}C(X_0,\delta)\to 0 \text{ as } \varepsilon\downarrow 0,\end{aligned} \tag{3.14}$$

where the convergence to 0 follows from (3.3).

Recall I from (3.4). We may conclude from (3.11) and (3.14) that $I \to 0$ as $\varepsilon \downarrow 0$, thus giving the L^2 convergence of $\exp(-\lambda X_{G_\varepsilon^x}(1)\psi_0(\varepsilon))$ to $\exp(-\lambda L^x)$ under $\mathbb{N}_{X_0}$. By Corollary 2.32 of Folland [5], for any sequence $\varepsilon_n \downarrow 0$, we may pick a subsequence $\varepsilon_{n_k} \downarrow 0$ so that

$$\lim_{\varepsilon_{n_k}\downarrow 0}\exp(-\lambda X_{G^x_{\varepsilon_{n_k}}}(1)\psi_0(\varepsilon_{n_k})) = \exp(-\lambda L^x), \quad \mathbb{N}_{X_0}\text{-a.e.} \tag{3.15}$$

We note the arguments in Folland [5] remain valid for our setting with the L^2 convergence under the σ-finite measure $\mathbb{N}_{X_0}$. It is immediate from (3.15) that

$$\lim_{\varepsilon_{n_k}\downarrow 0} X_{G^x_{\varepsilon_{n_k}}}(1)\psi_0(\varepsilon_{n_k}) = L^x, \quad \mathbb{N}_{X_0}\text{-a.e.} \tag{3.16}$$

At this stage, we may not conclude the convergence in measure due to the σ-finiteness of $\mathbb{N}_{X_0}$. This issue could be solved by noticing that the event $\{X_{G_\varepsilon^x} \neq 0 \text{ or } L^x \neq 0\}$ has only finite measure under $\mathbb{N}_{X_0}$. By using Proposition 2.1 (i), we get for any $0<\varepsilon<\delta/2$,

$$\begin{aligned}\mathbb{N}_{X_0}(\{L^x>0\}\cap\{X_{G_\varepsilon^x}(1)=0\}) &= \mathbb{N}_{X_0}(1_{\{X_{G_\varepsilon^x}(1)=0\}}\mathbb{N}_{X_0}(1_{\{L^x>0\}}|\mathcal{E}_{G_\varepsilon^x}))\\ &= \mathbb{N}_{X_0}(1_{\{X_{G_\varepsilon^x}(1)=0\}}\mathbb{E}_{X_{G_\varepsilon^x}}(L^x>0)) = 0,\end{aligned}$$

thus giving $\mathbb{N}_{X_0}$-a.e. $X_{G_\varepsilon^x}(1) = 0$ implies $L^x = 0$. Together with (3.9), we get for any $0 < \varepsilon < \delta/2$,

$$\mathcal{R}\cap S(X_0)^{>\delta/4} = \emptyset \text{ implies } L^x = 0 \text{ and } X_{G_\varepsilon^x}(1) = 0, \quad \mathbb{N}_{X_0}\text{-a.e.} \tag{3.17}$$

Therefore it follows that for any $\eta > 0$,

$$\begin{aligned}&\mathbb{N}_{X_0}\Big(|X_{G_\varepsilon^x}(1)\psi_0(\varepsilon) - L^x| > \eta\Big)\\&= \mathbb{N}_{X_0}\Big(\{|X_{G_\varepsilon^x}(1)\psi_0(\varepsilon) - L^x| > \eta\} \cap \{\mathcal{R} \cap S(X_0)^{>\delta/4} \neq \emptyset\}\Big),\end{aligned} \tag{3.18}$$

and so we may work with the finite measure $\mathbb{N}_{X_0}(\cdot \cap \{\mathcal{R} \cap S(X_0)^{>\delta/4} \neq \emptyset\})$ when considering the convergence in measure under $\mathbb{N}_{X_0}$. Apply Dominated Convergence Theorem with (3.16) and (3.18) to get

$$\lim_{\varepsilon_{n_k}\downarrow 0} \mathbb{N}_{X_0}\Big(|X_{G^x_{\varepsilon_{n_k}}}(1)\psi_0(\varepsilon_{n_k}) - L^x| > \eta\Big) = 0. \tag{3.19}$$

Hence for any sequence $\varepsilon_n \downarrow 0$, there is a subsequence $\varepsilon_{n_k} \downarrow 0$ such that (3.19) holds, thus completing the proof of convergence in measure under $\mathbb{N}_{X_0}$. For the $\mathbb{P}_{X_0}$ case, the above arguments work in a similar and even easier way, and so we omit the details.

Now we turn to the a.s. convergence in $d = 3$. For any $x \in S(X_0)^c$, set $r_0 = \delta/2$ where $\delta = d(x, S(X_0)) > 0$. In $d = 3$, by (6.10) of [8], for any $0 < \varepsilon < r_0$ we have

$$\mathbb{E}_{X_0}(X_{G_\varepsilon^x}(1)) = \mathbb{N}_{X_0}(X_{G_\varepsilon^x}(1)) = \int \frac{\varepsilon}{|x - x_0|} dX_0(x_0). \tag{3.20}$$

Hence for $0 < \varepsilon_2 < \varepsilon_1 < r_0$, we may apply Proposition 2.1(ii) to get

$$\mathbb{N}_{X_0}\Big(\frac{X_{G^x_{\varepsilon_2}}(1)}{\varepsilon_2}\Big|\mathcal{E}_{G^x_{\varepsilon_1}}\Big) = \mathbb{E}_{X_{G^x_{\varepsilon_1}}}\Big(\frac{X_{G^x_{\varepsilon_2}}(1)}{\varepsilon_2}\Big) = \frac{X_{G^x_{\varepsilon_1}}(1)}{\varepsilon_1}, \tag{3.21}$$

where the last equality follows by applying (3.20) with $X_0 = X_{G^x_{\varepsilon_1}}$ and by using the fact that the exit measure $X_{G^x_{\varepsilon_1}}$ is supported on $\partial G^x_{\varepsilon_1}$ by (1.10). Recall that in $d = 3$ we have $\psi_0(\varepsilon) = 1/(2\pi\varepsilon)$. Use (3.21) to conclude

$$\mathbb{N}_{X_0}\Big(X_{G^x_{\varepsilon_2}}(1)\psi_0(\varepsilon_2)\Big|\mathcal{E}_{G^x_{\varepsilon_1}}\Big) = X_{G^x_{\varepsilon_1}}(1)\psi_0(\varepsilon_1), \tag{3.22}$$

which implies $\{X_{G^x_{r_0-r}}(1)\psi_0(r_0 - r), 0 \leq r < r_0\}$ is a nonnegative martingale. Note that we always work with the càdlàg version of $X_{G^x_{r_0-r}}(1)$ on $0 \leq r < r_0$. Now we may apply the martingale convergence theorem to get $\mathbb{N}_{X_0}$-a.e. $\lim_{r\to r_0} X_{G^x_{r_0-r}}(1)\psi_0(r_0 - r)$ exists. Since we already have $X_{G_\varepsilon^x}(1)\psi_0(\varepsilon)$ converges to L^x in measure under $\mathbb{N}_{X_0}$ (see also (3.16)), we conclude that $\mathbb{N}_{X_0}$-a.e. $\lim_{\varepsilon\downarrow 0} X_{G_\varepsilon^x}(1)\psi_0(\varepsilon) = L^x$. The case for $\mathbb{P}_{X_0}$ follows in a similar way. ■

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